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► To cite this version:

Hiroaki Hamanaka, Koji Otaki. Fundamental task to generate the idea of proving by contradiction. Eleventh Congress of the European Society for Research in Mathematics Education, Utrecht University, Feb 2019, Utrecht, Netherlands. <hal-02398101>

HAL Id: hal-02398101

<https://hal.science/hal-02398101v1>

Submitted on 6 Dec 2019

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Fundamental task to generate the idea of proving by contradiction

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The teaching of proof by contradiction involves a didactical paradox: students' efficient use of this proving method is hard to achieve in mathematics classes, although students' argumentation using this method can occasionally be observed in extra-mathematical contexts. To address this issue, Antonini (2003) proposed a task of the non-example-related type that could lead students to produce indirect proofs. The aim of this paper is to propose a new type of tasks and situations related to counterexamples that can lead students to argument by contradiction, not by contraposition.

Keywords: Mathematical logic, proof by contradiction, indirect proof, counterexample.

Introduction

This paper reports some results of developmental research on the teaching and learning of proof by contradiction. Despite the various studies on the subject, in general, mathematical proving seems to remain difficult to learn for most students over the world, especially regarding proving by contradiction. In Japan, mathematical proof is learned in lower secondary school, while more delicate proof methods are learned in senior secondary school. Proof by contradiction, one such delicate method, can be rather difficult to learn, as reported by several authors (Antonini & Mariotti, 2008; Reid, 1998). In particular, from the didactical and cognitive perspective, it involves curious conflicting aspects. On one hand, it would be far from a desirable understanding of the subject just to learn the fact that proof by contradiction is a correct method and the manner to build such proofs, since students could not be convinced of the conclusions in such proofs unless they understood why this method works. On the other hand, it has been observed that students sometimes spontaneously use the method of proof by contradiction as argumentation in extra-mathematical contexts, even before they develop any notions regarding this method (e.g. Freudenthal, 1973, p. 629).

To manage this paradoxical issue, Antonini (2003) proposed tasks and situations that can help students to generate the idea of indirect argumentations and proof (i.e. tasks involving *non-examples*, which shall be addressed in the second section). This proposal is based on the notion of *cognitive unity*, which emphasizes the similarity between processes of argumentation and proof construction (Garuti, Boero, & Lemut, 1998). The developmental principle of cognitive unity claims the importance of preceding argumentations to produce the conjecture, before the stage of proof construction. In summary, Antonini (2003) conducted a task design for the teaching of indirect proof using of the principle of cognitive unity. Our study follows this same line.

First, we clarify the notion of proof by contradiction especially confirming how it differs from proof by contraposition. Then, we review the preceding result of Antonini (2003) and identify its focus on proving by contraposition rather than by contradiction. Therefore, the main objective of this paper is to propose new tasks and situations that can lead students to argumentation using the idea of proof by

contradiction, beyond proofs by contraposition. We present the task designed for this study and investigate some results of a teaching experiment with this task.

Preliminary analysis for the design of a new task with proving by contradiction

The authors' position on proof by contradiction

This section confirms the definition of proof by contradiction and summarizes its relation to indirect proof. As Chamberlain & Vidakovic (2017) point out, some studies do not distinguish proof by contradiction from that by contraposition. While Lin, Lee, & Wu Yu (2003) attribute proof by contradiction to the law of contraposition, Antonini & Maritotti (2008) refer to such proof methods as 'indirect proving', which may also include the method of proof by contraposition. Logically speaking, the method by contradiction to indirectly prove the statement ' $P \rightarrow Q$ ' is to directly prove the statement ' $P \wedge \neg Q \rightarrow \perp$ '¹. This method is based on the law of excluded middle, which claims that ' $Q \vee \neg Q$ ' is true. In addition, the method by contraposition to indirectly prove ' $P \rightarrow Q$ ' is to directly prove ' $\neg Q \rightarrow \neg P$ '. This method is based on the principle that ' $P \rightarrow Q$ ' and ' $\neg Q \rightarrow \neg P$ ' are equivalent. Here, we observe that $(P \rightarrow Q) \rightarrow (\neg Q \rightarrow \neg P)$ is true, even in the intuitionistic logic, although $(\neg Q \rightarrow \neg P) \rightarrow (P \rightarrow Q)$ requires the law of excluded middle. Moreover, it is known that the law of excluded middle can be verified conversely using this contraposition rule with the intuitionistic logic. Thus, from the *logical* perspective, we can consider these proving methods interchangeable, in the sense that adding either the law of excluded middle or the contraposition principle to the intuitionistic logic results in the same classical logic.

