Asymptotic properties of integrals of quotients, when the numerator oscillates and denominator degenerate

Sergei Kuksin

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Dedicated to V.A. Marchenko on the occasion of his 95-th birthday.

We study asymptotical expansion as $\nu \to 0$ for integrals over $\mathbb{R}^{2d} = \{(x,y)\}$ of quotients of the form $F(x,y)\cos(\lambda x\cdot y)/\big((x\cdot y)^2+\nu^2\big)$, where $\lambda \ge 0$ and F decays at infinity sufficiently fast. Integrals of this kind appear in the theory of wave turbulence.

Key words: asymptotic of integrals, oscillating integrals, four-waves interaction.

Mathematical Subject Classification 2010: 34E05, 34E10

1. Introduction

In the paper [1] we study asymptotical, as $\nu \to 0$, behaviour of integrals

$$I_\nu = \int_{\mathbb{R}^d\times\mathbb{R}^d} dx\,dy\ \frac{F(x,y)}{(x\cdot y)^2+(\nu\Gamma(x,y))^2}\,, \quad d\ge 2\,,\ 0<\nu\le 1,$$

where F and Γ are C^2-function, Γ is positive and the two satisfy certain conditions at infinity. In particular, if $\Gamma\equiv 1$, then

$$|\partial_z^\alpha F(z)| \le C'\langle z\rangle^{-N-|\alpha|} \quad \forall z=(x,y)\in\mathbb{R}^{2d},\ \forall\,|\alpha|\le 2\,, \tag{1.1}$$

where $C'>0$ and $N>2d-2$. Denote by

$$\Sigma\subset\mathbb{R}^{2d}=\mathbb{R}^d_x\times\mathbb{R}^d_y \tag{1.2}$$

the quadric $\{(x,y): x\cdot y=0\}$, and by Σ_* its regular part $\Sigma\setminus\{(0,0)\}$. It is proved in [1] (see [2] for related results) that

$$I_\nu = \pi\nu^{-1}\int_{\Sigma_*}\frac{F(z)}{|z|\,\Gamma(z)}\,d_{\Sigma_*}z + O(\chi_d(\nu)), \tag{1.3}$$

where

$$\chi_d(\nu)=\begin{cases}1, & d\ge 3\,,\\ \max\big(\ln(\nu^{-1}),1\big), & d=2\,,\end{cases}$$

d_{Σ_*} is the volume element on Σ_*, induced from the standard Riemann structure in $\mathbb{R}^{2d}$, and the integral in (1.3) converges absolutely. Integrals of this form appear in the study of the four-waves interaction. The wave turbulence (WT) limit in systems with the four-waves interaction leads to oscillating versions of the integrals above with constant functions Γ. Re-denoting $\nu\Gamma$ back to ν we write the integrals in question as

$$J_\nu = \int_{\mathbb{R}^{2d}} dz\ \frac{F(z)\cos(\lambda x\cdot y)}{(x\cdot y)^2+\nu^2}\,,\quad d\ge 2\,,\ \ \lambda\ge 0\,,\ 0<\nu\le 1 \tag{1.4}$$

(as before, $z=(x,y)$). We assume that F is a C^2–function, satisfying (1.1).

The aim of this work is to prove the following result, describing the asymptotical behaviour of J_ν when $\nu\to 0$, uniformly in $\lambda\ge 0$:

Theorem 1.1. *Let $0<\nu\le 1$ and $\lambda\ge 0$. Then the integral J_ν and the integral*

$$J_0 = \pi e^{-\nu\lambda}\int_{\Sigma_*} F(z)|z|^{-1}\,d_{\Sigma_*}z$$

converge absolutely and

$$\big|J_\nu-\nu^{-1}J_0\big|\le C\chi_d(\nu), \tag{1.5}$$

where C depends on d and the constants C', N in (1.1), but not on ν and λ.

Note that since C does not depend on λ, then relation (1.5) remains valid for integrals (1.4), where $\lambda=\lambda(\nu)$ is any function of ν. Concerning the imposed restriction $d\ge 2$ see item iv) in Section 5.

