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# Integrating Euclidean rationality of proving with a dynamic approach to validation of statements: The role of continuity of transformations

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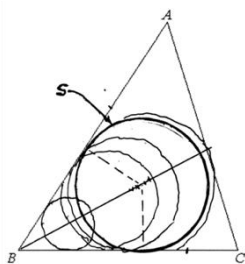
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*During a long-term teaching experiment aimed at developing 10<sup>th</sup> grade students' culture of theorems through a pathway in Euclidean plane geometry, some students' autonomous reasoning moved towards non-Euclidean proofs based on continuity of transformation of geometric figures. Based on the use of existing analytical tools to analyze such episodes, the aim of this paper is to outline a wider scope for synthetic geometry in order to make it more suitable for students' approach to the culture of theorems. Through the introduction of a continuity principle to legitimate such extension, the paper suggests how to exploit students' potential in transformational reasoning, and to bridge the gap between synthetic geometry and analytic geometry rationalities in classroom work.*

*Keywords: Euclidean geometry, transformational reasoning, continuity principle, rationality.*

## Introduction

Since 2016 we are engaged in designing and implementing in 10<sup>th</sup>-grade classes a teaching and learning pathway in Euclidean geometry aimed at promoting students' approach to the culture of theorems (Bartolini Bussi, Boero, Ferri, Garuti, & Mariotti, 2007) – i.e. at developing not only the knowledge of statements, proofs and their applications, but also autonomous proving and the awareness of crucial meta-knowledge about theorems (the role of hypotheses and thesis, the requirements of proof, etc.). Such aim is not easy to attain; the question that originated the study reported in this paper was: is it possible to lessen the students' difficulties with proving by legitimating some spontaneous ways of solving geometry problems and validating statements? Let us consider an episode, a single case of a wider phenomenon observed in the three classes where the experiments on our pathway have been performed. Its role is to put into evidence the potential inherent in students' dynamic approach to solving theoretical problems in geometry. Students have already learnt to construct a circle tangent to both sides of an angle, and to justify the construction. In interaction among them and with the teacher, students have learned to build up the circle with center chosen on the bisector and ray derived from the following construction: to draw the perpendicular line from the chosen center to one side of the angle; to consider its intersection with that side, and the segment joining it with the center as the ray of the circle. They have proved (with the help of the teacher) that the drawn circle is tangent to both sides of the angle through the steps of Euclid' validation of the same construction. Some weeks later they are asked to construct a circle tangent to the three sides of a given triangle. They have at their disposal a worksheet with a drawn triangle, the ruler and the compass.



**Figure 1**

One student (Ale) draws the bisector of the angle ABC, then he tries to choose a point on the bisector as the center of a circle by taking different points on the bisector and adapting the width of the compass to the sides of the triangle. Finally, he draws the circle S (the arrow and S are added to the original figure). Ale is not satisfied with his drawing. After a while he makes a free-hand drawing of a circle near to B, then other circles more and more near to AC. Finally, he stops and observes the worksheet for several seconds. Then he starts writing: “The solution of the problem is when the circle, which is tangent to the sides of the angle B, meets the third side”. The participant observer (PO: the first author of this paper) starts an interaction with Ale:

PO: May you explain your reasoning to me?

Ale: The tangent circle becomes bigger and bigger, and a certain moment a circle will meet the third side. It will be the solution!

PO: Why? Are you sure that it is tangent to the three sides of the triangle?

Ale: Yes, when I move the center on the bisector the circle is (pause) the circle becomes bigger and bigger, and remains tangent to the two sides of the bisector (pause)

PO: Are you sure that it becomes tangent to the third side?

Ale: Because the circle (pause) if I continue moving the point on the bisector, one part of the circle will go outside the triangle (pause) Therefore there will be ONE (emphasis) point to get the contact, (pause) the tangency with the third side.

PO: And if the circle becomes bigger and bigger?