In fact, we can show proof diagrams in which the two proving methods, support each other (Figure 1).

$$\begin{array}{c}
 \frac{\neg Q \rightarrow \neg P}{(P \wedge \neg Q) \rightarrow (P \wedge \neg P)} \\
 \frac{(P \wedge \neg Q) \rightarrow \perp}{P \rightarrow Q}
 \end{array}
 \qquad
 \begin{array}{c}
 \frac{\neg Q \wedge (\neg Q \rightarrow \perp) \rightarrow \perp}{\neg Q \rightarrow ((\neg Q \rightarrow \perp) \rightarrow \perp)} \\
 \frac{\neg Q \rightarrow \neg \neg \neg Q}{\neg Q \rightarrow \neg P} \\
 \frac{\neg Q \rightarrow \neg P}{P \rightarrow Q}
 \end{array}
 \qquad
 \begin{array}{c}
 \frac{P \wedge \neg Q \rightarrow \perp}{P \rightarrow (\neg Q \rightarrow \perp)} \\
 \frac{P \rightarrow (\neg Q \rightarrow \perp)}{P \rightarrow \neg \neg Q} \\
 \frac{P \rightarrow \neg \neg Q}{\neg \neg \neg Q \rightarrow \neg P}
 \end{array}$$

Figure 1: Two proving methods mutually supported

However, from the *cognitive* or *epistemological* perspective, we can point out certain differences. How should the argumentation by contraposition against the statement ' $P \rightarrow Q$ ' begin? It would be natural to begin the generation of argumentation by contraposition with the question 'what if Q is not true?', while it is desirable to begin argumentation by contradiction with the question 'is there any situation where " P " and "not Q " is possible?'. In addition, the middle processes of both types of argumentations have differences. In the case of contraposition, the argumentation starting from the assumption 'not Q ' would not result in any contradiction but in the conclusion 'not P '. Thus, the instances in this argumentation are possible and real under this assumption 'not Q '. On the other hand, the argumentation starting from the assumption 'if " P " and also "not Q " are possible' would lead to a contradiction. Therefore, the argumentation in the middle process deals with impossible cases. Moreover, we can see differences in the goal of the argumentations. In the case of contraposition, the conclusion to be reached can be specified as 'not P ' from the beginning, while, in the case of contradiction, the conclusion can be any type of contradiction. Therefore, the goal of this type of

¹ The symbol \perp means the contradiction, which is unconditionally false.

argumentation is not specified in the beginning and proof by contradiction seems to be more difficult than that by contraposition.

Despite these differences, both proving methods tend to be integrated as indirect proofs. We believe this to stem from a didactical reason, instead of a reason based on compatibility in logic. The principle of contraposition in the school mathematical context is not an axiom, but a meta-theorem that can be verified somehow. In fact, faced the question of why ' $P \rightarrow Q$ ' can be implied from ' $\text{not } Q \rightarrow \text{not } P$ ', they can possibly use the method by contradiction: 'if Q is not true under $P \dots$ '. Thus, it is didactically natural to verify the method of proof by contraposition based on contradiction, although the inverse is verifiable by logic. Therefore, from the didactic and cognitive perspectives, we consider that proof by contradiction is more fundamental and supports the method of proof by contraposition, which can be acknowledged as an application of proof by contradiction.

