



**HAL**  
open science

# The interplay of logical relations and their linguistic forms in proofs written in natural language

Kerstin Hein

► **To cite this version:**

Kerstin Hein. The interplay of logical relations and their linguistic forms in proofs written in natural language. Eleventh Congress of the European Society for Research in Mathematics Education, Utrecht University, Feb 2019, Utrecht, Netherlands. hal-02398105

**HAL Id: hal-02398105**

**<https://hal.science/hal-02398105>**

Submitted on 6 Dec 2019

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# The interplay of logical relations and their linguistic forms in proofs written in natural language

Kerstin Hein

TU Dortmund University, Germany; kerstin.hein@math.tu-dortmund.de

*The Toulmin model and the systemic functional grammar are combined to analyse logical relations and their linguistic forms in students' written proofs for identifying obstacles and possibilities to foster the understanding of proofs. The qualitative analysis of 63 students' products reveals a parallelism between syntactical and content-related explications and condensations. In particular, the use of conjunctions seems to support more options 1.) to make explicit logical relations between premise and warrant or conclusion, 2.) to combine several steps of a proof, and 3.) to recycle conclusions as new premises. The logical relation from the warrant to the conclusion is often only made explicit using causal prepositions as linguistic condensed forms of relations.*

*Keywords: Formal proof, logical relations, Toulmin model, linguistic analysis*

## Introduction

There is still a lack of knowledge about the concrete language demands for specific topics, although the importance of identifying academic language and the epistemic role of language has been explored (e.g. Schleppegrell, 2004). For these reasons, a linguistic analysis of the classroom discourse to support teaching-learning processes should be pursued for several subjects and topics, in particular mathematics (Schleppegrell, 2007). Especially, for the challenging topic of proof and proofing an analysis of students reasoning is important (e.g. Mariotti, Durand-Guerrier, & Stylianides, 2018, p. 80), in particular, the analysis of logical rules and their linguistic forms, which cannot be translated from formal language directly (Durand-Guerrier, 2004, p. 2). Within the logical structures, the "implicit logical relationships" (Schleppegrell, 2007, p. 141) in their linguistic forms are one of the major challenges in proof. Because of the high density of academic language, it is demanding for the students to be aware of and to understand the academic language. Therefore, it is important to unpack the meanings in more explicit language (O'Halloran, 1998). The suggestion for teachers is to be explicit and to hold syntactic control within their own language during teaching logic (Durand-Guerrier, Boero, Douek, Epp, & Tanguay, 2011).

As a first step, a linguistic analysis is needed for analysing the language demands of logical relations. The structural and linguistic analysis of students' products in this study follows the above mentioned, general suggestions to identify language demands in logical structures, here for logical relations. It pursues three research questions: 1) *How can logical relations and their linguistic forms be analysed?* 2) *Which pattern can be seen in the interplay between the logical relations and their linguistic forms?* 3) *Which linguistic forms of logical relations can be used to make logical relations explicit?* The first two sections present the theoretical background and methodology of the analysis. The outcome of the qualitative analysis of the written proofs is presented afterwards illustrated by case studies of two texts.

## **Theoretical background: Logical relations and their linguistic forms**

In line with Mesnil (2013), the interaction between logic and language and the importance of explicitness of the language in teaching and learning proof is assumed. This study focuses on the logical relations as one challenging part of the language of proof (Schleppegrell, 2007). For this reason, the study aims at identifying the linguistic forms of logical relations.

### **Logical elements**

For the analysis of proofs, the Toulmin model (1958) is often applied, although it was developed to describe everyday argumentations. Within mathematics education the structural analysis with the Toulmin model was also applied to analyse mathematical classrooms (Krummheuer, 1995), the structure gap between argumentation and proofs (Pedemonte, 2007), and for proofs with several steps (e.g. Knipping & Reid, 2015), even if there are limits for the analysis of proofs and proofing (also discussed in Mariotti et al., 2018, p. 78). In the structure model of Toulmin, the function of the logical elements as premise etc., not their relations, is crucial. This becomes more important when it is applied for the logical structure analysis of proof. This article refers to the short versions of Toulmin's model and considers premise, warrant and conclusion as the relevant logical elements within deductive steps. These logical elements are connected more or less implicitly by logical relations, which need to be unpacked, here.