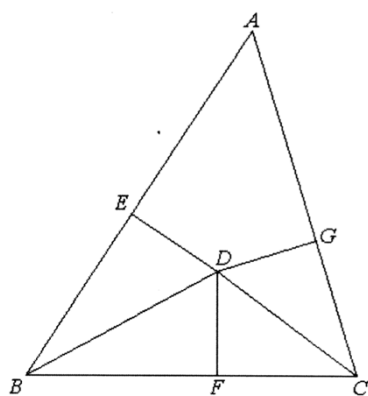
Ale: It will be no more tangent to the sides of the triangle, but... No, if I come back, the two intersections finally join in the tangency point.

Given that the circle exists (by such continuity considerations) it is easy to prove that its center belongs to the three bisectors - provided that students already know that if a point is equidistant from the sides of an angle, it belongs to the bisector of the angle, and that the tangent straight line is perpendicular to the ray in the point of tangency. Hence the ruler and compass construction may be easily made by using the intersection point of two bisectors and the perpendicular straight line from it to one side of the circle. In this case *the method of construction derives from the reasoning used to prove the existence of the tangent circle* and the knowledge of its properties. Note that performed exploration might result in the construction of a theoretical justification (*cognitive unity of theorems*: see later), once Ale’s reasoning by continuity would be legitimated.

In Book IV of the Elements (prop.4) Euclid describes how to construct a circle tangent to the three sides of a given triangle and provides a theoretical justification for it.

**PROPOSITION 4** (book 4, Heath’s translation)

In a given triangle to inscribe a circle.



**Figure 2**

Let ABC be the given triangle; thus it is required to inscribe a circle in the triangle ABC.

Let the angles ABC, ACB be bisected by the straight lines BD, CD [I. 9], and let these meet one another at the point D; from D let DE, DF, DG be drawn perpendicular to the straight lines AB, BC, CA.

Now, since the angle ABD is equal to the angle CBD, and the right angle BED is also equal to the right angle BFD, EBD, FBD are two triangles having two angles equal to two angles and one side equal to one side, namely that subtending one of the equal angles, which is BD common to the triangles; therefore they will also have the remaining sides equal to the remaining sides;

[I. 26], therefore DE is equal to DF. For the same reason DG is also equal to DF. Therefore the three straight lines DE, DF, DG are equal to one another; therefore the circle described with centre D and distance one of the straight lines DE, DF, DG will pass also through the remaining points, and will touch the straight lines AB, BC, CA, because the angles at the points E, F, G are right.

For, if it cuts them, the straight line drawn at right angles to the diameter of the circle from its extremity will be found to fall within the circle : which was proved absurd; [III. 16] therefore the circle described with centre D and distance one of the straight lines DE, DF, DG will not cut the straight lines AB, BC, CA; therefore it will touch them, and will be the circle inscribed in the triangle ABC. [IV. Def. 5]. Let it be inscribed, as FGE.

Euclid's line of thinking is different from Ale's. Euclid describes how to solve the problem of the circle inscribed in a triangle by *finding candidates to be the center of the circle and three of its rays*, then a *theoretical justification for the chosen solution follows*, which relies on definitions and previously proved theorems, and results in a proof of the existence of the inscribed circle. No continuity or transformational considerations are made.

Based on the above episode and other episodes that will be shortly presented later, and with reference to some constructs in mathematics education literature, integrated with a *principle of continuity to legitimate the widening of the scope of synthetic geometry*, we will present and discuss the potential inherent in students' transformational reasoning (Simon, 2006) in the approach to the culture of theorems, once that principle is assumed.

## **Theoretical background**

### **Transformational reasoning, and the continuity principle**

Transformational reasoning was defined by Simon (1996) as:

The mental or physical enactment of an operation or set of operations on an object or set of objects that allows one to envision the transformations that these objects undergo and the set of results of these operations. Central to transformational reasoning is the ability to consider, not a static state, but a dynamic process by which a new state or a continuum of states are generated (p. 201).

