

# ANALYSIS OF NON SCALAR CONTROL PROBLEMS FOR PARABOLIC SYSTEMS BY THE BLOCK MOMENT METHOD\*

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**Abstract.** This article deals with abstract linear time invariant controlled systems of parabolic type. In [*Annales Henri Lebesgue*, **3** (2020), pp. 717–793], with A. Benabdallah, we introduced the block moment method for scalar control operators. The principal aim of this method is to compute the minimal time needed to drive an initial condition (or a space of initial conditions) to zero, in particular in the case when spectral condensation occurs. The purpose of the present article is to push forward the analysis to deal with any admissible control operator. The considered setting leads to applications to one dimensional parabolic-type equations or coupled systems of such equations.

With such admissible control operator, the characterization of the minimal null control time is obtained thanks to the resolution of an auxiliary vectorial block moment problem (*i.e.* set in the control space) followed by a constrained optimization procedure of the cost of this resolution. This leads to essentially sharp estimates on the resolution of the block moment problems which are uniform with respect to the spectrum of the evolution operator in a certain class. This uniformity allows the study of uniform controllability for various parameter dependent problems. We also deduce estimates on the cost of controllability when the final time goes to the minimal null control time.

We illustrate how the method works on a few examples of such abstract controlled systems and then we deal with actual coupled systems of one dimensional parabolic partial differential equations. Our strategy enables us to tackle controllability issues that seem out of reach by existing techniques.

**Key words.** Control theory, parabolic partial differential equations, minimal null control time, block moment method

**AMS subject classifications.** 93B05, 93C20, 93C25, 30E05, 35K90, 35P10

## 1. Introduction.

### 1.1. Problem under study and state of the art.

In this paper we study the controllability properties of the following linear control system

$$(1.1) \quad \begin{cases} y'(t) + \mathcal{A}y(t) = \mathcal{B}u(t), \\ y(0) = y_0. \end{cases}$$

The assumptions on the operator  $\mathcal{A}$  (see Section 2.1) will lead to applications to linear parabolic-type equations or coupled systems of such equations mostly in the one dimensional setting. In all this article the Hilbert space of control will be denoted by  $U$  and the operator  $\mathcal{B}$  will be a general admissible operator.

The question we address is the characterization of the minimal null control time (possibly zero or infinite) from  $y_0$  that is: for a given initial condition  $y_0$ , what is the minimal time  $T_0(y_0)$  such that, for any  $T > T_0(y_0)$ , there exists a control  $u \in L^2(0, T; U)$  such that the associated solution of (1.1) satisfies  $y(T) = 0$ . A more precise definition of the minimal null control time is given in Definition 2.1 in Section 2.1.1.

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\*This work was supported by the Agence Nationale de la Recherche, Project TRECOS, under grant ANR-20-CE40-0009.

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39 For a presentation of null controllability of parabolic control problems as well as  
 40 the possible existence of a positive minimal null control time for such equations we  
 41 refer to [4] or [9, Section 1.1] and the references therein. Such a positive minimal  
 42 null control time is due either to insufficient observation of eigenvectors, or to con-  
 43 densation of eigenvalues or to the geometry of generalized eigenspaces, or even to a  
 44 combination of all those phenomena. Let us underline that this phenomenon is com-  
 45 pletely unrelated to the minimal control time arising from constraints on the state  
 46 or on the control as studied for instance in [31], or to the one arising in hyperbolic  
 47 problems due to intrinsic finite speed of propagation in the equation.

48 Under the considered assumptions on  $\mathcal{A}$ , the problem of characterizing the mini-  
 49 mal null control time has been solved for scalar controls ( $\dim U = 1$ ) in [9] where the  
 50 *block moment method* has been introduced in that purpose. The aim of the present  
 51 article is to push forward the analysis of [9] to extend it to any admissible control  
 52 operator. The new difficulties come from the interplay between spectral condensation  
 53 phenomena and the particular geometry of the control operator.

54 To present the general ideas, let us assume for simplicity that the operator  $\mathcal{A}^*$   
 55 has a sequence of real and positive eigenvalues  $\Lambda$  and that the associated eigenvectors  
 56  $\phi_\lambda$ , for  $\lambda \in \Lambda$ , form a complete family of the state space (the precise functional setting  
 57 is detailed in Section 2.1). Then, the solution of system (1.1) satisfies  $y(T) = 0$  if and  
 58 only if the control  $u \in L^2(0, T; U)$  solves the following moment problem

$$59 \quad (1.2) \quad \int_0^T e^{-\lambda t} \langle u(T-t), \mathcal{B}^* \phi_\lambda \rangle_U dt = -e^{-\lambda T} \langle y_0, \phi_\lambda \rangle, \quad \forall \lambda \in \Lambda.$$

60 • *Solving moment problems associated with a scalar control operator.*

61 In the scalar case ( $U = \mathbb{R}$ ), provided that  $\mathcal{B}^* \phi_\lambda \neq 0$ , the moment problem reduces  
 62 to

$$63 \quad (1.3) \quad \int_0^T e^{-\lambda t} u(T-t) dt = -e^{-\lambda T} \left\langle y_0, \frac{\phi_\lambda}{\mathcal{B}^* \phi_\lambda} \right\rangle, \quad \forall \lambda \in \Lambda.$$

64 This problem is usually solved by the construction of a biorthogonal family  $(q_\lambda)_{\lambda \in \Lambda}$   
 65 to the exponentials

$$66 \quad \{t \in (0, T) \mapsto e^{-\lambda t}; \lambda \in \Lambda\}$$

67 in  $L^2(0, T; U)$ , *i.e.*, a family  $(q_\lambda)_{\lambda \in \Lambda}$  such that

$$68 \quad \int_0^T q_\lambda(t) e^{-\mu t} dt = \delta_{\lambda, \mu}, \quad \forall \lambda, \mu \in \Lambda.$$

69 From [36], the existence of such biorthogonal family is equivalent to the summability  
 70 condition

$$71 \quad (1.4) \quad \sum_{\lambda \in \Lambda} \frac{1}{\lambda} < +\infty.$$

72 **REMARK 1.1.** *This condition (which will be assumed in the present article) is the*  
 73 *main restriction to apply the moment method. Indeed, due to Weyl's law it imposes*  
 74 *on many examples of partial differential equations of parabolic-type a restriction to the*  
 75 *one dimensional setting. However, in some particular multi-dimensional geometries,*  
 76 *the controllability problem can be transformed into a family of parameter dependent*  
 77 *moment problems, each of them satisfying such assumption (see for instance [8, 3, 15]*  
 78 *among others).*

79 With such a biorthogonal family, a formal solution of the moment problem (1.3)  
 80 is given by

$$81 \quad u(T-t) = - \sum_{\lambda \in \Lambda} e^{-\lambda T} \left\langle y_0, \frac{\phi_\lambda}{\mathcal{B}^* \phi_\lambda} \right\rangle q_\lambda(t), \quad t \in (0, T).$$

82 Thus if, for any  $y_0$ , the series defining  $u$  converges in  $L^2(0, T; U)$  one obtains null  
 83 controllability of system (1.1) in time  $T$ . To do so, it is crucial to prove upper bounds  
 84 on  $\|q_\lambda\|_{L^2(0, T)}$ .

85 Suitable bounds on such biorthogonal families were provided in the pioneering  
 86 work of Fattorini and Russell [21] in the case where the eigenvalues of  $\mathcal{A}^*$  are well  
 87 separated *i.e.* satisfy the classical gap condition:  $\inf \{|\lambda - \mu|; \lambda, \mu \in \Lambda, \lambda \neq \mu\} > 0$ .  
 88 When the eigenvalues are allowed to condensate we refer to the work [5] for almost  
 89 sharp estimates implying the condensation index of the sequence  $\Lambda$ . A discussion on  
 90 other references providing estimates on biorthogonal families is detailed below. These  
 91 results have provided an optimal characterization of the minimal null control time  
 92 when the eigenvectors of  $\mathcal{A}^*$  form a Riesz basis of the state space (and thus do not  
 93 condensate).

94 However, as analyzed in [9], there are situations in which the eigenvectors also  
 95 condensate and for which providing estimates on biorthogonal families is not sufficient  
 96 to characterize the minimal null control time. In [9], it is assumed that the spectrum  
 97  $\Lambda$  can be decomposed as a union  $\mathcal{G}$  of well separated groups of bounded cardinality.  
 98 Then, the control  $u$  is sought in the form

$$99 \quad u(T-t) = \sum_{G \in \mathcal{G}} v_G(t),$$

100 where, for any  $G \in \mathcal{G}$ , the function  $v_G \in L^2(0, T; U)$  solves the block moment problem

$$101 \quad (1.5) \quad \begin{cases} \int_0^T e^{-\lambda t} v_G(t) dt = e^{-\lambda T} \left\langle y_0, \frac{\phi_\lambda}{\mathcal{B}^* \phi_\lambda} \right\rangle, & \forall \lambda \in G, \\ \int_0^T e^{-\lambda t} v_G(t) dt = 0, & \forall \lambda \notin G. \end{cases}$$

102 This enables to deal with the condensation of eigenvectors: the eigenvectors  $(\phi_\lambda)_{\lambda \in \Lambda}$   
 103 are only assumed to form a complete family of the state space.

104 • *Solving moment problems associated with a non scalar control operator.*

105 When the control is not scalar there are less available results in the literature. Here  
 106 again, these results rely on the existence of a biorthogonal family to the exponentials  
 107 with suitable bounds. For instance, in [6], null controllability in optimal time is proved  
 108 using a subtle decomposition of the moment problem into two families of moment  
 109 problems. In a more systematic way, one can take advantage of the biorthogonality  
 110 in the time variable to seek for a solution  $u$  of the moment problem (1.2) in the form

$$111 \quad u(T-t) = - \sum_{\lambda \in \Lambda} e^{-\lambda T} \langle y_0, \phi_\lambda \rangle q_\lambda(t) \frac{\mathcal{B}^* \phi_\lambda}{\|\mathcal{B}^* \phi_\lambda\|_U^2}.$$

112 This strategy was introduced by Lagnese in [25] for a one dimensional wave equation  
 113 and used in the parabolic context for instance in [17, 2, 18, 3].

114 In the present article we deal with such general admissible control operators.  
 115 As the eigenvectors will only be assumed to form a complete family, for each initial

116 condition  $y_0$ , we study its null control time for system (1.1) by solving block moment  
 117 problems of the following form

$$118 \quad (1.6) \quad \begin{cases} \int_0^T \langle V_G(t), e^{-\lambda t} \mathcal{B}^* \phi_\lambda \rangle_U dt = \langle y_0, e^{-\lambda T} \phi_\lambda \rangle, & \forall \lambda \in G, \\ \int_0^T \langle V_G(t), e^{-\lambda t} \mathcal{B}^* \phi_\lambda \rangle_U dt = 0, & \forall \lambda \notin G. \end{cases}$$

119 Let us recall that, for pedagogical purposes, we have restricted this first introductory  
 120 subsection to the case of real simple eigenvalues. The general form of block moment  
 121 problems under study in this article is detailed in Section 2.2.

122 The strategy to solve such block moment problem and estimate its solution is  
 123 presented on an example in Section 1.3. Let us already notice that the geometry of  
 124 the finite dimensional space  $\text{Span}\{\mathcal{B}^* \phi_\lambda; \lambda \in G\}$  is crucial.

125 For instance, if this space is one dimensional, say generated by some  $b \in U$ , the  
 126 strategy of Lagnese can be adapted if one seeks for  $V_G$  solution of the block moment  
 127 problem (1.6) in the form

$$128 \quad V_G(t) = v_G(t)b,$$

129 where  $v_G \in L^2(0, T; \mathbb{R})$  solves a scalar block moment problem of the same form  
 130 as (1.5).

131 If, instead, the family  $(\mathcal{B}^* \phi_\lambda)_{\lambda \in G}$  is composed of linearly independent vectors  
 132 then it admits a biorthogonal family in  $U$  denoted by  $(b_\lambda^*)_{\lambda \in G}$ . Then, one can for  
 133 instance seek for  $V_G$  solution of the block moment problem (1.6) in the form

$$134 \quad V_G(t) = v_G(t) \left( \sum_{\lambda \in G} b_\lambda^* \right).$$

135 where  $v_G$  solves a scalar block moment problem of the form (1.5). An upper bound  
 136 of the minimal control time can then be obtained thanks to an estimate of the family  
 137  $(b_\lambda^*)_{\lambda \in G}$ , but without guarantee of optimality.

138 In the general setting, taking into account the geometry of the observations of  
 139 eigenvectors to solve block moment problems of the form (1.6) is a more intricate  
 140 question that we solve in this article, still under the summability condition (1.4).

141 Let us mention that we not only solve block moment problems of the form (1.6)  
 142 but we also provide estimates on their solutions to ensure that the series defining the  
 143 control converges. These estimates will lead to an optimal characterization of the  
 144 minimal null control time for each given problem.

145 We pay particular attention to these estimates so that they do not directly depend  
 146 on the sequence  $\Lambda$  but are uniform for classes of such sequences. This is an important  
 147 step to tackle uniform controllability for parameter dependent control problems. Esti-  
 148 mates of this kind have already proved their efficiency in various contexts such as: nu-  
 149 merical analysis of semi-discrete control problems [2], oscillating coefficients [32], anal-  
 150 ysis of degenerate control problems with respect to the degeneracy parameter [17, 18],  
 151 analysis of higher dimensional controllability problems by reduction to families of one  
 152 dimensional control problems [8, 1, 3, 15] or analysis of convergence of Robin-type  
 153 controls to Dirichlet controls [11].

154 Another important feature of the estimates we obtain is to track the dependency  
 155 with respect to the final time  $T$  when  $T$  goes to the minimal null control time. As pre-  
 156 sented in Remark 2.8, this allows applications in higher dimensions (with a cylindrical  
 157 geometry) or applications to nonlinear control problems.

- *An overview of some estimates on biorthogonal families.*

Finally, let us recall some classical results providing estimates for biorthogonal families to a sequence of exponentials.

Under the classical gap condition, uniform estimates for biorthogonal families were already obtained in [22] and sharp short-time estimates were obtained in [8]. In this setting, bounds with a detailed dependency with respect to parameters were given in [19]. In this work, the obtained bounds take into account the fact that the gap property between eigenvalues may be better in high frequencies. Similar results were also obtained in [26].

Under a weak-gap condition of the form (2.4), that is when the eigenvalues can be gathered in blocks of bounded cardinality with a gap between blocks (which is the setting of the present article), uniform estimates on biorthogonal sequences follow from the uniform estimates for the resolution of block moment problems proved in [9]. Similar estimates, but where the sharp dependency with respect to  $T$  of the different constants is tracked, were obtained in [24]. Using the strategy detailed in [12], the estimates of [9] can also be supplemented with such dependency with respect to  $T$  (see Theorem A.1). Let us mention that similar results were also obtained in [16] with stronger assumptions, namely with a weak-gap assumption on the square roots of the eigenvalues.

In the absence of any gap-type condition, estimates on biorthogonal families were first proved in [5] involving the condensation index and then later in [3] involving a local measure of the gap.

## 1.2. Structure of the article.

To ease the reading, let us give here the detailed outline of this article.

In Section 1.3 we detail, for a simple example, the obtained results as well as our strategy of proof. This allows to explain the contents of this article without introducing too many notations.

In Section 2.1, we detail the framework, assumptions and notations that will be used throughout this article. The main results concerning the resolution of block moment problems with a non scalar control are stated in Section 2.2. The application of these results to the characterization of the minimal null control time is stated in Section 2.3. We provide in Section 2.4 more explicit formulas to compute the minimal null control time. We also deduce from our study some estimates on the cost of controllability that are given in Section 2.5.

The results concerning the resolution of block moment problems are proved in Section 3. The application of these results to the characterization of the minimal null control time and the study of the cost of null controllability are then proved in Section 4. More explicit formulas for the computation of the minimal null control time are proved in Section 5.

Finally we apply these results to different examples. First we deal in Section 6 with academic examples. For these examples the computations are rather simple and this allows to highlight the different phenomena at stake in this minimal null control time study. We end this article with the analysis of null controllability for systems of coupled linear partial differential equations of parabolic type in Section 7.

## 1.3. Our analysis on a toy system.

To highlight the ideas we develop in this article (without drowning them in technicalities or notations), let us present our strategy of analysis of null controllability on an abstract simple example.

206 We consider  $X = L^2(0, 1; \mathbb{R})^2$  and  $\omega \subset (0, 1)$  a non empty open set. For a given  
 207  $a > 0$  we define

$$208 \quad \Lambda = \left\{ \lambda_{k,1} := k^2, \lambda_{k,2} := k^2 + e^{-ak^2}; k \geq 1 \right\},$$

209 and take  $(\varphi_k)_{k \geq 1}$  a Hilbert basis of  $L^2(0, 1; \mathbb{R})$  such that

$$210 \quad \inf_{k \geq 1} \|\varphi_k\|_{L^2(\omega)} > 0.$$

211 Let  $\phi_{k,1} := \begin{pmatrix} \varphi_k \\ \varphi_k \end{pmatrix}$  and  $\phi_{k,2} := \begin{pmatrix} 0 \\ \varphi_k \end{pmatrix}$ . We define the operator  $\mathcal{A}^*$  in  $X$  by

$$212 \quad \mathcal{A}^* \phi_{k,1} = \lambda_{k,1} \phi_{k,1}, \quad \mathcal{A}^* \phi_{k,2} = \lambda_{k,2} \phi_{k,2},$$

213 with

$$214 \quad \mathcal{D}(\mathcal{A}^*) = \left\{ \sum_{k \geq 1} a_{k,1} \phi_{k,1} + a_{k,2} \phi_{k,2}; \sum_{k \geq 1} \lambda_{k,1}^2 a_{k,1}^2 + \lambda_{k,2}^2 a_{k,2}^2 < +\infty \right\}.$$

215 The control operator  $\mathcal{B}$  is defined by  $U = L^2(0, 1; \mathbb{R})$  and

$$216 \quad \mathcal{B} : u \in U \mapsto \begin{pmatrix} 0 \\ \mathbf{1}_\omega u \end{pmatrix} \in X.$$

217 The condition  $\inf_{k \geq 1} \|\varphi_k\|_{L^2(\omega)} > 0$  yields

$$218 \quad (1.7) \quad \mathcal{B}^* \phi_{k,1} = \mathcal{B}^* \phi_{k,2} = \mathbf{1}_\omega \varphi_k \neq 0, \quad \forall k \geq 1.$$

219 This ensures approximate controllability of system (1.1).

220 We insist on the fact that the goal of this article is not to deal with this particular  
 221 example but to develop a general methodology to analyze the null controllability of  
 222 system (1.1). The general assumptions that will be considered in this article are  
 223 detailed in Section 2.1.

224 • Let  $y_0 \in X$ . From Proposition 2.1 and the fact that  $\{\phi_{k,1}, \phi_{k,2}; k \geq 1\}$  forms a  
 225 complete family of  $X$ , system (1.1) is null controllable from  $y_0$  at time  $T$  if and only  
 226 if there exists  $u \in L^2(0, T; U)$  such that for any  $k \geq 1$  and any  $j \in \{1, 2\}$ ,

$$227 \quad \int_0^T e^{-\lambda_{k,j}t} \langle u(T-t), \mathcal{B}^* \phi_{k,j} \rangle_U dt = -e^{-\lambda_{k,j}T} \langle y_0, \phi_{k,j} \rangle_X.$$

228 Following the idea developed in [9], we seek for a control  $u$  of the form

$$229 \quad (1.8) \quad u(t) = - \sum_{k \geq 1} v_k(T-t)$$

230 where, for each  $k \geq 1$ ,  $v_k \in L^2(0, T; U)$  solves the block moment problem

$$231 \quad (1.9) \quad \begin{cases} \int_0^T e^{-\lambda_{k,j}t} \langle v_k(t), \mathcal{B}^* \phi_{k,j} \rangle_U dt = e^{-\lambda_{k,j}T} \langle y_0, \phi_{k,j} \rangle_X, & \forall j \in \{1, 2\}, \\ \int_0^T e^{-\lambda_{k',j}t} \langle v_k(t), \mathcal{B}^* \phi_{k',j} \rangle_U dt = 0, & \forall k' \neq k, \forall j \in \{1, 2\}. \end{cases}$$

232 • To solve (1.9), for a fixed  $k$ , we consider the following auxiliary block moment  
 233 problem in the space  $U$

$$234 \quad (1.10) \quad \begin{cases} \int_0^T e^{-\lambda_{k,j}t} v_k(t) dt = \Omega_{k,j}, & \forall j \in \{1, 2\}, \\ \int_0^T e^{-\lambda_{k',j}t} v_k(t) dt = 0, & \forall k' \neq k, \forall j \in \{1, 2\}, \end{cases}$$

235 where  $\Omega_{k,j} \in U$  have to be precised. If we impose that  $\Omega_{k,1}$  and  $\Omega_{k,2}$  satisfy the  
 236 constraints

$$237 \quad (1.11) \quad \langle \Omega_{k,j}, \mathcal{B}^* \phi_{k,j} \rangle_U = e^{-\lambda_{k,j}T} \langle y_0, \phi_{k,j} \rangle_X, \quad \forall j \in \{1, 2\},$$

238 we obtain that the solutions of (1.10) also solve (1.9). The existence of  $\Omega_{k,1}$  and  
 239  $\Omega_{k,2}$  satisfying the constraints (1.11) is ensured by the approximate controllabil-  
 240 ity condition (1.7); however there exist infinitely many choices. A crucial point  
 241 is that, by orthogonal projection, there exists  $\Omega_{k,1}$  and  $\Omega_{k,2}$  in the space  $U_k =$   
 242  $\text{Span}\{\mathcal{B}^* \phi_{k,1}, \mathcal{B}^* \phi_{k,2}\}$  satisfying the constraints (1.11).

243 Then, for any  $\Omega_{k,1}, \Omega_{k,2} \in U_k$ , since the space  $U_k$  is of finite dimension, applying  
 244 the scalar results of [9] component by component leads to the existence of  $v_k \in$   
 245  $L^2(0, T; U)$  satisfying (1.10). It also gives the following estimate

$$246 \quad (1.12) \quad \|v_k\|_{L^2(0,T;U)}^2 \leq C_{T,\varepsilon} e^{\varepsilon \lambda_{k,1}} F(\Omega_{k,1}, \Omega_{k,2}),$$

247 with

$$248 \quad F : (\Omega_{k,1}, \Omega_{k,2}) \in U^2 \mapsto \|\Omega_{k,1}\|_U^2 + \left\| \frac{\Omega_{k,2} - \Omega_{k,1}}{\lambda_{k,2} - \lambda_{k,1}} \right\|_U^2.$$

249 Using (1.12) and minimizing the function  $F$  under the constraints (1.11) we obtain  
 250 that there exists  $v_k \in L^2(0, T; U)$  solution of the block moment problem (1.9) such  
 251 that

$$252 \quad (1.13) \quad \|v_k\|_{L^2(0,T;U)}^2 \leq C_{T,\varepsilon} e^{\varepsilon \lambda_{k,1}} \inf \{F(\Omega_{k,1}, \Omega_{k,2}) ; \Omega_{k,1}, \Omega_{k,2} \text{ satisfy (1.11)}\}.$$

253 The corresponding general statements of the resolution of block moment problems  
 254 are detailed in Section 2.2 (see Theorem 2.4) and proved in Section 3. Actually using  
 255 a refined version of the results in [9] (see Theorem A.1) we obtain sharper results  
 256 including dependency with respect to  $T$ .

257 • Now that we can solve the block moment problems (1.9), a way to characterize the  
 258 minimal null control time is to estimate for which values of  $T$  the series (1.8) defining  
 259 the control  $u$  converges in  $L^2(0, T; U)$ .

260 To achieve this goal, we isolate in the estimate (1.13) the dependency with respect  
 261 to  $T$ . Notice that the function  $F$  does not depend on  $T$  but that the constraints (1.11)  
 262 do.

263 For any  $k \geq 1$  and any  $\Omega_{k,1}, \Omega_{k,2} \in U_k$  we set

$$264 \quad \tilde{\Omega}_{k,j} := e^{\lambda_{k,j}T} \Omega_{k,j}, \quad \forall j \in \{1, 2\}.$$

265 Then, there is equivalence between the constraints (1.11) and the new constraints

$$266 \quad (1.14) \quad \left\langle \tilde{\Omega}_{k,j}, \mathcal{B}^* \phi_{k,j} \right\rangle_U = \langle y_0, \phi_{k,j} \rangle_X, \quad \forall j \in \{1, 2\}.$$

267 Now these constraints are independent of the variable  $T$ . From the mean value theo-  
 268 rem we obtain

$$\begin{aligned}
 269 \quad F(\Omega_{k,1}, \Omega_{k,2}) &= \left\| e^{-\lambda_{k,1}T} \tilde{\Omega}_{k,1} \right\|_U^2 + \left\| \frac{e^{-\lambda_{k,2}T} \tilde{\Omega}_{k,2} - e^{-\lambda_{k,1}T} \tilde{\Omega}_{k,1}}{\lambda_{k,2} - \lambda_{k,1}} \right\|_U^2 \\
 270 \quad &\leq e^{-2\lambda_{k,1}T} \left\| \tilde{\Omega}_{k,1} \right\|_U^2 + 2e^{-2\lambda_{k,2}T} \left\| \frac{\tilde{\Omega}_{k,2} - \tilde{\Omega}_{k,1}}{\lambda_{k,2} - \lambda_{k,1}} \right\|_U^2 \\
 271 \quad &\quad + 2 \left( \frac{e^{-\lambda_{k,2}T} - e^{-\lambda_{k,1}T}}{\lambda_{k,2} - \lambda_{k,1}} \right)^2 \left\| \tilde{\Omega}_{k,1} \right\|_U^2 \\
 272 \quad &\leq 2(1 + T^2) e^{-2\lambda_{k,1}T} F(\tilde{\Omega}_{k,1}, \tilde{\Omega}_{k,2}).
 \end{aligned}$$

274 The general statement of this estimate is given in Lemma 4.1.

275 Plugging this estimate into (1.12) and optimizing the function  $F$  under the con-  
 276 straints (1.14) yields

$$277 \quad (1.15) \quad \|v_k\|_{L^2(0,T;U)}^2 \leq C_{T,\varepsilon} e^{\varepsilon\lambda_{k,1}} e^{-2\lambda_{k,1}T} \mathcal{C}_k(y_0)$$

278 where  $\mathcal{C}_k(y_0)$  is the quantity, independent of  $T$ , given by

$$\begin{aligned}
 280 \quad (1.16) \quad \mathcal{C}_k(y_0) &:= \inf \left\{ \left\| \tilde{\Omega}_1 \right\|_U^2 + \left\| \frac{\tilde{\Omega}_2 - \tilde{\Omega}_1}{\lambda_{k,2} - \lambda_{k,1}} \right\|_U^2 ; \tilde{\Omega}_1, \tilde{\Omega}_2 \in U_k \text{ satisfy} \right. \\
 281 \quad &\quad \left. \left\langle \tilde{\Omega}_j, \mathcal{B}^* \phi_{k,j} \right\rangle_U = \langle y_0, \phi_{k,j} \rangle_X, \forall j \in \{1, 2\} \right\}.
 \end{aligned}$$

283 Estimate (1.15) proves that for any time  $T > 0$  such that

$$284 \quad T > \limsup_{k \rightarrow +\infty} \frac{\ln \mathcal{C}_k(y_0)}{2\lambda_{k,1}}$$

285 the series (1.8) defining the control  $u$  converges in  $L^2(0, T; U)$ . Thus, null controlla-  
 286 bility of (1.1) from  $y_0$  holds for such  $T$ .

287 We also prove that the obtained estimate (1.15) is sufficiently sharp so that it  
 288 characterizes the minimal null control time from  $y_0$  as

$$289 \quad (1.17) \quad T_0(y_0) = \limsup_{k \rightarrow +\infty} \frac{\ln \mathcal{C}_k(y_0)}{2\lambda_{k,1}}.$$

290 The corresponding general statements regarding the minimal null control time  
 291 together with bounds on the cost of controllability are detailed in Section 2.2 (see  
 292 Theorem 2.5) and proved in Section 4.

293 • At this stage we have characterized the minimal null control time as stated in (1.17).  
 294 However to be able to estimate the actual value of  $T_0(y_0)$  one should be able to  
 295 estimate the quantity  $\mathcal{C}_k(y_0)$  as defined in (1.16). This formula is not very explicit  
 296 and it does not get better in the general setting.

297 However, we notice that (1.16) is a finite dimensional optimization problem that  
 298 we explicitly solve in terms of the eigenelements of  $\mathcal{A}^*$  and their observations through  
 299  $\mathcal{B}^*$ .

