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Explanatoriness as a value in mathematics and mathematics teaching

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In this foremost philosophical paper, I propose a rather holistic idea of (re)establishing mathematical proof in school mathematics by explicating and enacting the particular merits of mathematical proof with respect to a variety of mathematical activities, e.g.: observation, systematic variation of example cases, informal argumentation, defining, hypothesising, inductive and abductive reasoning. Such activities are connected naturally to proving in practices of mathematics, with “practice” referring both to mathematics as a human endeavor in general, and to mathematics a (scientific) discipline. To this end, a concept of “mathematical value” from the philosophy of mathematical practice is employed. I focus exemplarily on the value of explanatoriness, which is both a core merit of good proof in scientific mathematical practice, and an important didactical goal of teaching proof.

Keywords: explanation, mathematical value, proof, philosophy, mathematical practice.

Introduction

From the perspectives of mathematics education research and curriculum development, it is widely agreed to date that mathematical proving shall be encouraged and fostered in school mathematics throughout all content areas and grade levels. At the same time, at least in Germany, standards of rigor and formalization in classroom proving changed rapidly in comparison to, e.g., the traditional Euclidean paradigm of proof in elementary geometry, or to algebraic proving in the $\varepsilon - \delta$ - calculus, and developed into content-oriented, depictive, or intuitive ways of argumentation. In spite of the efforts to make mathematical proving an integrated core element throughout the curriculum, the demand for more proving in German mathematics classrooms frequently produces teachers' complaints: the curriculum is too dense, time is too short, and after all, proof is not that relevant for the final exams; pupils don't like proof, they are afraid they cannot handle it and fall into blockade, and after all, proof is not that relevant for students' future lives. This phenomenon can, probably and partly, be explained as an enduring symptom of a perceived “false dichotomy” (Sierpinska, 2005) of mathematical proving on the one hand, and other kinds of mathematical activities on the other. Such a dichotomy may have been over-emphasized by both the traditional university part of mathematics teacher education (at least in Germany), and by the former, strong curricular focus on proofs in the context of elementary geometry, with a sustainable effect on teaching attitudes and beliefs that cannot be simply suppressed by declarations of curricular intent (compare also Knuth, 2002) for similar aspects of teachers' proof conceptions and attitudes towards teaching proof in the U.S.). Hence, there appears to be an ongoing need for clarification of a reasonable interpretation of the call for more proving in mathematics classrooms. This need was already expressed by Stylianides (2007), together with the proposal of defining a framing core concept of mathematical proof to be explicated and enacted in classroom (pp. 291f.), which can be coherently specified both with regard to different loops of the curricular spiral, and to different fields of mathematical content, and which allows for a clear distinction from inductive, “empirical“ (p. 298) arguments. In

recent years, the awareness of this need has also led to a deep discussion in mathematics education about diverging and common aspects of mathematical proof and argumentation (Mariotti et al. 2018). A substantial part of this debate is devoted to logical or cognitive aspects, including the role of deductive vs. other forms of inferential reasoning, of argumentation base, of language, symbolism, or other semiotic elements (ibid.), e.g., the analysis of structural and referential aspects of argumentations and proofs that provide cognitive unity in conjecturing-and-proving learning settings (Fiallo & Gutierrez 2017), or the examination of the coordination of different registers in the transition from argumentation to proving (Duval, 2007; with a different reading of “epistemic value” than I will use in the following).