If $\lambda=0$, the integral J_ν becomes a special case of I_ν (with $\Gamma=1$), and (1.5) follows from (1.3). Since $\sin^2(\frac{\lambda}{2}x\cdot y)=\frac12(1-\cos(\lambda x\cdot y))$, then combining (1.3) and (1.5) we get

Corollary 1.2. *As $\nu\to 0$,*

$$\begin{aligned}\int_{\mathbb{R}^d\times\mathbb{R}^d} dx\,dy\ \frac{F(x,y)\sin^2(\frac{\lambda}{2}x\cdot y)}{(x\cdot y)^2+\nu^2}\\ =\frac{\pi}{2}\,\nu^{-1}(1-e^{-\nu\lambda})\int_{\Sigma_*}\frac{F(z)}{|z|}\,d_{\Sigma_*}z+O(\chi_d(\nu))\,,\end{aligned} \tag{1.6}$$

uniformly in $\lambda\ge 0$.

Classically the WT considers singular versions of the integral in the l.h.s. of (1.6):

$$\int dx\,dy\ \frac{F(x,y)\sin^2(\frac{\lambda}{2}x\cdot y)}{(x\cdot y)^2}\,. \tag{1.7}$$

The theory deals with these integrals by performing certain formal calculations, see Section 6 of [3] (e.g. note there eq. (6.39)+(6.41)). Assertion (1.6) may be regarded as a regularisation of the integral (1.7). The factor $|z|^{-1}$ which it introduces in the limiting density is not present in the asymptotical description of integrals (1.7), used in the works on WT.

Theorem 1.1 is proved below in Sections 2–4, using the geometric approach of the paper [1], which also applies to various modifications of integrals I_ν and J_ν. Some of these applications are discussed in the last Section 5.

Notation. As usual, we denote $\langle z\rangle = \sqrt{|z|^2+1}$. For an integral $I = \int_{\mathbb{R}^{2d}} f(z)\,dz$ and a submanifold $M \subset \mathbb{R}^{2d}$, $\dim M = m \le 2d$, compact or not (if $m = 2d$, then M is an open domain in $\mathbb{R}^{2d}$) we write $\langle I, M\rangle = \int_M f(z)\,d_M(z)$, where $d_M(z)$ is the volume–element on M, induced from $\mathbb{R}^{2d}$. Similar $\langle |I|, M\rangle$ stands for the integral $\int_M |f(z)|\,d_M(z)$.

2. Geometry of the quadric $\{x\cdot y = 0\}$ and its vicinity.

2.1. The geometry of the quadric. The difficulty in studying the integral J_ν with small ν comes from the vicinity of the quadric $\Sigma = \{x\cdot y = 0\}$. To examine the integral's behaviour there we first analyse the geometry of the vicinity of the regular part of the quadric $\Sigma_* = \Sigma\setminus\{(0,0)\}$, following [1]. Example 5.1 at the end of the paper provides an elementary illustration to the objects, involved in this analysis.

The set Σ_* is a smooth submanifold of $\mathbb{R}^{2d}$ of dimension $2d-1$. We denote by ξ a local coordinate on Σ_* with the coordinate mapping $\xi \mapsto (x_\xi, y_\xi) = z_\xi \in \Sigma_*$, denote $|\xi| = |(x_\xi, y_\xi)|$ and denote $N(\xi) = (y_\xi, x_\xi)$. The latter is the normal to Σ_* at ξ of length $|\xi|$. For any $0 \le R_1 < R_2$ we set

$$S^{R_1} = \{z\in\mathbb{R}^{2d} : |z| = R_1\}\,, \quad \Sigma^{R_1} = \Sigma\cap S^{R_1}\,,$$
$$S^{R_2}_{R_1} = \{z : R_1 < |z| < R_2\}\,, \quad \Sigma^{R_2}_{R_1} = \Sigma\cap S^{R_2}_{R_1}\,,$$

and for $t>0$ denote by D_t the dilation operator

$$D_t : \mathbb{R}^{2d}\to\mathbb{R}^{2d}, \quad z\mapsto tz\,.$$

For $z=(x,y)$ we write $\omega(z) = x\cdot y$.

