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Task design for developing students' recognition of the roles of assumptions in mathematical activity

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Mathematics education researchers have highlighted the importance of assumptions in school mathematics given their vital roles in mathematical practice. However, there is scarcity of research aiming at enhancing students' recognition of different roles that assumptions play in mathematical activity. In this paper, we begin to address this issue by formulating two task design principles and reporting on the implementation of a classroom intervention where lower secondary school students in two classes worked, with the same teacher, on a task designed following the proposed principles. Our analysis shows how the task, together with the purposeful teacher's actions in implementing it, led to students' developing appreciation of two roles that assumptions play in mathematical activity.

Keywords: Assumptions, Task design principles, Classroom intervention, Teacher's role.

Introduction

Assumptions refer to statements that doers of mathematics use or accept (often implicitly) and on which their claims are based (Stylianides & Stylianides, 2017). Several researchers in mathematics education have highlighted the importance of helping students be more aware of the roles that assumptions play in mathematics, especially in relation to the notion of *proof*. For example, Fawcett (1938), who treated proof as a key notion in school mathematics, listed four criteria for checking students' understanding of the nature of proof, among which two are closely related to assumptions: students understand “[t]he necessity for assumptions or unproved propositions” and “[t]hat no demonstration proves anything that is not implied by assumptions” (p. 10).

Despite the importance of assumptions in school mathematics, assumptions have received little research and pedagogic attention in school mathematics. Furthermore, existing studies (e.g., Fawcett, 1938; Jahnke & Wambach, 2013; Komatsu, 2017) have mainly focused on geometry at the secondary school level. Research in other mathematical domains is needed that will build on and widen the scope of application of prior research in geometry, given also the recent calls for curriculum reform (see, e.g., NGA & CCSSO, 2010, in the United States) to introduce proof-related mathematical activity across content areas as well as in primary school (e.g., Stylianides, 2016).

To develop students' recognition of the roles of assumptions in mathematical activity, we focus on the design of mathematical tasks, which have potential to affect significantly students' experiences with and understanding of the subject. Specifically, in this paper we propose two principles that can be used to support task design in the area of assumptions, we report on an intervention implemented in two classes where lower secondary school students worked on a task designed following the proposed principles, and we discuss the influence on students' understanding of the roles of assumptions in mathematical activity.

Task design principles for the roles of assumptions

Types of assumptions

Assumptions often refer to statements that are accepted as true without proofs, and their typical examples are axioms like the fifth postulate in Euclidean geometry whose modification led to the formation of non-Euclidean geometries. To expand the opportunity for students to experience the relativity of truth in mathematics, we extend the mainstream notion of assumptions to include two additional types of assumptions, both related to mathematical tasks and their formulation: (1) *conditions* of tasks (including premises of statements mentioned in tasks), and (2) *definitions* of terms mentioned in tasks. This paper reports on an intervention concerning the former type in the context of a function task. We are in the process of designing another intervention addressing the latter type.

Roles of assumptions

Assumptions play multiple roles in the discipline of mathematics. Their primary role has been to introduce the relativity of truth, i.e., whether a proposition is true or false cannot be absolutely determined, but hinges on assumptions (Fawcett, 1938). For example, Euclidean geometry was originally accepted as the standard form of geometry that was deemed to be true in an absolute sense. However, following mathematicians' unsuccessful attempts to deduce the fifth postulate from other postulates, mathematicians created new geometries, the non-Euclidean geometries, by adopting axioms different from the fifth postulate. It was thus recognised that some propositions in Euclidean geometry are true only under the set of Euclidean axioms, and that the conclusions are different under different sets of axioms.

A second role is to mediate disputes that occur concerning the truth of conjectures; mathematicians have managed to resolve such disputes and reach consensus by delving into the assumptions underlying the conjectures. Consider, for example, the conjecture that the limit of any convergent series of continuous functions is itself continuous. According to Lakatos (1976), while Cauchy provided a proof for this conjecture in the 19th century, a counterexample to the conjecture was found in Fourier's work. Mathematicians attempted various explanations for this puzzling situation where the proof for and counterexamples against the conjecture coexisted. Eventually, Seidel analysed Cauchy's proof and discovered a hidden lemma, which once incorporated into the conjecture as a condition (related to the concept of uniform convergence), validated the conjecture.

Learning goals

Considering the importance of promoting authentic mathematical practice in school mathematics (Lampert, 1992), our study aims to help cultivate students' sense of the two roles of assumptions we discussed earlier. Specifically, we set up the following two learning goals for the intervention in our study (similar to the goals discussed in Stylianides & Stylianides, 2017, for prospective teachers):

Learning Goal 1: To recognise that a conclusion is dependent on the assumptions on which the argument that led to it was based.