300 Indeed the minimization problem (1.16) has a unique solution characterized by  
 301 the existence of multipliers  $m_1, m_2 \in \mathbb{R}$  such that for any  $H_1, H_2 \in U_k$  we have  
 (1.18)

$$302 \quad \left\langle H_1, \tilde{\Omega}_1 \right\rangle_U + \left\langle \frac{\tilde{\Omega}_2 - \tilde{\Omega}_1}{\lambda_{k,2} - \lambda_{k,1}}, \frac{H_2 - H_1}{\lambda_{k,2} - \lambda_{k,1}} \right\rangle_U = m_1 \langle H_1, \mathcal{B}^* \phi_{k,1} \rangle_U + m_2 \langle H_2, \mathcal{B}^* \phi_{k,2} \rangle_U.$$

303 Setting  $H_1 = H_2 = H$  for any  $H \in U_k$  implies

$$304 \quad \tilde{\Omega}_1 = m_1 \mathcal{B}^* \phi_{k,1} + m_2 \mathcal{B}^* \phi_{k,2}.$$

305 Setting  $H_1 = 0$  and  $H_2 = (\lambda_{k,2} - \lambda_{k,1})H$  for any  $H \in U_k$  implies

$$306 \quad \tilde{\Omega}_2 = m_1 \mathcal{B}^* \phi_{k,1} + m_2 \mathcal{B}^* \phi_{k,2} + m_2 (\lambda_{k,2} - \lambda_{k,1})^2 \mathcal{B}^* \phi_{k,2}.$$

307 Getting back to the constraints (1.14) we obtain

$$308 \quad (1.19) \quad \begin{pmatrix} \langle y_0, \phi_{k,1} \rangle_X \\ \langle y_0, \phi_{k,2} \rangle_X \end{pmatrix} = M \begin{pmatrix} m_1 \\ m_2 \end{pmatrix},$$

309 where the  $2 \times 2$  matrix  $M$  is defined by

$$310 \quad M = \text{Gram}_U(\mathcal{B}^* \phi_{k,1}, \mathcal{B}^* \phi_{k,2}) + \text{Gram}_U(0, (\lambda_{k,2} - \lambda_{k,1}) \mathcal{B}^* \phi_{k,2}).$$

311 Setting  $H_1 = \tilde{\Omega}_1$  and  $H_2 = \tilde{\Omega}_2$  in (1.18) and using (1.19) imply

$$312 \quad \mathcal{C}_k(y_0) = \left\| \tilde{\Omega}_1 \right\|_U^2 + \left\| \frac{\tilde{\Omega}_2 - \tilde{\Omega}_1}{\lambda_{k,2} - \lambda_{k,1}} \right\|_U^2 = \left\langle \begin{pmatrix} \langle y_0, \phi_{k,1} \rangle_X \\ \langle y_0, \phi_{k,2} \rangle_X \end{pmatrix}, \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \right\rangle \\ 313 \quad \quad \quad = \left\langle \begin{pmatrix} \langle y_0, \phi_{k,1} \rangle_X \\ \langle y_0, \phi_{k,2} \rangle_X \end{pmatrix}, M^{-1} \begin{pmatrix} \langle y_0, \phi_{k,1} \rangle_X \\ \langle y_0, \phi_{k,2} \rangle_X \end{pmatrix} \right\rangle. \\ 314$$

315 Thus, after computations, for the particular example we are considering here, the  
 316 obtained formula reads

$$317 \quad \mathcal{C}_k(y_0) = \frac{1}{\|\varphi_k\|_{L^2(\omega)}^2} \left\langle y_0, \begin{pmatrix} \varphi_k \\ \varphi_k \end{pmatrix} \right\rangle_X^2 + \frac{e^{2ak^2}}{\|\varphi_k\|_{L^2(\omega)}^2} \left\langle y_0, \begin{pmatrix} \varphi_k \\ 0 \end{pmatrix} \right\rangle_X^2.$$

318 Then, from (1.17), it comes that the minimal null control time from  $X$  of this example  
 319 is given by

$$320 \quad T_0(X) = a.$$

321 Notice, for instance, that this expression also gives that for a given  $y_0$  if the set

$$322 \quad \left\{ k \in \mathbb{N}^* ; \left\langle y_0, \begin{pmatrix} \varphi_k \\ 0 \end{pmatrix} \right\rangle_X \neq 0 \right\}$$

323 is finite, then null controllability from  $y_0$  holds in any positive time, *i.e.*  $T_0(y_0) = 0$ .

324 We obtain different explicit formula depending on the configuration for the multi-  
 325 plicity of the eigenvalues of the considered block. The general statements of an explicit  
 326 solution of the corresponding optimization problem are detailed in Section 2.4 (see  
 327 Theorem 2.8 and Theorem 2.10) and proved in Section 5.

328 **2. Main results.**

329 We state in this section the main results of this article concerning the resolution  
 330 of block moment problems and the application to the characterization of the minimal  
 331 null control time. We start by giving the functional setting and assumptions we use.

332 **2.1. Framework, spectral assumptions and notations.**

333 **2.1.1. Functional setting.**

334 The functional setting for the study of system (1.1) is the same as in [9]. For the  
 335 sake of completeness, let us briefly detail it. Unless explicitly stated, all the spaces  
 336 are assumed to be complex vector spaces.

337 We consider  $X$  a Hilbert space, whose inner product and norm are denoted by  
 338  $\langle \bullet, \bullet \rangle_X$  and  $\|\bullet\|_X$  respectively. The space  $X$  is identified to its anti-dual through the  
 339 Riesz theorem. Let  $(\mathcal{A}, D(\mathcal{A}))$  be an unbounded operator in  $X$  such that  $-\mathcal{A}$  generates  
 340 a  $C^0$ -semigroup in  $X$ . Its adjoint in  $X$  is denoted by  $(\mathcal{A}^*, D(\mathcal{A}^*))$ . Up to a suitable  
 341 translation, we can assume that 0 is in the resolvent set of  $\mathcal{A}$ .

342 We denote by  $X_1$  (resp.  $X_1^*$ ) the Hilbert space  $D(\mathcal{A})$  (resp.  $D(\mathcal{A}^*)$ ) equipped  
 343 with the norm  $\|x\|_1 := \|\mathcal{A}x\|_X$  (resp.  $\|x\|_{1^*} := \|\mathcal{A}^*x\|_X$ ) and we define  $X_{-1}$  as the  
 344 completion of  $X$  with respect to the norm

$$\|y\|_{-1} := \sup_{z \in X_1^*} \frac{\langle y, z \rangle_X}{\|z\|_{1^*}}.$$

346 Notice that  $X_{-1}$  is isometrical to the topological anti-dual of  $X_1^*$  using  $X$  as a pivot  
 347 space (see for instance [38, Proposition 2.10.2]). The corresponding duality bracket  
 348 will be denoted by  $\langle \bullet, \bullet \rangle_{-1,1^*}$  and satisfies

$$\langle y, cz \rangle_{-1,1^*} = \bar{c} \langle y, z \rangle_{-1,1^*}, \quad \forall y \in X_{-1}, \forall z \in X_1^*, \forall c \in \mathbb{C}.$$

350 The control space  $U$  is a Hilbert space (that we will identify to its anti-dual). Its inner  
 351 product and norm are denoted by  $\langle \bullet, \bullet \rangle_U$  and  $\|\bullet\|_U$  respectively. Let  $\mathcal{B} : U \rightarrow X_{-1}$  be  
 352 a linear continuous control operator and denote by  $\mathcal{B}^* : X_1^* \rightarrow U$  its adjoint in the  
 353 duality described above.

354 Let  $(X_\diamond^*, \|\cdot\|_{\diamond^*})$  be a Hilbert space such that  $X_1^* \subset X_\diamond^* \subset X$  with dense and  
 355 continuous embeddings. We assume that  $X_\diamond^*$  is stable by the semigroup generated by  
 356  $-\mathcal{A}^*$ . We also define  $X_{-\diamond}$  as the subspace of  $X_{-1}$  defined by

$$X_{-\diamond} := \left\{ y \in X_{-1}; \|y\|_{-\diamond} := \sup_{z \in X_1^*} \frac{\langle y, z \rangle_{-1,1^*}}{\|z\|_{\diamond^*}} < +\infty \right\},$$

358 which is also isometrical to the anti-dual of  $X_\diamond^*$  with  $X$  as a pivot space. The cor-  
 359 responding duality bracket will be denoted by  $\langle \bullet, \bullet \rangle_{-\diamond, \diamond^*}$ . Thus, we end up with the  
 360 following five functional spaces

$$X_1^* \subset X_\diamond^* \subset X \subset X_{-\diamond} \subset X_{-1}.$$

362 We say that the control operator  $\mathcal{B}$  is an admissible control operator for (1.1) with  
 363 respect to the space  $X_{-\diamond}$  if for any  $T > 0$  there exists  $C_T > 0$  such that

$$(2.1) \quad \int_0^T \left\| \mathcal{B}^* e^{-(T-t)\mathcal{A}^*} z \right\|_U^2 dt \leq C_T \|z\|_{\diamond^*}^2, \quad \forall z \in X_1^*.$$

365 Notice that if (2.1) holds for some  $T > 0$  it holds for any  $T > 0$ . The admissibility  
 366 condition (2.1) implies that, by density, we can give a meaning to the map

$$367 \quad (t \mapsto \mathcal{B}^* e^{-(T-t)\mathcal{A}^*} z) \in L^2(0, T; U),$$

368 for any  $z \in X_\diamond^*$ . Then, we end up with the following well-posedness result (see [9,  
 369 Proposition 1.2]).

370 **PROPOSITION 2.1.** *Assume that (2.1) holds. Then, for any  $T > 0$ , any  $y_0 \in X_{-\diamond}$ ,  
 371 and any  $u \in L^2(0, T; U)$ , there exists a unique  $y \in C^0([0, T]; X_{-\diamond})$  solution to (1.1)  
 372 in the sense that it satisfies for any  $t \in [0, T]$  and any  $z_t \in X_\diamond^*$ ,*

$$373 \quad \langle y(t), z_t \rangle_{-\diamond, \diamond} - \langle y_0, e^{-t\mathcal{A}^*} z_t \rangle_{-\diamond, \diamond} = \int_0^t \langle u(s), \mathcal{B}^* e^{-(t-s)\mathcal{A}^*} z_t \rangle_U ds.$$

374 Moreover there exists  $C_T > 0$  such that

$$375 \quad \sup_{t \in [0, T]} \|y(t)\|_{-\diamond} \leq C_T (\|y_0\|_{-\diamond} + \|u\|_{L^2(0, T; U)}).$$

376 **REMARK 2.1.** *By analogy with the semigroup notation, when  $u = 0$ , we set for  
 377 any  $t \in [0, T]$ ,  $e^{-t\mathcal{A}^*} y_0 := y(t)$ . This extends the semigroup  $e^{-\bullet\mathcal{A}^*}$  defined on  $X$  to  $X_{-\diamond}$   
 378 and implies that for any  $z \in X_{-\diamond}$ ,*

$$379 \quad (2.2) \quad \langle e^{-T\mathcal{A}^*} z, \phi \rangle_{-\diamond, \diamond} = \langle z, e^{-T\mathcal{A}^*} \phi \rangle_{-\diamond, \diamond}, \quad \forall \phi \in X_\diamond^*.$$

380 With this notion of solution at hand, we finally define the minimal null control  
 381 time from a subspace of initial conditions  $Y_0$ .

382 **DEFINITION 2.1.** *Let  $Y_0$  be a closed subspace of  $X_{-\diamond}$  and let  $T > 0$ . The sys-  
 383 tem (1.1) is said to be null controllable from  $Y_0$  at time  $T$  if for any  $y_0 \in Y_0$ , there  
 384 exists a control  $u \in L^2(0, T; U)$  such that the associated solution of (1.1) satisfies  
 385  $y(T) = 0$ .*

386 *The minimal null control time  $T_0(Y_0) \in [0, +\infty]$  is defined by*

- 387 • for any  $T > T_0(Y_0)$ , system (1.1) is null controllable from  $Y_0$  at time  $T$ ;
- 388 • for any  $T < T_0(Y_0)$ , system (1.1) is not null controllable from  $Y_0$  at time  $T$ .

389 To simplify the notations, for any  $y_0 \in X_{-\diamond}$ , we define  $T_0(y_0) := T_0(\text{Span}\{y_0\})$ .

### 390 **2.1.2. Spectral assumptions.**

391 In all this article we assume that the operators  $\mathcal{A}$  and  $\mathcal{B}$  satisfy the assumptions  
 392 of Section 2.1.1. Moreover to solve the control problem we will need some additional  
 393 spectral assumptions.

394 *★ Behavior of eigenvalues.*

395 We assume that the spectrum of  $\mathcal{A}^*$ , denoted by  $\Lambda$ , is only composed of (countably  
 396 many) eigenvalues. Moreover, we assume that the eigenvalues lie in a suitable sector  
 397 of the complex plane, i.e., there exists  $\tau > 0$  such that

$$398 \quad (2.3) \quad \Lambda \subset S_\tau$$

399 where

$$400 \quad S_\tau := \{z \in \mathbb{C} ; \Re z > 0 \text{ and } |\Im z| < (\sinh \tau) \Re z\}.$$

401 REMARK 2.2. In [9], the assumption on  $\Lambda$  was stronger. Namely, in that article  
 402 it was assumed that  $\Lambda \subset (1, +\infty)$ . The fact that  $\min \Lambda \geq 1$  was only used in the  
 403 lower bound on the solution of scalar block moment problems (see estimate (A.3)).  
 404 The extension to complex eigenvalues satisfying (2.3) was done in [12] and is stated  
 405 in Appendix A.

406 If necessary, one can replace the operator  $\mathcal{A}$  by  $\mathcal{A} + \sigma$  without modifying the  
 407 controllability properties. Then, in the different estimates, the behavior with respect  
 408 to  $\sigma$  can be carefully tracked if needed.

409 As in the case of a scalar control (see [9]) we assume that this spectrum satisfies  
 410 a weak-gap condition. Namely, there exists  $p \in \mathbb{N}^*$  and  $\varrho > 0$  such that

$$411 \quad (2.4) \quad \#(\Lambda \cap D(\mu, \varrho/2)) \leq p, \quad \forall \mu \in \mathbb{C},$$

412 where  $D(\mu, \varrho/2)$  denotes the open disk in the complex plane with center  $\mu$  and radius  
 413  $\varrho/2$ . This means that the eigenvalues are allowed to condensate by groups but the  
 414 cardinality of these groups should be bounded. To precise this, let us recall the notion  
 415 of groupings used in [9, Definition 1.6, Proposition 7.1] and extended to the complex  
 416 setting in [12, Proposition V.5.28].

417 PROPOSITION 2.2. Let  $p \in \mathbb{N}^*$  and  $\varrho > 0$ . Let  $\Lambda \subset \mathbb{C}$  be such that the weak-gap  
 418 condition (2.4) holds. Then, there exists a countable family  $\mathcal{G}$  of disjoint subsets of  $\Lambda$   
 419 satisfying

$$420 \quad (2.5) \quad \Lambda = \bigcup_{G \in \mathcal{G}} G$$

421 and each  $G \in \mathcal{G}$  satisfies

$$422 \quad (2.6) \quad \text{diam } G \leq \varrho,$$

423

$$424 \quad (2.7) \quad \#G \leq p,$$

425 and

$$426 \quad (2.8) \quad \text{dist}(\text{Conv } G, \Lambda \setminus G) \geq \frac{\varrho}{2 \times 4^{p-1}}.$$

427 Let us mention that the results do not depend on the particular construction done  
 428 in [12, Proposition V.5.28] and remain valid for any grouping  $\mathcal{G}$  satisfying (2.5)-(2.8).

429 Concerning the asymptotic behavior of the spectrum we will use the counting  
 430 function associated to  $\Lambda$  defined by

$$431 \quad N_\Lambda : r > 0 \mapsto \#\{\lambda \in \Lambda; |\lambda| \leq r\}.$$

432 We assume that there exists  $\kappa > 0$  and  $\theta \in (0, 1)$  such that

$$433 \quad (2.9) \quad N_\Lambda(r) \leq \kappa r^\theta, \quad \forall r > 0$$

434 and

$$435 \quad (2.10) \quad |N_\Lambda(r) - N_\Lambda(s)| \leq \kappa \times (1 + |r - s|^\theta), \quad \forall r, s > 0.$$

436 Notice that this condition is slightly stronger than the classical summability condi-  
 437 tion (1.4) used for instance in [22, 5, 9] and many other works.

438 REMARK 2.3. *Let us underline that if we do not assume (2.10) to hold all the*  
 439 *results of the present article still hold with a slight change in the estimates. To lighten*  
 440 *the writing we only detail this change for Theorem A.1 concerning the resolution of*  
 441 *block moment problems with a scalar control (see Remark A.1). However, as proved*  
 442 *in Section 7, the assumption (2.10) holds for many examples.*

443 Notice also that (2.9), with  $r = \inf |\Lambda|$ , implies the following lower bound on the  
 444 bottom of the spectrum

$$445 \quad \inf |\Lambda| \geq \kappa^{-\theta}.$$

446 Our goal is not only to study the controllability properties of our system but also to  
 447 obtain estimates that are uniform in a way to be precised. To do so, we define the  
 448 following class of sequences: let  $p \in \mathbb{N}^*$ ,  $\varrho, \tau, \kappa > 0$ ,  $\theta \in (0, 1)$  and consider the class

$$449 \quad (2.11) \quad \mathcal{L}_w(p, \varrho, \tau, \theta, \kappa) := \{\Lambda \subset S_\tau ; \Lambda \text{ satisfies (2.4), (2.9) and (2.10)}\}.$$

450 *★ Multiplicity of eigenvalues.*

451 In our study we allow both algebraic and geometric multiplicities for the eigenval-  
 452 ues. We assume that these multiplicities are finite and that the algebraic multiplicity  
 453 is globally bounded. More precisely, we assume that

$$454 \quad (2.12) \quad \gamma_\lambda := \dim \text{Ker}(\mathcal{A}^* - \lambda) < +\infty, \quad \forall \lambda \in \Lambda,$$

455 and that there exists  $\eta \in \mathbb{N}^*$  such that

$$456 \quad (2.13) \quad \text{Ker}(\mathcal{A}^* - \lambda)^\eta = \text{Ker}(\mathcal{A}^* - \lambda)^{\eta+1}, \quad \forall \lambda \in \Lambda.$$

457 For any  $\lambda \in \Lambda$  we denote by  $\alpha_\lambda$  the smallest integer such that

$$458 \quad \text{Ker}(\mathcal{A}^* - \lambda)^{\alpha_\lambda} = \text{Ker}(\mathcal{A}^* - \lambda)^{\alpha_\lambda+1}$$

459 and set

$$460 \quad E_\lambda := \text{Ker}(\mathcal{A}^* - \lambda)^{\alpha_\lambda}.$$

461 *★ (Generalized) eigenvectors.*

462 To study null-controllability, we assume that the Fattorini-Hautus criterion is  
 463 satisfied

$$464 \quad (2.14) \quad \text{Ker}(\mathcal{A}^* - \lambda) \cap \text{Ker} \mathcal{B}^* = \{0\}, \quad \forall \lambda \in \Lambda.$$

465 It is a necessary condition for approximate controllability. Note that, under additional  
 466 assumptions on  $\mathcal{A}$  and  $\mathcal{B}$  it is also a sufficient condition for approximate controllability  
 467 (see for instance [20, 34]). However, when studying null controllability of system (1.1)  
 468 for initial conditions in a closed strict subspace  $Y_0$  of  $X_{-\diamond}$  the condition (2.14) can be  
 469 too strong, see for instance Sections 7.1.2 and 7.1.3.

470 We assume that the family of generalized eigenvectors of  $\mathcal{A}^*$

$$471 \quad \Phi = \{\phi \in E_\lambda ; \lambda \in \Lambda\} = \bigcup_{\lambda \in \Lambda} E_\lambda$$

472 is complete in  $X_{\diamond}^*$  i.e. for any  $y \in X_{-\diamond}$ ,

$$473 \quad (2.15) \quad \left( \langle y, \phi \rangle_{-\diamond, \diamond} = 0, \quad \forall \phi \in \Phi \right) \implies y = 0.$$

474 In the following, to simplify the writing, we gather these assumptions and say  
 475 that the operators  $\mathcal{A}$  and  $\mathcal{B}$  satisfy (H) if there exists  $p \in \mathbb{N}^*$ ,  $\varrho, \tau, \kappa > 0$  and  $\theta \in (0, 1)$   
 476 such that

$$477 \quad (\text{H}) \quad \begin{cases} \mathcal{A} \text{ and } \mathcal{B} \text{ satisfy the assumptions of Section 2.1.1;} \\ \Lambda = \text{Sp}(\mathcal{A}^*) \text{ belongs to } \mathcal{L}_w(p, \varrho, \tau, \theta, \kappa) \text{ and satisfies (2.12) and (2.13);} \\ \text{the associated (generalized) eigenvectors satisfy (2.14) and (2.15).} \end{cases}$$

### 478 2.1.3. Notation.

479 We give here some notation that will be used throughout this article.

- 480 • For any  $a, b \in \mathbb{R}$ , we define the following subsets of  $\mathbb{N}$ :

$$481 \quad [a, b] := [a, b] \cap \mathbb{N}, \quad \llbracket a, b \llbracket := [a, b) \cap \mathbb{N}.$$

- 482 • In all the present paper,  $\langle \bullet, \bullet \rangle$  denotes the usual inner product in finite di-  
 483 mension i.e.

$$484 \quad \langle f, g \rangle = {}^t f \bar{g}.$$

- 485 • For any  $t \in \mathbb{R}$  we denote by  $e_t$  the exponential function

$$486 \quad \begin{aligned} e_t : \mathbb{C} &\rightarrow \mathbb{C} \\ z &\mapsto e^{-tz}. \end{aligned}$$

- 487 • We shall denote by  $C_{\nu_1, \dots, \nu_l} > 0$  a constant possibly varying from one line to  
 488 another but depending only on the parameters  $\nu_1, \dots, \nu_l$ .
- 489 • For any non empty subset  $\Gamma \subset \Lambda$ , we set

$$490 \quad (2.16) \quad r_\Gamma := \inf_{\lambda \in \Gamma} \Re \lambda.$$

491 Notice that assumptions (2.3) and (2.4) imply that  $r_\Gamma > 0$  for any  $\Gamma \subset \Lambda$ .

- 492 • For any multi-index  $\alpha \in \mathbb{N}^n$ , we denote its length by  $|\alpha| = \sum_{j=1}^n \alpha_j$  and its  
 493 maximum by  $|\alpha|_\infty = \max_{j \in \llbracket 1, n \rrbracket} \alpha_j$ .
- 494 For  $\alpha, \mu \in \mathbb{N}^n$ , we say that  $\mu \leq \alpha$  if and only if  $\mu_j \leq \alpha_j$  for any  $j \in \llbracket 1, n \rrbracket$ .
- 495 • In all this article the notation  $f[\dots]$  stands for (generalized) divided dif-  
 496 ferences of a set of values  $(x_j, f_j)$ . Let us recall that, for pairwise distinct  
 497  $x_1, \dots, x_n \in \mathbb{C}$  and  $f_1, \dots, f_n$  in any vector space, the divided differences are  
 498 defined by

$$499 \quad f[x_j] = f_j, \quad f[x_1, \dots, x_j] = \frac{f[x_2, \dots, x_j] - f[x_1, \dots, x_{j-1}]}{x_j - x_1}.$$

500 The two results that will be the most used in this article concerning divided  
 501 differences are the Leibniz formula

$$502 \quad (gf)[x_1, \dots, x_j] = \sum_{k=1}^j g[x_1, \dots, x_k] f[x_k, \dots, x_j],$$

503 and Jensen inequality stating that, when  $f_j = f(x_j)$  for an holomorphic  
 504 function  $f$ , we have

$$505 \quad |f[x_1, \dots, x_j]| \leq \frac{|f^{(j-1)}(z)|}{(j-1)!},$$

506 with  $z \in \text{Conv}\{x_1, \dots, x_j\}$ . For more detailed statements and other useful  
507 properties as well as their generalizations when  $x_1, \dots, x_n$  are not assumed to  
508 be pairwise distinct we refer the reader to [12, Appendix A.2] This general-  
509 ization is used in the present article whenever there are algebraically multiple  
510 eigenvalues.  
511 • For any closed subspace  $Y$  of  $X_{-\diamond}$  we denote by  $P_Y$  the orthogonal projection  
512 in  $X_{-\diamond}$  onto  $Y$ . We denote by  $P_Y^* \in L(X_\diamond^*)$  its adjoint in the duality  $X_{-\diamond}$ ,  
513  $X_\diamond^*$ .

## 514 2.2. Resolution of block moment problems.

515 ★ *Definition of block moment problems.*

516 Using the notion of solution given in Proposition 2.1 and the assumption (2.15),  
517 null controllability from  $y_0$  in time  $T$  reduces to the resolution of the following problem:  
518 find  $u \in L^2(0, T; U)$  such that

$$519 \quad (2.17) \quad \int_0^T \langle u(t), \mathcal{B}^* e^{-(T-t)\mathcal{A}^*} \phi \rangle_U dt = - \langle y_0, e^{-T\mathcal{A}^*} \phi \rangle_{-\diamond, \diamond}, \quad \forall \phi \in E_\lambda, \forall \lambda \in \Lambda.$$

520 Following the strategy initiated in [9] for scalar controls, we decompose this problem  
521 into block moment problems. Hence we look for a control of the form

$$522 \quad (2.18) \quad u = - \sum_{G \in \mathcal{G}} v_G(T - \bullet)$$

523 where  $\mathcal{G}$  is a grouping (as stated in Proposition 2.2) and, for every  $G \in \mathcal{G}$ ,  $v_G \in$   
524  $L^2(0, T; U)$  solves the moment problem in the group  $G$  i.e.

$$525 \quad (2.19a) \quad \int_0^T \langle v_G(t), \mathcal{B}^* e^{-t\mathcal{A}^*} \phi \rangle_U dt = \langle y_0, e^{-T\mathcal{A}^*} \phi \rangle_{-\diamond, \diamond}, \quad \forall \phi \in E_\lambda, \forall \lambda \in G,$$

$$526 \quad (2.19b) \quad \int_0^T \langle v_G(t), \mathcal{B}^* e^{-t\mathcal{A}^*} \phi \rangle_U dt = 0, \quad \forall \phi \in E_\lambda, \forall \lambda \in \Lambda \setminus G.$$

528 In fact it is sufficient to solve the following block moment problem

$$529 \quad (2.20a) \quad \int_0^T \langle v_G(t), \mathcal{B}^* e^{-t\mathcal{A}^*} \phi \rangle_U dt = \langle e^{-T\mathcal{A}} y_0, \phi \rangle_{-\diamond, \diamond}, \quad \forall \phi \in E_\lambda, \forall \lambda \in G,$$

$$530 \quad (2.20b) \quad \int_0^T v_G(t) t^l e^{-\bar{\lambda} t} dt = 0, \quad \forall \lambda \in \Lambda \setminus G, \forall l \in \llbracket 0, \eta \rrbracket$$

532 where  $e^{-T\mathcal{A}} y_0$  is defined in (2.2).

533 Indeed, for any  $\phi \in E_\lambda$ , from [9, (1.22)], it comes that

$$534 \quad (2.21) \quad e^{-t\mathcal{A}^*} \phi = e^{-\lambda t} \sum_{r \geq 0} \frac{(-t)^r}{r!} (\mathcal{A}^* - \lambda)^r \phi = \sum_{r \geq 0} e_t \left[ \lambda^{(r+1)} \right] (\mathcal{A}^* - \lambda)^r \phi,$$

535 where the sums are finite (and contains at most the first  $\alpha_\lambda$  terms). Thus, every  
536 solution of (2.20) solves (2.19). The orthogonality condition (2.20b) is more restrictive  
537 than (2.19b) but leads to negligible terms in the estimates.

538  $\star$  *Resolution of block moment problems.*

539 In our setting, the block moment problem (2.20) is proved to be solvable for any  
 540  $T > 0$ . The resolution will follow from the scalar study done in [9] and refined in [12]  
 541 (see Theorem A.1).

542 Due to (2.18), the main issue to prove null controllability of (1.1) is thus to sum  
 543 those contributions to obtain a solution of (2.17). This is justified thanks to a precise  
 544 estimate of the cost of the resolution of (2.20) for each group  $G$  which is the quantity

$$545 \quad \inf \{ \|v_G\|_{L^2(0,T;U)} ; v_G \text{ solution of (2.20)} \}.$$

546 To state this result, we introduce some additional notation.

547 To solve the moment problem (2.20) we propose to lift it into a ‘vectorial block  
 548 moment problem’ of the following form (see (3.1))

$$549 \quad \begin{cases} \int_0^T v_G(t) \frac{(-t)^l}{l!} e^{-\bar{\lambda}t} dt = \Omega_{\bar{\lambda}}^l, & \forall \lambda \in G, \forall l \in \llbracket 0, \alpha_\lambda \rrbracket, \\ \int_0^T v_G(t) t^l e^{-\bar{\lambda}t} dt = 0, & \forall \lambda \in \Lambda \setminus G, \forall l \in \llbracket 0, \eta \rrbracket, \end{cases}$$

550 where  $\Omega_{\bar{\lambda}}^l$  belongs to  $U$ . Following (2.21), to recover a solution of (2.20), we need  
 551 to impose some constraints on the right-hand side that are given in the following  
 552 definition.

