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To cite this version:
Fikret Cihan, Hatice Akkoç. Developing Pre-service Mathematics Teachers’ Pedagogical Content Knowledge of Proof Schemes: An Intervention Study. Eleventh Congress of the European Society for Research in Mathematics Education, Utrecht University, Feb 2019, Utrecht, Netherlands. hal-02398044

HAL Id: hal-02398044
https://hal.science/hal-02398044
Submitted on 6 Dec 2019

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Developing Pre-service Mathematics Teachers’ Pedagogical Content Knowledge of Proof Schemes: An Intervention Study

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The purpose of this paper is to examine the effects of an intervention study which aimed at improving pre-service mathematics teachers’ pedagogical content knowledge (PCK) of proof schemes. Sowder and Harel’s (1998) framework of proof schemes constitutes the conceptual framework of this paper. To explore twenty-two pre-service teachers’ PCK, we designed a survey using a scenario. The quantitative findings of the study revealed that the intervention had a meaningful and large effect on participants’ PCK of proof schemes. The qualitative findings of the study indicated that participants had difficulties identifying especially symbolic, example-based, transformational and axiomatic proof schemes prior to intervention, but they had overcome these difficulties after the intervention.

Keywords: Proof, proof schemes, pedagogical content knowledge, pre-service mathematics teachers

Introduction

Teaching proof is an extremely difficult matter for teachers to overcome (Heinze & Reiss, 2004). For teaching proof effectively, teachers should be able to determine types of justification used by their students and help them enhance their justification types to be able to reach the axiomatic level. In other words, they should have pedagogical content knowledge (PCK) of proof. In this paper, we are particularly interested in proof schemes which could be described as cognitive characteristics of proving processes (Harel & Sowder, 1998). However, there is a gap in the literature which explores or aims to develop PCK of proof schemes which is crucial for pre-service mathematics teachers. In order to address this problem, mathematics teacher educators should look for different ideas to teach proof in teacher preparation courses (Stylianides & Stylianides, 2017). Lack of such courses and instructional materials in pre-service mathematics teacher education programs call for intervention studies that foster PCK of proof.

Conceptual Framework of the Study: PCK of Proof Schemes

The conceptual framework of this study adopts two different frameworks: pedagogical content knowledge (Shulman, 1987) and proof schemes (Harel & Sowder, 1998; Sowder & Harel, 1998). Shulman (1987, p. 8) describes PCK as an “amalgam of content and pedagogy that is uniquely the province of teachers”. Various researchers describe different components of PCK. Among them, knowledge of students’ thinking has been extensively studied in teacher education literature (Depaepe, Verschaffel, & Kelchtermans, 2013). Knowledge of students’ thinking in a specific domain such as proof has a topic-specific dimension in it. With this regard, Lesseig (2016) created the “MKT for Proof” framework by adapting the Mathematical Knowledge for Teaching (MKT) framework developed by Ball, Thames, and Phelps (2008). She defines a subdomain of PCK as the
knowledge of content and students (KCS) for proof which involves “knowledge of students’ typical conceptions or misconceptions of proof as well as an understanding of developmental sequences” (p. 256). More specifically, it is the knowledge of “characteristics of external, empirical and deductive proof schemes, students’ tendency to rely on authority or empirical examples, typical progression from inductive to deductive proof” (p. 257). In this study, we focus on PCK of proof schemes, knowledge of students’ proving processes and identifying students’ proof schemes in particular.

Proof schemes are cognitive characteristics of the proving process and describe one’s methods of justification. Harel and Sowder (1998, p. 244) discovered undergraduate students’ categories of proof schemes each of which “represents a cognitive stage and intellectual ability in students’ mathematical development”. They offered three main categories and their sub-categories:

The first category, external proof schemes, points out to an external source that convinces the student. Also, students persuade others by referring to these external sources. When this source is an authority (e.g. a teacher or a textbook), it is called authoritarian proof scheme. The external source might also be the form or appearance of arguments e.g. proofs in geometry must be in two columns. In this case, it is called ritual proof scheme. The last sub-category of an external proof scheme is symbolic proof schemes which refer to meaningless manipulations of symbols (Harel, 2007).