In past research of our group this construct was already used, in particular, to characterize one of the types of generation of conditionality of statements (Boero, Garuti, & Lemut, 1999). In this paper

it will be used to describe processes of discovery of the reason why a statement is true, or strategies to solve a construction problem in Euclidean geometry.

In our toolkit we will integrate the construct of transformational reasoning with a *principle of continuity*. Continuity provides students not only with hints for the solution of a construction problem or a proving problem, but also, in some cases, substantial elements for the validity of the construction and the proof. For instance, in the episode presented in the Introduction, Ale “by continuity” proves the existence of the tangent circle. Thus a principle of continuity as a criterion for the epistemic validity of a conclusion derived through transformational reasoning might widen the scope of acceptable proofs, with positive consequences for the approach to the culture of theorems (see Discussion). How to formulate the continuity principle within the perspective of synthetic geometry (i.e. geometry theory based on constructions performed with ruler and compass)? Continuity axioms (including Archimedes’ axiom) were among the axioms added by Hilbert to Euclid’s axioms (Trudeau, 1987), but it is not easy to formulate a consequence of them which in simple operational terms accounts for the truth of the “existence theorem” proved by Ale. However, it is possible to provide a formulation of the continuity principle in operational terms by making reference to analytic geometry:

The continuity principle guarantees the solution of a synthetic geometry problem through transformational reasoning provided that the translation of the problem and the related transformational strategy into analytic terms (by using algebraic expressions) allows a treatment which brings to the solution thanks to the continuity of the set of real numbers.

As an example, consider the episode presented in the Introduction. As a generic example (Mason & Pimm, 1984) of the situation, we may assume  $B=(0,0)$ ,  $a$  represented by  $x=0$ ,  $b$  represented by  $y=0$ , and the side  $AC$  represented by  $y=-2x+3$ . Then the system  $y=-2x+3$  &  $(x-K)^2+(y-K)^2=K^2$ , will represent the circle centered in the point  $(K, K)$  of the bisector, which is tangent to  $a$  and  $b$ . By varying  $K$  we represent the situations of no intersection with the straight line  $y=-2x+3$ , of tangency (two possibilities, including tangency from the exterior), and of intersection in two points.

We will use the above criterion as a provisional solution for the characterization of the continuity principle; further research (and the analysis of more episodes!) is needed to formulate it in more precise and effective terms.

## Theorems

Mariotti (2001) defined a Theorem as a statement and its proof with reference to a theory (and related inference rules). With reference to Guala & Boero (2017), Mariotti’s definition

encompasses theorems related to various kinds of theories throughout history (e.g., Euclid’s, as well as Hilbert’s, geometry; graph theory, with its crucial reference to visual objects; 19th-century classical probability theory as well as Kolmogorov’s axiomatic theory, etc.), together with the different ways of considering proof since the Greeks (Grabiner, 2012) along with the cultures these ways came from (Siu, 2012) (p. 210).

In this paper we use the construct of Theorem to consider different, possible ways of proving the same theorem with reference to different theories and different inference rules.

## **Cognitive unity of theorems**

After having found some cases of theorems (in geometry, and in elementary arithmetic) for which students behaved in a similar way, Garuti, Boero and Lemut (1998) defined “cognitive unity of theorem” what happens for some theorems when:

during the production of the conjecture, the student progressively works out his/her statement through an intensive argumentative activity functionally intermingled with the justification of the plausibility of his/her choices. During the subsequent statement-proving stage, the student links up with this process in a coherent way, organizing some of the previously produced arguments according to a logical chain (p. 345).

The cognitive unity construct was also extended to the case of the relationships between the exploratory phase of proving a theorem, and the subsequent construction of a proof for that theorem (Garuti et al., 1998): indeed, the exploratory phase of proving shares some common aspects with conjecturing (as re-construction of the meaning, and appropriation, of a statement; and identification of elements for its validity).