### **Logical relations**

In this article, logical relations are understood as relations between logical elements. In deductive proofs, the relations between premise, warrant and conclusion are crucial (as described by Duval 1991, p. 235), but also within a warrant with the logical form of implication or equivalence (Selden & Selden, 1995). In particular, the logical relation from premise to warrant is crucial for the verification of the premises (Duval, 1991). These logical relations are often implicit in the language, creating an obstacle for students (Schleppegrell, 2007) as they have to be unpacked in order to understand them (Selden & Selden, 1995).

### **Different linguistic forms**

Logical relations are often expressed in natural language by logical connectives such as “because”, “due to” or “if..., then...” (Clarkson, 2004). These logical connectives can be classified as causal-conditional principal markers by the systemic functional grammar of Halliday (1985), describing language from a functional perspective. They have different linguistic forms such as causal conjunctions (“because”), conditional conjunctions (“if... then...”) or causal prepositions (“due to”). However, there is still a research lack of the analysis of the combination of logical relations and their linguistic forms (e.g. Durand-Guerrier et al., 2011; Schleppegrell, 2007).

## **Methodology for the analysis of written proofs**

### **Data collection**

The students' texts of written proofs have been generated within a teaching-learning arrangement on deductive reasoning in grade 8-12 all with angle sets (Hein & Prediger, 2017; Prediger & Hein, 2017). In the teaching-learning arrangement conjunctions were used to express logical relations. The students were also asked to reason why a theorem can be applied (premises are met). In this

paper, the learning process and the effects of the teaching-learning arrangement are not analysed. Instead, it focuses only on the written products of the teaching-learning arrangement in order to investigate the interplay between logical relations and their linguistic forms. The data corpus consists of 63 written texts from 48 students (20 in grade 8, 6 in grade 9, 4 in grade 10, 18 in grade 12). The result section presents the results of two cases with texts of the twelfth graders Linus and Petra.

### Methods for qualitative data analysis

The qualitative analysis of logical relations and their linguistic forms in the written products combines two analysis models: (1) *Toulmin model*: The Toulmin model in its short version (1958) is applied for the logical structure analysis to identify the addressed logical elements (premise, warrant, conclusion) (Pedemonte, 2007) and to disentangle the several steps of the proof (Knipping & Reid, 2015). (2) *Systemic functional grammar*: The linguistic analysis of the logical connectors as language means for logical relations draws upon systemic functional grammar (Halliday, 1985), which can also be used to identify linguistic challenges in mathematics education (Schlepppegrell, 2007). The analysis approach systematically identifies lexis used for logically connecting sentences or elements within sentences and classifies their syntactical forms as conjunction (con) respectively prepositions (pre). For these, the English functional grammar (in which the products are presented here) and the German functional grammar (the original language of the products) resonate with each other. For example, causal conjunctions can be described in English and German with conjunctions and prepositions and function in a similar way. Here, first the existence of a logical relation as links between logical elements and then their grammatical form (conjunctions such as “therefore” or preposition such as “according to”) are identified.

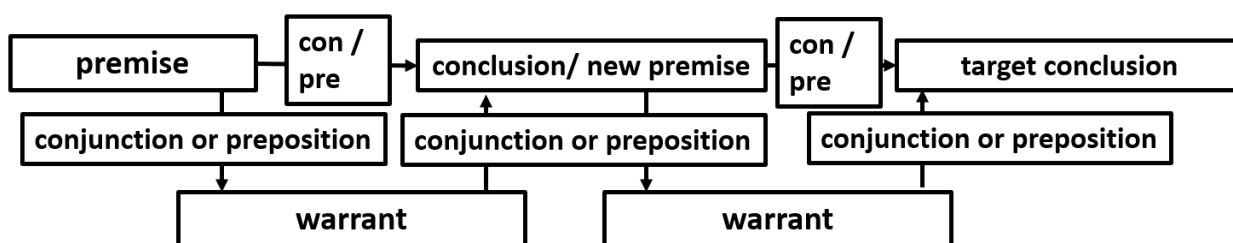


Figure 1: Analysis tool with Toulmin for several steps and both linguistic forms for logical relations