The following result is Lemma 3.1 from [1]:

Lemma 2.1. *1) There exists $\theta_0\in(0,1]$ such that a suitable neighbourhood $\Sigma^{nbh} = \Sigma^{nbh}(\theta_0)$ of Σ_* in $\mathbb{R}^{2d}\setminus\{0\}$, is invariant with respect to the dilations D_t, $t>0$, and may be uniquely parametrized as*

$$\Sigma^{nbh} = \{\pi(\xi,\theta) : \xi\in\Sigma_*,\ |\theta|<\theta_0\}\,,$$

where $\pi(\xi,\theta) = (x_\xi, y_\xi) + \theta N_\xi = (x_\xi, y_\xi) + \theta(y_\xi, x_\xi)$. In particular, $|\pi(\xi,\theta)|^2 = |\xi|^2(1+\theta^2)$.
2) If $\pi(\xi,\theta)\in\Sigma^{nbh}$, then

$$\omega\big(\pi(\xi,\theta)\big) = |\xi|^2\theta. \tag{2.1}$$

3) If $(x,y)\in S^R\setminus\Sigma^{nbh}$, then $|x\cdot y|\ge cR^2$ for some $c = c(\theta_0) > 0$.

For $0\le R_1 < R_2$ we will denote

$$(\Sigma^{nbh})^{R_2}_{R_1} = \pi(\Sigma^{R_2}_{R_1}\times\big(-\theta_0,\theta_0\big)).$$

Now we discuss the Riemann geometry of the domain $\Sigma^{nbh} = \Sigma^{nbh}(\theta_0)$, following [1].

The set Σ is a cone with the vertex in the origin, and $\Sigma_* = \{tz : t > 0, z \in \Sigma^1\}$. The set Σ^1 is a closed manifold of dimension $2d-2$. Let us cover it by a finite system of charts $\mathcal{N}_1, \dots, \mathcal{N}_{\tilde n}$, $\mathcal{N}_j = \{\eta^j = (\eta^j_1, \dots, \eta^j_{2d-2})\}$, and for any chart $\mathcal{N}_j$ denote by $m(d\eta^j)$ the volume element on Σ^1, induced from $\mathbb{R}^{2d}$. Below we write points in any chart $\mathcal{N}_j$ as η, and the volume element – as $m(d\eta)$.

The mapping

$$\Sigma^1 \times \mathbb{R}^+ \to \Sigma_*, \quad ((x_\eta, y_\eta), t) \to D_t(x_\eta, y_\eta)$$

is a diffeomorphism. Accordingly, we can cover Σ_* by the $\tilde n$ charts $\mathcal{N}_j \times \mathbb{R}_+$ with the coordinates $(\eta^j, t) =: (\eta, t)$. The coordinates (η, t, θ), where $\eta \in \mathcal{N}_j, t > 0$ and $|\theta| < \theta_0$, $1 \le j \le \tilde n$, make coordinate systems on $\Sigma^{nbh} = \Sigma^{nbh}(\theta_0)$. In the coordinates (η, t) the volume element on Σ_* is

$$d_{\Sigma_*} = t^{2d-2} m(d\eta)\, dt\,. \tag{2.2}$$

In the coordinates (η, t, θ) the volume elements in $\mathbb{R}^{2d}$ reeds

$$dx\, dy = t^{2d-1}\mu(\eta, \theta) m(d\eta) dt\, d\theta\,, \quad \text{where } \mu(\eta, 0) = 1 \tag{2.3}$$

(see [1]), a dilation map D_r, $r > 0$, reeds $D_r(\eta, t, \theta) = (\eta, rt, \theta)$, and by (2.1)

$$\omega(\eta, t, \theta) = t^2\theta\,. \tag{2.4}$$

Finally, since at a point $z = \pi(\xi, \theta) \in \Sigma^{nbh}$ we have $\frac{\partial}{\partial\theta} = \nabla_z \cdot (y_\xi, x_\xi)$, then in view of (1.1) for any (η, t, θ) and any $k \le 2$,

$$\big|\frac{\partial^k}{\partial\theta^k} F(\eta, t, \theta)\big| \le C\langle t\rangle^{-N}\,, \quad N > 2d-4. \tag{2.5}$$