Focus and rationale of this paper

The approach I want to propose here is, in a way, a complementary one. One main difference is that it does not aim at a closed definition of what mathematical proof is, nor at an emphasis of a clear-cut distinction of proving vs. non-deductive forms of mathematical activities, e.g., inductive or abductive argumentation based on variation of example cases. A second main difference is the idea that an investigation of mathematical proving and related activities that sticks to the logical and cognitive aspects, might not suffice to grasp all epistemologically relevant facets, which may partially conform to approaches employing Habermas’ model of rational behavior (reported in Mariotti et al. 2018). The term “false dichotomy” cited above refers to Michael Otte’s view of *complementary* facets of proving in mathematics (Otte, 1990, p. 59ff.), and emphasizes intuitive, social, and metaphorical facets of proof which cannot be properly modeled with regard to the logic and cognitive functioning of proof or argumentation. Several much more recent studies from different fields suggest that, in a similar vein, issues of epistemic values essentially connected to mathematical proving, like explanatoriness, transcend the scope of logical and cognitive matters, due to a discursive character, to non-reducible pragmatic and affective aspects, and also to the metaphysics of mathematical proving; compare, e.g., (Müller-Hill, 2011) on the epistemic role(s) of formalizability as a feature of discursive proving actions, (Novaes, 2018) on explanatoriness within a dialogical conceptualization of mathematical proof, (Andersen, 2018) on a socio-empirically informed view on acceptable gaps in proofs, (Johansen & Misfeldt, 2016) on argumentation and problem choice as mathematical values, or (Dawkins & Weber, 2017) on the integral role of values like “increasing mathematical understanding” for the “apprenticeship of students into proving practice” (p. 123). Thus, a focus on explanatoriness as an *epistemic* value does not mean to exclude other perspectives than the epistemological one, but is inclusive to, e.g., sociological, communicative, and pragmatic perspectives. The following sections aim at *indicating first steps into the direction* of a program which consists of (1) explicating the particular epistemic merits of the idea of proof that become manifest in the activity of proving in mathematical practice, (2) understanding these coherently in view of the whole process of hypothesizing, generating and communicating proof, (3) emphasizing aspects and features of what counts as proof in mathematical practice that play a particular role in achieving these merits, and (4) indicating elements of a variety of mathematical activities, particularly the generation, operation with, and variation of examples, that already correspond to these aspects and features of proof. In the next two sections, a number of useful concepts, conceptions and results have to be introduced in quite curtate descriptions. First, I

provide conceptual considerations as well as exemplary socio-empirical evidence for an understanding of explanatoriness as a core epistemic value of *scientific* mathematical practice in the sense described above. This goes hand in hand with the stance that the epistemic nature of the scientific discipline of mathematics, understood as a human practice and endeavor which generates, communicates and imparts mathematical knowledge and skills, is relevant to the teaching and learning of mathematics for principal reasons. I then sketch a philosophically informed didactical model of mathematical explanation, developed in (Müller-Hill, 2017), and a model of the interplay of deduction, induction, and abduction in explanation processes. I shortly review the socio-empirical insights on the background of these models, and finish with a glimpse on future work in enacting explanatoriness as a value in school mathematics.

Explanatoriness as an epistemic value in scientific mathematical practice

I employ a concept of *value* here that emerged in recent discussions within philosophy of mathematics (see Larvor, 2016). I follow Ernest's understanding of value as something that is "expressed through the action of valuing", and "manifested in both the prizing of certain characteristics and in the making of fundamental choices" (Ernest, 2016, p. 190), including besides ontic, aesthetic, and ethical values also *epistemic* values (p. 191). Ernest discusses the epistemic values of truth, provability, universalism, objectivism, and rationalism (pp. 193ff.). *Explanation* is a well-known and frequently discussed *function* of mathematical proof, both from the point of view of philosophy and of mathematics education. Frequently proposed features of explanatory proofs are, e.g., surveyability (Müller-Hill, 2013), reference to characteristic features of the involved mathematical entities and structures (Steiner, 1978), or width (Hanna, 2014). These are more or less intrinsic features of mathematical proofs as objects. Understanding explanatoriness as a fundamental *value*, which manifests itself particularly in activities related to mathematical proving, somehow changes the perspective: Explanatoriness is conceptualized rather dialectically by looking for *traces of valuing actions* (like prizing and choice-making) directed at proving in mathematical practice that may *indicate* this value. The issue of (epistemic) *values* in mathematics has, with exceptions like (Lorenzen, 1974, pp. 152ff.), rather been neglected by philosophy of mathematics until recently (Ernest 2016, p. 191) due to the methodological paradigm of classical analytical philosophy, which also dominated philosophy of mathematics for quite a long time. Accordingly, philosophy can contribute to an epistemology of mathematics by conceptually grasping epistemic features of proof through semantic analysis of, e.g., knowledge attributions, resulting in truth-conditions. Concerning the question of epistemic *values* in actual mathematical practice, a different approach is needed that can take concrete valuing actions like prizing and choice-making of practicing mathematicians into account. Valuing actions directed to mathematical proving may include the way in which examples are chosen and exploited during a proving process, the prizing of alternative hypotheses, the choice of definitions and methods, the degree of formalization and abstraction, the representations used, or the fine-grainedness of the argumentation. Philosophy of mathematical practice, a transdisciplinary endeavor at the intersection of philosophy and sociology of mathematics, has developed conceptual and methodological frameworks that allow appropriate investigation of such issues.