### Empirical insights into the cases of Petra’s and Linus’ written proofs

Petra’s and Linus’ texts were chosen for illustrating the results because their texts contain typical linguistic forms for the logical relations – condensed and non-condensed – which were also found in many other written proofs about angle sets. The students work on the proof for the sum of angles in a triangle (Figure 2). Before starting to write individually, they discuss the proof, identify which theorems have to be applied and draw the sketch printed in Figure 3 to which both texts refer.

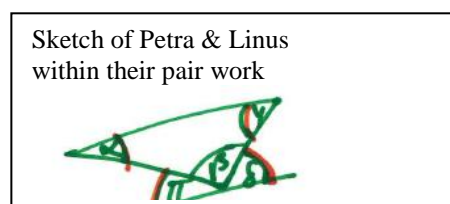
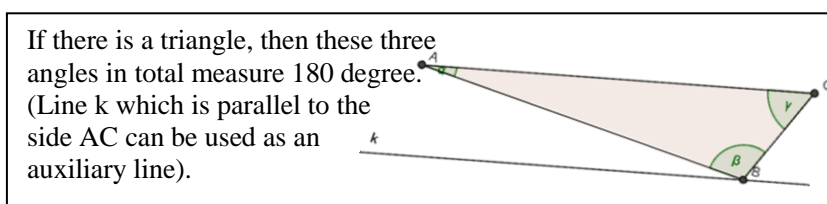


Figure 2: Mathematical statement to be proven by Petra and Linus

Figure 3: Students’ joint sketch

Analysis of activated mathematical statements: Both students refer to the same mathematical statements, the alternate interior theorem (Step 1 and 2), supplementary angle theorem (Step 3), calculating with angles theorem (Step 4), but both texts show substantial differences with respect to making the logical relations and their linguistic forms explicit. Presentation of the structural and linguistic analysis: The logical elements are illustrated with colours (green: premises (P); blue: warrant (W), orange: conclusion (C)), both in the text and the graphical representations. Only partly explicated logical elements are illustrated with \* in the text and with dashed boxes in the graphics. The logical relations (R) (conjunctions (con), prepositions (pre)) are illustrated in black and by arrows in the graphical presentations (continuous arrow for conjunctions, dashed arrow for prepositions).

### Linus' text with causal prepositions for the logical relations

Step	Translated (and original) text	Structural and linguistic analysis
1+2	<p>The triangle ABC has the angles <math>\alpha := \sphericalangle CAB</math>; <math>\beta := \sphericalangle ABC</math>, <math>\gamma := \sphericalangle BCA</math>.            Additionally, a line passes through point B.            Additionally, the angle <math>\pi</math> is located at line k and at <math>\overline{AB}</math> and <math>\delta</math> is located at line k and the side <math>\overline{BC}</math>. (p*: Parallelism of k to BC is missing.)            According to (R/pre) the alternate interior angle theorem (W), the angles <math>\alpha</math> and <math>\pi</math> have the same measure and <math>\delta</math> and <math>\gamma</math> have the same measure. (C)</p> <p>(German Original: Das Dreieck ABC hat die Winkel <math>\alpha := \sphericalangle CAB</math>, <math>\beta := \sphericalangle ABC</math>, <math>\gamma := \sphericalangle BCA</math>. Zusätzlich geht durch den Punkt B eine Gerade. Zusätzlich liegt der Winkel <math>\pi</math> an der Geraden k und an <math>\overline{AB}</math> und <math>\delta</math> liegt an der Geraden k und an der Seite <math>\overline{BC}</math>. Gemäß des Wechselwinkelarguments sind die Winkel <math>\alpha</math> und <math>\pi</math> gleich groß und <math>\delta</math> und <math>\gamma</math> gleich groß.)</p>	<p>The detailed explicated premises (without parallelism) are not verbally connected with the warrant.</p> <p>With "According to" the warrant (alternate interior angle theorem) is connected with the conclusion.</p> <p>"According to" is a preposition within one sentence.</p>
3	<p>According to (R/pre) the supplementary angle theorem (W), the angles <math>\beta</math>, <math>\pi</math> and <math>\delta</math> sum to 180 degree (C).</p> <p>(German Original: Gemäß des Nebenwinkelarguments sind die Winkel <math>\beta</math>, <math>\pi</math> und <math>\delta</math> zusammen gleich 180 Grad groß.)</p>	<p>"According to": analysis analogue to Step 1+2</p>
4	<p><math>\pi</math> can be substituted by <math>\alpha</math> and <math>\delta</math> can be substituted by <math>\gamma</math> (P*: Conclusion of step 3 as new premises is missing.).            According to (R/pre) the calculating argument (W), the interior angles <math>\alpha</math>, <math>\beta</math> and <math>\gamma</math> have a measure of 180 degree (C).</p> <p>(German Original: Für <math>\pi</math> kann <math>\alpha</math> eingesetzt werden und für <math>\delta</math> <math>\gamma</math> eingesetzt werden. Gemäß des Rechenarguments sind die Innenwinkel <math>\alpha</math>, <math>\beta</math> und <math>\gamma</math> gleich 180 Grad groß.)</p>	<p>linguistically, it is not marked as previous conclusion of Step 1+2 (implicit recycling). Conclusion from Step 3 is not explicitly used.</p> <p>"According to" connects the theorem with the conclusion with a preposition.</p>