553 **DEFINITION 2.2.** *For any  $\lambda \in \Lambda$  and any  $z \in X_{-\diamond}$ , we set*

554

$$555 \quad (2.22) \quad \mathcal{O}(\lambda, z) = \left\{ (\Omega^0, \dots, \Omega^{\alpha_\lambda-1}) \in U^{\alpha_\lambda} ; \right.$$

$$556 \quad \left. \sum_{l=0}^{\alpha_\lambda-1} \langle \Omega^l, \mathcal{B}^*(\mathcal{A}^* - \lambda)^l \phi \rangle_U = \langle z, \phi \rangle_{-\diamond, \diamond}, \quad \forall \phi \in E_\lambda \right\}.$$

557

558 *For a given group  $G$ , we set*

$$559 \quad (2.23) \quad \mathcal{O}(G, z) = \prod_{\lambda \in G} \mathcal{O}(\lambda, z) \subset U^{|\alpha|}$$

560 *where  $\alpha$  is the multi-index of the algebraic multiplicities of the eigenvalues.*

561 Consider any sequence of multi-indices  $(\mu^l)_{l \in \llbracket 0, |\alpha| \rrbracket}$  such that

$$562 \quad (2.24) \quad \begin{cases} \mu^{l-1} \leq \mu^l, & \forall l \in \llbracket 1, |\alpha| \rrbracket, \\ |\mu^l| = l, & \forall l \in \llbracket 0, |\alpha| \rrbracket, \\ \mu^{|\alpha|} = \alpha. \end{cases}$$

563 To measure the cost associated to the group  $G = \{\lambda_1, \dots, \lambda_g\}$  let us define the fol-  
 564 lowing functional

$$565 \quad (2.25) \quad F : \Omega = (\Omega_1^0, \dots, \Omega_1^{\alpha_1-1}, \dots, \Omega_g^0, \dots, \Omega_g^{\alpha_g-1}) \in U^{|\alpha|} \mapsto \sum_{l=1}^{|\alpha|} \left\| \Omega \left[ \bar{\lambda}^{\bullet(\mu^l)} \right] \right\|_U^2$$

566 with the convention

$$567 \quad \Omega \left[ \bar{\lambda}_j^{(l+1)} \right] = \Omega_j^l, \quad \forall j \in \llbracket 1, g \rrbracket, \forall l \in \llbracket 0, \alpha_j \rrbracket.$$

568 The use of such functional to measure the cost comes from the analysis conducted for  
569 scalar controls in [9] (see Proposition 3.3). It appears in the following lower bound  
570 for solutions of block moment problems.

571 PROPOSITION 2.3. Assume that the operators  $\mathcal{A}$  and  $\mathcal{B}$  satisfy the assumption (H)  
572 (see Section 2.1.2). Let  $T \in (0, +\infty)$ , and  $G \subset \Lambda$  be a group satisfying (2.7).

573 There exists  $C_{p,\eta,r_\Lambda} > 0$  such that, for any  $z \in X_{-\diamond}$ , any  $v_G \in L^2(0, T; U)$  solving

$$574 \int_0^T \left\langle v_G(t), \mathcal{B}^* e^{-t\mathcal{A}^*} \phi \right\rangle_U dt = \langle z, \phi \rangle_{-\diamond, \diamond}, \quad \forall \phi \in E_\lambda, \forall \lambda \in G$$

575 satisfies

$$576 (2.26) \quad \|v_G\|_{L^2(0, T; U)}^2 \geq C_{p,\eta,r_\Lambda} \mathcal{C}(G, z)$$

577 where

$$578 (2.27) \quad \mathcal{C}(G, z) := \inf \{F(\Omega) ; \Omega \in \mathcal{O}(G, z)\}$$

579 with  $F$  defined in (2.25) and  $\mathcal{O}(G, z)$  defined in Definition 2.2.

580 The first main result of this article concerns the resolution of block moment  
581 problems of the form (2.20). It roughly states that, up to terms that turns out to be  
582 negligible, the lower bound obtained in Proposition 2.3 is optimal.

583 THEOREM 2.4. Assume that the operators  $\mathcal{A}$  and  $\mathcal{B}$  satisfy the assumption (H)  
584 (see Section 2.1.2). Let  $T \in (0, +\infty)$ , and  $G \subset \Lambda$  be a group satisfying (2.6)–(2.8).

585 For any  $z \in X_{-\diamond}$ , there exists  $v_G \in L^2(0, T; U)$  solution of

$$586 (2.28a) \quad \int_0^T \left\langle v_G(t), \mathcal{B}^* e^{-t\mathcal{A}^*} \phi \right\rangle_U dt = \langle z, \phi \rangle_{-\diamond, \diamond}, \quad \forall \phi \in E_\lambda, \forall \lambda \in G,$$

$$587 (2.28b) \quad \int_0^T v_G(t) t^l e^{-\bar{\lambda}t} dt = 0, \quad \forall \lambda \in \Lambda \setminus G, \forall l \in \llbracket 0, \eta \rrbracket,$$

588 satisfying the following estimate

$$590 (2.29) \quad \|v_G\|_{L^2(0, T; U)}^2 \leq C \exp\left(\frac{C}{T^{1-\theta}}\right) \exp(Cr_G^\theta) \mathcal{C}(G, z).$$

591 In this estimate,  $\mathcal{C}(G, z)$  is defined in (2.27) and  $r_G$  is defined in (2.16). The constant  
592  $C > 0$  appearing in the estimate only depends on the parameters  $\tau, p, \varrho, \eta, \theta$  and  $\kappa$ .

593 Before giving the application of this resolution of block moment problems to the  
594 null controllability of our initial system (1.1), let us give some comments.

- 595 • As it was the case in [9], the considered setting allows for a wide variety  
596 of applications. In (2.15) the generalized eigenvectors are only assumed to  
597 form a complete family (and not a Riesz basis as in many previous works)  
598 which is the minimal assumption to use a moment method-like strategy. The  
599 weak gap condition (2.4) is also well adapted to study systems of coupled one  
600 dimensional parabolic equations (see Section 7).
- 601 • The main restriction is the assumption (2.9). As detailed in Section 1.1, this  
602 assumption is common to most of the results based on a moment-like method.  
603 Though restrictive, let us underline that the moment method is, to the best  
604 of our knowledge, the most suitable method to capture very sensitive features  
605 such as a minimal null control time for parabolic control problems without  
606 constraints.

- 607 • The main novelty of this theorem is to ensure solvability of block moment  
608 problems coming from control problems with control operators that are only  
609 assumed to be admissible. In particular, the space  $U$  can be of infinite di-  
610 mension. Results concerning block moment problems with more general right-  
611 hand sides, that is not necessarily coming from a controllability problem, are  
612 stated in Appendix C
- 613 • The estimate (2.29) does not explicitly depend on the sequence of eigenval-  
614 ues  $\Lambda$  but rather on some parameters such as the weak-gap parameters and  
615 the asymptotic of the counting function. As presented in Section 1.1, the  
616 uniformity of such bounds can be used to deal with parameter dependent  
617 problems.
- 618 • Let us also underline that the obtained estimate (2.29) tracks the depen-  
619 dency of the constants with respect to the controllability time  $T$ . This will  
620 be crucial to estimate the cost of controllability in Proposition 2.11. We re-  
621 fer to Remark 2.8 for possible applications of such estimates of the cost of  
622 controllability.
- 623 • Though quite general and useful for the theoretical characterization of the  
624 minimal null control time, the obtained estimate (2.29) still requires to be able  
625 to evaluate quantities of the form  $\mathcal{C}(G, z)$ , which can be intricate. We provide  
626 in Section 2.4 some explicit formulas that makes this estimation possible in  
627 many actual examples.

### 628 2.3. Determination of the minimal null control time.

629 The resolution of block moment problems stated in Theorem 2.4 allows to obtain  
630 the following characterization of the minimal null control time of our abstract control  
631 problem from a given initial condition.

632 **THEOREM 2.5.** *Assume that the operators  $\mathcal{A}$  and  $\mathcal{B}$  satisfy the assumption (H)*  
633 *(see Section 2.1.2) and let  $\mathcal{G}$  be an associated grouping as stated in Proposition 2.2.*  
634 *Then, for any  $y_0 \in X_{-\diamond}$ , the minimal null control time of (1.1) from  $y_0$  is given by*

$$635 \quad (2.30) \quad T_0(y_0) = \limsup_{G \in \mathcal{G}} \frac{\ln^+ \mathcal{C}(G, y_0)}{2r_G}$$

636 where  $\mathcal{C}(G, y_0)$  is defined in (2.27).

637 In this statement we have used the notation  $\ln^+ s = \max(0, \ln s)$ , for any  $s \geq 0$ .

638 If one considers a space of initial conditions (instead of a single initial condition),  
639 the characterization of the minimal null control time is given in the following corollary.

640 **COROLLARY 2.6.** *Let  $Y_0$  be a closed subspace of  $X_{-\diamond}$ . Then, under the assump-*  
641 *tions of Theorem 2.5, the minimal null control time from  $Y_0$  is given by*

$$642 \quad T_0(Y_0) = \limsup_{G \in \mathcal{G}} \frac{\ln^+ \mathcal{C}(G, Y_0)}{2r_G}$$

643 with

$$644 \quad \mathcal{C}(G, Y_0) := \sup_{\substack{y_0 \in Y_0 \\ \|y_0\|_{-\diamond} = 1}} \mathcal{C}(G, y_0).$$

645 **2.4. More explicit formulas.**

646 Assume that the operators  $\mathcal{A}$  and  $\mathcal{B}$  satisfy the assumption (H). Let  $G \subset \Lambda$  be  
 647 such that  $\sharp G \leq p$  and  $\text{diam } G \leq \varrho$ . We have seen in Theorem 2.5 that the key quantity  
 648 to compute the minimal null control time from  $y_0$  is

$$649 \quad \mathcal{C}(G, y_0) = \inf \{F(\Omega) ; \Omega \in \mathcal{O}(G, y_0)\}.$$

650 where the function  $F$  is defined in (2.25) and the constraints  $\mathcal{O}(G, y_0)$  are defined  
 651 in (2.23). Let us give more explicit formulas to compute such costs.

652 Notice that, for any  $z \in X_{-\infty}$ , the quantity  $\mathcal{C}(G, z)$  can be expressed as a finite  
 653 dimensional constrained problem. Indeed, for a given group  $G$  we consider the finite  
 654 dimensional subspace

$$655 \quad (2.31) \quad U_G = \mathcal{B}^* \text{Span} \{\phi \in E_\lambda ; \lambda \in G\}$$

656 and  $P_{U_G}$  the orthogonal projection in  $U$  onto  $U_G$ . Then, for any  $\Omega \in \mathcal{O}(G, z)$  it  
 657 comes that  $P_{U_G}\Omega \in \mathcal{O}(G, z)$  and  $F(P_{U_G}\Omega) \leq F(\Omega)$ . Thus, the optimization problem  
 658 defining  $\mathcal{C}(G, z)$  reduces to

$$659 \quad \mathcal{C}(G, z) = \inf \left\{ F(\Omega) ; \Omega \in \mathcal{O}(G, z) \cap U_G^{|\alpha|} \right\},$$

660 which is a finite dimensional optimization problem. From [9, Proposition 7.15], the  
 661 function  $F$  is coercive which implies that the infimum is actually attained:

$$662 \quad (2.32) \quad \mathcal{C}(G, z) = \min \left\{ F(\Omega) ; \Omega \in \mathcal{O}(G, z) \cap U_G^{|\alpha|} \right\}.$$

663 In this section, solving the optimization problem (2.32), we provide more explicit  
 664 formulas for this cost for some particular configurations for the multiplicities of the  
 665 eigenvalues in the group  $G$  (and only in that particular group).

666 *★ A group  $G$  of geometrically simple eigenvalues.*

667 First, assume that the eigenvalues in  $G = \{\lambda_1, \dots, \lambda_g\}$  are all geometrically simple  
 668 *i.e.*  $\gamma_\lambda = 1$  for every  $\lambda \in G$  where  $\gamma_\lambda$  is defined in (2.12).

669 For any  $j \in \llbracket 1, g \rrbracket$  we denote by  $\phi_j^0$  an eigenvector of  $\mathcal{A}^*$  associated to the eigen-  
 670 value  $\lambda_j$  and by  $(\phi_j^l)_{l \in \llbracket 0, \alpha_j \rrbracket}$  an associated Jordan chain *i.e.*

$$671 \quad (\mathcal{A}^* - \lambda_j)\phi_j^l = \phi_j^{l-1}, \quad \forall l \in \llbracket 1, \alpha_j \rrbracket.$$

672 To simplify the writing, we set

$$673 \quad b_j^l := \mathcal{B}^* \phi_j^l \in U, \quad \forall l \in \llbracket 0, \alpha_j \rrbracket, \forall j \in \llbracket 1, g \rrbracket.$$

674 Recall that the sequence of multi-index  $(\mu^l)_{l \in \llbracket 0, |\alpha| \rrbracket}$  satisfy (2.24) and let

$$675 \quad (2.33) \quad M := \sum_{l=1}^{|\alpha|} \Gamma_\mu^l$$

676 with

$$677 \quad \Gamma_\mu^l := \text{Gram}_U \left( \underbrace{0, \dots, 0}_{l-1}, b \left[ \lambda_\bullet^{(\mu^l - \mu^{l-1})} \right], \dots, b \left[ \lambda_\bullet^{(\mu^{|\alpha|} - \mu^{l-1})} \right] \right)$$

678 where for every  $u_1, \dots, u_n \in U$ ,  $\text{Gram}_U(u_1, \dots, u_n)$  denotes the Gram matrix whose  
679 entry on the  $i$ -th row and  $j$ -th column is  $\langle u_j, u_i \rangle_U$ . To explicit the cost  $\mathcal{C}(G, y_0)$ , we  
680 will use the inverse of this matrix. Its invertibility is guaranteed by the following  
681 proposition which is proved in Section 5.2.

682 PROPOSITION 2.7. *Under condition (2.14), the matrix  $M$  defined in (2.33) is in-*  
683 *vertible.*

684 The matrix  $M$  plays a crucial role in the computation of the cost  $\mathcal{C}(G, y_0)$ . Let us  
685 give some comments. It is a sum of Gram matrices whose construction is summarized  
686 in Figure 1 on an example with  $G = \{\lambda_1, \lambda_2\}$  with  $\alpha_1 = 3$  and  $\alpha_2 = 2$ . Each of these  
687 matrices is of size  $|\alpha|$  which is the number of eigenvalues (counted with their algebraic  
688 multiplicities) that belong to the group  $G$ . Thus, on actual examples (see Section 7),  
689 the size of these matrices is usually reasonably small.

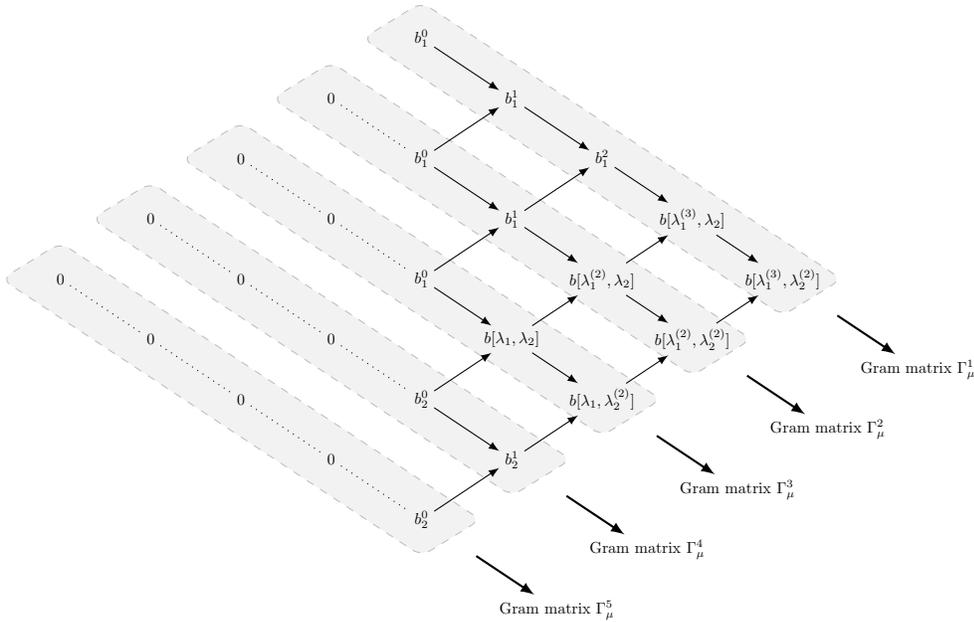


FIGURE 1. Construction of the Gram matrices  $\Gamma_\mu^l$  in the case of a group  $G = \{\lambda_1, \lambda_2\}$  with multiplicities  $\alpha = (3, 2)$  and the sequence of multi-indices  $\mu = ((0, 0), (1, 0), (2, 0), (3, 0), (3, 1), (3, 2))$

690 Then, we obtain the following formula for the cost of a group of geometrically  
691 simple eigenvalues.

692 THEOREM 2.8. *Assume that the operators  $\mathcal{A}$  and  $\mathcal{B}$  satisfy the assumption (H)*  
693 *(see Section 2.1.2). Let  $G = \{\lambda_1, \dots, \lambda_g\} \subset \Lambda$  be such that  $\sharp G \leq p$  and  $\text{diam } G \leq \rho$*   
694 *and assume that  $\gamma_\lambda = 1$  for every  $\lambda \in G$ . Then, for any  $y_0 \in X_{-\diamond}$ , we have*

$$695 \quad \mathcal{C}(G, y_0) = \langle M^{-1}\xi, \xi \rangle, \quad \text{where } \xi = \begin{pmatrix} \left\langle y_0, \phi \left[ \lambda_{\bullet}^{(\mu^1)} \right] \right\rangle_{-\diamond, \diamond} \\ \vdots \\ \left\langle y_0, \phi \left[ \lambda_{\bullet}^{(\mu^{|\alpha|})} \right] \right\rangle_{-\diamond, \diamond} \end{pmatrix} \in \mathbb{C}^{|\alpha|}$$

696 and  $M$  is defined in (2.33).

697 Moreover, if  $Y_0$  is a closed subspace of  $X_{-\diamond}$ ,

$$698 \quad (2.34) \quad \mathcal{C}(G, Y_0) = \rho(\text{Gram}_{X_\diamond^*}(\psi_1, \dots, \psi_{|\alpha|})M^{-1})$$

699 where  $\psi_j := P_{Y_0}^* \phi \left[ \lambda_{\bullet}^{(\mu^j)} \right]$  and, for any matrix  $M$ , the notation  $\rho(M)$  denotes the  
700 spectral radius of the matrix  $M$ .

701 **REMARK 2.4.** Notice that we do not choose any particular eigenvector or Jordan  
702 chain. To compute explicitly the cost  $\mathcal{C}(G, y_0)$  on actual examples, we will often choose  
703 them to satisfy

$$704 \quad \|b_j^0\|_U = 1, \quad \langle b_j^0, b_j^l \rangle_U = 0, \quad \forall l \in \llbracket 1, \alpha_j \rrbracket,$$

705 to simplify the Gram matrices. Obviously, as the quantity  $\mathcal{C}(G, y_0)$  is independent of  
706 this choice, we can choose any other specific Jordan chains or eigenvectors that are  
707 more suitable to each problem.

708 **REMARK 2.5.** In the case where the eigenvalues of the considered group  $G$  are  
709 also algebraically simple, then the expression of  $M$  given in (2.33) reduces to

$$710 \quad (2.35) \quad M = \sum_{l=1}^g \Gamma^l \quad \text{with} \quad \Gamma^l = \text{Gram}_U \left( \underbrace{0, \dots, 0}_{l-1}, b[\lambda_l], \dots, b[\lambda_l, \dots, \lambda_g] \right)$$

711 and the expression of  $\xi$  reduces to

$$712 \quad \xi = \begin{pmatrix} \langle y_0, \phi[\lambda_1] \rangle_{-\diamond, \diamond} \\ \vdots \\ \langle y_0, \phi[\lambda_1, \dots, \lambda_g] \rangle_{-\diamond, \diamond} \end{pmatrix}.$$

713  $\star$  A group  $G$  of semi-simple eigenvalues.

714 We now assume that all the eigenvalues in  $G$  are semi-simple i.e. for any  $\lambda \in G$   
715 we have  $\alpha_\lambda = 1$  where  $\alpha_\lambda$  is defined in (2.13).

716 For any  $j \in \llbracket 1, g \rrbracket$ , we denote by  $(\phi_{j,i})_{i \in \llbracket 1, \gamma_j \rrbracket}$  a basis of  $\text{Ker}(\mathcal{A}^* - \lambda_j)$ . To simplify  
717 the writing, we set

$$718 \quad b_{j,i} := \mathcal{B}^* \phi_{j,i}, \quad \forall j \in \llbracket 1, g \rrbracket, \forall i \in \llbracket 1, \gamma_j \rrbracket$$

719 and  $\gamma_G := \gamma_1 + \dots + \gamma_g$ .

720 For any  $i \in \llbracket 1, g \rrbracket$ , we set  $\delta_1^i := 1$  and

$$721 \quad (2.36) \quad \delta_j^i := \prod_{k=1}^{j-1} (\lambda_i - \lambda_k), \quad \forall j \in \llbracket 2, g \rrbracket.$$

722 Notice that  $\delta_j^i = 0$  as soon as  $j > i$ .

723 Let

$$724 \quad (2.37) \quad M = \sum_{l=1}^g \Gamma^l \quad \text{with} \quad \Gamma^l = \text{Gram}_U (\delta_l^1 b_{1,1}, \dots, \delta_l^1 b_{1,\gamma_1}, \dots, \delta_l^g b_{g,1}, \dots, \delta_l^g b_{g,\gamma_g}).$$

725 Here again, to explicit the cost  $\mathcal{C}(G, y_0)$  we will use the inverse of this matrix. Its  
726 invertibility is guaranteed by the following proposition which is proved in Section 5.3.

727 PROPOSITION 2.9. Under condition (2.14), the matrix  $M$  defined in (2.37) is in-  
 728 vertible.

729 Notice that the square matrix  $\Gamma^l$  is of size  $\gamma_G$  and can be seen as a block matrix where  
 730 the block  $(i, j)$  with  $\gamma_i$  rows and  $\gamma_j$  columns is

$$731 \begin{pmatrix} \langle \delta_l^j b_{j,1}, \delta_l^i b_{i,1} \rangle_U & \cdots & \langle \delta_l^j b_{j,\gamma_j}, \delta_l^i b_{i,1} \rangle_U \\ \vdots & & \vdots \\ \langle \delta_l^j b_{j,1}, \delta_l^i b_{i,\gamma_i} \rangle_U & \cdots & \langle \delta_l^j b_{j,\gamma_j}, \delta_l^i b_{i,\gamma_i} \rangle_U \end{pmatrix}.$$

732 Thus, the block  $(i, j)$  of  $\Gamma^l$  is identically 0 for  $i, j \in \llbracket 1, l \rrbracket$ .

733 Then, we obtain the following formula for the cost of a group made of semi-simple  
 734 eigenvalues.

735 THEOREM 2.10. Assume that the operators  $\mathcal{A}$  and  $\mathcal{B}$  satisfy the assumption (H)  
 736 (see Section 2.1.2). Let  $G = \{\lambda_1, \dots, \lambda_g\} \subset \Lambda$  be such that  $\sharp G \leq p$  and  $\text{diam } G \leq \rho$   
 737 and assume that  $\alpha_\lambda = 1$  for every  $\lambda \in G$ . Then, for any  $y_0 \in X_{-\diamond}$ , we have

$$738 \mathcal{C}(G, y_0) = \langle M^{-1} \xi, \xi \rangle$$

739 where

$$740 \xi = \begin{pmatrix} \langle y_0, \phi_{1,1} \rangle_{-\diamond, \diamond} \\ \vdots \\ \langle y_0, \phi_{1,\gamma_1} \rangle_{-\diamond, \diamond} \\ \vdots \\ \langle y_0, \phi_{g,1} \rangle_{-\diamond, \diamond} \\ \vdots \\ \langle y_0, \phi_{g,\gamma_g} \rangle_{-\diamond, \diamond} \end{pmatrix}$$

741 and  $M$  is defined in (2.37).

742 Moreover, if  $Y_0$  is a closed subspace of  $X_{-\diamond}$ ,

$$743 (2.38) \quad \mathcal{C}(G, Y_0) = \rho \left( \text{Gram}_{X_*}(\psi_{1,1}, \dots, \psi_{1,\gamma_1}, \dots, \psi_{g,1}, \dots, \psi_{g,\gamma_g}) M^{-1} \right)$$

744 where  $\psi_{j,i} := P_{Y_0}^* \phi_{j,i}$  and, for any matrix  $M$ , the notation  $\rho(M)$  denotes its spectral  
 745 radius.

746 REMARK 2.6. When the eigenvalues of the group  $G$  are geometrically and alge-  
 747 braically simple, Theorem 2.10 gives a characterization of the cost of the block  $\mathcal{C}(G, y_0)$   
 748 which is different from the one coming from Theorem 2.8 and detailed in Remark 2.5.  
 749 A direct proof of this equivalence (stated in Proposition D.3) using algebraic manipu-  
 750 lations is given in Appendix D.

751 *★ Dealing simultaneously with geometric and algebraic multiplicity.*

752 Combining Theorems 2.8 and 2.10, we can deal with operators  $\mathcal{A}^*$  which have  
 753 both groups of geometrically simple eigenvalues and groups of semi-simple eigenval-  
 754 ues. However, for technical reasons, in the case where both algebraic and geometric  
 755 multiplicities need to be taken into account into a group  $G$  we do obtain a general  
 756 formula for the cost of this group  $\mathcal{C}(G, y_0)$ . Nevertheless, if this situation occurs in  
 757 actual examples, computing this cost is a finite dimensional constrained optimization

758 problem which can be solved ‘by hand’. We present in Section 5.4 an example of such  
 759 resolution for a group  $G$  that does not satisfies the assumptions of Theorem 2.8 nor  
 760 of Theorem 2.10.

761 **2.5. Estimate of the cost of null controllability.**

762 When system (1.1) is null controllable, we obtain the following bound on the cost  
 763 of controllability.

764 PROPOSITION 2.11. *Assume that the operators  $\mathcal{A}$  and  $\mathcal{B}$  satisfy the assumption (H)  
 765 (see Section 2.1.2) and let  $\mathcal{G}$  be an associated grouping a as stated in Proposition 2.2.*

766 *Let  $y_0 \in X_{-\diamond}$  and let  $T > T_0(y_0)$ . There exists a control  $u \in L^2(0, T; U)$  such  
 767 that the associated solution of (1.1) initiated from  $y_0$  satisfies  $y(T) = 0$  and*

$$768 \quad \|u\|_{L^2(0, T; U)}^2 \leq C \exp\left(\frac{C}{(T - T_0(y_0))^{\frac{\theta}{1-\theta}}}\right) (1+T)^{2\eta} \sum_{G \in \mathcal{G}} e^{-(T - T_0(y_0))r_G} e^{-2r_G T_0(y_0)} \mathcal{C}(G, Y_0).$$

769 *The constant  $C > 0$  appearing in the estimate only depends on the parameters  $\tau$ ,  $p$ ,  
 770  $\varrho$ ,  $\eta$ ,  $\theta$  and  $\kappa$ .*

771 Though quite general the above formula is not very explicit. More importantly,  
 772 it is proved in [29, Theorem 1.1] that, with a suitable choice of  $\mathcal{A}$  and  $\mathcal{B}$  satisfying  
 773 our assumptions, any blow-up of the cost of controllability can occur. We give below  
 774 a setting (inspired from [29, Theorem 1.2]) in which this upper bound on the cost of  
 775 controllability is simpler and can have some applications (see Remark 2.8).

776 COROLLARY 2.12. *Assume that the operators  $\mathcal{A}$  and  $\mathcal{B}$  satisfy the assumption (H)  
 777 (see Section 2.1.2) and let  $\mathcal{G}$  be an associated grouping as stated in Proposition 2.2.  
 778 Let  $\beta > 0$ . For any  $y_0 \in X_{-\diamond}$  satisfying,*

$$779 \quad (2.39) \quad \mathcal{C}(G, y_0) \leq \beta e^{2r_G T_0(y_0)} \|y_0\|_{-\diamond}^2, \quad \forall G \in \mathcal{G},$$

780 *for any  $T > T_0(y_0)$  close enough to  $T_0(y_0)$ , there exists a control  $u \in L^2(0, T; U)$  such  
 781 that the associated solution of (1.1) satisfies  $y(T) = 0$  and*

$$782 \quad \|u\|_{L^2(0, T; U)} \leq C \exp\left(\frac{C}{(T - T_0(y_0))^{\frac{\theta}{1-\theta}}}\right) \|y_0\|_{-\diamond},$$

783 *where the constant  $C > 0$  only depends on the parameters  $\beta$ ,  $\tau$ ,  $p$ ,  $\varrho$ ,  $\eta$ ,  $\theta$  and  $\kappa$ .*

784 REMARK 2.7. *In the setting of Corollary 2.12, replacing the assumption (2.39)  
 785 by*

$$786 \quad \mathcal{C}(G, y_0) \leq \beta e^{\beta r_G^\sigma} e^{2r_G T_0(y_0)} \|y_0\|_{-\diamond}^2, \quad \forall G \in \mathcal{G},$$

787 *with  $\sigma \in (0, 1)$  leads to the following estimate*

$$788 \quad \|u\|_{L^2(0, T; U)} \leq C \exp\left(\frac{C}{(T - T_0(y_0))^{\frac{\max(\theta, \sigma)}{1 - \max(\theta, \sigma)}}}\right) \|y_0\|_{-\diamond}.$$

789 REMARK 2.8. *Giving the best possible estimate on the cost of small time null  
 790 controllability is a question that has drawn a lot of interest in the past years.*

791 *In classical cases, for instance for heat-like equations, null controllability holds  
 792 in any positive time and the cost of controllability in small time behaves like  $\exp\left(\frac{C}{T}\right)$   
 793 (see for instance [37]). There are two main applications of such estimate.*

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- *Controllability in cylindrical domains.*

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*It is proved in [8] that null controllability of parabolic problems in cylindrical geometries (with operators compatible with this geometry) with a boundary control located on the top of the cylinder can be proved thanks to null controllability of the associated problem in the transverse variable together with suitable estimates of the cost of controllability. Their proof relies on an adaptation of the classical strategy of Lebeau and Robbiano [28] and thus uses an estimate of the cost of controllability in small time of the form  $\exp(\frac{C}{T})$ . These ideas were already present in [10] and later generalized in an abstract setting in [1].*

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- *Nonlinear control problems.*

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*The source term method has been introduced in [30] to prove controllability of a nonlinear fluid-structure system (see also [7, Section 2] for a general presentation of this strategy). Roughly speaking it amounts to prove null controllability with a source term in suitable weighted spaces and then use a fixed point argument. The null controllability with a source term is here proved by an iterative process which strongly uses that the cost of controllability of the linearized system behaves like  $\exp(\frac{C}{T})$ .*

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*Notice that from the upper bound given in Corollary 2.12, the cost of controllability in small time can explode faster than  $\exp(\frac{C}{T})$ . Yet, as studied in [33] and in [35, Chapter 4], the arguments of the two previous applications can be adapted with an*

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*explosion of the cost of the form  $\exp\left(\frac{C}{T^{1-\theta}}\right)$  with  $\theta \in (0, 1)$ .*

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*However, these two applications uses a decomposition of the time interval  $[0, T]$  into an infinite number of sub-intervals (which explains the use of the asymptotic of the cost of controllability when the time goes to zero). Thus their extension in the case of a minimal null control time is an open problem.*

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### 3. Resolution of block moment problems.