Results from a socio-empirically informed epistemological study

In this paragraph, I will review exemplary empirical results from an interview study (Müller-Hill, 2011) conducted with mathematicians of high standing, from various fields of professional specialization areas, under the lenses of explanatoriness as an epistemic value in mathematical practice (for further methodological details and results of this study, see also (Müller-Hill, 2013)). In general, the interviewees thoroughly point to a strong relation between explanatory proofs and understanding why the proven theorem holds (ibid.; particularly aspects (2)-(4)). This is in line with the close relationship of explanation and understanding which is widely agreed upon in philosophical as well as in didactical discussions (see, e.g., the still prominent, well-differentiated discussion in (Sierpinska, 1994), or the very recent (Dawkins & Weber 2017)). Furthermore, selected quotes from (Müller-Hill, 2011) display that prizing and choice-making within processes of generating, communicating and assessing proofs in mathematical practice happens with respect to the *robustness* of hypotheses and arguments *under local and global attacks*, e.g., when “try[ing] to shoot holes in it” and “ask questions why it is true, why it works this way, not that way” (Interview 4, p. 246). This appears to be particularly relevant in review processes, as “the right way of understanding the contribution that [a] new proof brings” is to ask “what is really the new idea” that “makes something work that previously did not work”, and to find “the key argument that made the proof work”, or instead, “where it breaks”. If the proof is “non breaking” under attacks, one “would see the spark of [a] new idea that finally makes it work” (Int. 5, p. 246). It also includes *variation of perspective or context*, to look at a hypothesis or an argument “from different directions, through different angles” (Int. 4, p. 246). Also, “in the process of understanding it happens that you have a definition, and that you think you understand it, and that you come back to it for one reason, or somebody tells you something about it, and that you realize that you never understood it at all” (Int. 6, p. 246). In prizing and choice-making actions, the *balance between establishing a “big picture” and “the details”* is also valued. The “big picture” conveys “the real understanding of why it is true” (Int. 5, S. 199), a “global idea” of “why it works like that”, an “overview of the proof” (Int. 4, S. 179), which also affects the consistency with already established results. On the other hand, there are “the details, to follow the proof from step to step, the logical consequences” (Int. 4, S. 179), and “the process of writing down these details”, which are sometimes not valued as equally important to the global picture because “not all people are good in this“ though they “really know mathematical truths” (Int. 5, pp. 171, 199). A well-balanced proof, in this sense, also makes it easier to “be able to look at it at a later time, and still be able to reproduce the arguments” (Int. 4, p. 179), which is taken as an indicator for understanding (see also (Hanna, 2014)). The different aspects have a *context-dependent impact* on concrete valuing actions: “So one thing: maths is in a sense something personal. [...] When do I think I understand something?” (Int. 4, p. 246), which can be influenced by “deep personal involvement with [a] proof” (Int. 2, p. 313).

Explanatoriness within a didactical model of nomic mathematical explanation

Insights gained from the interview study appear to be in line with the concept of so-called *nomic mathematical explanation* (Müller-Hill, 2017). In this view, an explanation provides a general pattern, expressed by a general conditional sentence, from which a phenomenon can be inferred as a regular consequence under certain conditions. Moreover, an explanatory, nomic pattern has to fulfill certain conditions on form (ibid., p. 181) and content.