The second category is empirical proof schemes. For this scheme, “conjectures are validated, impugned, or subverted by appeals to physical facts or sensory experiences” (Harel & Sowder, 1998, p. 252). This could be in two ways: (a) relying on evidence from one or more examples (example-based proof schemes) or (b) relying on intuition or perception to convince or to be convinced (perceptual proof schemes) (Harel, 2007).

The third category is analytical proof schemes which is at the highest level of justification. In this case, conjectures are validated by means of logical deductions. It has two sub-categories: transformational and axiomatic proof schemes. Transformational proof schemes have three characteristics: generality, operational thought, and logical inference. (Harel, 2007). Generality is concerned with justifying “for all”. Operational thought takes place when a student “forms goals and subgoals and attempts to anticipate his/her outcomes during the proving process” (Harel, 2007, p. 67). Finally, logical inference requires mathematical justification based on the rules of logical inference. In addition to these three characteristics, in the axiomatic proof scheme, proving processes are built upon an axiomatic system, therefore must start from accepted principles (Harel, 2007).

The aim of the study and the research question

This study is part of a PhD thesis which aims to design an undergraduate course for developing pre-service mathematics teachers’ (PSMTs) view, content and pedagogical content knowledge of proof. The aim of this study is to report the findings of an intervention study which aims to develop pre-service teachers’ knowledge of proof schemes. As part of the course, a module on proof schemes, was implemented and our research question is as follows: “How does the module affect pre-service mathematics teachers’ pedagogical content knowledge of proof schemes?”
Methodology

This study used designed-based research (DBR) and specifically ADDIE (Analysis, Design, Development, Implementation, and Evaluation) model (Branch, 2009). In the analysis phase, the needs analysis was made based on the literature. Calls for the design and analysis of interventions that foster PCK of proof and proving, proof schemes in particular, in the context of pre-service teacher education were considered. The objectives and learning outcomes of the course were determined. One of them was related to proof schemes: “PMTs will be able to describe and identify students’ proof schemes”. In the design phase, a 15-week course consisting of various modules (i.e. modern components of proof, proof methods, identifying proof schemes, student difficulties with proving, reasons behind student difficulties, teaching strategies that can overcome student difficulties) was prepared and expert opinion was taken. In the development phase, learning and teaching situations were organized. For the module on proof schemes, the classification of proof schemes by Sowder and Harel (1998) were explained to participants using two scenarios. Participants worked both individually and in groups to identify students’ proof schemes using the scenarios. After the intervention, the learning outcome related to proof schemes was evaluated using a different scenario. In the evaluation phase, the effect of the module on the learning outcome related to proof schemes was evaluated.

Participants are twenty-two PMTs who are in the second year of a teacher education program in a state university in Istanbul, Turkey. To explore PCK of proof schemes, we designed a survey called Pedagogical Content Knowledge of Proof Survey (PCK-P survey) using a scenario (See Appendix). The scenario includes an excerpt of a hypothetical discussion among a mathematics teacher and ten 9th grade students (age of fifteen). Prior to the intervention the PMTs were asked to describe student thinking; after the intervention, the PMTs were specifically asked to describe students’ proof schemes. The topic is set theory which is a typical topic in 9th grade curriculum in Turkey and the class discusses the truth of a proposition. Sowder and Harel’s (1998) proof schemes are illustrated by excerpts of students (See Table 2 for which student has each scheme). For validity concerns, the number of students was chosen to be ten which is bigger than seven (which is the number of proof schemes) to prevent participants from matching students’ work to the proof schemes. The teacher presents a proposition and asks students whether this proposition is true or false and justify their answers: “Let $X$, $Y$, and $Z$ be sets. If $X \subset Y$ and $Y \subset Z$ then $X \subset Z$”. The survey includes the following questions concerning the scenario: “Describe how students S1,...,S10 justify their answers to the truth of the proposition. Is the proof correct”. To increase the validity of the findings, the scenario used as data collection tool was chosen to be different from the scenario used during the module. The topic was also different. PMTs filled the PCK-P survey before and after the intervention (fifteen weeks later). In the second implementation, PMTs were directly asked to identify proof schemes of students.

We analyzed qualitative data to explore the effectiveness of the module and obtained quantitative findings which will also be supported with written explanations of participants. Each participant identified ten students’ proof schemes, therefore there is a total of 220 answers. 220 answers from 22 participants were coded as “correct”, “incorrect” or “no response”. We used the Wilcoxon
Signed Rank Test (Wilcoxon, 1945) to investigate whether the module significantly affected participants’ PCK. The effect size was calculated using the formula $r = Z/\sqrt{n}$.