Table 1: Analysis of Linus' text

In Linus' first sentences, the premises for the Steps 1, 2 and 3 are almost explicated. In further sentences, the warrants and the conclusions for Step 1,

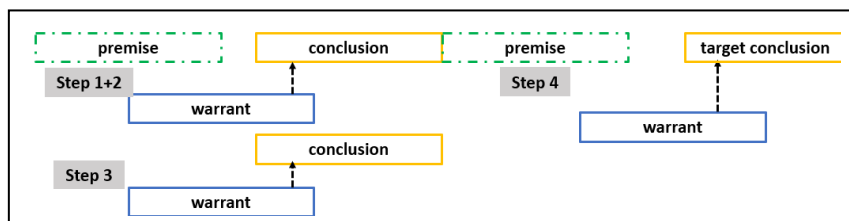


Figure 4: Summary of the analysis of Linus' text

2, 3 and 4 are explicated. The recycling of the conclusions from Step 1, 2 and 3 to new premises in Step 4 is not made explicit. However, logical relations between these elements are only explicated from the warrant to the conclusions, whereas the logical relations from the premises to the warrants or to the conclusions are not explicated. In each occurrence of logical connectives, Linus activates prepositions such as “according to”.

### Petra’s text with more causal conjunctions for the logical relations





Step	Translated (and original) text	Structural and linguistic analysis
1	<p>A triangle with <math>\alpha</math>, <math>\beta</math> and <math>\gamma</math> is given. The line <math>k</math> is parallel to line <math>k</math> and crosses point B. To identify the measures of the angles <math>\alpha</math> and <math>\gamma</math> (crossed out), at the point B to the angle <math>\beta</math> the angles <math>\pi</math> and <math>\delta</math> are defined (P). Because (R/con) the line <math>k</math> and the side AC are parallel and are crossed by the side AB (P), the angle theorem (W) can be applied. Therefore, (R/con) <math>\alpha = \pi</math> (C).</p> <p>(German Original: Es ist ein Dreieck mit <math>\alpha</math>, <math>\beta</math> und <math>\gamma</math> gegeben. Die Gerade <math>k</math> ist parallel zu der Geraden <math>k</math> und schneidet den Punkt B. Um die Größe der Winkel <math>\alpha</math> und <math>\gamma</math> (durchgestrichen) zu erfahren, definiert man an dem Punkt B zu dem Winkel <math>\beta</math> die Winkel <math>\pi</math> und <math>\delta</math>. Da die Gerade <math>k</math> und die Seite AC parallel sind und durch die Seite AB geschnitten wird, kann man das Winkelargument anwenden. Daraus folgt, dass <math>\alpha = \pi</math>.)</p>	 <p>„Because... can be applied“ connects the premise with the theorem. “Because” is a causal conjunction (within one sentence).</p> <p>“Therefore” connect the warrant (here angle theorem) with the conclusion.</p> <p>“Therefore” is a causal conjunction and connects two sentences.</p>