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In this section we prove Theorem 2.4 that is we solve the block moment problem (2.28). To do so, we first consider a vectorial block moment problem (see (3.1) below) which is proved to be equivalent to the block moment problem (2.28) in Proposition 3.1. This equivalence strongly relies on the constraints (2.22). Then we prove the lower bound for solutions of block moment problems stated in Proposition 2.3.

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Finally, in Section 3.2, we solve the vectorial block moment problem (3.1) which will conclude the proof of Theorem 2.4.

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#### 3.1. An auxiliary equivalent vectorial block moment problem.

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Let  $\Lambda \subset S_\tau$ ,  $G = \{\lambda_1, \dots, \lambda_g\} \subset \Lambda$ ,  $\eta \in \mathbb{N}^*$  and  $\alpha = (\alpha_1, \dots, \alpha_g) \in \mathbb{N}^g$  with  $|\alpha|_\infty \leq \eta$ . For any

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$$\Omega = (\Omega_1^0, \dots, \Omega_1^{\alpha_1-1}, \dots, \Omega_g^0, \dots, \Omega_g^{\alpha_g-1}) \in U^{|\alpha|},$$

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we consider the following auxiliary vectorial block moment problem : find  $v_G \in L^2(0, T; U)$  such that

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$$(3.1a) \quad \int_0^T v_G(t) \frac{(-t)^l}{l!} e^{-\bar{\lambda}_j t} dt = \Omega_j^l, \quad \forall j \in \llbracket 1, g \rrbracket, \forall l \in \llbracket 0, \alpha_j \rrbracket,$$

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$$(3.1b) \quad \int_0^T v_G(t) t^l e^{-\bar{\lambda} t} dt = 0, \quad \forall \lambda \in \Lambda \setminus G, \forall l \in \llbracket 0, \eta \rrbracket.$$

837 This block moment problem is said to be *vectorial*: the right-hand side  $\Omega$  belongs  
 838 to  $U^{|\alpha|}$  and its solution  $v_G(t)$  belongs to the control space  $U$  for almost every  $t$ . Its  
 839 resolution with (almost) sharp estimates is given in Proposition 3.3.

840 Through (2.21), when the right-hand side  $\Omega$  of (3.1) satisfy the constraints (2.23),  
 841 solving this vectorial block moment problem provides a solution of the original block  
 842 moment problem (2.28). More precisely we have the following proposition

843 PROPOSITION 3.1. *Let  $T > 0$  and  $z \in X_{-\diamond}$ . The following two statements are*  
 844 *equivalent:*

- 845 *i. there exists  $\Omega \in \mathcal{O}(G, z)$  such that the function  $v_G \in L^2(0, T; U)$  solves (3.1);*
- 846 *ii. the function  $v_G \in L^2(0, T; U)$  solves (2.28).*

847 *Proof.* Assume first that there exists  $\Omega \in \mathcal{O}(G, z)$  and let  $v \in L^2(0, T; U)$  be such  
 848 that (3.1) holds.

849 Then, using (2.21), for any  $j \in \llbracket 1, g \rrbracket$  and any  $\phi \in E_{\lambda_j}$  we have

$$\begin{aligned}
 850 \quad \int_0^T \langle v(t), \mathcal{B}^* e^{-t\mathcal{A}^*} \phi \rangle_U dt &= \int_0^T \left\langle v(t), e^{-\lambda_j t} \sum_{l=0}^{\alpha_j-1} \frac{(-t)^l}{l!} \mathcal{B}^* (\mathcal{A}^* - \lambda_j)^l \phi \right\rangle_U dt \\
 851 &= \sum_{l=0}^{\alpha_j-1} \left\langle \int_0^T v(t) \frac{(-t)^l}{l!} e^{-\bar{\lambda}_j t} dt, \mathcal{B}^* (\mathcal{A}^* - \lambda_j)^l \phi \right\rangle_U \\
 852 &= \sum_{l=0}^{\alpha_j-1} \langle \Omega_j^l, \mathcal{B}^* (\mathcal{A}^* - \lambda_j)^l \phi \rangle_U. \\
 853
 \end{aligned}$$

854 Since  $(\Omega_j^0, \dots, \Omega_j^{\alpha_j-1}) \in \mathcal{O}(\lambda_j, z)$  (see (2.22)), this leads to

$$855 \quad \int_0^T \langle v(t), \mathcal{B}^* e^{-t\mathcal{A}^*} \phi \rangle_U dt = \langle z, \phi \rangle_{-\diamond, \diamond}, \quad \forall j \in \llbracket 1, g \rrbracket, \forall \phi \in E_{\lambda_j},$$

856 which proves that  $v$  solves (2.28).

857 Assume now that  $v \in L^2(0, T; U)$  solves (2.28). Setting

$$858 \quad \Omega_j^l := \int_0^T v(t) \frac{(-t)^l}{l!} e^{-\bar{\lambda}_j t} dt$$

859 we obtain that  $v$  solves (3.1). As in the previous step, the identity (2.21) implies that  
 860  $\Omega \in \mathcal{O}(G, z)$ .  $\square$

861 Using this vectorial block moment problem allows to prove the lower bound stated  
 862 in Proposition 2.3.

863 *Proof (of Proposition 2.3).* Let  $v_G \in L^2(0, T; U)$  be any solution of (2.28a). Let

$$864 \quad \Omega_j^l := \int_0^T v_G(t) \frac{(-t)^l}{l!} e^{-\bar{\lambda}_j t} dt = \int_0^T v_G(t) e_t \left[ \bar{\lambda}_j^{(l+1)} \right] dt, \quad \forall j \in \llbracket 1, g \rrbracket, \forall l \in \llbracket 0, \alpha_j \rrbracket.$$

865 As in the proof of Proposition 3.1, the use of (2.21) implies that

$$866 \quad \Omega = (\Omega_1^0, \dots, \Omega_1^{\alpha_1-1}, \dots, \Omega_g^0, \dots, \Omega_g^{\alpha_g-1}) \in \mathcal{O}(G, z).$$

867 Thus,

$$868 \quad (3.2) \quad \mathcal{C}(G, z) \leq F(\Omega) = \sum_{l=1}^{|\alpha|} \left\| \Omega \left[ \bar{\lambda}^{(\mu^l)} \right] \right\|_U^2.$$

869 Notice that

$$870 \quad \Omega \left[ \bar{\lambda}_{\bullet}^{(\mu^l)} \right] = \int_0^T v_G(t) e_t \left[ \bar{\lambda}_{\bullet}^{(\mu^l)} \right] dt, \quad \forall l \in \llbracket 0, |\alpha| \rrbracket.$$

871 Using Jensen inequality [9, Proposition 6.1] yields,

$$872 \quad \left| e_t \left[ \bar{\lambda}_{\bullet}^{(\mu^l)} \right] \right| = \left| e_t \left[ \lambda_{\bullet}^{(\mu^l)} \right] \right| \leq \frac{t^{l-1} e^{-r_G t}}{(l-1)!}.$$

873 Together with Cauchy-Schwarz inequality this implies

$$874 \quad \left\| \Omega \left[ \bar{\lambda}_{\bullet}^{(\mu^l)} \right] \right\|_U \leq \left( \int_0^{+\infty} \frac{t^{l-1} e^{-r_G t}}{(l-1)!} dt \right)^{\frac{1}{2}} \|v_G\|_{L^2(0,T;U)}.$$

875 Then, as  $r_G \geq r_\Lambda$  and  $|\alpha| \leq p\eta$ , estimate (3.2) ends the proof of Theorem 2.4.  $\square$

### 876 3.2. Solving the original moment problem.

877 In view of Proposition 3.1, to solve (2.28), we prove that there exists at least one  
878  $\Omega$  satisfying the constraints (2.22).

879 PROPOSITION 3.2. *Let  $\lambda \in \Lambda$  and  $z \in X_{-\diamond}$ . Then, under assumption (2.14), we  
880 have*

$$881 \quad \mathcal{O}(\lambda, z) \neq \emptyset.$$

882 *Proof.* Let  $T > 0$ . The finite dimensional space  $E_\lambda$  is stable by the semi-  
883 group  $e^{-\bullet \mathcal{A}^*}$  (see for instance (2.21)). Using the approximate controllability assump-  
884 tion (2.14) we have that

$$885 \quad \phi \in E_\lambda \mapsto \left\| \mathcal{B}^* e^{-\bullet \mathcal{A}^*} \phi \right\|_{L^2(0,T,U)}$$

886 is a norm on  $E_\lambda$ . Then, the equivalence of norms in finite dimension implies that the  
887 following HUM-type functional

$$888 \quad J : \phi \in E_\lambda \mapsto \frac{1}{2} \left\| \mathcal{B}^* e^{-\bullet \mathcal{A}^*} \phi \right\|_{L^2(0,T,U)}^2 - \Re \langle z, \phi \rangle_{-\diamond, \diamond}$$

889 is coercive. Let  $\tilde{\phi} \in E_\lambda$  be such that

$$890 \quad J(\tilde{\phi}) = \inf_{\phi \in E_\lambda} J(\phi)$$

891 and  $v := \mathcal{B}^* e^{-\bullet \mathcal{A}^*} \tilde{\phi}$ . The optimality condition gives (paying attention to the fact that  
892  $E_\lambda$  is a complex vector space)

$$893 \quad (3.3) \quad \int_0^T \left\langle v(t), \mathcal{B}^* e^{-t \mathcal{A}^*} \phi \right\rangle_U dt = \langle z, \phi \rangle_{-\diamond, \diamond}, \quad \forall \phi \in E_\lambda.$$

894 Finally, we set  $\Omega := (\Omega^0, \dots, \Omega^{\alpha_\lambda - 1})$  with

$$895 \quad \Omega^l := \int_0^T v(t) \frac{(-t)^l}{l!} e^{-\bar{\lambda} t} dt, \quad \forall l \in \llbracket 0, \alpha_\lambda \rrbracket.$$

896 Using (3.3) and following the computations of Proposition 3.1 we obtain that  $\Omega \in$   
897  $\mathcal{O}(\lambda, z)$ .  $\square$

898 We now turn to the resolution of the vectorial block moment problem (3.1).

899 PROPOSITION 3.3. Let  $p \in \mathbb{N}^*$ ,  $\varrho, \tau, \kappa > 0$  and  $\theta \in (0, 1)$ . Assume that

900 
$$\Lambda \in \mathcal{L}_w(p, \varrho, \tau, \theta, \kappa).$$

901 Let  $G = \{\lambda_1, \dots, \lambda_g\} \subset \Lambda$  be a group satisfying (2.6)–(2.8). Let  $T \in (0, +\infty)$  and  
 902  $\eta \in \mathbb{N}^*$ . For any multi-index  $\alpha \in \mathbb{N}^g$  with  $|\alpha|_\infty \leq \eta$  and any

903 
$$\Omega = (\Omega_1^0, \dots, \Omega_1^{\alpha_1-1}, \dots, \Omega_g^0, \dots, \Omega_g^{\alpha_g-1}) \in U^{|\alpha|},$$

904 there exists  $v_G \in L^2(0, T; U)$  solution of (3.1) such that

905 
$$\|v_G\|_{L^2(0, T; U)}^2 \leq C \exp\left(\frac{C}{T^{1-\theta}}\right) \exp(Cr_G^\theta) F(\Omega),$$

906 where  $F$  is defined in (2.25) and  $r_G$  is defined in (2.16). The constant  $C > 0$  appearing  
 907 in the estimate only depends on the parameters  $\tau, p, \varrho, \eta, \theta$  and  $\kappa$ .

908 *Proof.* Let  $(e_j)_{j \in \llbracket 1, d \rrbracket}$  be an orthonormal basis of the finite dimensional subspace  
 909 of  $U$  given by

910 
$$\text{Span} \left\{ \Omega_j^l ; j \in \llbracket 1, g \rrbracket, l \in \llbracket 0, \alpha_j \rrbracket \right\}.$$

911 Then, for any  $j \in \llbracket 1, g \rrbracket$  and  $l \in \llbracket 0, \alpha_j \rrbracket$ , there exists  $\left( a_i \left[ \overline{\lambda_j}^{(l+1)} \right] \right)_{i \in \llbracket 1, d \rrbracket} \in \mathbb{C}^{|\alpha|}$  such  
 912 that the decomposition of  $\Omega_j^l$  reads

913 
$$\Omega_j^l = \sum_{i=1}^d a_i \left[ \overline{\lambda_j}^{(l+1)} \right] e_i.$$

914 From Theorem A.1, for any  $i \in \llbracket 1, d \rrbracket$ , there exists  $v_i \in L^2(0, T; \mathbb{C})$  such that

915 
$$\begin{cases} \int_0^T v_i(t) \frac{(-t)^l}{l!} e^{-\overline{\lambda_j} t} dt = a_i \left[ \overline{\lambda_j}^{(l+1)} \right], & \forall j \in \llbracket 1, g \rrbracket, \forall l \in \llbracket 0, \alpha_j \rrbracket, \\ \int_0^T v_i(t) t^l e^{-\overline{\lambda} t} dt = 0, & \forall \lambda \in \Lambda \setminus G, \forall l \in \llbracket 0, \eta \rrbracket, \end{cases}$$

916 and

917 
$$\|v_i\|_{L^2(0, T; \mathbb{C})}^2 \leq C e^{CT^{-\frac{\theta}{1-\theta}}} e^{Cr_G^\theta} \max_{\substack{\mu \in \mathbb{N}^g \\ \mu \leq \alpha}} \left| a_i \left[ \overline{\lambda_1}^{(\mu_1)}, \dots, \overline{\lambda_g}^{(\mu_g)} \right] \right|^2.$$

918 Setting

919 
$$v := \sum_{i=1}^d v_i e_i,$$

920 we get that  $v$  solves (3.1) and using [9, Proposition 7.15]

$$\begin{aligned}
921 \quad \|v\|_{L^2(0,T;U)}^2 &= \sum_{i=1}^d \|v_i\|_{L^2(0,T;\mathbb{C})}^2 \\
922 \quad &\leq C e^{CT^{-\frac{\theta}{1-\theta}}} e^{Cr_G^\theta} \sum_{i=1}^d \max_{\substack{\mu \in \mathbb{N}^g \\ \mu \leq \alpha}} \left| a_i \left[ \bar{\lambda}_1^{(\mu_1)}, \dots, \bar{\lambda}_g^{(\mu_g)} \right] \right|^2 \\
923 \quad &\leq C_{p,\varrho,\eta} C e^{CT^{-\frac{\theta}{1-\theta}}} e^{Cr_G^\theta} \sum_{p=1}^{|\alpha|} \left( \sum_{i=1}^d \left| a_i \left[ \bar{\lambda}_\bullet^{(\mu^p)} \right] \right|^2 \right) \\
924 \quad &= C e^{CT^{-\frac{\theta}{1-\theta}}} e^{Cr_G^\theta} \sum_{p=1}^{|\alpha|} \left\| \Omega \left[ \bar{\lambda}_\bullet^{(\mu^p)} \right] \right\|^2. \\
925
\end{aligned}$$

926 This ends the proof of Proposition 3.3.  $\square$

927 We now have all the ingredients to prove Theorem 2.4.

928 *Proof (of Theorem 2.4).* From Proposition 3.2, we have  $\mathcal{O}(G, z) \neq \emptyset$ . Recall  
929 that, from (2.32), the optimization problem defining  $\mathcal{C}(G, z)$  can be reduced to a  
930 finite dimensional optimization problem for which the infimum is attained. Thus, let  
931  $\Omega \in \mathcal{O}(G, z)$  be such that

$$932 \quad F(\Omega) = \mathcal{C}(G, z).$$

933 Let  $v_G \in L^2(0, T; U)$  be the solution of (3.1) given by Proposition 3.3 with  $\Omega$  as right-  
934 hand side. As  $\Omega \in \mathcal{O}(G, z)$ , from Proposition 3.1 we deduce that  $v_G$  solves (2.28).  
935 The upper bound (2.29) on  $\|v_G\|_{L^2(0,T;U)}$  is given by Proposition 3.3.  $\square$

#### 936 4. Application to the determination of the minimal null control time.

937 This section is dedicated to the consequences of Theorem 2.4 on the null control-  
938 lability properties of system (1.1).

939 From Theorem 2.4, the resolution of block moment problems (2.20) associated  
940 with null controllability of (1.1) will involve the quantity  $\mathcal{C}(G, e^{-T\mathcal{A}}y_0)$ . To formulate  
941 the minimal null control time we isolate the dependency with respect to the variable  
942  $T$  leading to quantities involving  $\mathcal{C}(G, y_0)$ . The comparison between these two costs  
943 is detailed in Section 4.1.

944 Then, this leads to the formulation of the minimal null control time stated in  
945 Theorem 2.5. We then prove the estimates on the cost of null controllability stated  
946 in Proposition 2.11 and Corollary 2.12. This is detailed in Section 4.2.

##### 947 4.1. Relating the different costs.

948 Let us prove that the cost  $\mathcal{C}(G, e^{-T\mathcal{A}}z)$  appearing in Theorem 2.4 roughly behaves  
949 like  $e^{-2r_G T} \mathcal{C}(G, z)$ . More precisely, we have the following estimates.

950 LEMMA 4.1. *Assume that the operators  $\mathcal{A}$  and  $\mathcal{B}$  satisfy the assumption (H) (see*  
951 *Section 2.1.2). There exists  $C_{p,\varrho,\eta} > 0$  such that for any  $G \subset \Lambda$  with  $\sharp G \leq p$  and*  
952 *diam  $G \leq \varrho$ , for any  $T > 0$  and any  $z \in X_{-\varrho}$ ,*

$$953 \quad (4.1) \quad \mathcal{C}(G, e^{-T\mathcal{A}}z) \leq C_{p,\varrho,\eta} (1+T)^{2|\alpha|} e^{-2r_G T} \mathcal{C}(G, z)$$

954 and

$$955 \quad (4.2) \quad e^{-2r_G T} \mathcal{C}(G, z) \leq C_{p,\varrho,\eta} (1+T)^{2|\alpha|} e^{2\varrho T} \mathcal{C}(G, e^{-T\mathcal{A}}z).$$

956 *Proof.* Recall that from (2.2) we have

$$957 \quad \left\langle y_0, e^{-T\mathcal{A}^*} \phi \right\rangle_{-\diamond, \diamond} = \left\langle e^{-T\mathcal{A}} y_0, \phi \right\rangle_{-\diamond, \diamond}, \quad \forall \phi \in X_\diamond^*.$$

958 We set  $G = \{\lambda_1, \dots, \lambda_g\}$ .

959 • We start with the proof of (4.1).

960 From (2.32), let  $\tilde{\Omega} \in \mathcal{O}(G, z)$  be such that  $F(\tilde{\Omega}) = \mathcal{C}(G, z)$ . We define  $\Omega$  by

$$961 \quad \Omega_j^l := (e_T \tilde{\Omega}) \left[ \overline{\lambda_j^{(l+1)}} \right], \quad \forall j \in \llbracket 1, g \rrbracket, \forall l \in \llbracket 0, \alpha_j \rrbracket,$$

962 with the convention

$$963 \quad \tilde{\Omega} \left[ \overline{\lambda_j^{(l+1)}} \right] = \tilde{\Omega}_j^l, \quad \forall j \in \llbracket 1, g \rrbracket, \forall l \in \llbracket 0, \alpha_j \rrbracket.$$

964 Let us prove that  $\Omega \in \mathcal{O}(G, e^{-T\mathcal{A}}z)$ . For any  $j \in \llbracket 1, g \rrbracket$  and any  $\phi \in E_{\lambda_j}$ , using [9,  
965 Definition 7.12] we obtain

$$\begin{aligned} 966 \quad \sum_{l \geq 0} \langle \Omega_j^l, \mathcal{B}^*(\mathcal{A}^* - \lambda_j)^l \phi \rangle_U &= \sum_{l \geq 0} \sum_{r=0}^l e_T \left[ \overline{\lambda_j^{(r+1)}} \right] \langle \tilde{\Omega}_j^{l-r}, \mathcal{B}^*(\mathcal{A}^* - \lambda_j)^l \phi \rangle_U \\ 967 &= \sum_{r \geq 0} e_T \left[ \overline{\lambda_j^{(r+1)}} \right] \sum_{l \geq r} \langle \tilde{\Omega}_j^{l-r}, \mathcal{B}^*(\mathcal{A}^* - \lambda_j)^l \phi \rangle_U \\ 968 &= \sum_{r \geq 0} e_T \left[ \overline{\lambda_j^{(r+1)}} \right] \sum_{l \geq 0} \langle \tilde{\Omega}_j^l, \mathcal{B}^*(\mathcal{A}^* - \lambda_j)^{l+r} \phi \rangle_U. \end{aligned}$$

970 Since  $\tilde{\Omega} \in \mathcal{O}(G, z)$  and  $e_T \left[ \overline{\lambda_j^{(r+1)}} \right] = \overline{e_T \left[ \lambda_j^{(r+1)} \right]}$  for any  $r \geq 0$ , using (2.21) this  
971 yields

$$\begin{aligned} 972 \quad \sum_{l \geq 0} \langle \Omega_j^l, \mathcal{B}^*(\mathcal{A}^* - \lambda_j)^l \phi \rangle_U &= \sum_{r \geq 0} \overline{e_T \left[ \lambda_j^{(r+1)} \right]} \langle z, ((\mathcal{A}^* - \lambda_j)^r \phi) \rangle_{-\diamond, \diamond} \\ 973 \quad (4.3) \quad &= \left\langle z, e^{-T\mathcal{A}^*} \phi \right\rangle_{-\diamond, \diamond} = \left\langle e^{-T\mathcal{A}} z, \phi \right\rangle_{-\diamond, \diamond}. \end{aligned}$$

975 This proves the claim.

976 Applying Leibniz formula [9, Proposition 7.13] and Jensen inequality [9, Propo-  
977 sition 6.1] we obtain,

$$\begin{aligned} 978 \quad \left\| \Omega \left[ \overline{\lambda_\bullet^{(\mu^l)}} \right] \right\|_U &= \left\| \sum_{q=1}^l e_T \left[ \overline{\lambda_\bullet^{(\mu^l - \mu^{q-1})}} \right] \tilde{\Omega} \left[ \overline{\lambda_\bullet^{(\mu^q)}} \right] \right\| \\ 979 &\leq C_{p, \varrho, \eta} (1+T)^{|\alpha|} e^{-r_G T} \left( \sum_{q=1}^l \left\| \tilde{\Omega} \left[ \overline{\lambda_\bullet^{(\mu^q)}} \right] \right\|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

981 Thus,

$$982 \quad F(\Omega) \leq C_{p, \varrho, \eta} (1+T)^{2|\alpha|} e^{-2r_G T} F(\tilde{\Omega}) = C_{p, \varrho, \eta} (1+T)^{2|\alpha|} e^{-2r_G T} \mathcal{C}(G, z).$$

983 As  $\Omega \in \mathcal{O}(G, e^{-T\mathcal{A}}z)$ , this proves (4.1).

984 • The proof of (4.2) uses the same ingredients.  
 985 From (2.32), let  $\Omega \in \mathcal{O}(G, e^{-T\mathcal{A}}z)$  be such that  $F(\Omega) = \mathcal{C}(G, e^{-T\mathcal{A}}z)$ . For any  
 986  $j \in \llbracket 1, g \rrbracket$  and any  $l \in \llbracket 0, \alpha_j \rrbracket$ , let

$$987 \quad \tilde{\Omega}_j^l := (e_{-T}\Omega) \left[ \bar{\lambda}_j^{(l+1)} \right]$$

988 where

$$989 \quad \Omega \left[ \bar{\lambda}_j^{(l+1)} \right] := \Omega_j^l.$$

990 As previously, applying Leibniz formula [9, Proposition 7.13] and Jensen inequality [9,  
 991 Proposition 6.1], since  $\lambda_j$  satisfies  $\Re \lambda_j \leq r_G + \varrho$  for any  $j \in \llbracket 1, g \rrbracket$ , we obtain

$$992 \quad \left\| \tilde{\Omega} \left[ \bar{\lambda}_{\bullet}^{(\mu^l)} \right] \right\|_U \leq C_{p,\varrho,\eta} (1+T)^{2|\alpha|} e^{(r_G+\varrho)T} \left( \sum_{q=1}^l \left\| \Omega \left[ \bar{\lambda}_{\bullet}^{(\mu^q)} \right] \right\|^2 \right)^{\frac{1}{2}}.$$

993 The same computations as in (4.3) give that  $\Omega \in \mathcal{O}(G, z)$ . Thus

$$994 \quad \mathcal{C}(G, z) \leq F(\tilde{\Omega}) \leq C_{p,\varrho,\eta} (1+T)^{2|\alpha|} e^{2(r_G+\varrho)T} F(\Omega) \\
 995 \quad = C_{p,\varrho,\eta} (1+T)^{2|\alpha|} e^{2(r_G+\varrho)T} \mathcal{C}(G, e^{-T\mathcal{A}}z)$$

997 and (4.2) is proved.  $\square$

#### 998 4.2. The minimal null control time.

999 This section is dedicated to the proof of Theorem 2.5 and Corollary 2.6 concerning  
 1000 the minimal null control time. Proposition 2.11 and Corollary 2.12 concerning the  
 1001 cost of null controllability will follow from the estimates obtained in the proof of  
 1002 Theorem 2.5. This is discussed at the end of the current section.

1003 *Proof (of Theorem 2.5).*

1004 • We start with the proof of null controllability in time  $T > T_0(y_0)$ .

1005 We set  $\varepsilon = T - T_0(y_0) > 0$ . Let  $G \in \mathcal{G}$  and let  $v_G \in L^2(0, \varepsilon; U)$  be the solution  
 1006 of the block moment problem (2.28) in time  $\varepsilon$  associated with  $z = e^{-T\mathcal{A}}y_0$  given by  
 1007 Theorem 2.4 *i.e.*

$$1008 \quad \int_0^\varepsilon \left\langle v_G(t), \mathcal{B}^* e^{-t\mathcal{A}^*} \phi \right\rangle_U dt = \langle e^{-T\mathcal{A}}y_0, \phi \rangle_{-\diamond, \diamond}, \quad \forall \phi \in E_\lambda, \forall \lambda \in G,$$

$$1009 \quad \int_0^\varepsilon v_G(t) t^l e^{-\bar{\lambda}t} dt = 0, \quad \forall \lambda \in \Lambda \setminus G, \forall l \in \llbracket 0, \eta \rrbracket.$$

1011 We still denote by  $v_G \in L^2(0, T; U)$  the extension of  $v_G$  by 0. Thus,  $v_G$  satisfies

$$1012 \quad \int_0^T \left\langle v_G(t), \mathcal{B}^* e^{-t\mathcal{A}^*} \phi \right\rangle_U dt = \langle e^{-T\mathcal{A}}y_0, \phi \rangle_{-\diamond, \diamond}, \quad \forall \phi \in E_\lambda, \forall \lambda \in G,$$

$$1013 \quad \int_0^T v_G(t) t^l e^{-\bar{\lambda}t} dt = 0, \quad \forall \lambda \in \Lambda \setminus G, \forall l \in \llbracket 0, \eta \rrbracket.$$

1015 From (2.21), this implies that  $v_G$  solves (2.19). Thus, the only point left is to prove  
 1016 that the series (2.18) defining the control  $u$  converges in  $L^2(0, T; U)$ .