From investigation of non-example to non-existence of counterexample

As mentioned in the first section, Antonini (2003) proposed tasks involving non-examples against almost the same research question. Given a property A , non-example is an instance that negates A , while an ordinary example is one that verifies A . Antonini (2003) argues that, faced with a question such as 'given a hypothesis A , what can you deduce?', students tend to generate examples and (or) non-examples and, through the observation of non-examples that verifies B but does not satisfy A , students can assume that 'if B is true, A is not true'. From this argumentation, students may be led to obtain 'if A is true, B cannot be true', which is an indirect argumentation. Although such tasks and processes are reasonable enough, we would like to indicate room for further improvements: the argumentations that would be observed in these processes are proofs by contraposition, not by contradiction. Such argumentations certainly include ways of thinking such as '... if it were not so, it would happen that...'. However, because this argumentation starts from the observation of a non-example (not satisfying A), such thinking necessarily has the form 'if B is true, A cannot be true', which certainly corresponds to an argumentation by contraposition.

In contrast, our hypothesis is that the *principle of non-existence of counterexample* yields argumentation by contradiction. Generally speaking, the intension to *disprove* some statement may lead to the pursuit of its counterexample. However, it is difficult to prompt disproving activities in mathematics classes, which are usually filled by the requirement for *proof*. In the following subsection, we discuss a favourable condition involving counterexample towards argumentation by contradiction.

Judgement of truth and proof by contradiction

In the former part of this section, we defined the method of proof by contradiction. However, it is written in propositional logic, while there are many propositions of predicative logic that will be addressed in mathematics in upper secondary school or higher. In fact, in Japan, many common propositions to be proved in mathematics are universal propositions. Let S be a proposition such as 'for any x , if $P(x)$ is true, $Q(x)$ is true', and consider a task such as 'is this proposition S true?', that is, the judgement of the proposition S . Facing this type of task, what types of argumentations occur naturally? As a form of naïve argumentation, one may try to verify S in an empirical way: each instance x verifying $P(x)$ is examined to verify whether it satisfies $Q(x)$ or not. Then, the more

instances one checks, the more plausible S would be. In addition, this operative work might prompt students to understand what the negation of the statement S is. If the above examination of each instance x fails even once, in other words, if one finds an instance x satisfying $P(x)$ but not $Q(x)$, S turns out to be false. In class, the task of proving S makes it unquestionable, because such a task *implicitly* means that S is true. However, against the task of judgement, one may suspect its negation quite naturally. Moreover, in this type of task, which will be proposed in the following section, this suspicion is important and can lead students to the idea of proof by contradiction, as this suspicion is the initial point of the proving method by contradiction: if one doubts the truth of S , he or she can start argumentation from the assumption of the counterexample's existence, that is, an instance x satisfying $P(x)$ but not $Q(x)$. If this argumentation comes to a contradiction, the non-existence of a counterexample can be deduced; furthermore, it claims the truth of S , which is an argumentation by contradiction.

Regarding the proving by contradiction, Antonini & Mariotti (2010) point out two difficulties: the treatment of impossible mathematical objects and the link between the contradiction and the statement to be proved. In fact, when proving by contradiction, one should think of impossible and absurd cases and deduce a contradiction. One may not make sense of a contradiction from an absurd assumption that looks trivial and from which nothing is likely to be deduced. This is similar to the low assessment by students of proof of a *trivial* theorem, such as 'the base angles of an isosceles triangle are equal'. Likewise, in proof by contradiction, if the provisional assumption is apparently absurd, the deduced contradiction would make no persuasive argument. Thus, to promote students' argumentation by contradiction, it is important that the provisional situation, where the hypothesis is true but the conclusion is false, is unknown to be possible or impossible, and its possibility should be the focus of the discussion. This is exactly the case in the proposed tasks of judgement.