2	<p>Additionally, the side BC crosses the parallel line <math>k</math> to the side AC (P), therefore (R/con) <math>\gamma = \delta</math> (C) reasoned by (R/prep) the alternate interior angles theorem (W).</p> <p>(German Original: Außerdem schneidet die Seite BC die parallele Gerade <math>k</math> zu der Seite AC, deshalb ist <math>\gamma = \delta</math> begründet durch das Wechselwinkel-argument.)</p>	 <p>With „therefore“ (causal conjunction), the conclusion is drawn. “Reasoned by” is used at the end of the sentence to add the logical relation from the warrant to the conclusion.</p>
3	<p>By the fact that (R/prep) <math>\Omega</math> and <math>\delta</math> are supplementary angles (<math>\Omega = \beta + \pi</math>) (P), <math>\beta</math>, <math>\pi</math> and <math>\delta = 180^\circ</math> (C), according to (R/prep) the supplementary angle argument (W).</p> <p>(German Original: Dadurch dass <math>\Omega</math> und <math>\delta</math> Nebenwinkel sind <math>\Omega = \beta + \pi</math>, sind <math>\beta</math>, <math>\pi</math> und <math>\delta = 180^\circ</math>, laut dem Nebenwinkelargument.)</p>	 <p>With „By the fact that...“, the conclusion from the premise is drawn. “By” is a preposition. The premises are condensed to an object with “as a fact” which is explained in a sub-clause. “According to” (preposition) is used at the end of the sentences to add the logical relation from the warrant to the conclusion.</p>
4	<p>So, if (R/ con) <math>\beta + \pi + \delta = 180</math> degree and <math>\alpha = \pi</math> and <math>\gamma = \delta</math> (P), then (R/ con), according to (R/ prep) the calculating argument (W), <math>\beta + \gamma + \alpha = 180</math> degree (C).</p> <p>(German Original: Wenn also <math>\beta + \pi + \delta = 180</math> Grad und <math>\alpha = \pi</math> und <math>\gamma = \delta</math>, dann sind nach dem Rechenargument <math>\beta + \gamma + \alpha = 180</math> Grad.)</p>	 <p>With „If..., then” (conditional conjunction) the warrant is applied to the case. With “so” there is a reference to the previous conclusion, which is used as new premise (recycling). For “According to...”, see Step 3.</p>

Table 2: Analysis of Petra’s text

In her text, Petra addresses the premises, the warrant with the name of the theorems and the conclusions as Linus. For making the logical relations explicit, she connects the premises with the warrants or the conclusions with causal conjunctions such as “because” and “there-

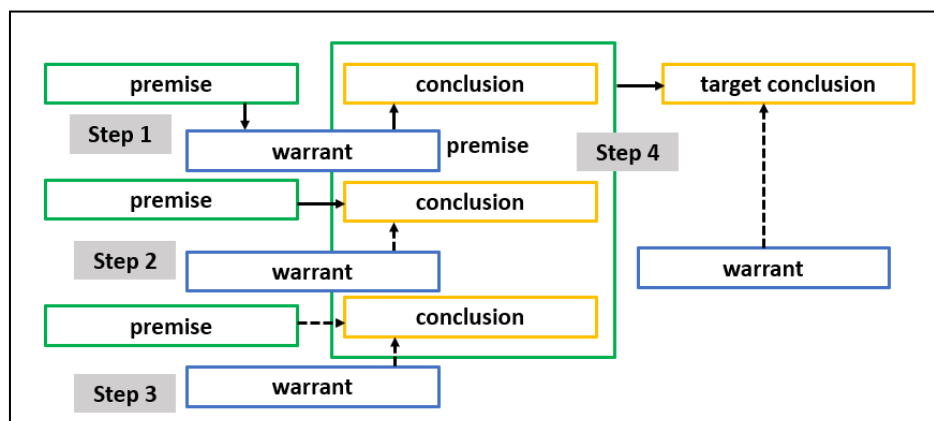


Figure 5: Summary of the analysis of Petra’s text

fore”. In one case, Petra connects the premise with the conclusion by a preposition (Step 3). Petra also uses prepositions such as “due to” by adding the expression “due to the ...-argument” as last part of the sentences after connecting the premise to the conclusion (Step 3 and 4).