The cognitive unity construct may be used to account for what happens when students produce, thanks to transformational reasoning, arguments that may be re-arranged in a proof (be it Euclidean, or based on a continuity principle: see the episode in the Introduction).

## **Rationality**

Many cultural activities (including mathematical ones) may be described as discursive activities sharing some common features: First, criteria to establish truth and falsity of propositions, and validity of reasoning. Second, strategies to attain the goal of the activity, which can be evaluated. Third, a specific language for social interaction and self- dialogue.

The *rationality* construct elaborated by Habermas (1998) may be exploited to move from such a superficial description to a deeper treatment of discursive activities. According to Habermas’ construct, rational behavior is characterized by: conscious taking in charge of truth and validity criteria (*epistemic rationality*), of strategies to attain the goal (*teleological rationality*), and of communication means (*communicative rationality*); and by dynamic links between knowing, doing and communicating in the rationality perspective (for a discussion of potential and limitations of Habermas’ construct as it was adapted to mathematics education, see Boero & Planas, 2014).

## **Where the episodes come from**

We think that it is important to put into evidence some salient features of the long term teaching experiment that provided us with the elements for the theoretical elaboration of this paper (in particular, Ale’s and other episodes like those presented here); in particular we agree with Simon (1996) when he says that transformational reasoning is a natural way of thinking, which needs to be “nurtured” - thus suitable cultural and educational choices must be performed in order to allow students to develop it.

## **A teaching-learning pathway to the culture of theorems**

We have chosen to base the approach to the culture of theorems in grade X of high school on the Euclid’s plane geometry for the following reasons: First, in Italy, since the end of the XIX century and for several decades Euclidean geometry was the main subject intended to allow secondary

students to meet theorems, proofs, proving. Second, Euclidean geometry offers the possibility to approach different kinds of proof (including proof by contradiction – the preferred indirect proof in Euclid's *Elements*). Third, through the alternation of “theorems” and “constructions to be validated” in Euclidean geometry the teacher is offered the opportunity of stressing the relevance of theoretical thinking in mathematics – in particular the distinction between a drawing and a geometric figure, and the need of moving from visual truth to theoretical truth (even if in Euclid's *Elements* the goal of going beyond visual evidence to validate statements is not completely accomplished: see Trudeau, 1987. However, in our design we have not followed any path taken as such from Euclid's *Elements*; we have chosen only some constructions and some theorems. In the case of theorems with a thesis consisting of more than one claim we have chosen only one claim (this is in the case of the theorem considered in the second and third episodes below).

### **The educational context of the episodes**

The episodes presented in this paper happened within a pathway to the culture of theorems in the domain of Euclidean geometry that had been experimented for the first time in the year 2016-17 in a 10<sup>th</sup>-grade class of a scientific oriented high school, and in the year 2017-18 in two 10<sup>th</sup>-grade classes of the same school, with two 50' lessons per week from October to May (out of 5 hours devoted also to other mathematics subjects: algebra, probability and statistics). Classroom activities were based on an alternation of individual activities, in some cases preceded by a constructive interaction with the teacher, and collective activities. Individual activities concerned six kinds of tasks: solution of construction problems, conjecturing, proving, analyzing, evaluating and improving some schoolfellow's productions, close activities regarding proof texts. Collective discussions were orchestrated by the teacher and in most cases concerned the comparison and the critical analysis of a few students' individual productions selected by the teacher. According to the aim of developing competencies related to the culture of theorems, the assessment method consisted of: the individual revision of individual work at the end of each of the three parts in which the pathway was divided, with careful identification and remediation of “what does not work” in each individual production, and an overall synthesis on the individual itinerary concerning difficulties met, still obscure points, reasons for mistakes, emotional problems, etc. This evaluation method was derived and adapted from a similar method currently adopted in several Genoa University courses for pre-service teacher education aimed at developing professional competencies of cultural analysis of the content to be taught (see Guala & Boero, 2017, for more information on the assessment method and its motivations).