2.2. The volume element d_{Σ_*} and the measure $|z|^{-1}d_{\Sigma_*}$. Theorem 1.1 and the result of [1] (see (1.3)) show that the manifold Σ_*, equipped with the measure $|z|^{-1}d_{\Sigma_*}$, is crucial to study asymptotic of integrals I_ν, J_ν and their similarities (cf. Section 6 of [1] and Section 5 below). The coordinates (η, t) and the presentation (2.2) for the volume element are sufficient for the purposes of this work. But the quadric Σ is reach in structures and admits more instrumental coordinate systems. In particular, if $d = 2$ we can introduce in the space $\mathbb{R}^2_x$ in (1.2) the polar coordinates (r, φ). Then for any fixed non-zero vector $x = (r, \varphi) \in \mathbb{R}^2_x$ the set $\{y \in \mathbb{R}^2_y : (x, y) \in \Sigma_*\}$ is the line in $\mathbb{R}^2_y$, perpendicular to x, and having the angle $\varphi + \pi/2$ with the horizontal axis. Parametrizing it by the length-coordinate l we get on Σ_* the coordinates $(r, l, \varphi) \in \mathbb{R}^+ \times \mathbb{R} \times S^1$, $S^1 = \mathbb{R}/2\pi\mathbb{Z}$, with the coordinate mapping

$$\Phi : (r, l, \varphi) \mapsto \big(x = (r\cos\varphi, r\sin\varphi), y = (-l\sin\varphi, l\cos\varphi)\big)$$

(this map is singular at $r = 0$). Since

$$|\partial\Phi/\partial r|^2 = 1,\ |\partial\Phi/\partial l|^2 = 1,\ |\partial\Phi/\partial\varphi|^2 = r^2 + l^2,$$
$$\langle\partial\Phi/\partial r, \partial\Phi/\partial l\rangle = \langle\partial\Phi/\partial r, \partial\Phi/\partial\varphi\rangle = \langle\partial\Phi/\partial l, \partial\Phi/\partial\varphi\rangle = 0,$$

then in these coordinates the volume element on Σ_* reeds as $\sqrt{r^2+l^2}\,drdld\varphi$, and the measure $|z|^{-1}d_{\Sigma_*}$ – as $dr\,dl\,d\varphi$. Consider the fibering

$$\Pi : \mathbb{R}^2_x \times \mathbb{R}^2_y \supset \Sigma_* \to \mathbb{R}^2_x, \quad (x, y) \mapsto x.$$

It has a singular fiber $\Pi^{-1}0 = \{0\} \times \mathbb{R}^2_y$, and for any non-zero x the fiber $\Pi^{-1}x$ equals $\{x\} \times x^\perp$, where $x^\perp$ is the line in $\mathbb{R}^2_y$, perpendicular to x. Since $dx = rdr\,d\varphi$, then the given above presentation for the measure $|z|^{-1}d_{\Sigma_*}$ implies that its restriction to the regular part Σ^+_* of the fibered manifold Σ_*, $\Sigma^+_* = \Sigma_* \setminus (\{0\} \times \mathbb{R}^2_y)$, disintegrates by the foliation Π as

$$(|z|^{-1}d_{\Sigma_*})\,|_{\Sigma^+_*} = |x|^{-1}dx\,d_{x^\perp}y, \quad x \neq 0,\ y \in x^\perp, \tag{2.6}$$

where $d_{x^\perp}$ is the length on the euclidean line $x^\perp \subset \mathbb{R}^2_y$.

We do not undertake the job of getting a right analogy of this result for the multidimensional case $d > 2$, but note that a straightforward modification of the construction above leads to the observation that for any $d \geq 2$ the measure $|z|^{-1}d_{\Sigma_*}$, restricted to Σ^+_*, disintegrates as

$$p_d(x, y)dx\,d_{x^\perp}y, \quad x \in \mathbb{R}^d \setminus \{0\},\ y \in x^\perp, \tag{2.7}$$

where $x^\perp$ is the orthogonal complement to x in R^d_y, , $d_{x^\perp}$ is the volume element on this euclidean space, and the function p_d satisfies the estimate $p_d \leq C(|x| + |y|)^{d-2}|x|^{1-d}$.

3. Integral over the vicinity of Σ

To study the behaviour of the integral over a neighbourhood of Σ we first prove that the integral, evaluated over the vicinity of the singular point $(0, 0)$ is small, and next study the integral over the vicinity of the regular part Σ_* of the quadric.