Learning Goal 2: To recognise that making the underlying assumptions explicit is crucial for reaching a consensus on the conclusion.

Task design principles

Based on our previous studies (Komatsu, 2017; Komatsu & Jones, 2019; Stylianides, 2016, chapter 4; Stylianides & Stylianides, 2017), we formulated the task design principles below in order to address the above learning goals:

Principle 1: Designing tasks that are subject to different legitimate assumptions by leaving purposefully some of the assumptions of tasks implicit or unspecified.

Principle 2: Allowing tasks to have different legitimate answers based on different legitimate assumptions about the tasks.

At the centre of these principles is the deliberate act of keeping some of the assumptions of the tasks implicit or unspecified. The typical way of showing a task to students and promptly clarifying its assumptions eliminates the possibility of multiple interpretations of these assumptions and reinforces the intuitive connection between the clarified assumptions and the expected answer. In our on going research (part of which we report here), we intend to break this norm by creating a situation where an assumption of a task is purposefully unstated by the teacher so that the students can work on the task under their own (implicit and potentially varied) assumptions. We anticipate that the disagreement in the proposed answers likely to emerge in the whole-class discussion will prompt the students to explore the reasons for the inconsistency. This discussion, purposefully orchestrated by the classroom teacher, is expected to help students realise that their answers depend on their respective assumptions and that an explicit assumption is needed to justify each answer.

Method

Task design

Based on the principles stated above, we designed a task shown in Figure 1, which we adapted from a task implemented in the 2016 National Assessment of Academic Ability in Japan (NIEPR, 2016). With respect to Principle 1, the functional relationship between x and y is not specified in the task so that students can work on the task under different assumptions about functional relationships. With respect to Principle 2, the answer to the task is, for example, $y = 6$ if y is inversely proportional to x , and $y = -6$ if y is a linear function of x . Despite the divergence between these answers (as well as an infinite number of other legitimate answers not discussed here), both of them are, or can be, correct as long as the respective functional relationship is assumed.

The table below represents y -values corresponding to the given x -values. Find the y -value when $x = 6$.

x	...	2	3	...	6	...
y	...	18	12

Figure 1: The designed task, which we adapted from a task derived from NIEPR (2016)

Participants

We conducted an intervention study using the task in Figure 1 in two ninth-grade classes (approximately 40 students in each class, aged 14–15 years) in a Japanese lower secondary school

affiliated with a national university. The intervention consisted of one lesson (50 minutes) and was implemented in the two classes by the same teacher who had five years of teaching experience.

We worked together with the teacher in planning the intervention. Specifically, the first author showed to the teacher the task in its original form from the national assessment, and also discussed the intended learning goals and task design principles we presented in the previous section. The teacher then created a lesson plan and discussed it with the first author. Based on the discussion, we decided to use the task as in Figure 1 that was modified from the original form. The teacher then implemented the task following closely the lesson plan as outlined below. The participating students were not introduced to the notion of assumptions before, but they were familiar with the definition of functions and several kinds of functions: proportion ($y = ax$), inverse proportion ($y = a/x$), linear function ($y = ax + b$), a special case of quadratic function ($y = ax^2$), and step function.

Data collection and analysis

The data include the transcripts of the videotaped lessons, the students' worksheets, and the field notes taken by the first author during the lessons. The data analysis aimed to determine whether the learning goals described earlier were achieved in each class. We focused our analysis on the whole-class discussions to examine the learning trajectory of each class as a whole, rather than attempting to trace individual students' thinking. Due to space limitations, results from analysis of students' worksheets are referred to only for triangulation, complementing the results pertaining to the whole-class activity. Because the two classes followed nearly identical paths, we report only the results from one class (38 students) in this paper. All student names are pseudonyms.

Classroom intervention

The relative correctness of answers

The teacher began the lesson by presenting the task in Figure 1; the students individually worked on it for about five minutes. During this work, some students provided the answer $y = 6$, while others the answer $y = -6$. This disagreement surfaced during the subsequent whole-class discussion. Misaki first said, "I think y is 6", but a student objected to her answer, "That's wrong". Aoi then said, "[$y = -6$]", but Shun questioned her answer saying "What?" After that, the teacher posed to the students the pre-planned question of which answer, $y = 6$ or $y = -6$, was actually correct:

- 31 Teacher: Well, now there are two answers. Which is a correct answer?
32 Students: Both are correct answers.
33 Teacher: Really? Are both correct answers? Please think everyone. [...] [The students start to discuss with their neighbours.]
34 Nanami: What? What is the conclusion?
35 Takumi: I don't know.