### Comparison of the case studies of Linus and Petra

In both texts, almost all contents of the elements of the short Toulmin Model are explicated, even Linus only mentions the premises partly at the beginning, respectively not completely in Step 4. The conclusions as new premises in Linus’ example are not explicated in their function, in particular, the logical relations from the premises to the warrant or conclusion. The linguistic analysis shows that *causal conjunctions* such as “because” and “therefore” of logical relations are only used in Petra’s text, where also the logical relations from the premises to the warrant respectively conclusions and the recycling of previous conclusions are expressed. These findings are in line with those of other written products (see Prediger & Hein, 2017). According to Halliday (1985) conjunctions are non-condensed forms for relations as in everyday language (called by him as *coherent*). In this case, these conjunctions were also used to make explicit the logical relations from the premises to the warrant or from the premises to the conclusion. With *causal prepositions* such as “according to”, “due to” or “by”, virtually nothing but the logical relations between the warrant and the conclusion are made explicit. This was also found in other texts (Prediger & Hein, 2017). Only in Step 3 of Petra’s text, the relation from the premises to the conclusion is made explicit with a preposition (“by the fact that...”). Here, the content of the premises is expressed by a sub-clause. One reason for virtually explicating nothing but the logical relation from the warrant to the conclusion (exception: Step 3 in Petra’s text) may be that prepositions need nominalizations and here only the warrant is condensed to a nominalization by its name and can be easily integrated by a preposition (“due to the ...-theorem”). By adding the phrase with the preposition (“.... according to the ...-argument.”) at the end of the sentences (such as in Step 3 and Step 4 of Petra’s text), the logical relations from the premises are also made explicit in the same sentence. In all other cases, in sentences with prepositions only the logical relation from the warrant to the conclusion is expressed. Prepositions are linguistically condensed forms of logical relations (called *metaphorical* by Halliday, 1985). This kind of linguistic phenomena is one of the most important characteristics of academic language to increase lexical density. However, these condensed forms (e.g. prepositions for logical relations) are challenging for students (Martin, 1999) and have to be



unpacked first into non-condensed forms before students can understand their meaning (O'Halloran, 1998, p. 382). In the context of this case study, mainly the logical relations between warrant and conclusion are articulated by condensed forms of prepositions instead of conjunctions. The qualitative analysis shows that the prepositions provide not only challenges to understand the logical relations, they also seem to hinder the explication of logical relations, from the premises to the warrant or to the conclusion or within a theorem. Prepositions also seem to be obstacles for combining several logical elements and steps.

## **Conclusion**

The analysis of students' products reveals three main findings: 1) The Toulmin model has not only limits for analysing proofs in general (Mariotti et al., 2018, p. 78), but in particular, for capturing the logical relations. 2) The linguistic analysis here suggests that in non-condensed forms some aspects are more often expressed than in condensed forms. This finding can be used to be more explicit on language while teaching proof as Mesnil (2013) has recommended. 3.) For these reasons, it might be useful to first offer non-condensed linguistic forms (conjunctions), before condensed forms (prepositions) are used in the classroom. This approach resonates with Martin's (1999) observations regarding the challenging condensed forms in academic language.

Of course, this case study has significant methodological limits as only 63 texts were analysed, that were produced within the specific setting and content. It has not yet been taken into account how the setting and the content influence the students' articulations. Future research is required to overcome these limits. Furthermore, the texts are written in German and not in English, so every interpretation can only be made in the original.

## **Acknowledgment**

I would like to thank my supervisor Susanne Prediger for the great support and intense cooperation. The doctoral thesis on which the article is based was carried out with the financial support of the Foundation of German Business.