For $0 < \delta \leq 1$ denote

$$K_\delta = \{z = (x, y) : |x| \leq \delta, |y| \leq \delta\} \subset \mathbb{R}^d \times \mathbb{R}^d\,.$$

An upper bound for the integral over K_δ follows from Lemma 2.1 of [1]:

$$|\langle|J_\nu|, K_\delta\rangle| \leq \int_{K_\delta} \frac{|F(z)|\,dz}{(x \cdot y)^2 + \nu^2} \leq C\nu^{-1}\delta^{2d-2}\,. \tag{3.1}$$

Now we estimate the integral over the neighbourhood Σ^{nbh} of Σ_*. For this end, using (2.3), for $0 \leq A < B \leq \infty$ we disintegrate $\langle J_\nu, (\Sigma^{nbh})^B_A\rangle$ as

$$\begin{aligned}\langle J_\nu, (\Sigma^{nbh})^B_A\rangle &= \int_{\Sigma^1} m(d\eta) \int_A^B dt\,t^{2d-1} \int_{-\theta_0}^{\theta_0} d\theta\,\frac{F(\eta, t, \theta)\mu(\eta, \theta)\cos(\lambda x \cdot y)}{t^4\theta^2 + \nu^2} \\ &= \int_{\Sigma^1} m(d\eta) \int_A^B dt\,t^{2d-1}\Upsilon_\nu(\eta, t)\,,\end{aligned} \tag{3.2}$$

where

$$\Upsilon_\nu(\eta,t)=t^{-4}\int_{-\theta_0}^{\theta_0}\frac{F(\eta,t,\theta)\mu(\eta,\theta)\cos(\lambda t^2\theta)\,d\theta}{\theta^2+\varepsilon^2}\,,\quad \varepsilon=\nu t^{-2}\,.$$

To study Υ_ν we first consider the integral Υ^0_ν, obtained from Υ_ν by frozening $F\mu$ at $\theta=0$. Since $\mu(\eta,0)=1$, then

$$\Upsilon^0_\nu=2t^{-4}F(\eta,t,0)\int_0^{\theta_0}\frac{\cos(\lambda t^2\theta)\,d\,\theta}{\theta^2+\varepsilon^2}=2\nu^{-1}t^{-2}F(\eta,t,0)\int_0^{\theta_0/\varepsilon}\frac{\cos(\nu\lambda w)\,dw}{w^2+1}\,.$$

Consider the integral

$$2\int_0^{\theta_0/\varepsilon}\frac{\cos(\nu\lambda w)\,dw}{w^2+1}=2\int_0^{\infty}\frac{\cos(\nu\lambda w)\,dw}{w^2+1}-2\int_{\theta_0/\varepsilon}^{\infty}\frac{\cos(\nu\lambda w)\,dw}{w^2+1}=:I_1-I_2\,.$$

Since

$$2\int_0^{\infty}\frac{\cos(\xi w)\,dw}{w^2+1}=\int_{-\infty}^{\infty}\frac{e^{i\xi w}\,dw}{w^2+1}=\pi e^{-|\xi|},$$

then $I_1=\pi e^{-\nu\lambda}$. For I_2 we have an obvious bound $|I_2|\le 2\varepsilon/\theta_0=C_1\nu t^{-2}$. So

$$\Upsilon^0_\nu(\eta,t)=\pi\nu^{-1}t^{-2}F(\eta,t,0)(e^{-\nu\lambda}+\Delta_t)\,,\qquad |\Delta_t|\le C\nu t^{-2}\,. \tag{3.3}$$

Now we estimate the difference between Υ_ν and Υ^0_ν. Writing $(F\mu)(\eta,t,\theta)-(F\mu)(\eta,t,0)$ as $A(\eta,t)\theta+B(\eta,t,\theta)\theta^2$, where $|A|,|B|\le C\langle t\rangle^{-N}$ in view of (2.5), we have

$$\Upsilon_\nu-\Upsilon^0_\nu=t^{-4}\int_{-\theta_0}^{\theta_0}\frac{(A\theta+B\theta^2)\cos(\lambda t^2\theta)\,d\theta}{\theta^2+\varepsilon^2}\,.$$

Since the first integrand is odd in θ, then its integral vanishes, and

$$|\Upsilon_\nu-\Upsilon^0_\nu|\le C\langle t\rangle^{-N}t^{-4}\int_{-\theta_0}^{\theta_0}\frac{\theta^2\,d\theta}{\theta^2+\varepsilon^2}\le 2C\langle t\rangle^{-N}t^{-4}\theta_0\,.$$

So by (3.3)

$$\begin{aligned}&|\Upsilon_\nu(\eta,t)-\pi\nu^{-1}t^{-2}F(\eta,t,0)e^{-\nu\lambda}|\\&\qquad\le C\langle t\rangle^{-N}\big(t^{-4}+\nu^{-1}t^{-2}\,\nu t^{-2}\big)\le C'\langle t\rangle^{-N}t^{-4}\,.\end{aligned}\tag{3.4}$$