1017 From Theorem 2.4 we have that

$$1018 \quad \|v_G\|_{L^2(0, T; U)}^2 = \|v_G\|_{L^2(0, \varepsilon; U)}^2 \leq C e^{C\varepsilon^{-\frac{\theta}{1-\theta}}} e^{Cr_G^\theta} \mathcal{C}(G, e^{-T\mathcal{A}}y_0).$$

1019 By the Young inequality we get

$$1020 \quad e^{Cr_G^\theta} \leq e^{C\varepsilon^{-\frac{\theta}{1-\theta}}} e^{\varepsilon r_G}$$

1021 where in the right-hand side the constant  $C > 0$  has changed but still depend on the  
1022 same parameters. Thus,

$$1023 \quad \|v_G\|_{L^2(0,T;U)}^2 \leq C e^{C\varepsilon^{-\frac{\theta}{1-\theta}}} e^{\varepsilon r_G} \mathcal{C}(G, e^{-TA}y_0).$$

1024 Using (4.1) we obtain

$$1025 \quad \|v_G\|_{L^2(0,T;U)}^2 \leq C e^{C\varepsilon^{-\frac{\theta}{1-\theta}}} (1+T)^{2|\alpha|} e^{-\varepsilon r_G} e^{-2r_G(T-\varepsilon)} \mathcal{C}(G, y_0).$$

1026 Recalling that  $\varepsilon = T - T_0(y_0)$  this gives  
1027 (4.4)

$$1027 \quad \|v_G\|_{L^2(0,T;U)}^2 \leq C \exp\left(\frac{C}{(T - T_0(y_0))^{\frac{\theta}{1-\theta}}}\right) (1+T)^{2|\alpha|} e^{-(T-T_0(y_0))r_G} e^{-2r_G T_0(y_0)} \mathcal{C}(G, y_0).$$

1028 Recall that in (2.30) we have defined  $T_0(y_0)$  by

$$1029 \quad T_0(y_0) = \limsup_{G \in \mathcal{G}} \frac{\ln^+ \mathcal{C}(G, y_0)}{2r_G}.$$

1030 Thus, when  $r_G$  is sufficiently large, we have

$$1031 \quad e^{-2r_G T_0(y_0)} \mathcal{C}(G, y_0) \leq \exp\left(\frac{T - T_0(y_0)}{2} r_G\right).$$

1032 Together with (4.4) this implies, for  $r_G$  sufficiently large,

$$1033 \quad \|v_G\|_{L^2(0,T;U)}^2 \leq C \exp\left(\frac{C}{(T - T_0(y_0))^{\frac{\theta}{1-\theta}}}\right) (1+T)^{2|\alpha|} \exp\left(-\frac{T - T_0(y_0)}{2} r_G\right)$$

1034 and proves that the series

$$1035 \quad (4.5) \quad u = \sum_{G \in \mathcal{G}} v_G(T - \bullet)$$

1036 converges in  $L^2(0, T; U)$ . This proves null controllability of (1.1) from  $y_0$  in any time  
1037  $T > T_0(y_0)$ .

1038 • We now end the proof of Theorem 2.5 by proving that null controllability does not  
1039 hold in time  $T < T_0(y_0)$ . The proof mainly relies on the optimality of the resolution  
1040 of the block moment problems given in Proposition 2.3 (see (2.26)).

1041 Let  $T > 0$ . Assume that problem (1.1) is null controllable from  $y_0$  in time  $T$ .  
1042 Thus there exists  $u \in L^2(0, T; U)$  such that  $y(T) = 0$  and

$$1043 \quad \|u\|_{L^2(0,T;U)} \leq C_T \|y_0\|_{-\diamond}.$$

1044 Let  $v := -u(T - \bullet)$ . Then, for any  $G \in \mathcal{G}$ ,  $v$  satisfies (2.28a) with  $z = e^{-TA}y_0$ .  
1045 From (2.26), this implies

$$1046 \quad (4.6) \quad C_T^2 \|y_0\|_{-\diamond}^2 \geq \|u\|_{L^2(0,T;U)}^2 = \|v\|_{L^2(0,T;U)}^2 \geq C_{p,\eta,r_\Lambda} \mathcal{C}(G, e^{-TA}y_0).$$

1047 Applying (4.2) we obtain

$$1048 \quad \mathcal{C}(G, y_0) \leq C_{T,p,\varrho,\eta} e^{2r_G T} \mathcal{C}(G, e^{-TA} y_0).$$

1049 Together with (4.6) this implies

$$1050 \quad (4.7) \quad \mathcal{C}(G, y_0) \leq C_{T,p,\varrho,\eta,r_\Lambda} \|y_0\|_{-\diamond}^2 e^{2r_G T}.$$

1051 Getting back to the definition of  $T_0(y_0)$  given in (2.30), this implies that  $T \geq T_0(y_0)$   
 1052 and ends the proof of Theorem 2.5.  $\square$

1053 **REMARK 4.1.** *It is worth noticing that the solution  $v_G$  constructed is only active*  
 1054 *on the time interval  $(0, T - T_0(y_0))$ . Thus, whenever  $T_0(y_0) > 0$ , the series (4.5)*  
 1055 *defining the control  $u$  proves that it is possible to control  $y_0$  to 0 in any time  $T > T_0(y_0)$*   
 1056 *with a control that is identically vanishing on the time interval  $(0, T_0(y_0))$ .*

1057 We now turn to the proof of Corollary 2.6.

1058 *Proof (of Corollary 2.6).* By definition, we have  $T_0(Y_0) = \sup_{y_0 \in Y_0} T_0(y_0)$ . Using  
 1059 the definition of  $\mathcal{C}(G, Y_0)$  and Theorem 2.5, it directly comes that

$$1060 \quad T_0(Y_0) \leq \limsup_{G \in \mathcal{G}} \frac{\ln^+ \mathcal{C}(G, Y_0)}{2r_G}.$$

1061 We now focus on the converse inequality. Let  $T > 0$  such that

$$1062 \quad T < \limsup_{G \in \mathcal{G}} \frac{\ln^+ \mathcal{C}(G, Y_0)}{2r_G}$$

1063 and let us prove that  $T \leq T_0(Y_0)$ .

1064 There exists  $\varepsilon > 0$  and a sequence of groups  $(G_k)_{k \in \mathbb{N}} \in \mathcal{G}^{\mathbb{N}}$  such that for any  
 1065  $k \in \mathbb{N}^*$ , there exists  $y_{0,k} \in Y_0$  with  $\|y_{0,k}\|_{-\diamond} = 1$  satisfying

$$1066 \quad (4.8) \quad T + \varepsilon < \frac{\ln \mathcal{C}(G_k, y_{0,k})}{2r_{G_k}}.$$

1067 By contradiction, assume that for any  $y_0 \in Y_0$ , we have  $T > T_0(y_0)$ . Thus,  
 1068 from (4.7), there exists  $C_{T,p,\varrho,\eta,r_\Lambda} > 0$  such that for any  $k \in \mathbb{N}^*$

$$1069 \quad \frac{\ln \mathcal{C}(G_k, y_{0,k})}{2r_{G_k}} \leq \frac{\ln C_{T,p,\varrho,\eta,r_\Lambda}}{2r_{G_k}} + T.$$

1070 Taking  $k$  sufficiently large, this is in contradiction with (4.8).  $\square$

1071 We end this section with the proof of Proposition 2.11 and Corollary 2.12 concerning  
 1072 the cost of null controllability.

1073 A careful inspection of the proof of null controllability in time  $T > T_0(y_0)$  detailed  
 1074 in Section 4.2 allows to give a bound on the cost of controllability.

1075 *Proof (of Proposition 2.11 and Corollary 2.12).* The proof of Proposition 2.11 follows  
 1076 directly from (2.18) and (4.4).

1077 The proof of Corollary 2.12 then follows directly from Proposition 2.11, assumption  
 1078 (2.39) and the estimate

$$1079 \quad \sum_{G \in \mathcal{G}} e^{-r_G x} \leq \frac{C_{\theta,\kappa}}{x^\theta}, \quad \forall x > 0,$$

1080 proved in [12, Proposition A.5.38].  $\square$

1081 **5. Computation of the cost of a block.**

1082 In this section we prove more explicit formulas to estimate the cost  $\mathcal{C}(G, y_0)$   
 1083 of the resolution of a block moment problem depending on the assumptions on the  
 1084 eigenvalues in the group  $G$ . More precisely, we prove here Theorems 2.8 and 2.10.  
 1085 For pedagogical purpose, we start in Section 5.1 with Theorem 2.8 for algebraically  
 1086 (and geometrically) simple eigenvalues *i.e.* when  $\alpha_\lambda = \gamma_\lambda = 1$  for any  $\lambda \in G$ . Then,  
 1087 in Section 5.2, we prove the general statement of Theorem 2.8 that is when all the  
 1088 eigenvalues in the group are geometrically simple *i.e.*  $\gamma_\lambda = 1$  for any  $\lambda \in G$ .

1089 The formula for the cost  $\mathcal{C}(G, y_0)$  when all the eigenvalues in the group  $G$  are  
 1090 semi-simple (*i.e.*  $\alpha_\lambda = 1$  for any  $\lambda \in G$ ) stated in Theorem 2.10 is then proved in  
 1091 Section 5.3. The extension to spaces of initial conditions (2.34) and (2.38) does not  
 1092 depend on the matrix  $M$  and follows directly from Lemma B.1. Thus, their proofs  
 1093 are not detailed here.

1094 When both algebraic and geometric multiplicities appear in the same group we  
 1095 do not get a general formula but describe the procedure on an example in Section 5.4.

1096 Recall that from (2.32), computing  $\mathcal{C}(G, y_0)$  is a finite dimensional optimization  
 1097 problem given by

$$1098 \quad \mathcal{C}(G, y_0) = \min \left\{ F(\Omega) ; \Omega \in \mathcal{O}(G, y_0) \cap U_G^{|\alpha|} \right\}$$

1099 where the function  $F$  is defined in (2.25), the constraints associated with  $\mathcal{O}(G, y_0)$  are  
 1100 defined in (2.23) and  $U_G$  is defined in (2.31).

1101 **5.1. The case of simple eigenvalues.**

1102 In all this section, we consider the simpler case where  $\alpha_\lambda = \gamma_\lambda = 1$  for every  
 1103  $\lambda \in G$ . Thus, in the rest of this section, we drop the superscript 0 associated to  
 1104 eigenvectors.

1105 We start with the proof of the invertibility of the matrix  $M$  stated in Proposi-  
 1106 tion 2.7.

1107 *Proof.* Recall that, as  $\alpha_\lambda = \gamma_\lambda = 1$ , the positive semi-definite matrix  $M$  is defined  
 1108 in (2.35). Let  $\tau \in \mathbb{C}^g$  be such that  $\langle M\tau, \tau \rangle = 0$ . Then, for each  $l \in \llbracket 1, g \rrbracket$ , we have

$$1109 \quad \langle \Gamma^l \tau, \tau \rangle = 0.$$

1110 We prove that  $\tau = 0$ . By contradiction let

$$1111 \quad l = \max\{j \in \llbracket 1, g \rrbracket ; \tau_j \neq 0\}.$$

1112 Then from (2.35) this leads to  $\langle \Gamma^l \tau, \tau \rangle = \|b[\lambda_l]\|_U^2 |\tau_l|^2$ . Using (2.14) implies  $\tau_l = 0$ .  
 1113 This is in contradiction with the definition of  $l$  which proves the invertibility of  $M$ .  $\square$

1114 We now prove Theorem 2.8.

1115 *Proof.* First of all, notice that the function  $F$  to minimize reduces to

$$1116 \quad F(\Omega) = \sum_{j=1}^g \|\Omega[\bar{\lambda}_1, \dots, \bar{\lambda}_j]\|^2$$

1117 and, as  $\gamma_\lambda = \alpha_\lambda = 1$ , the constraints defining the set  $\mathcal{O}(\lambda_j, y_0)$  reduce to

$$1118 \quad \langle \Omega_j, b_j \rangle_U = \langle y_0, \phi_j \rangle_{-\infty, \infty}.$$

1119 Thus, the minimization problem reduces to

1120

1121 (5.1)  $\mathcal{C}(G, y_0) = \min \left\{ F(\Omega) ; \Omega = (\Omega_1, \dots, \Omega_g) \in U_G^g \text{ such that}$

1122  
1123

$$\langle \Omega_j, b_j \rangle_U = \langle y_0, \phi_j \rangle_{-\diamond, \diamond}, \forall j \in \llbracket 1, g \rrbracket \}.$$

1124 For the sake of generality, let us consider for this proof any  $\omega_1, \dots, \omega_g \in \mathbb{C}$  and the  
1125 more general constraints

1126 (5.2) 
$$\langle \Omega_j, b_j \rangle_U = \omega_j, \quad \forall j \in \llbracket 1, g \rrbracket.$$

1127 Using the formalism of divided differences, this is equivalent to the family of con-  
1128 straints

1129 (5.3) 
$$\langle \Omega, b \rangle_U [\bar{\lambda}_1, \dots, \bar{\lambda}_j] = \omega [\bar{\lambda}_1, \dots, \bar{\lambda}_j], \quad \forall j \in \llbracket 1, g \rrbracket.$$

1130 We consider the constrained complex minimization problem

1131 
$$\min \{ F(\Omega) ; \Omega = (\Omega_1, \dots, \Omega_g) \in U_G^g \text{ such that (5.3) holds} \}.$$

1132 It has a unique solution, which is characterised by the existence of multipliers  $(m_j)_{j \in \llbracket 1, g \rrbracket} \subset$   
1133  $\mathbb{C}$  such that

1134 (5.4) 
$$\sum_{j=1}^g \langle H[\bar{\lambda}_1, \dots, \bar{\lambda}_j], \Omega[\bar{\lambda}_1, \dots, \bar{\lambda}_j] \rangle_U = \sum_{j=1}^g \bar{m}_j \langle H, b \rangle_U [\bar{\lambda}_1, \dots, \bar{\lambda}_j],$$

1135 for any  $H_1, \dots, H_g \in U_G$ .

1136 Then, for a given  $q \in \llbracket 1, g \rrbracket$ , using Leibniz formula [12, Proposition A.2.11], the  
1137 constraints (5.3) can be rewritten as

1138 (5.5) 
$$\omega [\bar{\lambda}_1, \dots, \bar{\lambda}_q] = \langle \Omega, b \rangle_U [\bar{\lambda}_1, \dots, \bar{\lambda}_q] = \sum_{j=1}^q \langle \Omega[\bar{\lambda}_1, \dots, \bar{\lambda}_j], b[\lambda_j, \dots, \lambda_q] \rangle_U$$

1139 To relate (5.5) and (5.4), we look for  $H_1, \dots, H_g \in U_G$  such that, for a given  $q \in \llbracket 1, g \rrbracket$   
1140 we have

1141 
$$H[\bar{\lambda}_1, \dots, \bar{\lambda}_j] = \begin{cases} b[\lambda_j, \dots, \lambda_q], & \text{for } j \leq q, \\ 0, & \text{for } j > q. \end{cases}$$

1142 This can be done by setting  $H_1 = b[\lambda_1, \dots, \lambda_q]$  and, from the interpolation formula [9,  
1143 Proposition 7.6], by defining  $H_j$  by the formula

1144 
$$H_j = \sum_{i=1}^j \left( \prod_{k=1}^{j-1} (\bar{\lambda}_i - \bar{\lambda}_k) \right) H[\bar{\lambda}_1, \dots, \bar{\lambda}_i], \quad \forall j \in \llbracket 2, g \rrbracket.$$

1145 Then, from (5.5) we obtain

1146 
$$\omega [\bar{\lambda}_1, \dots, \bar{\lambda}_q] = \sum_{j=1}^g \langle \Omega[\bar{\lambda}_1, \dots, \bar{\lambda}_j], H[\bar{\lambda}_1, \dots, \bar{\lambda}_j] \rangle_U.$$

1147 Now relation (5.4) leads, after conjugation, to

$$1148 \quad \overline{\omega[\bar{\lambda}_1, \dots, \bar{\lambda}_q]} = \sum_{j=1}^g \bar{m}_j \langle H, b \rangle_U [\bar{\lambda}_1, \dots, \bar{\lambda}_j].$$

1149 The application of Leibniz formula [12, Proposition A.2.11] yields

$$1150 \quad \overline{\omega[\bar{\lambda}_1, \dots, \bar{\lambda}_q]} = \sum_{j=1}^g \bar{m}_j \left( \sum_{l=1}^j \langle H[\bar{\lambda}_1, \dots, \bar{\lambda}_l], b[\lambda_l, \dots, \lambda_j] \rangle_U \right)$$

$$1151 \quad = \sum_{j=1}^g \bar{m}_j \left( \sum_{l=1}^{\min(j,q)} \langle b[\lambda_l, \dots, \lambda_q], b[\lambda_l, \dots, \lambda_j] \rangle_U \right).$$

1152

1153 Conjugating this relation leads to

$$1154 \quad \omega[\bar{\lambda}_1, \dots, \bar{\lambda}_q] = \sum_{j=1}^g m_j \left( \sum_{l=1}^{\min(j,q)} \langle b[\lambda_l, \dots, \lambda_j], b[\lambda_l, \dots, \lambda_q] \rangle_U \right)$$

$$1155 \quad = \sum_{l=1}^g \sum_{j=1}^g m_j \Gamma_{q,j}^l = (Mm)_q,$$

1156

1157 where  $\Gamma^l$  and  $M$  are defined in (2.35).

1158 Let

$$1159 \quad \xi := \begin{pmatrix} \omega[\bar{\lambda}_1] \\ \vdots \\ \omega[\bar{\lambda}_1, \dots, \bar{\lambda}_g] \end{pmatrix} \in \mathbb{C}^g.$$

1160 We have just proved that  $m = M^{-1}\xi$ . Getting back to (5.4) with  $H = \Omega$  together  
1161 with the constraints (5.3), we obtain

$$1162 \quad F(\Omega) = \sum_{j=1}^g \bar{m}_j \langle \Omega, b \rangle_U [\bar{\lambda}_1, \dots, \bar{\lambda}_j] = \langle M^{-1}\xi, \xi \rangle.$$

1163 With the specific choice,  $\omega_j = \langle y_0, \phi_j \rangle_{-\diamond, \diamond}$ , this ends the proof of Theorem 2.8 with  
1164 the extra assumption that  $\alpha_\lambda = 1$  for all  $\lambda \in G$ . Indeed, by anti-linearity we have

$$1165 \quad \omega[\bar{\lambda}_1, \dots, \bar{\lambda}_j] = \langle y_0, \phi[\lambda_1, \dots, \lambda_j] \rangle_{-\diamond, \diamond}, \quad \forall j \in \llbracket 1, g \rrbracket. \quad \square$$

1166 **REMARK 5.1.** As mentioned in Remark 2.4, estimate (5.1) implies that the cost  
1167 of the block  $G$  (i.e. the quantity  $\langle M^{-1}\xi, \xi \rangle$ ) can be estimated using any eigenvectors:  
1168 there is no normalization condition.

1169 **REMARK 5.2.** Rewriting the constraints in the form (5.3) is not mandatory but,  
1170 as the function to minimize involves divided differences, it leads to more exploitable  
1171 formulas and will ease the writing when dealing with algebraic multiplicity of eigen-  
1172 values. Dealing directly with (5.2) would lead to the expression (D.9) for the cost of  
1173 the block  $G$  as it will appear in the proof of Theorem 2.10.

1174 **5.2. The case of geometrically simple eigenvalues.**

1175 The proof of Proposition 2.7 and Theorem 2.8 under the sole assumption  $\gamma_\lambda = 1$   
 1176 for any  $\lambda \in G$  follows closely the proof done in Section 5.1. The main difference is the  
 1177 use of generalized divided differences (see [9, Section 7.3]) instead of classical divided  
 1178 differences as detailed below.

1179 *Proof (of Proposition 2.7).* Due to (2.24), for any  $l \in \llbracket 1, |\alpha| \rrbracket$  the multi-index  
 1180  $\mu^l - \mu^{l-1}$  is composed of only one 1 and  $g-1$  zeros. Thus,

1181 
$$b \left[ \lambda_{\bullet}^{(\mu^l - \mu^{l-1})} \right] = b_j^0$$

1182 for a certain  $j \in \llbracket 1, g \rrbracket$ . From (2.14) it comes that

1183 
$$b \left[ \lambda_{\bullet}^{(\mu^l - \mu^{l-1})} \right] \neq 0, \quad \forall l \in \llbracket 1, |\alpha| \rrbracket.$$

1184 The rest of the proof follows as in Section 5.1. □

1185 *Proof (of Theorem 2.8).* As  $\gamma_\lambda = 1$ , the constraints defining the set  $\mathcal{O}(\lambda_j, y_0)$   
 1186 reduce to

1187 
$$\sum_{r=0}^l \langle \Omega_j^r, b_j^{l-r} \rangle_U = \sum_{r=0}^l \langle \Omega_j^r, \mathcal{B}^*(\mathcal{A}^* - \lambda_j)^r \phi_j^l \rangle_U$$
  
 1188 
$$= \langle y_0, \phi_j^l \rangle_{-\diamond, \diamond}, \quad \forall l \in \llbracket 0, \alpha_j \rrbracket.$$
  
 1189

1190 By definition of  $\langle \Omega, b \rangle_U \left[ \overline{\lambda}_j^{(l+1)} \right]$ , this is equivalent to

1191 
$$\langle \Omega, b \rangle_U \left[ \overline{\lambda}_j^{(l+1)} \right] = \langle y_0, \phi_j^l \rangle_{-\diamond, \diamond}, \quad \forall l \in \llbracket 0, \alpha_j \rrbracket.$$

1192 Thus,

1193 (5.6) 
$$\mathcal{C}(G, y_0) = \min \left\{ F(\Omega); \Omega = (\Omega_1^0, \dots, \Omega_1^{\alpha_1-1}, \dots, \Omega_g^0, \dots, \Omega_g^{\alpha_g-1}) \in U_G^{|\alpha|} \right.$$
  
 1194 
$$\left. \text{such that } \langle \Omega, b \rangle_U \left[ \overline{\lambda}_j^{(l+1)} \right] = \langle y_0, \phi_j^l \rangle_{-\diamond, \diamond}, \quad \forall j \in \llbracket 1, g \rrbracket, \forall l \in \llbracket 0, \alpha_j \rrbracket \right\}.$$
  
 1195  
 1196

1197 For the sake of generality, let us consider for this proof any

1198 
$$(\omega_1^0, \dots, \omega_1^{\alpha_1-1}, \dots, \omega_g^0, \dots, \omega_g^{\alpha_g-1}) \in \mathbb{C}^{|\alpha|}$$

1199 and the more general constraints

1200 
$$\langle \Omega, b \rangle_U \left[ \overline{\lambda}_j^{(l+1)} \right] = \omega_j^l, \quad \forall j \in \llbracket 1, g \rrbracket, \forall l \in \llbracket 0, \alpha_j \rrbracket.$$

1201 From (2.24), this is equivalent to the family of constraints

1202 
$$\langle \Omega, b \rangle_U \left[ \overline{\lambda}_{\bullet}^{(\mu^p)} \right] = \omega \left[ \overline{\lambda}_{\bullet}^{(\mu^p)} \right], \quad \forall p \in \llbracket 1, |\alpha| \rrbracket,$$

1203 and we proceed as in Section 5.1. The only difference is the use of generalized divided  
 1204 differences. For instance, the equation (5.4) now reads

1205 
$$\sum_{l=1}^{|\alpha|} \left\langle H \left[ \overline{\lambda}_{\bullet}^{(\mu^l)} \right], \Omega \left[ \overline{\lambda}_{\bullet}^{(\mu^l)} \right] \right\rangle_U = \sum_{l=1}^{|\alpha|} \overline{m}_l \langle H, b \rangle_U \left[ \overline{\lambda}_{\bullet}^{(\mu^l)} \right], \quad \forall H = (H_j^l) \in U_G^{|\alpha|}.$$

1206 The rest of the proof remains unchanged. □

1207 REMARK 5.3. As mentioned in Remark 2.4, estimate (5.6) implies that the cost  
 1208 of the block  $G$  (i.e. the quantity  $\langle M^{-1}\xi, \xi \rangle$ ) can be estimated using any eigenvectors  
 1209 and any associated Jordan chains.

### 1210 5.3. The case of semi-simple eigenvalues.

1211 We start with the proof of Proposition 2.9.

1212 *Proof (of Proposition 2.9).* Recall that the positive semi-definite matrix  $M$  is  
 1213 defined in (2.37). Let  $\tau \in \mathbb{C}^{\gamma_G}$  be such that  $\langle M\tau, \tau \rangle = 0$ . Then, for any  $l \in \llbracket 1, g \rrbracket$ ,  
 1214  $\langle \Gamma^l \tau, \tau \rangle = 0$ . We prove that  $\tau = 0$ . By contradiction let

$$1215 \quad \tilde{l} = \max\{j \in \llbracket 1, \gamma_G \rrbracket ; \tau_j \neq 0\}$$

1216 and  $l \in \llbracket 1, g \rrbracket$  be such that

$$1217 \quad \gamma_1 + \cdots + \gamma_{l-1} < \tilde{l} \leq \gamma_1 + \cdots + \gamma_l$$

1218 with the convention that  $l = 1$  when  $\tilde{l} \leq \gamma_1$ . We denote by  $\sigma \in \mathbb{C}^{\gamma_l}$  the  $l^{\text{th}}$  block of  $\tau$   
 1219 i.e.

$$1220 \quad \sigma = \begin{pmatrix} \tau_{\gamma_1 + \cdots + \gamma_{l-1} + 1} \\ \vdots \\ \tau_{\gamma_1 + \cdots + \gamma_l} \end{pmatrix}.$$

1221 From (2.36) we have  $\delta_i^l = 0$  when  $i < l$ . Thus all the blocks  $(i, j)$  of  $\Gamma^l$  are equal to 0  
 1222 when  $i, j \in \llbracket 1, l \rrbracket$ . This leads to

$$1223 \quad \langle \Gamma^l \tau, \tau \rangle = |\delta_l^l|^2 \langle \text{Gram}_U(b_{l,1}, \dots, b_{l,\gamma_l}) \sigma, \sigma \rangle.$$

1224 As the eigenvalues  $\lambda_1, \dots, \lambda_g$  are distinct it comes that  $\delta_l^l \neq 0$  (see (2.36)) which  
 1225 implies

$$1226 \quad \langle \text{Gram}_U(b_{l,1}, \dots, b_{l,\gamma_l}) \sigma, \sigma \rangle = 0.$$

1227 From (2.14), we have that  $b_{l,1}, \dots, b_{l,\gamma_l}$  are linearly independent. This proves the  
 1228 invertibility of  $\text{Gram}_U(b_{l,1}, \dots, b_{l,\gamma_l})$  and gives  $\sigma = 0$ . This is in contradiction with  
 1229 the definition of  $\tilde{l}$  which proves the invertibility of  $M$ .  $\square$

1230 We now turn to the proof of Theorem 2.10.

1231 *Proof (of Theorem 2.10).* First of all, notice that the function  $F$  to minimize  
 1232 reduces to

$$1233 \quad F(\Omega) = \sum_{j=1}^g \|\Omega[\bar{\lambda}_1, \dots, \bar{\lambda}_j]\|^2$$

1234 and, as  $\alpha_\lambda = 1$ , the constraints defining the set  $\mathcal{O}(\lambda_j, y_0)$  reduce to

$$1235 \quad \langle \Omega_j, \mathcal{B}^* \phi \rangle_U = \langle y_0, \phi \rangle_{-\diamond, \diamond}, \quad \forall \phi \in \text{Ker}(\mathcal{A}^* - \lambda_j).$$

1236 To simplify the writing, let us consider the linear maps

$$1237 \quad B_j := \begin{pmatrix} \langle \bullet, \mathcal{B}^* \phi_{j,1} \rangle_U \\ \vdots \\ \langle \bullet, \mathcal{B}^* \phi_{j,\gamma_j} \rangle_U \end{pmatrix} \in \mathcal{L}(U, \mathbb{C}^{\gamma_j}).$$

1238 Then the constraints defining  $\mathcal{O}(\lambda_j, y_0)$  can be rewritten as the equality

$$1239 \quad (5.7) \quad B_j \Omega_j = \begin{pmatrix} \langle y_0, \phi_{j,1} \rangle_{-\diamond, \diamond} \\ \vdots \\ \langle y_0, \phi_{j,\gamma_j} \rangle_{-\diamond, \diamond} \end{pmatrix}.$$

1240 Thus,

1241

$$1242 \quad (5.8) \quad \mathcal{C}(G, y_0) = \min \left\{ F(\Omega) ; \Omega = (\Omega_1, \dots, \Omega_g) \in U_G^g \right.$$

1243

1244

such that (5.7) holds for any  $j \in \llbracket 1, g \rrbracket$  }.

1245 For the sake of generality, let us consider for this proof, for any  $j \in \llbracket 1, g \rrbracket$ , any  $\omega_j \in \mathbb{C}^{\gamma_j}$

1246 and the more general constraints

$$1247 \quad (5.9) \quad B_j \Omega_j = \omega_j, \quad \forall j \in \llbracket 1, g \rrbracket.$$

1248 As the  $\omega_j$ 's have different sizes we avoid in this proof the use of divided differences  
1249 to rewrite the constraints. This is why we end up with the formula (2.37) rather than  
1250 an adaptation of (2.35) (see also the discussion in Remark 5.2).

1251 Arguing as before, the solution of our optimisation problem satisfies

$$1252 \quad (5.10) \quad \sum_{j=1}^g \langle H[\bar{\lambda}_1, \dots, \bar{\lambda}_j], \Omega[\bar{\lambda}_1, \dots, \bar{\lambda}_j] \rangle_U = \sum_{j=1}^g \langle B_j H_j, m_j \rangle, \quad \forall H_1, \dots, H_g \in U_G,$$

1253 for some  $m_j \in \mathbb{C}^{\gamma_j}, j = 1, \dots, g$ .