**Findings**

Table 1 below presents the frequencies and percentages for correct, incorrect and no response categories. Percentages were calculated out of a total of 220 answers. As can be seen in Table 1, findings indicate a development of PCK of proof schemes. The percentage of correct answers is 61.4% before the intervention and it increases to 96.4% after the module implementation. The number of correct answers increased by 77 which represents 35%.

<table>
<thead>
<tr>
<th></th>
<th>Correct (%)</th>
<th>Incorrect (%)</th>
<th>No Response (%)</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre-intervention</td>
<td>135 (61.4)</td>
<td>82 (37.3)</td>
<td>3 (1.4)</td>
<td>220</td>
</tr>
<tr>
<td>Post-intervention</td>
<td>212 (96.4)</td>
<td>8 (3.6)</td>
<td>0 (0.0)</td>
<td>220</td>
</tr>
</tbody>
</table>

**Table 1: Frequencies and percentages for answers before and after the intervention**

Mean, standard deviation, minimum and maximum points were also calculated (over 10 which is the number of questions) considering answers to the survey before and after the intervention. The mean value increased from 6.14 to 9.64, minimum value increased from 3.00 to 8.00 and maximum value increased from 9.00 to 10.00. Standard deviation decreased to 0.58 from 1.49.

<table>
<thead>
<tr>
<th>Student no</th>
<th>Type of Proof Scheme</th>
<th>Correct (Pre-/Post-)</th>
<th>Incorrect (Pre-/Post-)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Authoritarian</td>
<td>19/18</td>
<td>2/4</td>
</tr>
<tr>
<td>2</td>
<td>Authoritarian</td>
<td>19/22</td>
<td>3/0</td>
</tr>
<tr>
<td>3</td>
<td>Perceptual</td>
<td>17/22</td>
<td>5/0</td>
</tr>
<tr>
<td>4</td>
<td>Ritual</td>
<td>18/22</td>
<td>4/0</td>
</tr>
<tr>
<td>5</td>
<td>Symbolic</td>
<td>0/20</td>
<td>22/2</td>
</tr>
<tr>
<td>6</td>
<td>Example-based (a single example of finite sets)</td>
<td>21/22</td>
<td>1/0</td>
</tr>
<tr>
<td>7</td>
<td>Example-based (a single example of infinite sets)</td>
<td>13/21</td>
<td>9/1</td>
</tr>
<tr>
<td>8</td>
<td>Example-based (multiple)</td>
<td>14/21</td>
<td>8/1</td>
</tr>
<tr>
<td>9</td>
<td>Transformational</td>
<td>2/22</td>
<td>19/0</td>
</tr>
<tr>
<td>10</td>
<td>Axiomatic</td>
<td>12/22</td>
<td>9/0</td>
</tr>
</tbody>
</table>

**Table 2. Frequencies of answers for proof schemes before and after intervention**

We used the Wilcoxon Signed Rank Test to decide whether differences in PCK scores is significant. A Wilcoxon signed rank test showed that there was a significant difference ($Z = -4.144, p < 0.001$) between scores obtained before and after the intervention. Outputs for mean ranks of
difference scores and sum of ranks imply that this significant effect is in favor of positive ranks, in other words, post-intervention. In addition, all participants’ scores of PCK after intervention are higher than their scores prior to intervention. Effect size ($r$) was found as $-0.88$. Since the absolute value of effect size is 0.88 which is greater than 0.50, it can be said that the module has a large effect size on the scores of PCK with regard to proof schemes in favor of post-intervention (Cohen, 1988).

As Table 2 shows, participants had difficulties to identify symbolic, example-based, transformational and axiomatic proof schemes before the intervention. They had overcome most of these difficulties after the intervention. Below, we present examples of these cases.

Student 5 has a symbolic proof scheme. He justified his answer considering the number of elements (See the lines 21-22 in Appendix): “$S(X) < S(Z)$ then $X \subset Z$”. Using this statement which is wrong, Student 5 uses a shallow symbolic manipulation. None of the participants correctly identified that this student has a symbolic proof scheme because participants also thought that it is a valid proof. After the intervention, 20 out of 22 participants identified Student 5’s proof scheme correctly. They used the terminology of proof scheme framework.