### **Further episodes**

We have chosen three further episodes. Like that presented in the Introduction, they concern relations among circles and straight lines. They show different roles that may be played by transformational reasoning and continuity in the field of plane geometry, and how to legitimate them within the proposed theoretical framework, in particular through the continuity principle.

#### **The rolling circle**

Students are requested to find if it exists a position of a given circle, such that the circle is tangent to two sides of an angle, and to justify the answer. Some students imagine to roll the circle on one line towards the second line and they discover that “Yes, it exists, because there is a moment in which

the rolling circle starts to touch the second line; in that moment it is tangent to both lines”. This intuition of the rolling circle facilitates also the discovery that, in the found position of the circle, the center of the circle is the point of intersection of the two straight lines that are parallel to the sides of the angle at the distance of the ray of the circle. Students “see” the movement of the center of the circle in parallel with the first line, and (after the tangency position) with the second line.

By considering the generic case of the straight lines  $y=0$  and  $y=x$  and of the rolling circle  $(x-K)^2+y^2=1$ , the continuity principle may be applied to legitimate the solution found by the students.

All this suggests a way both to find the center of the circle when it is tangent to both lines (a heuristic function for the solution of a construction problem), and to explain why in that position it is tangent to both lines (a proving function, once the continuity principle is adopted) – in a perspective of cognitive unity of theorems.

### Comparing the length of the chords of a circle

The diameter of a circle is its longest chord, and the length of the chord increases when its middle point approaches the center of the circle.

This is part of proposition 15 of Book 3 of the Elements:

Of straight lines in a circle the diameter is greatest, and of the rest the nearer to the centre is always greater than the more remote.

Euclid proof is rather complex- it needs the proof of 6 intermediate statements.

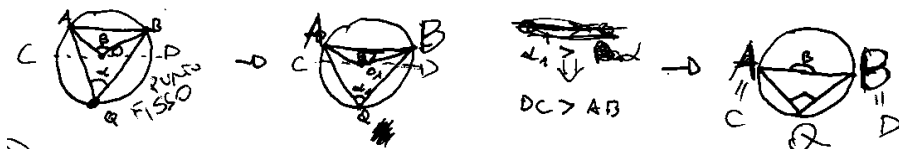


Figure 3

A student produces a reasoning that may be reported this way: he fixes a point on the circle and considers a chord on that side of the circle “which is opposite to it”, with corresponding angles at the center of the circle. He imagines to move the chord towards the center and he observes that the nearer the chord is to the center, the bigger is the corresponding angle –till when the chord becomes the diameter by collapsing on the two aligned rays. This is not yet a proof of the theorem, but the student’s line of reasoning might be integrated with elements that allow to prove that the length of the chord increases when the chord approaches the diameter. Indeed, the triangles obtained by joining the extremities of the chord with the center of the circle have two sides of equal length (the ray of the circle) and the width of the angle between them increases, thus also the length of the side opposite to the center increases. This may suggest to exploit the triangular inequality –the chord is shorter than the sum of the rays– to prove that the diameter (two rays long) is the longest chord; and it may suggest also to use Pythagoras’ theorem in order to prove that the length of the chords increases when the distance between the chord and the center of the circle decreases, or to find the length

$2\sqrt{1-K^2}$  of the chord intersected by  $y=K$  on the generic circle  $x^2+y^2=1$ : the length of the chord attains its maximum value 2 when  $K=0$ , i.e. when the chord becomes a diameter.



## **Tangency between a straight line and a circle.**

By composing two statements of Euclid's Elements, we get the following statement:

Given a circle centered in C and a straight line a intersecting the circle in the point T, the straight line a is tangent to the circle (i.e. T is their only common point) if and only if the straight line is perpendicular to CT.