1254 Recall that in (2.36) we defined the numbers

$$1255 \quad \delta_j^i = \prod_{k \in \llbracket 1, j \llbracket} (\lambda_i - \lambda_k), \quad \forall j \in \llbracket 2, g \rrbracket.$$

1256 Then, from the interpolation formula [9, Proposition 7.6], we obtain that

$$1257 \quad (5.11) \quad \Omega_i = \sum_{l=1}^i \delta_l^i \Omega[\bar{\lambda}_1, \dots, \bar{\lambda}_l].$$

1258 For any  $H \in U_G$  and  $i \in \llbracket 1, g \rrbracket$ , let us design  $H_1^{(i)}, \dots, H_g^{(i)} \in U_G$  such that

$$1259 \quad (5.12) \quad H^{(i)}[\bar{\lambda}_1, \dots, \bar{\lambda}_l] = \delta_l^i H, \quad \forall l \in \llbracket 1, i \rrbracket.$$

1260 To do so, we set  $H_1^{(i)} = H$  then, using the interpolation formula [9, Proposition 7.6],  
1261 we define recursively

$$1262 \quad H_j^{(i)} = \sum_{l=1}^j \delta_l^j H^{(i)}[\bar{\lambda}_1, \dots, \bar{\lambda}_l] = \left( \sum_{l=1}^j \delta_l^i \delta_l^j \right) H = a_j^{(i)} H$$

1263 with

$$1264 \quad (5.13) \quad a_j^{(i)} := \sum_{l=1}^g \delta_l^i \delta_l^j = \sum_{l=1}^{\min(i,j)} \delta_l^i \delta_l^j.$$

1265 This ensures (5.12). Plugging this set of values  $H_j^{(i)}, j = 1, \dots, g$  in (5.10) and taking  
 1266 into account (5.11), leads to

$$\begin{aligned}
 1267 \quad \sum_{j=1}^g a_j^{(i)} \langle B_j H, m_j \rangle &= \sum_{j=1}^g \langle B_j H_j^{(i)}, m_j \rangle \\
 1268 &= \sum_{j=1}^g \delta_j^i \langle H, \Omega[\bar{\lambda}_1, \dots, \bar{\lambda}_j] \rangle_U \\
 1269 &= \left\langle H, \sum_{j=1}^g \bar{\delta}_j^i \Omega[\bar{\lambda}_1, \dots, \bar{\lambda}_j] \right\rangle_U \\
 1270 &= \langle H, \Omega_i \rangle_U.
 \end{aligned}$$

1272 This being true for any  $H \in U_G$ , we end up with

$$1273 \quad (5.14) \quad \Omega_i = \sum_{j=1}^g \overline{a_j^{(i)}} B_j^* m_j.$$

1274 Together with (5.9), using (5.13), we obtain that

$$\begin{aligned}
 1275 \quad \omega_i &= \sum_{j=1}^g \overline{a_j^{(i)}} B_i B_j^* m_j \\
 1276 &= \sum_{l=1}^g \sum_{j=1}^g (\bar{\delta}_l^i B_i) (\bar{\delta}_l^j B_j)^* m_j \\
 1277 &= (Mm)_i
 \end{aligned}$$

1279 where  $M$  is defined in (2.37) and  $(Mm)_i \in \mathbb{C}^{\gamma_i}$  denotes the  $i^{\text{th}}$  block of  $Mm \in \mathbb{C}^{\gamma_G}$ .

1280 Finally, if we set

$$1281 \quad (5.15) \quad \xi := \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_g \end{pmatrix} \in \mathbb{C}^{\gamma_G},$$

1282 we have proved that the multiplier is given by  $m = M^{-1}\xi$ . Applying (5.10) with  
 1283  $H_j = \Omega_j$  and using the constraints (5.7) leads to

$$1284 \quad F(\Omega) = \sum_{j=1}^g \|\Omega[\bar{\lambda}_1, \dots, \bar{\lambda}_j]\|^2 = \langle M^{-1}\xi, \xi \rangle,$$

1285 which proves the claim. □

1286 **REMARK 5.4.** From (5.14) and the equality  $m = M^{-1}\xi$  it comes that

1287

$$1288 \quad \mathcal{C}(G, y_0) = \min \left\{ F(\Omega) ; \Omega = (\Omega_1, \dots, \Omega_g) \in U_G^g \right.$$

1289  
 1290

such that (5.9) holds for any  $j \in \llbracket 1, g \rrbracket$  }

1291 where the minimum is attained for

$$1292 \quad \Omega_i = \sum_{j=1}^g \left( \sum_{l=1}^g \overline{\delta_l^i} \delta_l^j \right) B_j^*(M^{-1}\xi)_j$$

1293 with  $\xi$  defined by (5.15).

1294 **REMARK 5.5.** As mentioned in Remark 2.4, estimate (5.8) implies that the cost  
 1295 of the block  $G$  (i.e. the quantity  $\langle M^{-1}\xi, \xi \rangle$ ) can be estimated using any basis of  
 1296 eigenvectors.

#### 1297 5.4. Dealing simultaneously with algebraic and geometric multiplici- 1298 ties.

1299 The proof of Theorem 2.8 strongly relies on the use of divided differences to rewrite  
 1300 the constraints whereas the proof of Theorem 2.10 is based on the vectorial writing  
 1301 of the constraints through the operators  $B_j \in \mathcal{L}(U; \mathbb{C}^{\gamma_j})$ . As the target spaces of  
 1302 these operators do not have the same dimension, one cannot directly compute divided  
 1303 differences. Thus, the setting we developed to compute the cost of a given block does  
 1304 not lead to a general formula when both kind of multiplicities need to be taken into  
 1305 account in the same group. However, for actual problems, the computation of this  
 1306 cost is a finite dimensional constrained optimization problem which can be explicitly  
 1307 solved.

1308 Let us give an example of such a group that does not fit into Theorem 2.8 nor  
 1309 into Theorem 2.10 but for which we manage to compute the cost *by hand*. To simplify  
 1310 a little the presentation, we give this example in the case of real Hilbert spaces and  
 1311 real eigenvalues.

1312 We consider a group  $G = \{\lambda_1, \lambda_2\}$  of two distinct eigenvalues such that  $\gamma_{\lambda_1} =$   
 1313  $\alpha_{\lambda_1} = 2$  and  $\gamma_{\lambda_2} = \alpha_{\lambda_2} = 1$ . Let  $(\phi_{1,1}^0, \phi_{1,2}^0)$  be a basis of  $\text{Ker}(\mathcal{A}^* - \lambda_1)$  and  $\phi_{2,1}^0$  be  
 1314 an eigenvector of  $\mathcal{A}^*$  associated to the eigenvalue  $\lambda_2$ . Assume that the generalized  
 1315 eigenvector  $\phi_{1,1}^1$  is such that

$$1316 \quad (\mathcal{A}^* - \lambda_1) \phi_{1,1}^1 = \phi_{1,1}^0,$$

1317 and that  $\{\phi_{1,1}^0, \phi_{1,1}^1, \phi_{1,2}^0\}$  forms a basis of  $\text{Ker}(\mathcal{A}^* - \lambda_1)^2$ .

1318 For this group, in the same spirit as in Theorems 2.8 and 2.10, we obtain the  
 1319 following result.

1320 **PROPOSITION 5.1.** For any  $y_0 \in X_{-\diamond}$ , we have

$$1321 \quad \mathcal{C}(G, y_0) = \langle M^{-1}\xi, \xi \rangle \quad \text{where} \quad \xi = \begin{pmatrix} \langle y_0, \phi_{1,1}^0 \rangle_{-\diamond, \diamond} \\ \langle y_0, \phi_{1,2}^0 \rangle_{-\diamond, \diamond} \\ \langle y_0, \phi_{1,1}^1 \rangle_{-\diamond, \diamond} \\ \langle y_0, \phi_{2,1}^0 \rangle_{-\diamond, \diamond} \end{pmatrix}$$

1322 and  $M$  is the invertible matrix defined by

$$1323 \quad M = \text{Gram}_U(b_{1,1}^0, b_{1,2}^0, b_{1,1}^1, b_{2,1}^0) \\
 1324 \quad + \text{Gram}_U(0, 0, b_{1,1}^0, \delta b_{2,1}^0) \\
 1325 \quad + \text{Gram}_U(0, 0, 0, \delta^2 b_{2,1}^0) \\
 1326$$

1327 with  $\delta = \lambda_2 - \lambda_1$ .

1328 *Proof.* Let

$$1329 \quad (\omega_{1,1}^0, \omega_{1,2}^0, \omega_{1,1}^1, \omega_{2,1}^0)^t \in \mathbb{R}^4.$$

1330 As in the proofs of Theorems 2.8 and 2.10, the goal is to compute the minimum of  
1331 the function

$$1332 \quad F : (\Omega_1^0, \Omega_1^1, \Omega_2^0) \in U_G^3 \mapsto \|\Omega_1^0\|^2 + \|\Omega_1^1\|^2 + \|\Omega[\lambda_1^{(2)}, \lambda_2]\|^2,$$

1333 under the 4 constraints

$$1334 \quad \langle \Omega_j^0, b_{j,i}^0 \rangle_U = \omega_{j,i}^0, \quad \forall i \in \llbracket 1, \gamma_j \rrbracket, \forall j \in \llbracket 1, 2 \rrbracket,$$

$$1335 \quad \langle \Omega_1^0, b_{1,1}^1 \rangle_U + \langle \Omega_1^1, b_{1,1}^0 \rangle_U = \omega_{1,1}^1.$$

1337 Then, the Lagrange multipliers  $m_{1,1}^0, m_{1,2}^0, m_{1,1}^1$  and  $m_{2,1}^0$  satisfy the equations

1338

$$1339 \quad (5.16) \quad \langle \Omega_1^0, H_1^0 \rangle_U + \langle \Omega_1^1, H_1^1 \rangle_U + \langle \Omega[\lambda_1^{(2)}, \lambda_2], H[\lambda_1^{(2)}, \lambda_2] \rangle_U = m_{1,1}^0 \langle H_1^0, b_{1,1}^0 \rangle_U$$

$$1340 \quad + m_{1,2}^0 \langle H_1^0, b_{1,2}^0 \rangle_U + m_{1,1}^1 \left( \langle H_1^0, b_{1,1}^1 \rangle_U + \langle H_1^1, b_{1,1}^0 \rangle_U \right) + m_{2,1}^0 \langle H_2^0, b_{2,1}^0 \rangle_U,$$

1342 for every  $H_1^0, H_1^1, H_2^0 \in U_G$ . Considering successively

$$1343 \quad H_1^0 = b_{1,1}^0, \quad H_1^1 = 0, \quad H_2^0 = b_{1,1}^0,$$

$$1344 \quad H_1^0 = b_{1,2}^0, \quad H_1^1 = 0, \quad H_2^0 = b_{1,2}^0,$$

$$1345 \quad H_1^0 = b_{1,1}^1, \quad H_1^1 = b_{1,1}^0, \quad H_2^0 = b_{1,1}^1 + (\lambda_2 - \lambda_1)b_{1,1}^0,$$

1348 and

$$1349 \quad H_1^0 = b_{2,1}^0, \quad H_1^1 = (\lambda_2 - \lambda_1)b_{2,1}^0, \quad H_2^0 = (1 + (\lambda_2 - \lambda_1)^2 + (\lambda_2 - \lambda_1)^4) b_{2,1}^0,$$

1350 and plugging it into (5.16), we obtain that  $\begin{pmatrix} \omega_{1,1}^0 \\ \omega_{1,2}^0 \\ \omega_{1,1}^1 \\ \omega_{2,1}^0 \end{pmatrix} = M \begin{pmatrix} m_{1,1}^0 \\ m_{1,2}^0 \\ m_{1,1}^1 \\ m_{2,1}^0 \end{pmatrix}$ . Then, the same  
1351 argument as in the proofs of Theorems 2.8 and 2.10 ends the proof.  $\square$

## 1352 6. Application to the study of null controllability of academic exam- 1353 ples.

1354 In this section we provide examples to illustrate how to use the formulas obtained  
1355 in Theorems 2.5, 2.8 and 2.10 in order to compute the minimal null control time.

1356 We start with academic examples for which computations are simpler. Then, in  
1357 Section 7, we study coupled systems of actual partial differential equations of parabolic  
1358 type.

### 1359 6.1. Setting and notations.

1360 Let  $A$  be the unbounded Sturm-Liouville operator defined in  $L^2(0, 1; \mathbb{R})$  by

$$1361 \quad (6.1) \quad D(A) = H^2(0, 1; \mathbb{R}) \cap H_0^1(0, 1; \mathbb{R}), \quad A = -\partial_x(\gamma \partial_x \bullet) + c \bullet,$$

1362 with  $c \in L^\infty(0, 1; \mathbb{R})$  satisfying  $c \geq 0$  and  $\gamma \in C^1([0, 1]; \mathbb{R})$  satisfying  $\inf_{[0,1]} \gamma > 0$ .

1363 The operator  $A$  admits an increasing sequence of eigenvalues denoted by  $(\nu_k)_{k \in \mathbb{N}^*}$ .  
1364 The associated normalized eigenvectors  $(\varphi_k)_{k \in \mathbb{N}^*}$  form a Hilbert basis of  $L^2(0, 1; \mathbb{R})$ .

1365       REMARK 6.1. *The assumption  $c \geq 0$  ensures that for any  $k \geq 1$ , the eigenvalues*  
1366 *satisfies  $\nu_k > 0$ . From Remark 2.2, the controllability results proved in the present*  
1367 *article still hold when the function  $c$  is bounded from below.*

1368       To lighten the notations, for any  $I \subset (0, 1)$  we set  $\|\bullet\|_I = \|\bullet\|_{L^2(I)}$ .

1369       Let  $f : \text{Sp}(A) \rightarrow \mathbb{R}$  be a bounded function. Associated to this function we consider  
1370 the operator  $f(A)$  defined on  $D(A)$  by the spectral theorem by

$$1371 \quad (6.2) \quad f(A) = \sum_{k \geq 1} f(\nu_k) \langle \bullet, \varphi_k \rangle_{L^2(0,1;\mathbb{R})} \varphi_k.$$

### 1372       6.2. Spectral properties of Sturm-Liouville operators.

1373       Let  $A$  be the Sturm-Liouville operator defined in (6.1). All the examples studied  
1374 in this article are based on this operator. We recall here some spectral properties that  
1375 will be used in our study.

1376       From [2, Theorem 1.1 and Remark 2.1], there exist  $\varrho > 0$  and  $C > 0$  such that

$$1377 \quad (6.3) \quad \varrho < \nu_{k+1} - \nu_k, \quad \forall k \geq 1,$$

1378

$$1379 \quad (6.4) \quad \frac{1}{C} \sqrt{\nu_k} \leq |\varphi'_k(x)| \leq C \sqrt{\nu_k}, \quad \forall x \in \{0, 1\}, \forall k \geq 1,$$

1380 and, for any non-empty open set  $\omega \subset (0, 1)$ ,

$$1381 \quad (6.5) \quad \inf_{k \geq 1} \|\varphi_k\|_\omega > 0.$$

1382 Moreover, using [12, Theorem IV.1.3], the associated counting function satisfies

$$1383 \quad (6.6) \quad N_{(\nu_k)_k}(r) \leq C\sqrt{r}, \quad \forall r > 0,$$

1384 and

$$1385 \quad (6.7) \quad |N_{(\nu_k)_k}(r) - N_{(\nu_k)_k}(s)| \leq C \left(1 + \sqrt{|r - s|}\right), \quad \forall r, s > 0.$$

1386 We also recall the classical Lebeau-Robbiano spectral inequality

$$1387 \quad (6.8) \quad \left\| \sum_{k \leq K} a_k \varphi_k \right\|_\Omega \leq C e^{C\sqrt{\nu_K}} \left\| \sum_{k \leq K} a_k \varphi_k \right\|_\omega, \quad \forall K \geq 1, \forall (a_k)_k \subset \mathbb{R}.$$

1388 Indeed, as detailed for instance in [12, Theorem IV.2.19], the proof of this spectral  
1389 inequality given in [27] directly extends to the low regularity coefficients considered  
1390 here.

### 1391       6.3. Perturbation of a 2x2 Jordan block.

1392       Let  $\omega \subset (0, 1)$  be a non-empty open set and  $U = L^2(\Omega)$ . Let  $A$  be the Sturm-  
1393 Liouville operator defined in (6.1) and  $f(A)$  be the operator defined in (6.2) with  
1394  $f : \text{Sp}(A) \rightarrow \mathbb{R}$  satisfying

$$1395 \quad |f(\nu_k)| < \frac{\varrho}{2}, \quad \forall k \geq 1.$$

1396 We consider the operator  $\mathcal{A}$  on  $X = L^2(0, 1; \mathbb{R})^2$  defined by

1397 (6.9) 
$$\mathcal{A} = \begin{pmatrix} A & I \\ 0 & A + f(A) \end{pmatrix}, \quad D(\mathcal{A}) = D(A) \times D(A),$$

1398 and

1399 (6.10) 
$$\mathcal{B} : u \in U \mapsto \begin{pmatrix} 0 \\ \mathbf{1}_\omega u \end{pmatrix}.$$

1400 Then,

1401 
$$\mathcal{B}^* : \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \in X \mapsto \mathbf{1}_\omega \varphi_2.$$

1402 It is easy to see that  $(-\mathcal{A}, D(\mathcal{A}))$  generates a  $\mathcal{C}_0$ -semigroup on  $X$  and that  $\mathcal{B} : U \rightarrow X$   
 1403 is bounded. Thus we consider for this example that  $X_\diamond^* = X = X_{-\diamond}$ .

1404 **PROPOSITION 6.1.** *Let us consider the control system (1.1) with  $\mathcal{A}$  and  $\mathcal{B}$  given*  
 1405 *by (6.9)-(6.10). Then, null-controllability from  $X_{-\diamond}$  holds in any time i.e.  $T_0(X_{-\diamond}) =$*   
 1406 *0.*

1407 *Proof.* The spectrum of  $(\mathcal{A}^*, D(\mathcal{A}))$  is given by

1408 
$$\Lambda = \{\nu_k ; k \geq 1\} \cup \{\nu_k + f(\nu_k) ; k \geq 1\}.$$

1409 Recall that  $(\nu_k)_{k \geq 1}$  satisfies (6.3), (6.6) and (6.7). From [12, Lemma V.4.20] it comes  
 1410 that there exists  $\kappa > 0$  such that  $\Lambda \in \mathcal{L}_w(2, \frac{\kappa}{2}, \frac{1}{2}, \kappa)$ .

1411 An associated grouping is given by

1412 
$$\begin{cases} G_k := \{\lambda_{k,1} := \nu_k, \lambda_{k,2} := \nu_k + f(\nu_k)\}, & \text{if } f(\nu_k) \neq 0, \\ G_k := \{\lambda_{k,1} := \nu_k\}, & \text{if } f(\nu_k) = 0. \end{cases}$$

1413 If  $f(\nu_k) \neq 0$  the eigenvalues  $\lambda_{k,1}$  and  $\lambda_{k,2}$  are simple and we consider the associated  
 1414 eigenvectors

1415 
$$\phi_{k,1}^0 = \begin{pmatrix} -f(\nu_k) \\ 1 \end{pmatrix} \varphi_k, \quad \phi_{k,2}^0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varphi_k.$$

1416 If  $f(\nu_k) = 0$  the eigenvalue  $\lambda_{k,1}$  is algebraically double and we consider the associated  
 1417 Jordan chain

1418 
$$\phi_{k,1}^0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varphi_k, \quad \phi_{k,1}^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \varphi_k.$$

1419 From (6.5) it comes that (2.14) and (2.15) are satisfied. Thus, from Theorem 2.5, we  
 1420 obtain that for any  $y_0 \in X_{-\diamond}$ ,

1421 
$$T_0(y_0) = \limsup_{k \rightarrow +\infty} \frac{\ln^+ \mathcal{C}(G_k, y_0)}{2 \min G_k}.$$

1422 Let us now conclude by estimating  $\mathcal{C}(G_k, y_0)$ .

1423 • Consider first that  $f(\nu_k) \neq 0$ . Then,  $\phi[\lambda_{k,1}, \lambda_{k,2}] = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \varphi_k$  and

1424 
$$b[\lambda_{k,1}, \lambda_{k,2}] = \mathcal{B}^* \phi[\lambda_{k,1}, \lambda_{k,2}] = \frac{\mathbf{1}_\omega \varphi_k - \mathbf{1}_\omega \varphi_k}{f(\nu_k)} = 0.$$

1425 From Theorem 2.8 it comes that

$$1426 \quad \mathcal{C}(G_k, y_0) = \langle M^{-1}\xi, \xi \rangle$$

1427 with

$$1428 \quad M = \text{Gram}(b[\lambda_{k,1}], b[\lambda_{k,1}, \lambda_{k,2}]) + \text{Gram}(0, b[\lambda_{k,2}]) = \begin{pmatrix} \|\varphi_k\|_\omega^2 & 0 \\ 0 & \|\varphi_k\|_\omega^2 \end{pmatrix}$$

1430 and

$$1431 \quad \xi = \begin{pmatrix} \langle y_0, \phi[\lambda_{k,1}] \rangle_{-\diamond, \diamond} \\ \langle y_0, \phi[\lambda_{k,1}, \lambda_{k,2}] \rangle_{-\diamond, \diamond} \end{pmatrix} = \begin{pmatrix} \left\langle y_0, \begin{pmatrix} -f(\nu_k) \\ 1 \end{pmatrix} \varphi_k \right\rangle_{-\diamond, \diamond} \\ \left\langle y_0, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \varphi_k \right\rangle_{-\diamond, \diamond} \end{pmatrix}.$$

1432 Thus,

$$1433 \quad \mathcal{C}(G_k, y_0) = \left\langle y_0, \begin{pmatrix} -f(\nu_k) \\ 1 \end{pmatrix} \frac{\varphi_k}{\|\varphi_k\|_\omega} \right\rangle_{-\diamond, \diamond}^2 + \left\langle y_0, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{\varphi_k}{\|\varphi_k\|_\omega} \right\rangle_{-\diamond, \diamond}^2.$$

1434 • Consider now that  $f(\nu_k) = 0$ . Then,  $b[\lambda_{k,1}^{(2)}] = 0$ . From Theorem 2.8 it comes  
1435 that

$$1436 \quad \mathcal{C}(G_k, y_0) = \langle M^{-1}\xi, \xi \rangle$$

1437 with

$$1438 \quad M_k = \text{Gram}(b[\lambda_{k,1}], b[\lambda_{k,1}^{(2)}]) + \text{Gram}(0, b[\lambda_{k,1}]) = \begin{pmatrix} \|\varphi_k\|_\omega^2 & 0 \\ 0 & \|\varphi_k\|_\omega^2 \end{pmatrix}.$$

1440 and

$$1441 \quad \xi = \begin{pmatrix} \langle y_0, \phi[\lambda_{k,1}] \rangle_{-\diamond, \diamond} \\ \langle y_0, \phi[\lambda_{k,1}^{(2)}] \rangle_{-\diamond, \diamond} \end{pmatrix} = \begin{pmatrix} \left\langle y_0, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varphi_k \right\rangle_{-\diamond, \diamond} \\ \left\langle y_0, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \varphi_k \right\rangle_{-\diamond, \diamond} \end{pmatrix}.$$

1442 As previously,

$$1443 \quad \mathcal{C}(G_k, y_0) = \left\langle y_0, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \frac{\varphi_k}{\|\varphi_k\|_\omega} \right\rangle_{-\diamond, \diamond}^2 + \left\langle y_0, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{\varphi_k}{\|\varphi_k\|_\omega} \right\rangle_{-\diamond, \diamond}^2.$$

1444 Gathering both cases and using estimate (6.5) we obtain, for any  $y_0 \in X_{-\diamond}$ ,

$$1445 \quad \mathcal{C}(G_k, y_0) \leq C \|y_0\|_{-\diamond}^2, \quad \forall k \geq 1.$$

1446 Thus,

$$1447 \quad T_0(y_0) = \limsup_{k \rightarrow +\infty} \frac{\ln^+ \mathcal{C}(G_k, y_0)}{2 \min G_k} = 0. \quad \square$$

1448 **6.4. Competition between different perturbations.**

1449 Let  $\omega_1, \omega_2 \subset (0, 1)$  be two open sets with  $\omega_1 \neq \emptyset$  and  $U = L^2(\Omega)^2$ . Let  $B_1, B_2 \in$   
 1450  $\mathbb{R}^3$ . To simplify the computations, we assume that

$$1451 \quad B_i = \begin{pmatrix} 0 \\ B_{i,2} \\ B_{i,3} \end{pmatrix}.$$

1452 Let  $\alpha, \beta > 0$  with  $\alpha \neq \beta$  and  $f, g : \text{Sp}(A) \rightarrow \mathbb{R}$  be defined by

$$1453 \quad f(\nu_k) = \frac{\rho}{2} e^{-\alpha\nu_k}, \quad g(\nu_k) = \frac{\rho}{2} e^{-\beta\nu_k}.$$

1454 As previously, we consider the associated operators  $f(A)$  and  $g(A)$  defined by the  
 1455 spectral theorem and we define the evolution operator  $\mathcal{A}$  on  $X = L^2(0, 1; \mathbb{R})^3$  by

$$1456 \quad (6.11) \quad \mathcal{A} = \begin{pmatrix} A & I & 0 \\ 0 & A + f(A) & 0 \\ 0 & 0 & A + g(A) \end{pmatrix}, \quad D(\mathcal{A}) = D(A)^3,$$

1457 and the control operator by

$$1458 \quad (6.12) \quad \mathcal{B} : \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in U \mapsto \mathbf{1}_{\omega_1} u_1 B_1 + \mathbf{1}_{\omega_2} u_2 B_2.$$

1459 Then, the observation operator reads

$$1460 \quad \mathcal{B}^* : \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} \in X \mapsto \begin{pmatrix} \mathbf{1}_{\omega_1} (B_{1,2}\varphi_2 + B_{1,3}\varphi_3) \\ \mathbf{1}_{\omega_2} (B_{2,2}\varphi_2 + B_{2,3}\varphi_3) \end{pmatrix}.$$

1461

1462 **PROPOSITION 6.2.** *Let us consider the control system (1.1) with  $\mathcal{A}$  and  $\mathcal{B}$  given*  
 1463 *by (6.11)-(6.12).*

1464 *i. If  $\omega_2 = \emptyset$ , we assume that*

$$1465 \quad (6.13) \quad B_{1,2}B_{1,3} \neq 0.$$

1466 *Then,*

$$1467 \quad T_0(X_{-\diamond}) = \beta + \min\{\alpha, \beta\}.$$

1468 *ii. If  $\omega_2 \neq \emptyset$ , we assume that*

$$1469 \quad (6.14) \quad (B_{1,2}^2 + B_{2,2}^2) (B_{1,3}^2 + B_{2,3}^2) \neq 0.$$

1470 *(a) If  $B_1$  and  $B_2$  are linearly independent, then,*

$$1471 \quad T_0(X_{-\diamond}) = 0.$$

1472 *(b) If  $B_1$  and  $B_2$  are not linearly independent, then,*

$$1473 \quad T_0(X_{-\diamond}) = \beta + \min\{\alpha, \beta\}.$$

1474 *Proof.* It is easy to see that  $(-\mathcal{A}, D(\mathcal{A}))$  generates a  $\mathcal{C}_0$ -semigroup on  $X$  and that  
 1475  $\mathcal{B} : U \rightarrow X$  is bounded. Thus we consider for this example that  $X_\diamond^* = X = X_{-\diamond}$  and  
 1476  $Y_0 = X_{-\diamond}$ .

1477 The spectrum of  $(\mathcal{A}^*, D(\mathcal{A}))$  is given by  $\Lambda = \bigcup_{k \geq 1} G_k$  where

$$1478 \quad G_k := \{\lambda_{k,1} := \nu_k, \lambda_{k,2} := \nu_k + f(\nu_k), \lambda_{k,3} := \nu_k + g(\nu_k)\}.$$

1479 Again, since  $(\nu_k)_{k \geq 1}$  satisfies (6.3), (6.6) and (6.7), the application of [12, Lemma  
 1480 V.4.20] yields the existence of  $\kappa > 0$  such that  $\Lambda \in \mathcal{L}_w(3, \frac{\rho}{2}, \frac{1}{2}, \kappa)$ . The sequence  
 1481  $(G_k)_{k \geq 1}$  is an associated grouping.