In the scenario, we prepared three different cases of an example-based proof scheme using (a) a single example of finite sets (Student 6), (b) a single example of infinite sets (Student 7) and (c) multiple examples (student 8). Before the intervention, 21 out of 22 participants noticed that Student 6 relied on only one example. After the intervention, all participants identified the proof scheme of Student 6 correctly. For the cases of (b) and (c), frequencies of correct answers increased considerably after the intervention (See Table 2). For (b), after the teacher called out for a more general example, Student 7 justified his answer using the sets $N$, $Z$, and $R$ which are infinite (See the lines 32-34 in Appendix). We consider this as an “example-based proof scheme using a single example” as in the case of (a) except the fact that $N$, $Z$, and $R$ are infinite sets. However, before the intervention, nine participants could not identify the proof scheme in the case of (b) because they thought that this is a generalization. Since they considered the student’s scheme as a generalization rather than example-based scheme, we coded their responses as incorrect. However, after the intervention, they improved in identifying this scheme (21 out of 22 participants correctly answered).

Student 8 suggested each student find one example so that there would be many examples to justify the truth of the proposition (See the lines 36-37 in Appendix). 8 out of 22 participants could not identify Student 8’s scheme correctly as “example-based” before the intervention. The main reason is that they thought multiple examples are convincing for a generalization:

PMT13: He reaches a generalization by a different example for each one in the class.

PMT17: He justifies by making a generalization and uses many examples that show the truth of the proposition so many times.

After the intervention, 21 out of 22 participants correctly identified Student 8’s justification as “example-based”.

Student 9 has **transformational proof scheme** since he reached a generalization through operational thought based on logical inference (See the lines 42-43 in Appendix). Before the intervention, only two participants could identify the proof scheme correctly, because others did not refer to any components of this scheme (generalization, operational thought or logical inference) in their explanations about Student 9’s justification. After the intervention, all participants identified the proof scheme correctly.

Student 10 has **axiomatic scheme** since he started the proof by using the definition of a subset and successfully completed the proof as can be seen in the scenario in Appendix (See the lines 47-48). Before the intervention, 12 out of 22 students could identify this scheme. Others just mentioned that it was a mathematical proof. After the intervention, all participants identified the proof scheme correctly because Student 10 used the modern components of an axiomatic system.

**Discussion and Conclusion**

Data indicated that the course module had significantly affected participants’ scores of PCK of proof schemes. The intervention was effective especially in overcoming difficulties with identifying symbolic, example-based, transformational and axiomatic proof schemes. Before the intervention, participants could not identify students’ shallow symbolic manipulation. Instead, they were probably convinced that the proof was valid just because it included symbols. For symbolic proof scheme, the course module included discussions of many examples of meaningless symbolic manipulation. After the intervention, participants improved in identifying this scheme. In the scenario, we expanded the notion of example-based proof schemes and included three cases (a single example of finite sets or infinite sets and multiple examples) and participants performed differently in each case. Before the intervention, participants found it more convincing compared to a proof with a single example which includes finite sets. In sum, although they identified an example-based proof scheme, they had difficulties in identifying proof schemes of students who used infinite sets and multiple examples. These findings indicate the importance of teachers’ awareness of how students may view examples and noticing aspects of example use (Tsamir, Tirosh, Barkai, & Levenson, 2017). The module which included different cases of example-based proofs helped them overcome their difficulties.

The module was also effective for overcoming participants’ difficulties in identifying transformational proof scheme by focusing on practices of generalizations using rules of logical inference and operational thought. Participants were more successful with identifying axiomatic proof schemes when compared to transformational scheme probably because they were more familiar with the modern components of proof such as definitions and axioms. However, before the intervention, they did not refer to these components to explain students’ proving processes.

Considering the potential of scenarios to investigate PCK of proof schemes as implied by the findings of this study, we suggest that future studies could design scenarios focusing on proofs in different content areas. We also suggest that scenarios could be used in transition courses in undergraduate mathematics programs as well as teacher preparation programs. However, one should consider potential limitations of assessing the knowledge of identifying proof schemes using scenarios which could not reflect the complexity of a classroom. The second cycle of intervention
could focus on teaching and learning situations in real classroom settings. Using Sowder ve Harel’s (1998) notion of proof schemes which is a psychological construct, we focused on the psychological aspects of proof. Future studies could consider epistemological and sociological aspects of a mathematical proof.