Proposal of the task and situations

First, it is necessary to recall the characterisations of a parallelogram (i.e. sufficient conditions for a quadrilateral to be a parallelogram). Although Japanese lower secondary schools do not use the word 'sufficient', they adopt the following five conditions in classes as 'conditions for a parallelogram':

- Two pairs of opposite sides are parallel. (Definition)
- Two pairs of opposite sides are equal in length.
- Two pairs of opposite angles are equal in measure.
- One pair of opposite sides are parallel and equal in length.
- The diagonals bisect each other.

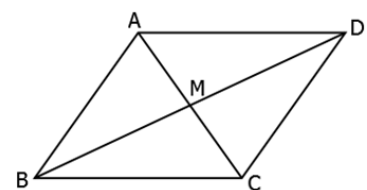


Figure 2: Parallelogram with Labels

Each of these five conditions consists of two conditions out of the following:

- | | |
|--|---|
| – a pair of opposite sides is parallel | – a pair of opposite sides is equal in length |
| – a diagonal bisects the other | – a pair of opposite angles is equal in measure |

Apart from the previous five sufficient conditions, various combinations of two out of the conditions specified above are possible. In particular, we adopt the following combination and the task in

connection with it: for a quadrilateral $ABCD$ with its diagonals crossing at M , does the condition of $\angle A = \angle C$ and $BM = MD$ make this quadrilateral a parallelogram? (See Figure 2.)

Since this condition is actually a necessary condition to make the quadrilateral a parallelogram, the figure of a parallelogram may give an impression that this condition is also sufficient. Thus, one may try to prove its sufficiency directly. In such an attempt to prove it, it would be natural to focus on a congruence of triangles, namely $\triangle AMB$ and $\triangle CMD$. Verifiable facts concerning these triangles from the given condition are $BM = DM$ and $\angle AMB = \angle CMD$. If one could claim that $AM = MC$, it is easy to conclude that this quadrilateral is a parallelogram using the congruence of triangles, although no more properties concerning $\triangle AMB$ and $\triangle CMD$ can be obtained directly from the assumption.

If this was against the task of proving a statement suggested by a teacher, which would be considered correct previously, then this proving attempt would end here and no further argumentation would hardly continue. However, in the task of judgement, one can proceed this argumentation as such a stagnation of verifying argumentation implies the possibility of a counterexample. A student can suspect the congruence of $\triangle AMB$ and $\triangle CMD$. In particular, if $AM = MC$ is true, they are congruent and the quadrilateral $ABCD$ turns out to be a parallelogram. Thus, one can imagine an instance satisfying $\angle A = \angle C$ and $BM = MD$, but $AM \neq MC$, or $AM < MC$. If one tries to draw a figure of an instance satisfying $BM = MD$, $AM < MC$ and also $\angle A = \angle C$, the impossibility of such an instance comes apparent. This might be seen by intuition (See Figure 3) and can also be verified as follows: if one take C' on MC with $AM = MC'$, we can see $\angle A = \angle BC'D > \angle C$ and obtain contradiction. This is proof by contradiction.

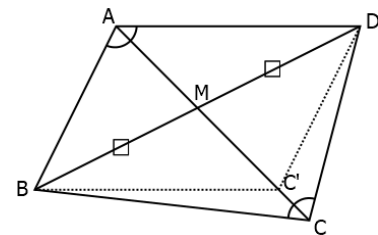


Figure 1: Impossibility of a counterexample

Teaching experiment

In this section, we report a teaching experiment in a typical upper secondary school in Japan. This experiment was mainly designed to investigate the learning about wide logical concepts through inquiries, not limited to the idea of proof by contradiction. The 40 students in our study are first years at a typical upper secondary school in Japan (15-16 years old) and, in general, students in this school are not highly competent in mathematics. The students were about to study logical notions in the small unit of 'sets' and had never studied proof by contradiction, not even the concepts of counterexample, sufficient condition, or necessary condition. They were divided into 12 groups of three or four students each. A main element of this experiment is the inquiry against the following task (Figure 4) given to the groups (cf. Hamanaka & Otaki, to appear).