As seen in this interaction, an anticipated sense of confusion emerged among students. While some of them considered both answers to be correct (line 32), others had no idea about which the correct answer was (lines 34, 35). To address this confusion, the teacher then asked students to think more about whether both answers were correct and to write their thoughts on their worksheets. The students then shared their thoughts in the whole-class discussion:

- 39 Ren: Um, I think both [answers] are correct. These two. Since we don't know what the function of this graph is, since we can interpret it as both an inverse proportion and a linear function, I think both are correct.
- 40 Teacher: You've described your opinion that both are correct. Well, let's listen to another student. Riko, can you share [your opinion]? What do you think?
- 41 Riko: Um, I also think both are correct. If we change how we read the table, it becomes an inverse proportion and also a linear function.

Ren and Riko's responses (lines 39, 41) are representative of students' responses and are relevant to Learning Goal 1: recognising that a conclusion is dependent on the assumptions on which the respective argument that led to it was based. That is, these students understood that the correctness of their answers was relative to their assumptions, implying that y is 6 if y is inversely proportional to x , and y is -6 if y is a linear function of x . All students in the class agreed that both answers could be correct and most students shared or wrote similar thoughts. For example, Mizuki wrote on her worksheet, "the answer is 6 if we interpret it as an inverse proportion, but -6 if it is a linear function". Similarly, Nanami wrote, "The graph becomes a curve where the y -value becomes 6, and also becomes a line where the y -value becomes -6 , so I think both values are correct depending on how we think about it". These students' responses show that the task has afforded them the opportunity to recognise the connection between different answers and respective assumptions.

Two ways for pinning down the answer

While the students recognised the relativity of their answers, they started to feel dissatisfied with the task they worked on: Nanami said, "That's not good. I don't know what I am solving". The teacher responded to the students' feeling of dissatisfaction by posing two pre-planned questions (written on the blackboard): "(1) What are the reasons for the answer to be ambiguous? (2) What can we do to address the ambiguity?" The students were given approximately ten minutes to tackle these questions, before sharing their thoughts in the whole-class discussion. The students' responses can be divided into two groups, of which Kenta's and Shun's contributions are representative:

- 75 Kenta: Um, um, [regarding question] number 1, [this] is because there is no explanation about whether y is proportional to x or inversely proportional. Um, if there are only points (2, 18) and (3, 12), equations are possible both for $y = 6$ and -6 . Um, if we interpret it as the graph of the inverse proportion of $y = 36/x$, um, we have 6. If we interpret it as the graph of the proportion of $-6x + 30$, we have -6 , I think. Well, [regarding question] number 2, I think, we should add an explanation, like when y is proportional to x or when y is inversely proportional to x . [Kenta misspoke here. He meant linear functions when he talked about proportions.]
- 88 Shun: Um, because there are only two given points, the answer changes depending on whether we connect two points with a line or connect [them] with a curve. [Regarding question] Number 2, I think, if we determine the values of more than two points, the answer will be [uniquely] determined.

According to first group of students (illustrated by Kenta's comment), the reason for the answer to the task to be ambiguous was the absence from the task description a specification of the functional relationship between x and y ; the way this group suggested to address the ambiguity was to clarify the

relationship in the task. According to the second group of students (illustrated by Shun's comment), the ambiguity was because only two pairs of values were given in the task; the way this group suggested to address the ambiguity was to provide in the task more points with given values.

Recognition of the necessity of assumptions

The teacher had anticipated these two kinds of student responses and made a planned move to address them, beginning with the second group's suggestion to increase the number of points with given x - and y -values. He plotted the points (2, 18) and (3, 12) on the coordinate plane on the blackboard and said: "Suppose that in the case of [...] 4, we have 6 [meaning $y = 6$ for $x = 4$]. Is this a linear function? We know three points." Here the teacher selected (4, 6), because the y -value is 6 when $x = 4$, if the function is assumed to be linear.

The alternative possibility of step functions was then proposed by Shota. Shota's suggestion greatly surprised other students, who later began to agree with him and acknowledged that, even if three pairs of x - and y -values were given, the function in the task could not be uniquely determined. Below is the whole-class discussion after Shota drew the graph shown in Figure 2.