1482 The eigenvalues are simple and the corresponding eigenvectors are given by

$$1483 \quad \phi_{k,1}^0 = \begin{pmatrix} -f(\nu_k) \\ 1 \\ 0 \end{pmatrix} \varphi_k, \quad \phi_{k,2}^0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \varphi_k, \quad \phi_{k,3}^0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \varphi_k.$$

1484 Thus, the assumption (2.15) hold. Moreover,

$$1485 \quad (6.15) \quad b_1 = b_2 = \begin{pmatrix} \mathbf{1}_{\omega_1} \varphi_k B_{1,2} \\ \mathbf{1}_{\omega_2} \varphi_k B_{2,2} \end{pmatrix}, \quad b_3 = \begin{pmatrix} \mathbf{1}_{\omega_1} \varphi_k B_{1,3} \\ \mathbf{1}_{\omega_2} \varphi_k B_{2,3} \end{pmatrix}$$

1486 From (6.5) and (6.13) or (6.14) (depending on the assumption on  $\omega_2$ ) it comes  
 1487 that (2.14) is satisfied. Thus, from Theorem 2.5, it comes that for any  $y_0 \in X_{-\diamond}$ ,

$$1488 \quad T_0(y_0) = \limsup_{k \rightarrow +\infty} \frac{\ln^+ \mathcal{C}(G_k, y_0)}{2 \min G_k}.$$

1489 Let us now estimate  $\mathcal{C}(G_k, y_0)$ . From Theorem 2.8 it comes that

$$1490 \quad \mathcal{C}(G_k, y_0) = \langle M^{-1} \xi, \xi \rangle$$

1491 with

$$1492 \quad M = \text{Gram}(b[\lambda_{k,1}], b[\lambda_{k,1}, \lambda_{k,2}], b[\lambda_{k,1}, \lambda_{k,2}, \lambda_{k,3}]) \\ 1493 \quad + \text{Gram}(0, b[\lambda_{k,2}], b[\lambda_{k,2}, \lambda_{k,3}]) + \text{Gram}(0, 0, b[\lambda_{k,3}])$$

1495 and

$$1496 \quad \xi = \begin{pmatrix} \langle y_0, \phi[\lambda_{k,1}] \rangle_{-\diamond, \diamond} \\ \langle y_0, \phi[\lambda_{k,1}, \lambda_{k,2}] \rangle_{-\diamond, \diamond} \\ \langle y_0, \phi[\lambda_{k,1}, \lambda_{k,2}, \lambda_{k,3}] \rangle_{-\diamond, \diamond} \end{pmatrix}.$$

1497 Explicit computations yield

$$1498 \quad \phi[\lambda_{k,1}] = \begin{pmatrix} -f(\nu_k) \\ 1 \\ 0 \end{pmatrix} \varphi_k, \quad \phi[\lambda_{k,1}, \lambda_{k,2}] = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \varphi_k,$$

1499 and

$$1500 \quad \phi[\lambda_{k,1}, \lambda_{k,2}, \lambda_{k,3}] = \frac{1}{g(\nu_k)(g(\nu_k) - f(\nu_k))} \begin{pmatrix} f(\nu_k) - g(\nu_k) \\ -1 \\ 1 \end{pmatrix} \varphi_k.$$

1501 i. Assume that  $\omega_2 = \emptyset$ .  
 1502 After the change of variables

$$1503 \quad z = \text{diag} \left( \frac{1}{B_{1,2}}, \frac{1}{B_{1,2}}, \frac{1}{B_{1,3}} \right) y,$$

1504 the system under study reads

$$1505 \quad \begin{cases} \partial_t z + \begin{pmatrix} A & I & 0 \\ 0 & A + f(A) & 0 \\ 0 & 0 & A + g(A) \end{pmatrix} z = \mathbf{1}_{\omega_1} u_1(t, x) \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \\ z(t, 0) = z(t, 1) = 0. \end{cases}$$

1506 This leads to

$$1507 \quad b[\lambda_{k,1}] = b[\lambda_{k,2}] = b[\lambda_{k,3}] = \mathbf{1}_{\omega_1} \varphi_k.$$

1508 Thus,  $M = \|\varphi_k\|_{\omega_1}^2 I_3$  and

$$1509 \quad \mathcal{C}(G_k, y_0) = \left\langle y_0, \begin{pmatrix} -f(\nu_k) \\ 1 \\ 0 \end{pmatrix} \frac{\varphi_k}{\|\varphi_k\|_{\omega_1}} \right\rangle_{-\diamond, \diamond}^2 + \left\langle y_0, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \frac{\varphi_k}{\|\varphi_k\|_{\omega_1}} \right\rangle_{-\diamond, \diamond}^2$$

$$1510 \quad + \left( \frac{1}{g(\nu_k)(g(\nu_k) - f(\nu_k))} \right)^2 \left\langle y_0, \begin{pmatrix} f(\nu_k) - g(\nu_k) \\ -1 \\ 1 \end{pmatrix} \frac{\varphi_k}{\|\varphi_k\|_{\omega_1}} \right\rangle_{-\diamond, \diamond}^2.$$

1511 From (6.5), we obtain for any  $y_0 \in X_{-\diamond}$ ,

$$1512 \quad \mathcal{C}(G_k, y_0) \leq C \|y_0\|_{-\diamond}^2 \left( 1 + \left( \frac{1}{g(\nu_k)(g(\nu_k) - f(\nu_k))} \right)^2 \right).$$

1513 This leads to

$$1514 \quad T_0(X_{-\diamond}) \leq \limsup_{k \rightarrow +\infty} \frac{-\ln^+ |g(\nu_k)(g(\nu_k) - f(\nu_k))|}{\nu_k}.$$

1515 Conversely, with the particular choice

$$1516 \quad y_0 = \sum_{k \geq 1} \frac{1}{\nu_k} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \varphi_k,$$

1517 we have

$$1518 \quad \mathcal{C}(G_k, y_0) = \frac{1}{\nu_k^2 \|\varphi_k\|_{\omega_1}^2} \left( \frac{1}{g(\nu_k)(g(\nu_k) - f(\nu_k))} \right)^2.$$

1519 Thus, from (6.5), we obtain

$$1520 \quad T_0(X_{-\diamond}) \geq T_0(y_0) = \limsup_{k \rightarrow +\infty} \frac{-\ln |g(\nu_k)(g(\nu_k) - f(\nu_k))|}{\nu_k}$$

1523 which gives

$$1524 \quad T_0(X_{-\diamond}) = \limsup_{k \rightarrow +\infty} \frac{-\ln |g(\nu_k)(g(\nu_k) - f(\nu_k))|}{\nu_k}.$$

1525 Then, the same computations as [9, Section 5.1.3] yield

$$1526 \quad T_0(X_{-\diamond}) = \beta + \min\{\alpha, \beta\}.$$

1527 ii. We now consider the case  $\omega_2 \neq \emptyset$ .

1528 (a) Assume that  $B_1$  and  $B_2$  are linearly independent. If necessary, we con-  
 1529 sider smaller control sets so that  $\omega_1 \cap \omega_2 = \emptyset$ . As we will prove that  
 1530  $T_0(X_{-\diamond}) = 0$ , this is not a restrictive assumption.  
 1531 To ease the reading we drop the index  $k$  in what follows. As pre-  
 1532 viously, the vector  $\xi$  is not bounded. Let us consider the dilatation  
 1533  $D_\epsilon = \text{diag}(1, 1, \epsilon)$  with

$$1534 \quad \epsilon = g(\nu)(g(\nu) - f(\nu))$$

1535 and  $\tilde{\xi} = D_\epsilon \xi$ . Then, from Section D.1, it comes that

$$1536 \quad \mathcal{C}(G, y_0) = \left\langle \tilde{M}^{-1} \tilde{\xi}, \tilde{\xi} \right\rangle$$

1537 with

$$1538 \quad \tilde{M} = \text{Gram}(b[\lambda_1], b[\lambda_1, \lambda_2], \epsilon b[\lambda_1, \lambda_2, \lambda_3]) \\ 1539 \quad + \text{Gram}(0, b[\lambda_2], \epsilon b[\lambda_2, \lambda_3]) + \text{Gram}(0, 0, \epsilon b[\lambda_3]).$$

1541 As  $\|\tilde{\xi}\|$  is bounded, we simply give a lower bound on the smallest eigen-  
 1542 value of  $\tilde{M}$ . Using (6.15), it comes that

$$1543 \quad b[\lambda_1, \lambda_2] = 0, \quad b[\lambda_2, \lambda_3] = \frac{b_3 - b_1}{g(\nu) - f(\nu)}, \quad b[\lambda_1, \lambda_2, \lambda_3] = \frac{1}{\epsilon}(b_3 - b_1).$$

1544 Thus,

$$1545 \quad \tilde{M} = \text{Gram}(b_1, 0, b_3 - b_1) + \text{Gram}(0, b_1, g(\nu)(b_3 - b_1)) \\ 1546 \quad + \text{Gram}(0, 0, \epsilon b_3).$$

1548 This gives that, for any  $\tau \in \mathbb{R}^3$ , we have

$$1549 \quad (6.16) \quad \left\langle \tilde{M} \tau, \tau \right\rangle = \|\tau_1 b_1 + \tau_3 (b_3 - b_1)\|_U^2 + \|\tau_2 b_1 + g(\nu) \tau_3 (b_3 - b_1)\|_U^2 \\ 1550 \quad + \epsilon^2 \|\tau_3 b_3\|_U^2.$$

1553 To obtain a lower bound on this quantity we use the following lemma.

1554 LEMMA 6.3. *There exists  $C > 0$  (independent of  $k$ ) such that for any*  
 1555  $\theta_1, \theta_3 \in \mathbb{R}$ ,

$$1556 \quad \|\theta_1 b_1 + \theta_3 b_3\|_U^2 \geq C (\theta_1^2 + \theta_3^2).$$

1557

*Proof.* As  $\omega_1 \cap \omega_2 = \emptyset$ ,

1558

$$\|\theta_1 b_1 + \theta_3 b_3\|_U^2 = (B_{1,2}\theta_1 + B_{1,3}\theta_3)^2 \|\varphi_k\|_{\omega_1}^2 + (B_{2,2}\theta_1 + B_{2,3}\theta_3)^2 \|\varphi_k\|_{\omega_2}^2.$$

1559

1561

Using (6.5) it comes that

1562

$$\begin{aligned} \|\theta_1 b_1 + \theta_3 b_3\|_U^2 &\geq C \left( (B_{1,2}\theta_1 + B_{1,3}\theta_3)^2 + (B_{2,2}\theta_1 + B_{2,3}\theta_3)^2 \right) \\ &= \left\| \begin{pmatrix} B_{1,2} & B_{1,3} \\ B_{2,2} & B_{2,3} \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_3 \end{pmatrix} \right\|^2. \end{aligned}$$

1563

1564

1565

Since  $B_1$  and  $B_2$  are linearly independent, this ends the proof.  $\square$

1566

Applying this lemma twice to (6.16) yield

1567

$$\begin{aligned} \langle \widetilde{M}\tau, \tau \rangle &\geq C \left( (\tau_1 - \tau_3)^2 + \tau_3^2 + (\tau_2 - g(\nu)\tau_3)^2 + g(\nu)^2 \tau_3^2 + \epsilon^2 \tau_3^2 \right) \\ &\geq C \left( (\tau_1 - \tau_3)^2 + \tau_3^2 + (\tau_2 - g(\nu)\tau_3)^2 \right). \end{aligned}$$

1568

1570

Taking into account that  $0 < g(\nu) < \frac{1}{2}$  for  $\nu$  large enough, the study of this quadratic form in  $\mathbb{R}^3$  leads to

1571

1572

$$\langle \widetilde{M}\tau, \tau \rangle \geq C (\tau_1^2 + \tau_2^2 + \tau_3^2).$$

1573

Thus the smallest eigenvalue of  $\widetilde{M}$  is bounded from below. This leads to the boundedness of  $\langle \widetilde{M}^{-1}\widetilde{\xi}, \widetilde{\xi} \rangle$  which concludes the proof of case ii (a).

1574

1575

- (b) Assume now that  $B_1$  and  $B_2$  are not linearly independent. Then, there exist  $x_1, x_2 \in \mathbb{R}$  such that

1576

1577

$$\begin{cases} x_1 B_{1,2} + x_2 B_{1,3} = 0 \\ x_1 B_{2,2} + x_2 B_{2,3} = 0. \end{cases}$$

1578

Up to a change of normalization of the eigenvectors (independent of  $k$ ) we obtain

1579

1580

$$b_1 = b_2 = b_3 = \begin{pmatrix} \mathbf{1}_{\omega_1} \varphi_k x_1 B_{1,2} \\ \mathbf{1}_{\omega_2} \varphi_k x_1 B_{2,2} \end{pmatrix}$$

1581

and this amounts to case i.  $\square$

1582

## 7. Analysis of controllability for systems of partial differential equations.

1583

1584

We now turn to the analysis of null controllability of actual partial differential equations. We consider here coupled systems of two linear one dimensional parabolic equations.

1585

1586

1587

### 7.1. Coupled heat equations with different diffusion coefficients.

1588

In this application, we consider the Sturm-Liouville operator  $A$  defined in (6.1) and we define in  $X = L^2(0, 1; \mathbb{R})^2$  the operator

1589

1590

$$\mathcal{A} = \begin{pmatrix} A & I \\ 0 & dA \end{pmatrix}, \quad D(\mathcal{A}) = D(A)^2,$$

1591 with  $d > 0$ . We will assume  $d \neq 1$ , since the case  $d = 1$  is much simpler and already  
 1592 studied in the literature: see the computations of Section 6.3 in the case  $f = 0$  or, for  
 1593 instance, [23] for a more general study based on Carleman estimates.

1594 We will consider two cases : the case where two boundary controls are applied to  
 1595 the system, and the case where we consider the same distributed control in the two  
 1596 equations of the system.

1597 **7.1.1. Spectrum of  $\mathcal{A}^*$ .** Let  $\Lambda_1 := \text{Sp}(A) = \{\nu_k ; k \geq 1\}$  and  $\Lambda_2 := d\Lambda_1$ .

1598 The spectrum of  $\mathcal{A}^*$  is given by  $\Lambda = \Lambda_1 \cup \Lambda_2$  which belongs to  $\mathcal{L}_w(2, \varrho, \frac{1}{2}, \kappa)$  for  
 1599 some  $\varrho, \kappa > 0$  (see [12, Lemma V.4.20]).

1600 For any  $\lambda \in \Lambda$ , there are two non mutually exclusive cases:

1601 • If  $\lambda = \nu_k \in \Lambda_1$ , then we can associate an eigenvector given by

1602 
$$\phi_{\lambda,1} = \begin{pmatrix} 1 \\ \varepsilon_k \end{pmatrix} \varphi_k,$$

1603 with  $\varepsilon_k = \frac{1}{\nu_k(1-d)}$ . Note that  $\varepsilon_k$  tends to zero when  $k$  goes to infinity.

1604 • If  $\lambda = d\nu_l \in \Lambda_2$ , then we can associate an eigenvector given by

1605 
$$\phi_{\lambda,2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varphi_l.$$

1606 It clearly appears that the elements in  $\Lambda_1 \cap \Lambda_2$  (if this set is not empty) are ge-  
 1607 ometrically double eigenvalues of  $\mathcal{A}^*$ , since in that case  $\phi_{\lambda,1}$  and  $\phi_{\lambda,2}$  are linearly  
 1608 independent.

1609 Note that (2.15) holds for the choices of  $X_\diamond^*$  that we will make in the sequel, since  
 1610  $(\varphi_k)_{k \geq 1}$  is a Hilbert basis of  $L^2(0, 1; \mathbb{R})$ .

1611 **7.1.2. Two boundary controls.** In this section, we study the following bound-  
 1612 ary control system

1613 (7.1) 
$$\begin{cases} \partial_t y + \mathcal{A}y = 0, & t \in (0, T), \\ y(t, 0) = B_0 u_0(t), \quad y(t, 1) = B_1 u_1(t), & t \in (0, T), \end{cases}$$

1614 with

1615 (7.2) 
$$B_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad B_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

1616 The control operator  $\mathcal{B}$  is defined in a weak sense as in [38]. The expression of its  
 1617 adjoint is given by

1618 
$$\mathcal{B}^* : \begin{pmatrix} f \\ g \end{pmatrix} \in X_1^* \mapsto \begin{pmatrix} -B_0^* \begin{pmatrix} f'(0) \\ g'(0) \end{pmatrix} \\ B_1^* \begin{pmatrix} f'(1) \\ g'(1) \end{pmatrix} \end{pmatrix} = \begin{pmatrix} -(f'(0) + g'(0)) \\ g'(1) \end{pmatrix}.$$

1619 Considering  $X_\diamond^* = H_0^1(0, 1; \mathbb{R})^2$ , we obtain that  $\mathcal{B}$  is admissible with respect to  $X_{-\diamond} =$   
 1620  $H^{-1}(0, 1; \mathbb{R})^2$ .

1621 **PROPOSITION 7.1.** *For any  $d \neq 1$ , there exists  $Y_0$  a closed subspace of  $H^{-1}(0, 1; \mathbb{R})^2$*   
 1622 *of finite codimension such that*

1623 • *for any  $y_0 \notin Y_0$ , system (7.1) is not approximately controllable;*

1624 • for any  $y_0 \in Y_0$ , system (7.1) is null controllable in any time  $T > 0$ .

1625 REMARK 7.1. The situation with a single control is quite different. Indeed, con-  
 1626 sidering  $B_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  and  $B_1 = 0$ , it is proved in [5] that, when  $A$  is the Dirichlet  
 1627 Laplace operator, approximate controllability holds if and only  $\sqrt{d} \notin \mathbb{Q}$  and in this  
 1628 case that

$$1629 \quad T_0(X_{-\diamond}) = \limsup_{\substack{\lambda \rightarrow \infty \\ \lambda \in \Lambda}} \frac{\ln^+ \left( \frac{1}{\text{dist}(\lambda, \Lambda \setminus \{\lambda\})} \right)}{\lambda}.$$

1630 With this formula the authors prove that, for any  $\tau \in [0, +\infty]$ , there exists a diffusion  
 1631 ratio  $d > 0$  such that the minimal null control time of system (7.1) satisfies  $T_0(X_{-\diamond}) =$   
 1632  $\tau$ .

1633 REMARK 7.2. From the definition of  $Y_0$  in the following proof, we directly obtain  
 1634 that in the case where  $\mathcal{A}$  is the Dirichlet Laplace operator on the interval  $(0, 1)$ , then  
 1635  $Y_0 = H^{-1}(0, 1; \mathbb{R})^2$ .

1636 REMARK 7.3. The particular choice of  $B_0$  and  $B_1$  is done to simplify the com-  
 1637 putations. Notice that with this choice, it is not possible to steer to zero the second  
 1638 equation and then control the first equation. This would be the case with the simpler  
 1639 choice

$$1640 \quad B_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad B_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

1641 *Proof.* Let us compute the observations associated to the eigenvectors of  $\mathcal{A}^*$ .

1642 For any  $k \geq 1$ , we define  $s_k \in \mathbb{R}$  be such that  $\varphi'_k(1) = s_k \varphi'_k(0)$ . From (6.4), there  
 1643 exists  $C > 0$  such that

$$1644 \quad (7.3) \quad \frac{1}{C} \leq |s_k| \leq C, \quad \forall k \geq 1.$$

1645 • For any  $\lambda = \nu_k \in \Lambda_1$ , we have

$$1646 \quad (7.4) \quad \mathcal{B}^* \phi_{\lambda,1} = -\varphi'_k(0) \begin{pmatrix} 1 + \varepsilon_k \\ -s_k \varepsilon_k \end{pmatrix}.$$

1647 • For any  $\lambda = d\nu_l \in \Lambda_2$ , we have

$$1648 \quad (7.5) \quad \mathcal{B}^* \phi_{\lambda,2} = -\varphi'_l(0) \begin{pmatrix} 1 \\ -s_l \end{pmatrix}.$$

1649 Due to (6.4) and (7.3), it comes that (2.14) holds for any simple eigenvalue  $\lambda \in$   
 1650  $(\Lambda_1 \setminus \Lambda_2) \cup (\Lambda_2 \setminus \Lambda_1)$ .

1651 However, for a geometrically double eigenvalue  $\lambda \in \Lambda_1 \cap \Lambda_2$ , there can be non-  
 1652 observable modes. Indeed, let  $k$  and  $l$  such that  $\lambda = \nu_k = d\nu_l$ . Then, the condition

$$1653 \quad \text{Ker}(\mathcal{A}^* - \lambda) \cap \text{Ker} \mathcal{B}^* \neq \{0\}$$

1654 is equivalent to the fact that  $\mathcal{B}^* \phi_{\lambda,1}$  and  $\mathcal{B}^* \phi_{\lambda,2}$  given by (7.4)-(7.5) are linearly  
 1655 independent, which is itself equivalent to the condition

$$1656 \quad (7.6) \quad s_k \varepsilon_k = s_l (1 + \varepsilon_k).$$

1657 Due to the asymptotics  $\varepsilon_k \xrightarrow[k \rightarrow +\infty]{} 0$  it turns out that the set

$$1658 \quad \Theta := \{\lambda = \nu_k = d\nu_l \in \Lambda_1 \cap \Lambda_2; (7.6) \text{ holds}\},$$

1659 is finite.

1660 For any  $\lambda \in \Theta$ , we can find  $\psi_\lambda \in \text{Span}(\phi_{\lambda,1}, \phi_{\lambda,2})$  such that  $\mathcal{B}^* \psi_\lambda = 0$  and  $\psi_\lambda \neq 0$ ,  
1661 that is a non observable mode.

1662 Finally, we introduce the set

$$1663 \quad Y_0 := \left\{ y_0 \in X_{-\diamond}; \langle y_0, \psi_\lambda \rangle_{-\diamond, \diamond} = 0, \forall \lambda \in \Theta \right\}$$

1664 which is, by construction, of finite codimension. For  $y_0 \in Y_0$ , the associated moment  
1665 problem reduces to the one where the geometrically double eigenvalues  $\lambda \in \Theta$  are now  
1666 considered as simple eigenvalues with associated eigenvector  $\phi_{\lambda,2}$ , since the moment  
1667 equation is automatically satisfied for the other eigenvector  $\psi_\lambda$ .

1668 We consider now a grouping  $\mathcal{G}$  as given by Proposition 2.2, with  $p = 2$  and  $\varrho > 0$   
1669 small enough such that for  $i \in \{1, 2\}$  we have

$$1670 \quad (7.7) \quad |\lambda - \mu| > \varrho, \quad \forall \lambda, \mu \in \Lambda_i, \lambda \neq \mu.$$

1671 Hence, Theorem 2.5 gives the formula

$$1672 \quad T_0(y_0) = \limsup_{G \in \mathcal{G}} \frac{\ln^+ \mathcal{C}(G, y_0)}{2r_G}.$$

1673 We will prove in the sequel, analyzing the different possible blocks, that

$$1674 \quad (7.8) \quad \sup_{G \in \mathcal{G}} \mathcal{C}(G, y_0) < +\infty,$$

1675 which will let us conclude the claim, that is  $T_0(y_0) = 0$ .

1676 • Blocks of a simple eigenvalue.

We immediately obtain

$$\mathcal{C}(G, y_0) = \begin{cases} \frac{|\langle y_0, \phi_{\lambda,1} \rangle_{-\diamond, \diamond}|^2}{((1 + \varepsilon_k)^2 + s_k^2 \varepsilon_k^2) |\varphi'_k(0)|^2} & , \text{ if } \lambda = \nu_k, \\ \frac{|\langle y_0, \phi_{\lambda,2} \rangle_{-\diamond, \diamond}|^2}{(1 + s_l^2) |\varphi'_l(0)|^2} & , \text{ if } \lambda = d\nu_l. \end{cases}$$

1677 Using again (6.4) the estimate (7.3) and the fact that  $(\varepsilon_k)_k$  goes to 0 as  $k$  goes  
1678 to infinity, we observe that the blocks consisting of a single simple eigenvalue  
1679 do not contribute to the minimal time: the quantity  $\mathcal{C}(G, y_0)$  is bounded  
1680 independently of  $G$ .

1681 Moreover, by the discussion above, the blocks consisting of a single double  
1682 eigenvalue belonging to  $\Theta$  do not contribute either.

1683 • Blocks of two simple eigenvalues:  $G = \{\lambda_1 := \nu_k\} \cup \{\lambda_2 := d\nu_l\}$ .

1684 From Theorem 2.10 we obtain

$$1685 \quad \mathcal{C}(G, y_0) = \langle M^{-1} \xi, \xi \rangle$$

1686 with

$$1687 \quad M = \text{Gram}(b[\lambda_1], b[\lambda_2]) + \text{Gram}(0, (\lambda_2 - \lambda_1)b[\lambda_2])$$

1688 and

$$1689 \quad \xi = \begin{pmatrix} \langle y_0, \phi_{\lambda_1,1} \rangle_{-\diamond, \diamond} \\ \langle y_0, \phi_{\lambda_2,2} \rangle_{-\diamond, \diamond} \end{pmatrix}.$$

1690 To ease the reading, we use the following change of normalization for the  
1691 eigenvectors

$$1692 \quad \tilde{\phi}_{\lambda_1} := \frac{\phi_{\lambda_1,1}}{-\varphi'_k(0)}, \quad \tilde{\phi}_{\lambda_2} := \frac{\phi_{\lambda_2,2}}{-\varphi'_l(0)},$$

1693 and we denote by  $\tilde{M}$  and  $\tilde{\xi}$  the associated quantities. Notice that, due to (6.4),  
1694 the quantity  $\|\tilde{\xi}\|$  is bounded. Thus, to estimate  $\mathcal{C}(G, y_0)$  we give a lower bound  
1695 on the smallest eigenvalue of  $\tilde{M}$ . We have

$$1696 \quad \begin{aligned} \tilde{M} &= \text{Gram}(\tilde{b}[\lambda_1], \tilde{b}[\lambda_2]) + \text{Gram}(0, (\lambda_2 - \lambda_1)\tilde{b}[\lambda_2]) \\ 1697 \quad &= \underbrace{\begin{pmatrix} \epsilon_k^2 s_k^2 + (1 + \epsilon_k)^2 & 1 + \epsilon_k + \epsilon_k s_k s_l \\ 1 + \epsilon_k + \epsilon_k s_k s_l & 1 + s_l^2 \end{pmatrix}}_{=\Gamma^1} + \begin{pmatrix} 0 & 0 \\ 0 & (\lambda_2 - \lambda_1)^2(1 + s_l^2) \end{pmatrix}. \end{aligned}$$

1699 For any  $\tau \in \mathbb{R}^2$ ,  $\langle \tilde{M}\tau, \tau \rangle \geq \langle \Gamma^1\tau, \tau \rangle$ . Then,

$$1700 \quad \min \text{Sp}(\Gamma^1) \geq \frac{\det(\Gamma^1)}{\text{tr}(\Gamma^1)} = \frac{((1 + \epsilon_k)s_l - \epsilon_k s_k)^2}{1 + (1 + \epsilon_k)^2 + \epsilon_k^2 s_k^2 + s_l^2}$$

1701 From (7.3), it comes that, for  $k$  large enough,  $\min \text{Sp}(\Gamma^1)$  is bounded from  
1702 below by a positive constant independent of  $G$ .

1703 • Blocks made of a geometrically double eigenvalue which does not belong to  
1704  $\Theta$ :

1705 Consider  $G = \{\lambda\}$  with  $\lambda = \nu_k = d\nu_l \in \Lambda_1 \cap \Lambda_2$ . With the same notations as  
1706 previously, Theorem 2.10 implies that

$$1707 \quad \mathcal{C}(G, y_0) = \langle \tilde{M}^{-1}\tilde{\xi}, \tilde{\xi} \rangle$$

1708 where

$$1709 \quad \tilde{\xi} = \begin{pmatrix} \langle y_0, \frac{\phi_{\lambda,1}}{-\varphi'_k(0)} \rangle_{-\diamond, \diamond} \\ \langle y_0, \frac{\phi_{\lambda,2}}{-\varphi'_l(0)} \rangle_{-\diamond, \diamond} \end{pmatrix}$$

1710 and

$$1711 \quad \tilde{M} = \text{Gram} \left( \frac{\mathcal{B}^* \phi_{\lambda,1}}{-\varphi'_k(0)}, \frac{\mathcal{B}^* \phi_{\lambda,2}}{-\varphi'_l(0)} \right) = \Gamma^1.$$

1713 Notice that since  $\lambda \notin \Theta$ , we have  $\det(\Gamma^1) = ((1 + \epsilon_k)s_l - \epsilon_k s_k)^2 > 0$ .

1714 Thus, the study of the previous item proves that, for  $\lambda$  large enough,  $\min \text{Sp}(\Gamma^1)$   
1715 is bounded from below by a positive constant independent of  $\lambda$ .

1716 Gathering all cases, we deduce (7.8) and the proof is complete.  $\square$

1717 **7.1.3. Simultaneous distributed control.** Let us now consider the following  
 1718 control problem

$$1719 \quad (7.9) \quad \begin{cases} \partial_t y + \mathcal{A}y = \mathbf{1}_\omega \begin{pmatrix} 1 \\ 1 \end{pmatrix} u(t, x), & t \in (0, T), \\ y(t, 0) = y(t, 1) = 0, & t \in (0, T). \end{cases}$$

1720 In that case, the observation operator  $\mathcal{B}^*$  is given by

$$1721 \quad \mathcal{B}^* : \begin{pmatrix} f \\ g \end{pmatrix} \in X_1^* \mapsto \mathbf{1}_\omega(f + g),$$

1722 and is clearly admissible with respect to the pivot space  $X$ . Our result concerning  
 1723 this example is very similar to Proposition 7.1 and reads as follows.

1724 **PROPOSITION 7.2.** *For any  $d \neq 1$ , there exists  $Y_0$  a closed subspace of  $H^{-1}(0, 1; \mathbb{R})^2$   
 1725 of codimension less or equal than 1 such that*

- 1726 • for any  $y_0 \notin Y_0$ , system (7.9) is not approximately controllable;
- 1727 • for any  $y_0 \in Y_0$ , system (7.9) is null controllable in any time  $T > 0$ .