## **References**

- Clarkson, P. C. (2004). Researching the language for explanations in mathematics teaching and learning. In P. Jeffrey (Ed.), *Proceedings of the Australian Association of Research in Education*. Melbourne: AARE.
- Durand-Guerrier, V. (2004). Logic and mathematical reasoning from a didactical point of view. A model-theoretic approach. In M. A. Mariotti (Ed.), *Proceedings of the Third Conference of the European Society for Research in Mathematics Education (CERME 3)*. Bellaria, Italy: University of Pisa/ERME.
- Durand-Guerrier, V., Boero, P., Douek, N., Epp, S. S., & Tanguay, D. (2011). Examining the Role of Logic in Teaching Proof. In G. Hanna & M. de Villiers (Eds.), *Proof and Proving in Mathematics Education: The 19th ICMI Study* (pp. 369–389). Dordrecht: Springer.
- Duval, R. (1991). Structure du Raisonnement Deductif et Apprentissage de la Demonstration. *Educational Studies in Mathematics*, 22(3), 233–261.
- Halliday, M. A. K. (1985). *Introduction to functional grammar*. London: Arnold.



- Hein, K., & Prediger, S. (2017). Fostering and investigating students' pathways to formal reasoning: A design research project on structural scaffolding for 9th graders. In T. Dooley & G. Gueudet (Eds.), *Proceedings of the Tenth Congress of the European Society for Research in Mathematics Education* (pp. 163–170). Dublin, Ireland: DCU/ERME.
- Knipping, C., & Reid, D. (2015). Reconstructing Argumentation Structures. In A. Bikner-Ahsbabs, C. Knipping, & N. Presmeg (Eds.), *Approaches to Qualitative Research in Mathematics Education: Examples of Methodology and Methods* (pp. 75–101). Dordrecht: Springer.
- Krummheuer, G. (1995). The Ethnography of Argumentation. In P. Cobb & H. Bauersfeld (Eds.), *The Emergence of Mathematical Meaning: Interaction in Classroom Cultures*. Hillsdale, New Jersey: Lawrence Erlbaum.
- Mariotti, M. A., Durand-Guerrier, V., & Stylianides, G. J. (2018). Argumentation and proof. In T. Dreyfus, M. Artigue, D. Potari, S. Prediger, & K. Ruthven (Eds.), *Developing Research in Mathematics Education - Twenty Years of Communication, Cooperation and Collaboration in Europe* (pp. 75–89). Oxon: Routledge.
- Martin, J. R. (1999). Mentoring semogenesis: 'genre-based' literacy pedagogy. In F. Christie (Ed.), *Pedagogy and the shaping of consciousness* (pp. 123–155). London: New York: Continuum.
- Mesnil, Z. (2013). A reference for studying the teaching of logic teaching in high school in France: A complex request for teachers. In B. Ubuz, Ç. Haser, & M. A. Mariotti (Eds.), *Proceedings of the Eighth Congress of the European Society for Research in Mathematics Education* (pp. 166–175). Ankara, Turkey: Middle East Technical University/ERME.
- O'Halloran, K. L. (1998). *Classroom Discourse in Mathematics: A Multisemiotic Analysis*. *Linguistics and Education*, 10(3), 359–388.
- Pedemonte, B. (2007). How can the relationship between argumentation and proof be analysed? *Educational Studies in Mathematics*, 66(1), 23–41.
- Prediger, S., & Hein, K. (2017). Learning to meet language demands in multi-step mathematical argumentations: Design research on a subject-specific genre. *European Journal of Applied Linguistics*, 5(2), 309–335.
- Schleppegrell, M. J. (2004). *The language of schooling: a functional linguistics perspective*. Mahwah, N.J: Lawrence Erlbaum Associates.
- Schleppegrell, M. J. (2007). The Linguistic Challenges of Mathematics Teaching and Learning: A Research Review. *Reading & Writing Quarterly*, 23(2), 139–159.
- Selden, J., & Selden, A. (1995). Unpacking the logic of mathematical statements. *Educational Studies in Mathematics*, 29(2), 123–151.
- Toulmin, S. E. (1958). *The Uses of Arguments*. Cambridge: Cambridge University Press.