<p>There is a quadrilateral $ABCD$ with its diagonals crossing at M. Find combinations of two out of the following eight conditions as many as you can, each of which makes this quadrilateral $ABCD$ a parallelogram: (a) $AB = CD$, (b) $AD = BC$, (c) $\angle A = \angle C$, (d) $\angle B = \angle D$, (e) $AM = MC$, (f) $BM = MD$, (g) $AB \parallel DC$, (h) $AD \parallel BC$.</p>	
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Figure 2: Task given in the teaching experiment

Among the 28 possible combinations of conditions, 16 are sufficient to make a quadrilateral a parallelogram, while 12 are not. Among the sufficient combinations, $\{(c), (f)\}$ and $\{(d), (e)\}$ are relatively difficult to be justified, and their proofs could involve proof by contradiction. The teaching experiment was carried through two teaching terms. While the first term was dedicated to students to do their own inquiries around the task, the second term was devoted to their reflection.

In fact, because of its difficulty, we did not expect the generation of the idea of proof by contradiction at the beginning. However, we could observe the generation of argumentation using that method in their inquiries as we report below. The following is a part of the conversation during the inquiries by a group of three students (A, B, and C) in the first term, which was mainly recorded by a voice recorder and partly by a video recorder. We were not able to identify their voices clearly and the speaker of the speech below was partly unspecified (Indicated as Student X). They were considering the case where $\angle A = \angle C$ and $BM = MD$.

Student X: As these lengths [BM and MD] are specified from the beginning and if we set that these angles [$\angle A$ and $\angle C$] are the same, I think it must be a parallelogram, right?

Student A: If we change these lengths [AM and MC], these angles [$\angle A$ and $\angle C$] must also change...

Student A: So, if we change these lengths [AM and MC] and also fix these angles [$\angle A$ and $\angle C$], these edges cannot meet.

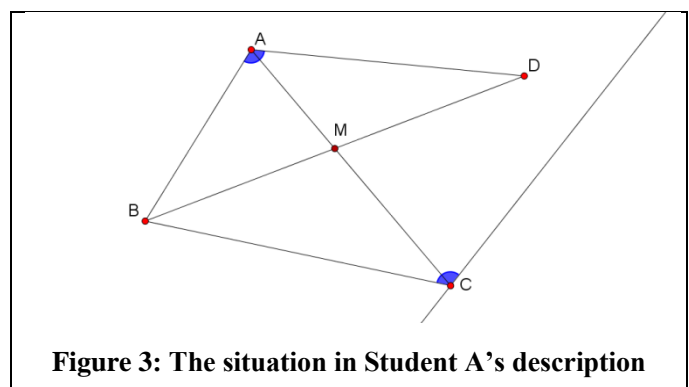
Student A: Wait, yes. If the angles [$\angle A$ and $\angle C$] are specified and the lengths [AM and MC] are different, it cannot be. Under these conditions [$\angle A = \angle C$ and $BM = MD$], it is impossible.

Student B: Well, why is it impossible? (A short silence follows.)

Student A: Well, listen. If we change the lengths [AM and MC], they [the edges CD and AD] cannot meet, can they? Let's see, if we keep these angles [$\angle A$ and $\angle C$] having the same measure, and also if we make this part [MC] longer...

Student C: Oh, they [CD and AD] will not meet... it [CD] cannot reach [D].

The final worksheet did not retain the figures used at the time of the above argumentation. It is assumed that Figure 4 illustrates the situation described by Student A. At that time, she was looking for a counterexample, that is, an instance that satisfies the given condition but is not a parallelogram. This idea led her to the argumentation 'if MC is longer than AM'. Their argumentation continued as follows:



Student B: Then, let's draw the figure (of the considered situation).