4. End of the proof of Theorem 1.1

1) In view of (3.2), (3.4) and since $N>2d-2$, for $\delta\in(0,1]$ we have

$$\begin{aligned}\big|\langle J_\nu,\big(\Sigma^{nbh}\big)_\delta^\infty\rangle-\pi\nu^{-1}e^{-\nu\lambda}\int_{\Sigma^1}m\,d\eta\int_\delta^\infty dt\,t^{2d-3}F(\eta,t,0)\,\big|\\\le C\int_\delta^\infty t^{2d-5}\langle t\rangle^{-N}\,dt\le C_1\chi_d(\delta)\,.\end{aligned}$$

2) Since $d \geq 2$ and $N > 2d-2$, then by estimate (2.5) the integral $\int_{\Sigma^1} m\, d\eta \int_0^\infty dt\, t^{2d-3} F(\eta, t, 0)$ converges absolutely, and by (2.2) it equals

$$\int_{\Sigma^1} m\, d\eta \int_0^\infty dt\, t^{2d-3} F(\eta, t, 0) = \int_{\Sigma_*} |z|^{-1} F(z)\, d_{\Sigma_*} z\,.$$

3) Applying 1) and 2) to F replaced by $F_0 = C' \langle z \rangle^{-N}$ and using that $|F| \leq |F_0|$ by (1.1) we find that the integral $\langle J_\nu, \big(\Sigma^{nbh}\big)_\delta^\infty \rangle$ also converges absolutely.

4) As $|\pi(\xi, \theta)| \leq \sqrt{2}\, |\xi|$, then $(\Sigma^{nbh})_0^\delta \subset S_0^{\sqrt{2}\delta} \subset K_{\sqrt{2}\delta}$. Therefore by (3.1)

$$\begin{aligned} &\big| \langle J_\nu, \big(\Sigma^{nbh}\big)_0^\delta \rangle - \pi \nu^{-1} e^{-\nu\lambda} \int_{\Sigma^1} m\, d\eta \int_0^\delta dt\, t^{2d-3} F(\eta, t, 0) \,\big| \\ &\leq \langle |J_\nu|, K_{\sqrt{2}\delta} \rangle + \pi \nu^{-1} e^{-\nu\lambda} \int_{\Sigma^1} m\, d\eta \int_0^\delta dt\, t^{2d-3} |F(\eta, t, 0)| \\ &\leq C_1 \nu^{-1} \delta^{2d-2} + C_2 \nu^{-1} \delta^2\,, \end{aligned}$$

for any $0 < \delta \leq 1$. Choosing $\delta = \sqrt{\nu}$, from here and 1)-3) we find that

$$\big| \langle J_\nu, \Sigma^{nbh} \rangle - \pi \nu^{-1} e^{-\nu\lambda} \int_{\Sigma^1} m\, d\eta \int_0^\infty dt\, t^{2d-3} F(\eta, t, 0) \,\big| \leq C \chi_d(\nu),$$

and that the integral $\langle J_\nu, \Sigma^{nbh} \rangle$ converges absolutely.

5) Finally, let us estimate the integral over $\mathbb{R}^{2d} \setminus \Sigma^{nbh}$:

$$\langle |J_\nu|, \mathbb{R}^{2d} \setminus \Sigma^{nbh} \rangle \leq \int_{\{|z| \leq \sqrt{\nu}\}} \frac{|F|\, dz}{\omega^2 + \nu^2} + C_d \int_{\sqrt{\nu}}^\infty dr\, r^{2d-1} \int_{S^r \setminus \Sigma^{nbh}} \frac{|F(z)|\, d_{S^r}}{\omega^2 + \nu^2}.$$

By item 3) of Lemma 2.1, $|\omega| \geq C r^2$ in $S^r \setminus \Sigma^{nbh}$. Jointly with (3.1) this implies that

$$\big| \langle |J_\nu|, \mathbb{R}^{2d} \setminus \Sigma^{nbh} \rangle \big| \leq C + C \int_{\sqrt{\nu}}^\infty r^{2d-1} r^{-4} \langle r \rangle^{-N}\, dr \leq C_1 \chi_d(\nu).$$

So the integral J_ν converges absolutely and, in view of 4) and 2),

$$\begin{aligned} &\big| J_\nu - \pi \nu^{-1} e^{-\nu\lambda} \int_{\Sigma^1} m\, d\eta \int_0^\infty dt\, t^{2d-3} F(\eta, t, 0) \,\big| \\ &\quad = \big| J_\nu - \pi \nu^{-1} e^{-\nu\lambda} \int_{\Sigma_*} |z|^{-1} F(z)\, d_{\Sigma_*} z \,\big| \leq C \chi_d(\nu)\,. \end{aligned} \tag{4.1}$$

This proves Theorem 1.1.