While trying to prove the “if” part of the statement, students were suggested by the teacher to reason by contradiction; they were also invited to consider a second point of intersection T'. Some students reacted to the discovery of a contradiction (an isosceles triangle with two rectangular angles) by imagining to make CT' collapse on CT by rotating it around C. This movement might be exploited to expand their reasoning through the consideration of the isosceles triangle TCT' and its height CH. The identity  $CH = \sqrt{(CT^2 - TH^2)}$  (Pythagoras theorem) allows to prove (*by continuity*) that TT' is 0 (i.e. there is only one point of intersection – which means tangency) if and only if the height CH of the triangle collapse on CT, i.e. if and only if CT is perpendicular to the straight line a.

In this case transformational reasoning produced by some students might play a double heuristic role: first, to suggest the *necessity* of the condition of perpendicularity for the tangency while proving its *sufficiency*; second, to suggest the way to get a proof (by the continuity principle) of the “if and only if” statement – still, cognitive unity might allow to arrange a valid, simple proof.

## **Discussion**

Transformational reasoning combined with the continuity principle may play three roles for the approach to the culture of theorems and, more generally, for the development of students' mathematical rationalities: first, it may help solving a construction problem and simplifying its proof, in comparison with the Euclid's proof (like in the episode presented in the introduction). Second, it may play a heuristic function by suggesting a method of proving in Euclidean geometry, or eventually a method of proving by relying on the continuity principle (like in the last two episodes). Third, it may allow to compare different ways of proving the same theorem in synthetic geometry, thus contributing to the development of the culture of theorems and at the same time to an initial understanding of the fact that a theorem may be tackled with different strategies and according to different criteria of truth and of validity of proving methods. Concerning the second and the third role, we may observe how the adoption of a principle of continuity would imply changes in the rationality of the discursive activity of proving, in comparison with Euclidean rationality. On the epistemic side: a new criterion of truth is introduced. On the teleological side: for a construction problem, the existence of the solution may be got through transformational reasoning; for the proof of a theorem, transformational reasoning in several cases (see our episodes) guarantees the possibility of the cognitive unity between the exploration phase and the proving phase. On the communicative side, new verbs and expressions are necessary to account for the specificities of transformational reasoning, in comparison with the language of Euclidean geometry. In the perspective of rationality, the principle of continuity might allow to compare (for some theorems) methods of proof corresponding to Euclid's ones with methods of proof which depend not only on Euclid's axioms and theorems, but also on the additional principle, thus contributing to the culture of theorems and prepare students to move to other theories – in particular, Analytic geometry – and help to “cross the borders” between the two domains (and the inherent rationalities: see Boero, Guala, & Morselli, 2013). We observe how, in the problem situations of our episodes,

the principle of continuity may be easily related to the algebraic modelization of the situations at stake. For instance, in the case of the last episode we may consider the circle of center (0,0) and ray 1, whose equation is  $x^2+y^2=1$ , and its intersections with the straight line of equation  $y=kx+1$ , which meets the circle in the point (0,1). The algebraic treatment of the system of the two equations offers an immediate answer to the problem of tangency by discussing the solutions of the second degree equation:  $(1+k^2)x^2+2kx=0$  derived from the system. The equation has the solutions (x-coordinates of the intersection points)  $x=0$ ,  $x=k/(1+k^2)$ . They are coincident (i.e. tangency, in terms of analytic geometry) for  $k=0$ , i.e., when the straight line of equation  $y=kx+1$  becomes perpendicular to the ray of the circle with extremities (0,0) and (0,1). Conversely, including the principle of continuity in synthetic geometry might allow to give sense to the algebraic expressions that are used in the algebraic modeling process of geometry problems, particularly as regards the role of the variable(s) – which not only are signs to be dealt with according to the syntactic rules of the algebraic language, but also represent dynamic phenomena (in our case, geometric transformations).

In this section we have used the expressions “may”, “might allow” several times to outline possible directions for studies, intended to develop the culture of theorems in the classroom on the reflective and on the operational sides, once the continuity principle (possibly, after further elaboration) is assumed to legitimate an extension of the scope of synthetic geometry and the inherent rationality.

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