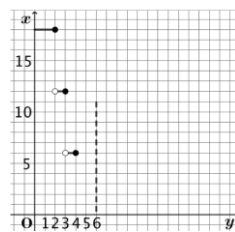


Figure 2: Step function drew by Shota (the dotted vertical line was added later by the teacher)

- 138 Teacher: Well, in this case, what happens? This, this, well, what did you draw?
- 139 Shota: A step function.
- 140 Teacher: Oh, yes, thank you. Is this a linear function?
- 141 Students: No.
- 142 Teacher: Yeah. [...] In the case of 6, this 5, 6, the point in this case... [The teacher added the dotted vertical line in Figure 2, and suggested a question asking the y -value when $x = 6$.]
- 143 Nanami: Anything [would be] okay [meaning that the y -value can be any values].
- 144 Teacher: Can you predict [the value of y]?
- 145 Nanami: Anything [would be] okay.
- 146 Moe: Like quite suddenly [it is] changing.
- 147 Nanami: It could change suddenly, it is changing.

In the above interaction, the students stated that even if another point (4, 6) was added to the table (Figure 1), the function could be a step function rather than linear, and that in this case, the y -value when $x = 6$ could not be uniquely determined (lines 143, 145–147). The class thus realised that increasing the number of points was not a viable option to address the ambiguity in the task. The consideration of step functions motivated the class to seek another way to pin down the answer, returning to the way that was represented earlier by Kenta (line 75), i.e., specifying the functional relationship between x and y in the task. At the end, the teacher summarised the lesson as follows:

194 Teacher: Making this clear [meaning to specify the functional relationship] is good. I introduce this [pointing “in the case of a linear function” and “in the case of an inverse proportion” written on the blackboard], this, the one shown at the beginning is, [...] this is called an *assumption, assumption*. So, I believe you see that the answer becomes definite by this assumption.

In this comment, the teacher introduced the term *assumption* and clarified that the conditions, such as “ y is a linear function of x ”, were assumptions one could make based on the task’s phrasing. He also mentioned that making the assumptions explicit was crucial for reaching a consensus answer. This is relevant to Learning Goal 2, which was reached by the class as illustrated by students’ comments at the end of the lesson when they summarised their learning. For example, Takumi said, “If there is no assumption [made explicit], even if we know many points, the graph cannot be fixed into one function’s graph”. Similarly, Yuka said, “If there is no assumption, different functions can be considered. So, when we want to make the answer definite, we write the assumption”.

Discussion

In this paper, we described a classroom intervention involving ninth-grade students with a particular mathematical task (Figure 1), designed based on two principles (Principles 1 and 2). Our analysis showed that the task and its purposeful implementation were useful for stimulating the students to recognise that the correctness of their answers was relative to their respective assumptions (Learning Goal 1), and that it was necessary to make the assumption of the task explicit to reach a consensus answer (Learning Goal 2). Hence, the principles can be regarded as helpful for designing tasks that aim to help students appreciate the roles of assumptions in mathematical activity.

Although in this paper we focused on the role of task design in promoting particular learning goals, the results of our intervention highlight also the important role that the teacher played for achieving these goals. Prior to the intervention, we held meetings with the teacher to discuss the task, the two design principles, and to explain the task’s intended purpose. The teacher well appreciated the learning goals of the task and was able to devise a detailed lesson plan where he anticipated students’ responses to the task and planned some questions strategically, preparing himself to capitalise on these responses during the lesson so as to effectively manage students’ contributions and steer class progression towards the learning goals. The intervention played out in both classrooms as expected, thereby limiting the need for teacher to improvise and make in-the-moment decisions; the lesson naturally progressed towards the learning goals, while the students’ contributions were respected and were integrated into the discussions (Sherin, 2002).

A significant aspect of the teacher’s role was asking probing questions. For example, in response to students’ different answers ($y = 6$ and -6), the teacher asked, “What are the reasons for the answer to be ambiguous? What can we do to address the ambiguity?” These questions triggered students’ interest in exploring ways to obtain a consensus answer, and created an “intellectual need” (Harel, 1998) for the class to be introduced to the notion of assumptions. Another important aspect of the teacher’s role was orchestrating whole-class discussions (Stein, Engle, Smith, & Hughes, 2008). When planning the intervention, the teacher had predicted that some students would come up with responses similar to Kenta’s (line 75) and Shun’s (line 88). When implementing the intervention, the teacher ‘filtered’ and ‘sequenced’ the students’ responses (Sherin, 2002; Stein et al., 2008) so that

Shun's and Kenta's ideas were examined, one at a time, in the whole-class discussion. When the students understood that Shun's idea was not viable, he helped students recognize the advantage of Kenta's idea (line 75) to make explicit the functional relationship in the task.

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