Acknowledgment

This study is part of a research project sponsored by Marmara University Scientific Research Projects Committee with project number EGT-C-DRP-120418-0202.

References


Appendix. The scenario used in the PCK-P survey

Teacher: Is the proposition below true or false? If true, why? If false, why? Justify your answer.
Proposition: “Let \( X, Y \) and \( Z \) be sets. If \( X \subseteq Y \) and \( Y \subseteq Z \) then \( X \subseteq Z \)”

Student 1: I think, it is false. We’ve seen a lot of rules about sets. But I don’t remember this one.
Teacher: OK. We haven’t seen it in a lesson. Couldn’t it be still true?
Student 1: I’ve never heard of a rule like this. Therefore, I think it’s false.
Student 2: Teacher, it is true. Because this theorem is in our maths textbook. So, it is definitely true.
Teacher: Do you think this is enough for a justification? You didn’t write or do anything about it.
Student 2: I think it’s enough. Why do we need another kind of justification if it’s in the textbook?
Teacher: It’s very important for us to reason about truth or falsity of a proposition.
Student 3: Teacher. May I draw a picture?
Teacher: Of course, you can.
Student 3: I think it’s obvious from the picture.
Teacher: (heading towards the class) Is it enough for a proof? Just to draw a picture?
Student 4: Well, in fact. Every element in \( X \) is also an element of set \( Y \). Every element in set \( Y \) is also an element of \( Z \). I can express truth of the theorem. But we should do something mathematical. But I can’t do it. Theorems should be proven using mathematical statements. But it shouldn’t be. Verbal expressions, like I use, convince me much more.
Teacher: How did you come to this conclusion that proofs consist of mathematical statements only?
Student 4: Because proofs I’ve seen so far are just like that.
Teacher: Is there anyone who could use mathematical statements?
Student 5: If \( X \subseteq Y \) then \( S(X) < S(Y) \) and if \( Y \subseteq Z \) then \( S(Y) < S(Z) \). Therefore \( S(X) < S(Z) \) that is \( X \subseteq Z \).
Teacher: If the number of elements of a set is smaller than the number of elements of another set, then does it mean that the first set is a subset of the second set?
Student 6: I think not. I think that there is an easier way.
Teacher: What is that?
Student 6: Let \( X \subseteq Y \) and \( Y \subseteq Z \). Let \( X = \{1,2\}, Y = \{1,2,3\} \) and \( Z = \{1,2,3,4\} \). Since \( \{1,2\} \subseteq \{1,2,3\} \subseteq \{1,2,3,4\} \) then \( X \subseteq Z \).
Teacher: Well. Do you think that this example is sufficient?
Student 6: Now it is true. I think it is sufficient.
Teacher: (heading towards the class) Do you think that this is sufficient?
Student 7: Not that example. But it would be sufficient if we justify with a more general example.
Teacher: For example?
Student 7: \( N \subseteq Z \) and \( Z \subseteq R \). Therefore \( N \subseteq R \).
Teacher: That is a more general example. But still, it is not sufficient for generality issue of a proof.
Student 8: Teacher! If each one of us in the class finds an example to show the truth (of the proposition), then we can reach a generalization.

Teacher: When I talk about a generalization, it means it is true for all $X, Y$ and $Z$. We can reach a generalisation through the rules of logical inference and operational thought. That is, using other rules we should reach a judgement from a hypothesis through operational thought. Is there anyone who could reach a generalization using what we’ve done in our previous lessons?

Student 9: Let $X \subset Y$ and $Y \subset Z$. Considering the rules we mentioned in our lessons, if $Y \subset Z$ then $Y \cup Z = Z$. $X \subset Y$ then $X \cup Z = Z$. If $X \cup Z = Z$ then $X \subset Z$.

Teacher: That is correct. However, it is better if we think of the modern components of proof. It is appropriate to start a proof with definitions and axioms. Is there anyone who could prove it using the definition of a subset?

Student 10: Let $X \subset Y$ and $Y \subset Z$. In this case, from the definition of a subset, if $X \subset Y$ then for $\forall a \in X \ a \in Y$. If $Y \subset Z$ then for $\forall a \in X \ a \in Z$. Therefore, since for $\forall a \in X \ a \in Z$ then $X \subset Z$. It’s proven.