Conditions on form: Basic structure of an explanatory pattern

Whenever manifestation condition(s) $\mathcal{M} = \{M_1, M_2, \dots, M_n\}$ would be fulfilled, the event E would take place (as a manifestation of certain properties of the involved objects) **and** whenever manifestation condition(s) $\mathcal{M} = \{M_1, M_2, \dots, M_n\}$ would not be fulfilled (but condition(s) \mathcal{M}' instead), the event E would not take place (but the event E').

The second part of the pattern is usually not formulated explicitly. An example pattern discussed in (ibid., p. 202) was reconstructed from a student’s explanation of why the sum of three consecutive natural numbers is divisible by three: *Whenever we would take three consecutive numbers a, b, c , and move 1 from the biggest number c to a , we would get three times b .*

Conditions on content: Basic invariance criteria for an explanatory pattern

Some philosophical contributions on the *concept* of *mathematical* explanation (see Hanna, 2018) provide ontic conditions, like reference to “a characterizing property” or “salient feature“ of the involved mathematical entities, for an argument to qualify as explanatory. Such conditions serve to explicate the conceptual meaning of „explanation“. The account of nomic mathematical explanation developed in (Müller-Hill, 2017) complements the ontic conditions by spelling out *epistemic* invariance conditions as *search keys, test and quality conditions* for explanatory patterns. It is to a good extent compatible with ideas of explanatory unification and causal explanation that are at the heart of well-established views of scientific explanation (Kitcher 1989). Table 1 reports and illustrates conditions that are relevant to grasp the aspects of explanatoriness resulting from the

Functional invariance		Invariance under intervention at the manifestation conditions: pattern gives correct information, if \mathcal{M} is fulfilled or not (e.g., regarding the example pattern above: <i>What if I intervene at M_i by taking non-consecutive numbers a, b, c.</i>
Scope invariance	<i>Object invariance</i>	Pattern also gives correct information when the involved objects are varied (e.g., by varying the chosen triplet (a, b, c)).
	<i>Contextual & representational invariance</i>	Pattern also gives correct information when the used contexts and/or employed representations are varied (e.g., by changing from symbolic to figurative representations of a, b, c).
	<i>Embedding invariance</i>	Pattern is consistent with corresponding patterns for an embedded or expanded pool of related phenomena (e.g., with the transition from natural numbers a, b, c to integers).

interview study.

Table 1: Invariance criteria for explanatory patterns (Müller-Hill 2017)

Functional invariance is *necessary* for an explanatory pattern. However, there can be explanatory patterns with different degrees of functional invariance, which affects the pattern’s *informativity*: the more differentiated the intervention relation, the more informative the pattern. Scope invariance has different specificities (object, contextual/representational, embedding), each of which can also

occur with different degrees. The degrees of scope invariance affect a pattern's *systematization power*.

The process of explanation generation

The *process* of generating a nomic explanation can be described as an iterative cycle of phases of abduction, deduction and induction (Figure 1), starting with *abducting* a potential explanatory pattern.

The abduction may be creative, that is, on the basis of experience gained by working with and varying test cases, the subject identifies a (by her) formerly unknown pattern to explain the phenomenon (e.g., *by exploring* $1 + 2 + 3, 2 + 3 + 4, \dots$). But it may also be rather selective, that is, the subject already knows one or several alternative patterns that are explanatory for a class of somehow related, familiar phenomena, and identifies (*by partial analogy, e.g., to familiar opposite-change task formats on addition*) one of these that could potentially also explain the phenomenon in question (Magnani, 2001).