5. Comments

i) The only part of the proof, where we use that $N > 2d-2$ is Step 2) in Section 4: there this relation is evoked to establish the absolute convergence of the integral J_0; everywhere else it suffices to assume that $N > 2d-4$. Accordingly,

if F satisfies (1.1) with $N > 2d-4$ and $\langle |F|, \Sigma_1^\infty \rangle < \infty$, then (1.5) holds, since $\langle |F|, \Sigma_0^1 \rangle < \infty$, see Step 4) Section 4.

ii) Our approach does not apply to study integrals (1.4), where the divisor $(x\cdot y)^2+\nu^2$ is replaced by $(x\cdot y)^2+(\nu\Gamma(x,y))^2$ and $\Gamma \neq \text{Const}$. But it applies to integrals

$$J_\nu^s = \int_{\mathbb{R}^{2d}} dz\ \frac{F(z)\sin(\lambda x\cdot y)}{(x\cdot y)^2+\nu^2}\,,$$

under certain restrictions on λ. E.g., if $1 \le \lambda \le \nu^{-1}$ and $d \ge 3$, then $J_\nu^s = O(1)$ as $\nu \to 0$, and the leading term again is given by an integral over Σ_*. The case $d=2$ is a bit more complicated.

iii) The approach allows to study integrals (1.4), where the quadratic form $z \mapsto x\cdot y$ is replaced by any non-degenerate indefinite quadratic form of $z \in \mathbb{R}^M$, $M \ge 4$.

iv) The restriction $M \ge 4$ in iii) (and $d \ge 2$ in the main text, where $\dim z = 2d$) was imposed since near the origin the disparity (4.1) is controlled by the integral $\int_0 t^{M-5}\,dt$, which strongly diverges if $M<4$. The difficulty disappears if F vanishes near zero. This may be illustrated by the following easy example:

Example 5.1. Consider

$$J'_\nu = \int_{\mathbb{R}^2} \frac{F(x,y)\cos(\lambda xy)}{x^2y^2+\nu^2}\,dxdy\,,$$

where $F \in C_0^2(\mathbb{R}^2)$ vanishes near the origin. Now $2d=2$, the quadric $\Sigma' = \{xy = 0\}$ is one dimensional, has a singularity at the origin and its smooth part $\Sigma'^* = \Sigma' \setminus 0$ has four connected components. Consider one of them: $\mathcal{C}_1 = \{(x,y) : y=0, x>0\}$. Now the coordinate ξ is a point in $\mathbb{R}_+$ with $(x_\xi, y_\xi) = (\xi, 0)$ and with the normal $N(\xi) = (0,\xi)$, the set $\Sigma_1 \cap \mathcal{C}_1$ is the single point $(1,0)$ and the coordinate (η, t, θ) in the vicinity of $\mathcal{C}_1$ degenerates to (t,θ), $t>0$, $|\theta|<\theta_0$, with the coordinate-map $(t,\theta) \mapsto (t, t\theta)$. The relations (2.2) and (2.3) are now obvious, and the integral (3.1) vanishes if $\delta > 0$ is sufficiently small. Interpreting $z=(x,y)$ as a complex number, we write the assertion of Theorem 1.1 as

$$\Big| J'_\nu - \pi\nu^{-1}e^{-\nu\lambda}\int_{\Sigma'} \frac{F(z)}{|z|}\,dz \Big| \le C\,,$$

where the integral is a contour integral in the complex plane.

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Sergei Kuksin,

CNRS, Institut de Mathémathiques de Jussieu–Paris Rive Gauche, UMR 7586, Université Paris Diderot, Sorbonne Paris Cité, F-75013, Paris, France & School of Mathematics, Shandong University, Shanda Nanlu, 27, 250100, PRC,
E-mail: Sergei.Kuksin@imj-prg.fr