Student A: No way. We cannot draw it because it is impossible.

Student B: Oh, right. I see.

Student C: Oh, we cannot draw it...

These protocols clearly show that their argumentation involves contradiction: a situation that can be considered but cannot be possible. Then, she organized her idea, draw a new figure (see Figure 6), and spoke as follows:

Student A: The answer is yes [sufficient] for this case. Because ... Let us use this line [set AC on this line], [under the assumption $BM=MD$] then look at these lengths [AM and MC]. If these lengths are different and if one is even one block longer, then these angles [$\angle A$ and $\angle C$] cannot be the same in measure. Thus, if we assume that these angles [$\angle A$ and $\angle C$] are same, these lengths [AM and MC] are *definitely* the same, too!

This shows how confident Student A finally was at the correctness of her argumentation and illustrates the possibility of generating the idea of proof by contradiction through this type of argumentation that judges the truth of a universal proposition.

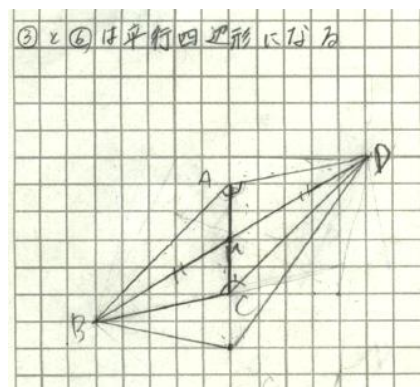


Figure 4: The final diagram
in their work sheet

Discussion and conclusion

Although we admit the weakness of our empirical evidence as our teaching experiment was not limited to proof by contradiction, we could observe small but specific data of successful argumentation in the teaching experiment. We would like to discuss how this proposed task relates to the generation of such argumentation. Moreover, this research is highly inspired by the research by Antonini (2003), which also describes that the non-example argumentation is not the only process for the generation of indirect proving. There is also a request for further research that develop other processes and related tasks. Since this paper is one of possible responses to such request, we also compare and discuss the differences between the argumentations by contradiction observed above and the indirect argumentations involving non-examples proposed by Antonini (2003).

As mentioned by Antonini (2003), in the students' argumentations, non-examples may work as generic examples; their observation, which indicates 'if B is true, it cannot be an example', can be generalized or expanded to the universal proposition of 'for any case, if A is true, B cannot be true'. This generalization is possible because the instances in consideration are real. On the other hand, although the proposed argumentation in this paper is also related to a universal proposition, nothing in the argumentation works as a generic example. The principle of this argumentation is the assertion of the non-existence of counterexamples, and it does not involve the effects of generalization or expansion. In fact, they deal with an impossible instance which can hardly be generalized or expanded. Thus, these two types of argumentations are entirely different. This difference comes from the fact that the proposed argumentation is directly based on proof by contradiction, not by contraposition.

In addition, the proposed argumentation essentially requires predicate logic: it is necessary to consider the negation of the predicate statement ‘for any x , $P(x)$ implies $Q(x)$ ’, that is, ‘for some x , $P(x)$ and not $Q(x)$ are true’, while, in the non-example argumentation, the phrase ‘for any x ’ is hidden or always prepositional. Although it seems rather complex at first glance, such an argumentation is supported by the features of the related task: first, this is not a proving task but a judgement task; second, this task may prompt the empirical verification activity that can stimulate the understanding of the negation of a universal proposition. This fact seems to be a reason of the didactic paradox of proof by contradiction mentioned in the introduction: between spontaneous proving by contradiction in extra-mathematical situations and the difficulty of understanding it in school mathematical situations. In the classroom, both the judgement task (or disproving task) and the empirical verification usually live in a narrower space than the proving task with theoretical ways.

Acknowledgment

This work is supported by JSPS KAKENHI (JP15H03501 and JP17H02694).

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