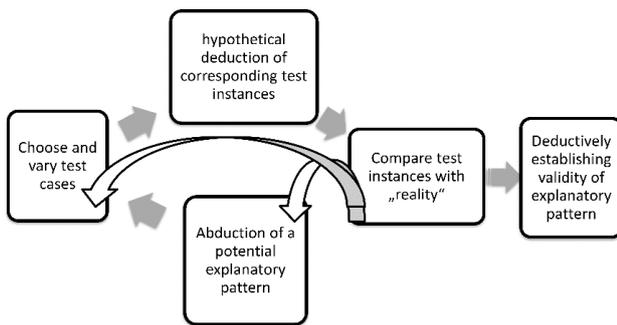


Figure 1: Interplay of abduction, induction and deduction

The cycle continues with choosing and varying test cases for the pattern (e.g., for “Whenever we would take three consecutive natural numbers a, b, c , and take 1 from the biggest number c and pass it to a , we would get three times b ”). Via *hypothetical deduction*, instances of the proposed pattern are produced for each test case (e.g., $(1 + 1) + 2 + (3 - 1)$ should be equal to $3 \cdot 2, \dots$). These test instances are then compared with “reality”, thereby *inductively backing* or weakening the pattern (e.g., $(1 + 1) + 2 + (3 - 1) = 2 + 2 + 2 = 3 \cdot 2, \dots$). If the pattern appears to be sufficiently backed up by inductive evidence, one can exploit the experiences already made to collect ideas for a *deductive justification* of the pattern’s general validity (e.g., *by replacing concrete test values for a, b , and c by generic figurative or general symbolic representations*), or enter the cycle again with a variation of the former test cases to find a more general or more informative pattern (e.g., *by trying out equidistant instead of consecutive number tripels*). If not, one may iterate the whole cycle.

Reviewing the results from the interview study

We can now distinguish between explanatoriness in a situated sense, and explanatoriness in a systematic sense (see Table 2), which are complementary.

	Abduction	Quality of invariance	Informativity/systematization
<i>Explanatoriness in a situated sense</i>	creative	Emphasis on functional invariance, detailed and detailed differentiated manifestation conditions	More informative, less systematizing power

<i>Explanatoriness in a systematic sense</i>	selective	Emphasis on scope invariance, generalized manifestation conditions	Less informative, more systematizing power
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Table 2: Situated and systematic sense of explanatoriness

Employing this distinction, the interview study results reported above suggest that prizing and decision making within mathematical activities related to proving in scientific mathematical practice happens with regard to explanatoriness *both* in the situated and in the systematic sense, with context dependent weighting. This includes, particularly, the balancing of informativity and systematization power, as two complementing aspects of explanatoriness.

Outlook and future work: Enacting explanatoriness in class

Regarding educational issues, the framework presented in this paper is meant both as a model and tool for researchers to reconstruct, analyze and understand explanation and valuing processes in mathematics and mathematics teaching, and as a basis for task design, and for initiating such processes in classroom. In this sense, the program formulated at the beginning of this paper entails at least two core messages: under an epistemic reading, „mathematical proof” in any strong sense is an ideal construct that is paradigmatic regarding the virtues and validity claims of the discipline of mathematics (that is one reason why we should still teach about it). However, various kinds of actual mathematical activities directed to providing proofs can and should already be driven and regulated with respect to these virtues and validity claims (that is why they work effectively in mathematical practice, and that is how we should teach them). The paper provides a rough impression of how these programmatic aims can be pursued, regarding explanatoriness as an epistemic value connected to proving. Though tersely presented, the reported results already indicate some opportunities and types of activities related to mathematical proving that can be appropriate to enact explanatoriness as a value in class. As a simple example, think of tasks like the following that could be tackled when introducing the concept of mean value (in German Gymnasium, e.g., in grade six): *Starting with a given data set of natural numbers, think of ways to vary the set in order to increase, lower, or leave invariant its mean value. Justify your answers.* Pilot runs with corresponding learning environments in the form of written tasks produced data that we actually evaluate to revise the learning environment design. However, it needs well-trained teachers to exploit such opportunities properly (compare, e.g., Lesseig, 2016a; 2016b). The next step in our work is to gain and analyse video-data from implementations of the revised learning environments. The results will also be used for developing a video-vignette based teacher training for secondary in-service teachers.

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