1728 **REMARK 7.4.** *During the proof it will appear that there exists a countable set  
 1729  $D \subset (1, +\infty)$  such that for any  $d \notin D \cup \{1\}$ , we have  $Y_0 = H^{-1}(0, 1; \mathbb{R})^2$ , which  
 1730 means that our system is null-controllable at any time  $T > 0$  for any initial data. In  
 1731 particular, it is noticeable that this property holds for any  $d < 1$ , that is in the case  
 1732 where the diffusion coefficient is lower in the second equation (the one which does not  
 1733 contain coupling terms).*

1734 *Proof.* We start by computing the observations related to the eigenelements of  
 1735  $\mathcal{A}^*$

- 1736 • For any  $\lambda = \nu_k \in \Lambda_1$ , we have

$$1737 \quad (7.10) \quad \mathcal{B}^* \phi_{\lambda,1} = (1 + \varepsilon_k) \varphi_k \mathbf{1}_\omega.$$

- 1738 • For any  $\lambda = d\nu_l \in \Lambda_2$ , we have

$$1739 \quad (7.11) \quad \mathcal{B}^* \phi_{\lambda,2} = \varphi_l \mathbf{1}_\omega.$$

If for some  $k$  we have  $1 + \varepsilon_k = 0$ , then we clearly get that (2.14) does not hold. We  
 can thus introduce the set

$$\Theta := \{\lambda = \nu_k ; 1 + \varepsilon_k = 0\},$$

1740 which is of cardinal less or equal than 1 (by definition of the sequence  $(\varepsilon_k)_k$ , see  
 1741 Section 7.1.1). Note also that for  $d < 1$ , we always have  $\varepsilon_k > 0$ , so that  $\Theta = \emptyset$ , see  
 1742 Remark 7.4.

1743 We notice however that, for any  $\lambda = d\nu_l$ , we have  $\mathcal{B}^* \phi_{\lambda,2} \neq 0$  and that if  $\lambda =$   
 1744  $\nu_k = d\nu_l \in \Lambda_1 \cap \Lambda_2$ , with  $\lambda \notin \Theta$ , then  $\mathcal{B}^* \phi_{\lambda,1}$  and  $\mathcal{B}^* \phi_{\lambda,2}$  are linearly independent.

Let us introduce

$$Y_0 := \{y_0 \in X; \text{ s.t. } \langle y_0, \phi_{\lambda,1} \rangle_X = 0, \forall \lambda \in \Theta\}.$$

1745 By definition of this set, for any initial data in  $Y_0$ , the moment equation (1.2) related  
 1746 to the eigenvector  $\phi_{\lambda,1}$  for  $\lambda \in \Theta$  is automatically satisfied for any control since both  
 1747 members are equal to zero.

1748 As in the proof of Proposition 7.1, we consider a grouping  $\mathcal{G}$  as given by Propo-  
 1749 sition 2.2, with  $p = 2$  and  $\varrho > 0$  small enough such that for  $i \in \{1, 2\}$  we have

$$1750 \quad |\lambda - \mu| > \varrho, \quad \forall \lambda, \mu \in \Lambda_i, \lambda \neq \mu.$$

1751 Hence, Theorem 2.5 gives the formula

$$1752 \quad T_0(y_0) = \limsup_{G \in \mathcal{G}} \frac{\ln^+ \mathcal{C}(G, y_0)}{2r_G}.$$

1753 Let us now evaluate the quantities  $\mathcal{C}(G, y_0)$  for every possible block.

- 1754 • Blocks made of a simple eigenvalue that does not belong to  $\Theta$ .  
 We immediately obtain

$$\mathcal{C}(G, y_0) = \begin{cases} \frac{|\langle y_0, \phi_{\lambda,1} \rangle_X|^2}{(1 + \varepsilon_k)^2 \|\varphi_k\|_\omega^2} & , \text{ if } \lambda = \nu_k, \\ \frac{|\langle y_0, \phi_{\lambda,2} \rangle_X|^2}{\|\varphi_l\|_\omega^2} & , \text{ if } \lambda = d\nu_l, \end{cases}$$

1755 which is a bounded quantity thanks to (6.5) and the fact that  $(\varepsilon_k)_k$  tends to  
 1756 zero at infinity.

- 1757 • Blocks made of two eigenvalues:  $G = \{\lambda_1 := \nu_k\} \cup \{\lambda_2 := d\nu_l\}$ . Note that  
 1758 the proof below works exactly the same in the case where  $\lambda_1 \neq \lambda_2$ , that is if  
 1759 the two eigenvalues are simple, or in the case where  $\lambda_1 = \lambda_2$ , that is if there  
 1760 is only a geometrically double eigenvalue.

1761 By the discussion above, we can assume that  $\lambda_1$  does not belong to  $\Theta$  (if  
 1762 not, this block has to be considered as a block containing only the simple  
 1763 eigenvalue  $\lambda_2$ ).

Thanks to Theorem 2.10 we have  $\mathcal{C}(G, y_0) \leq \langle \widetilde{M}^{-1} \xi, \xi \rangle$  where

$$\widetilde{M} = \text{Gram}(\mathbf{1}_\omega \varphi_k, \mathbf{1}_\omega \varphi_l),$$

$$\xi = \begin{pmatrix} \langle y_0, \phi_{\lambda_1,1} \rangle_X \\ 1 + \varepsilon_k \\ \langle y_0, \phi_{\lambda_2,2} \rangle_X \end{pmatrix}.$$

By using the Lebeau-Robbiano inequality (6.8), and the fact that  $|\lambda_1 - \lambda_2| \leq \varrho$ , we have that

$$\langle \widetilde{M}^{-1} \xi, \xi \rangle \leq C_1 e^{C_1 \sqrt{r_G}} \|\xi\|^2 \leq C_2 e^{C_1 \sqrt{r_G}} \|y_0\|_X^2,$$

1764 where  $C_1, C_2$  only depends on  $\varrho, \omega$  and on the operator  $\mathcal{A}$ .  
 All in all, we have obtained that

$$\ln^+ \mathcal{C}(G, y_0) \leq C (1 + \sqrt{r_G}).$$

1765 Gathering all cases, we conclude that  $T_0(y_0) = 0$ . □

1766 **7.2. Other applications.**

1767 Let us consider the following control system

(7.12)

$$\begin{cases} \partial_t y + \begin{pmatrix} -\partial_{xx} + c_1(x) & 1 \\ 0 & -\partial_{xx} + c_2(x) \end{pmatrix} y = \begin{pmatrix} 0 \\ \mathbf{1}_\omega u(t, x) \end{pmatrix}, & (t, x) \in (0, T) \times (0, 1), \\ y(t, 0) = y(t, 1) = 0, & t \in (0, T), \\ y(0, x) = y_0(x), \end{cases}$$

1768

1769 where  $c_1, c_2 \in L^2(0, 1; \mathbb{R})$ .

1770 With the technics developed in this article, one can prove the following control-  
1771 lability result.

1772 **PROPOSITION 7.3.** *For any non-negative potentials  $c_1, c_2$ , system (7.12) is null*  
1773 *controllable in any time  $T > 0$  from  $L^2(0, 1; \mathbb{R})^2$ .*

1774 The proof follows closely the computations done for the same system with a boundary  
1775 control in [9, Section 5.2.1]. The only difference is that the contributions of terms of  
1776 the form  $\|\mathcal{B}^* \bullet\|_U = \|\bullet\|_\omega$  are estimated using (6.5).

1777 As the result stated in Proposition 7.3 is already known (it is for instance an  
1778 application of [23] with a proof based on Carleman estimates), we do not detail the  
1779 proof here to lighten this article.

1780 With the technics developed in this article we can also analyze null controllability  
1781 for the following control system

(7.13)

$$\begin{cases} \partial_t y + \begin{pmatrix} A & q(x) \\ 0 & A \end{pmatrix} y = \begin{pmatrix} 0 \\ \mathbf{1}_\omega u(t, x) \end{pmatrix}, & (t, x) \in (0, T) \times (0, 1), \\ y(t, 0) = y(t, 1) = 0, & t \in (0, T), \\ y(0, x) = y_0(x), \end{cases}$$

1782

1783 where the coupling function  $q$  belongs to  $L^\infty(0, 1; \mathbb{R})$  and  $\omega \subset (0, 1)$  is a non empty  
1784 open set. We manage to characterize the value of the minimal null-control time  
1785 without any other assumption on  $q$  and  $\omega$ .

1786 This analysis extends previous results of [14] where approximate controllability  
1787 was studied and those of [6] where null controllability was studied in the particular  
1788 case where  $\mathcal{A}$  is the Dirichlet Laplace operator and  $\omega$  is an interval disjoint of  $\text{Supp } q$ .  
1789 Our formalism also allows us to recover null controllability in any time when  $q$  has a  
1790 strict sign on a subdomain of  $\omega$  as proved in [23] by means of Carleman estimates.

1791 Since the analysis of this example makes use of refined spectral properties of  
1792 the underlying operator whose proofs are rather intricate, we will develop it in the  
1793 forthcoming paper [13].

1794 **Appendix A. Some refinements in the case of scalar controls.**

1795 In [9], the block moment method was introduced to solve null controllability  
1796 problems with scalar controls ( $U = \mathbb{R}$ ). With respect to block moment problems, the  
1797 main result of this paper is [9, Theorem 4.1]. In this work there were no assumptions  
1798 on the counting function. The spectrum  $\Lambda$  was only assumed to satisfy  $\Lambda \subset [1, +\infty)$   
1799 and

1800 
$$\sum_{\lambda \in \Lambda} \frac{1}{\lambda} < +\infty.$$

1801 Using the slightly more restrictive condition (2.9) on the asymptotics of the count-  
1802 ing function we allow the eigenvalues to be complex valued and we obtain sharper

1803 estimates together with the explicit dependency of the constants with respect to the  
1804 final time  $T$  (see Remark 2.8 for possible applications of such estimates). This im-  
1805 proved resolution of scalar block moment problems reads as follow and is proved in [12,  
1806 Theorem V.4.26].

1807 THEOREM A.1. *Let  $p \in \mathbb{N}^*$ ,  $\varrho, \tau, \kappa > 0$  and  $\theta \in (0, 1)$ . Assume that*

$$1808 \quad \Lambda \in \mathcal{L}_w(p, \varrho, \tau, \theta, \kappa).$$

1809 *Let  $G = \{\lambda_1, \dots, \lambda_g\} \subset \Lambda$  be a group satisfying (2.6)–(2.8). Let  $T \in (0, +\infty)$  and*  
1810  *$\eta \in \mathbb{N}^*$ . For any multi-index  $\alpha \in \mathbb{N}^g$  with  $|\alpha|_\infty \leq \eta$  and any*

$$1811 \quad \omega = (\omega_1^0, \dots, \omega_1^{\alpha_1-1}, \dots, \omega_g^0, \dots, \omega_g^{\alpha_g-1}) \in \mathbb{C}^{|\alpha|},$$

1812 *there exists  $v_G \in L^2(0, T; \mathbb{C})$  satisfying*

$$1813 \quad (\text{A.1a}) \quad \int_0^T v_G(t) \frac{(-t)^l}{l!} e^{-\bar{\lambda}_j t} dt = \omega_j^l, \quad \forall j \in \llbracket 1, g \rrbracket, \forall l \in \llbracket 0, \alpha_j \rrbracket,$$

$$1814 \quad (\text{A.1b}) \quad \int_0^T v_G(t) \frac{(-t)^l}{l!} e^{-\bar{\lambda} t} dt = 0, \quad \forall \lambda \in \Lambda \setminus G, \forall l \in \llbracket 0, \eta \rrbracket.$$

1815  
1816 *The solution  $v_G$  satisfies the following estimate*

$$1817 \quad (\text{A.2}) \quad \|v_G\|_{L^2(0, T; U)} \leq C \exp\left(\frac{C}{T^{1-\theta}}\right) \exp(Cr_G^\theta) \max_{\substack{\mu \in \mathbb{N}^g \\ \mu \leq \alpha}} \left| \omega \left[ \bar{\lambda}_\bullet^{(\mu)} \right] \right|,$$

1818 *where  $r_G$  is defined in (2.16) and with the convention*

$$1819 \quad \omega \left[ \bar{\lambda}_j^{(l+1)} \right] = \omega_j^l, \quad \forall j \in \llbracket 1, g \rrbracket, \forall l \in \llbracket 0, \alpha_j \rrbracket.$$

1820 *The constant  $C > 0$  appearing in the estimate only depends on the parameters  $\tau, p,$   
1821  $\varrho, \eta, \theta$  and  $\kappa$ .*

1822 *Moreover, there exists a constant  $C_{p, \eta, r_\Lambda} > 0$  such that any  $v_G \in L^2(0, T; U)$   
1823 *solution of (A.1a) satisfy**

$$1824 \quad (\text{A.3}) \quad \|v_G\|_{L^2(0, T; \mathbb{C})} \geq C_{p, \eta, r_\Lambda} \max_{\substack{\mu \in \mathbb{N}^g \\ \mu \leq \alpha}} \left| \omega \left[ \bar{\lambda}_\bullet^{(\mu)} \right] \right|.$$

1825 REMARK A.1. *If every assumption hold except (2.10) in the definition of the class*  
1826  *$\mathcal{L}_w(p, \varrho, \tau, \theta, \kappa)$ , Theorem A.1 remains valid replacing  $\theta$  in estimate (A.2) by any*  
1827  *$\theta' \in (\theta, 1)$  (see [12, Theorem V.4.26]).*

1828 *Since every estimate on the resolution of block moment problems proved in this*  
1829 *paper follows from (A.2), this remark holds in the whole current paper. Notably it*  
1830 *applies to Theorem 2.4 and to the estimates of the cost of controllability stated in*  
1831 *Proposition 2.11 and Corollary 2.12.*

## 1832 Appendix B. An auxiliary optimization argument.

1833 LEMMA B.1. *Let  $Y$  be a closed subspace of  $X_{-\diamond}$ . Let  $g \in \mathbb{N}^*$  and  $\psi_1, \dots, \psi_g \in$   
1834  $P_Y^* X_\diamond^*$ . For any  $y \in Y$ , let*

$$1835 \quad \xi_y = \begin{pmatrix} \langle y, \psi_1 \rangle_{-\diamond, \diamond} \\ \vdots \\ \langle y, \psi_g \rangle_{-\diamond, \diamond} \end{pmatrix}.$$

1836 Then, for any positive semi-definite hermitian square matrix  $M \in \mathcal{M}_g(\mathbb{C})$ , we have

1837 (B.1) 
$$\sup_{\substack{y \in Y \\ \|y\|_{-\diamond} = 1}} \langle M\xi_y, \xi_y \rangle = \rho(\mathbf{G}_\psi M)$$

1838 with  $\mathbf{G}_\psi = \text{Gram}_{X_\diamond^*}(\psi_1, \dots, \psi_g)$ .

In the course of the proof we will use that there exists an isometric linear bijection  $I : X_{-\diamond} \mapsto X_\diamond^*$  such that

$$\langle y, \varphi \rangle_{-\diamond, \diamond} = (Iy, \varphi)_{\diamond^*}, \quad \forall y \in X_{-\diamond}, \forall \varphi \in X_\diamond^*.$$

Note that it satisfies

$$(Iy, \varphi)_{\diamond^*} = (y, I^{-1}\varphi)_{-\diamond}, \quad \forall y \in X_{-\diamond}, \forall \varphi \in X_\diamond^*.$$

*Proof.* Let  $S$  be the value of the supremum in the left-hand side of (B.1). By assumption on the  $(\psi_i)_i$ , we first observe that the supremum can be taken on the whole space  $X_{-\diamond}$  instead of  $Y$  without changing its value. Then, for any  $1 \leq i \leq g$ , we have

$$\langle y, \psi_i \rangle_{-\diamond, \diamond} = (y, I^{-1}\psi_i)_{-\diamond},$$

and therefore the value of  $S$  does not change if we take the supremum over the set

$$\tilde{\Psi} = \text{Span}(\tilde{\psi}_1, \dots, \tilde{\psi}_g) \subset X_{-\diamond},$$

1839 with

1840 (B.2) 
$$\tilde{\psi}_i = I^{-1}\psi_i.$$

We write any element  $y \in \tilde{\Psi}$  as follows  $y = \sum_{i=1}^g x_i \tilde{\psi}_i$ , with  $x = (x_j)_{j \in \llbracket 1, g \rrbracket} \in \mathbb{C}^g$  so that we can compute

$$\left( y, \tilde{\psi}_i \right)_{-\diamond} = \sum_{j=1}^g x_j \left( \tilde{\psi}_j, \tilde{\psi}_i \right)_{-\diamond} = (\mathbf{G}_{\tilde{\psi}} x)_i, \quad \forall i \in \llbracket 1, g \rrbracket,$$

$$(y, y)_{-\diamond} = \sum_{i=1}^g \sum_{j=1}^g \bar{x}_i x_j \left( \tilde{\psi}_j, \tilde{\psi}_i \right)_{-\diamond} = \left\langle \mathbf{G}_{\tilde{\psi}} x, x \right\rangle,$$

where  $\mathbf{G}_{\tilde{\psi}}$  is the Gram matrix in  $X_{-\diamond}$  of the family  $\{\tilde{\psi}_1, \dots, \tilde{\psi}_g\}$ . Using that  $I$  is an isometry from  $X_{-\diamond}$  onto  $X_\diamond^*$  it actually appears that

$$\mathbf{G}_{\tilde{\psi}} = \mathbf{G}_\psi.$$

Finally, we have proved that

$$\xi_y = \mathbf{G}_\psi x, \quad \text{and} \quad \|y\|_{-\diamond}^2 = \langle \mathbf{G}_\psi x, x \rangle.$$

The supremum we are looking for thus reads

$$S = \sup_{\substack{x \in \mathbb{C}^g \\ \langle \mathbf{G}_\psi x, x \rangle = 1}} \langle M \mathbf{G}_\psi x, \mathbf{G}_\psi x \rangle.$$

- By compactness, we know that this supremum is actually achieved at some point  $x_0 \in \mathbb{C}^g$ , that is

$$\langle MG_\psi x_0, G_\psi x_0 \rangle = S, \quad \text{and} \quad \langle G_\psi x_0, x_0 \rangle = 1.$$

1841 The Lagrange multiplier theorem gives that there exists  $\lambda \in \mathbb{C}$  such that

$$1842 \quad (\text{B.3}) \quad \langle MG_\psi x_0, G_\psi h \rangle = \lambda \langle G_\psi x_0, h \rangle, \quad \forall h \in \mathbb{C}^g.$$

Taking  $h = x_0$  in this equation, we get

$$\langle MG_\psi x_0, G_\psi x_0 \rangle = \lambda \langle G_\psi x_0, x_0 \rangle = \lambda,$$

1843 and thus  $\lambda = S$ , in particular  $\lambda$  is a non negative real number.

From (B.3), we deduce

$$G_\psi M G_\psi x_0 = \lambda G_\psi x_0.$$

and since  $G_\psi x_0 \neq 0$  (we recall that  $\langle G_\psi x_0, x_0 \rangle = 1$ ), we conclude that  $\lambda$  is an eigenvalue of  $G_\psi M$  and therefore

$$S = \lambda \leq \rho(G_\psi M).$$

We have thus proved that

$$S \leq \rho(G_\psi M).$$

- If  $\rho(G_\psi M) = 0$ , the claim is proved. If not, we set

$$\lambda = \rho(G_\psi M) = \rho(MG_\psi) = \rho\left(G_\psi^{\frac{1}{2}} M G_\psi^{\frac{1}{2}}\right),$$

which is a positive number which is an eigenvalue of the three matrices above. In particular, there exists  $x_0 \in \mathbb{C}^g \setminus \{0\}$  such that

$$M G_\psi x_0 = \lambda x_0.$$

Taking the inner product with  $G_\psi x_0$  we obtain

$$\langle M G_\psi x_0, G_\psi x_0 \rangle = \lambda \langle x_0, G_\psi x_0 \rangle,$$

and since  $\langle x_0, G_\psi x_0 \rangle = \left\| G_\psi^{\frac{1}{2}} x_0 \right\|^2$  cannot be equal to zero, we deduce that

$$\lambda \leq S,$$

1844 and the proof is complete.  $\square$

### 1845 **Appendix C. Solving general block moment problems.**

1846 As this paper is oriented towards control theory we do not deal with the most  
 1847 general block moment problems. Indeed, in Theorem 2.4, the considered block mo-  
 1848 ment problems have a specific right-hand side which is a linear form. This formalism  
 1849 is chosen in order to avoid exhibiting a particular basis of the generalized eigenspaces.  
 1850 The price to pay is this restriction on the considered right-hand sides. However the  
 1851 proofs detailed in Sections 3 and 5 directly lead to the following more general results.

1852 The study with a group composed of geometrically simple eigenvalues (see Sec-  
 1853 tions 5.1 and 5.2) leads to the following theorem.

1854 THEOREM C.1. Let  $p \in \mathbb{N}^*$ ,  $\varrho, \tau, \kappa > 0$  and  $\theta \in (0, 1)$ . Assume that

1855 
$$\Lambda \in \mathcal{L}_w(p, \varrho, \tau, \theta, \kappa).$$

1856 Recall that this class of sequences is defined in (2.11). Let  $G = \{\lambda_1, \dots, \lambda_g\} \subset \Lambda$  be  
 1857 a group satisfying (2.6)–(2.8). Let  $T \in (0, +\infty)$  and  $\eta \in \mathbb{N}^*$ . For any multi-index  
 1858  $\alpha \in \mathbb{N}^g$  with  $|\alpha|_\infty \leq \eta$ , any

1859 
$$\omega = (\omega_1^0, \dots, \omega_1^{\alpha_1-1}, \dots, \omega_g^0, \dots, \omega_g^{\alpha_g-1}) \in \mathbb{C}^{|\alpha|},$$

1860 and any  $b \in U^{|\alpha|}$  with

1861 
$$b_j^0 \neq 0, \quad \forall j \in \llbracket 1, g \rrbracket,$$

1862 there exists  $v_G \in L^2(0, T; U)$  satisfying

1863 (C.1a) 
$$\int_0^T \left\langle v_G(t), (e_t b) \left[ \bar{\lambda}_j^{(l+1)} \right] \right\rangle_U dt = \omega_j^l, \quad \forall j \in \llbracket 1, g \rrbracket, \forall l \in \llbracket 0, \alpha_j \rrbracket,$$

1864 (C.1b) 
$$\int_0^T v_G(t) t^l e^{-\bar{\lambda} t} dt = 0, \quad \forall \lambda \in \Lambda \setminus G, \forall l \in \llbracket 0, \eta \rrbracket.$$

1866 The solution  $v_G$  satisfies the following estimate

1867 
$$\|v_G\|_{L^2(0, T; U)} \leq C \exp\left(\frac{C}{T^{1-\theta}}\right) \exp(Cr_G^\theta) \langle M^{-1} \xi, \xi \rangle,$$

1868 where

1869 
$$\xi := \begin{pmatrix} \omega \left[ \bar{\lambda}_\bullet^{(\mu^1)} \right] \\ \vdots \\ \omega \left[ \bar{\lambda}_\bullet^{(\mu^{|\alpha|})} \right] \end{pmatrix},$$

1870 the sequence  $(\mu^p)_{p \in \llbracket 0, |\alpha| \rrbracket}$  is defined in (2.24), the associated matrix  $M$  is defined  
 1871 in (2.33),  $r_G$  is defined in (2.16) and with the convention

1872 
$$\omega \left[ \bar{\lambda}_j^{(l+1)} \right] = \omega_j^l, \quad \forall j \in \llbracket 1, g \rrbracket, \forall l \in \llbracket 0, \alpha_j \rrbracket.$$

1873 The constant  $C > 0$  appearing in the estimate only depends on the parameters  $\tau, p,$   
 1874  $\varrho, \eta, \theta$  and  $\kappa$ .

1875 Moreover, there exists a constant  $C_{p, \eta, r_\Lambda} > 0$  such that any  $v_G \in L^2(0, T; U)$   
 1876 solution of (C.1a) satisfy

1877 
$$\|v_G\|_{L^2(0, T; U)} \geq C_{p, \eta, r_\Lambda} \langle M^{-1} \xi, \xi \rangle.$$

1878 REMARK C.1. As detailed in Remark 2.5, when the eigenvalues in  $G$  are also  
 1879 algebraically simple, i.e.  $\alpha_\lambda = \gamma_\lambda = 1$  for any  $\lambda \in G$ , the expression of  $\xi$  reduces to

1880 
$$\xi := \begin{pmatrix} \omega \left[ \bar{\lambda}_1 \right] \\ \vdots \\ \omega \left[ \bar{\lambda}_1, \dots, \bar{\lambda}_g \right] \end{pmatrix},$$

1881 and the expression of  $M$  reduces to the one given in (2.35).

1882 The study with a group composed of semi-simple eigenvalues (see Section 5.3)  
 1883 leads to the following theorem.

1884 THEOREM C.2. Let  $p \in \mathbb{N}^*$ ,  $\varrho, \tau, \kappa > 0$  and  $\theta \in (0, 1)$ . Assume that

1885 
$$\Lambda \in \mathcal{L}_w(p, \varrho, \tau, \theta, \kappa).$$

1886 Recall that this class of sequences is defined in (2.11). Let  $G = \{\lambda_1, \dots, \lambda_g\} \subset \Lambda$  be a  
 1887 group satisfying (2.6)–(2.8). Let  $\gamma_1, \dots, \gamma_g \in \mathbb{N}^*$  and  $\gamma_G = \gamma_1 + \dots + \gamma_g$ . Let  $\eta \in \mathbb{N}^*$   
 1888 and  $T \in (0, +\infty)$ .

1889 For any  $(\omega_{j,i})_{j \in \llbracket 1, g \rrbracket, i \in \llbracket 1, \gamma_j \rrbracket} \in \mathbb{C}^{\gamma_G}$  and any  $(b_{j,i})_{j \in \llbracket 1, g \rrbracket, i \in \llbracket 1, \gamma_j \rrbracket} \in U^{\gamma_G}$  such  
 1890 that  $b_{j,1}, \dots, b_{j,\gamma_j}$  are linearly independent for every  $j \in \llbracket 1, g \rrbracket$ , there exists  $v_G \in$   
 1891  $L^2(0, T; U)$  satisfying

1892 (C.2a) 
$$\int_0^T \langle v_G(t), e^{-\bar{\lambda}_j t} b_{j,i} \rangle_U dt = \omega_{j,i}, \quad \forall j \in \llbracket 1, g \rrbracket, \forall i \in \llbracket 1, \gamma_j \rrbracket,$$

1893 (C.2b) 
$$\int_0^T v_G(t) t^l e^{-\bar{\lambda} t} dt = 0, \quad \forall \lambda \in \Lambda \setminus G, \forall l \in \llbracket 0, \eta \rrbracket.$$
  
 1894

1895 The solution  $v_G$  satisfies the following estimate

1896 
$$\|v_G\|_{L^2(0, T; U)} \leq C \exp\left(\frac{C}{T^{1-\theta}}\right) \exp(Cr_G^\theta) \langle M^{-1}\xi, \xi \rangle,$$

1897 where  $\xi \in \mathbb{C}^{\gamma_G}$  is defined by blocks with

1898 
$$\xi_j := \begin{pmatrix} \omega_{j,1} \\ \vdots \\ \omega_{j,g} \end{pmatrix},$$

1899 the associated matrix  $M$  is defined in (2.37) and  $r_G$  is defined in (2.16). The constant  
 1900  $C > 0$  appearing in the estimate only depends on the parameters  $\tau, p, \varrho, \eta, \theta$  and  $\kappa$ .

1901 Moreover, there exists a constant  $C_{p,\eta,r_\Lambda} > 0$  such that any  $v_G \in L^2(0, T; U)$   
 1902 solution of (C.2a) satisfy

1903 
$$\|v_G\|_{L^2(0, T; U)} \geq C_{p,\eta,r_\Lambda} \langle M^{-1}\xi, \xi \rangle.$$

1904 **Appendix D. Post-processing formulas.**

1905 The minimal null control time given in Theorem 2.5, together with the compu-  
 1906 tation of the contribution of each group given in Theorems 2.8 and 2.10, allow to  
 1907 answer the question of minimal null control time for a wide variety of one dimensional  
 1908 parabolic control problems. However, for a given problem, the precise estimate of the  
 1909 quantity of interest  $\langle M^{-1}\xi, \xi \rangle$  can remain a tricky question.

1910 There is no normalization condition on the eigenvectors and no uniqueness of  
 1911 the considered Jordan chains. Thus, it happens that there are choices for which the  
 1912 quantity of interest  $\langle M^{-1}\xi, \xi \rangle$  is easier to compute (see for instance Remark 2.4). We  
 1913 gather here some results that are used in Sections 6 and 7 to estimate such quantities.

1914 We will make an intensive use of the following reformulation. Let  $n \in \mathbb{N}^*$  and let  
 1915  $T, M \in \text{GL}_n(\mathbb{C})$ . For any  $\xi \in \mathbb{C}^n$ , let  $\tilde{\xi} := T\xi$ . Then,

1916 (D.1) 
$$\langle M^{-1}\xi, \xi \rangle = \langle M^{-1}T^{-1}\tilde{\xi}, T^{-1}\tilde{\xi} \rangle = \langle \widetilde{M}^{-1}\tilde{\xi}, \tilde{\xi} \rangle$$

1917 where

$$1918 \quad (D.2) \quad \widetilde{M} := TMT^*.$$

1919 As the matrix  $M$  is a sum of Gram matrices we will also use the following lemma.

1920 LEMMA D.1. *Let  $X$  be a Hilbert space. Let  $n \in \mathbb{N}^*$  and  $e = (e_1, \dots, e_n) \in X^n$ .*  
 1921 *Let  $T \in \mathcal{M}_n(\mathbb{C})$ . Then,*

$$1922 \quad T\text{Gram}_X(e_1, \dots, e_n)T^* = \text{Gram}_X((\overline{T}e)_1, \dots, (\overline{T}e)_n)$$

1923 where, for any  $i \in \llbracket 1, n \rrbracket$ ,  $(\overline{T}e)_i$  is defined by

$$1924 \quad (\overline{T}e)_i := \sum_{j=1}^n \overline{T_{i,j}} e_j.$$

1925 *Proof.* For any  $\omega \in \mathbb{C}^n$ , it comes that

$$1926 \quad (D.3) \quad \langle T\text{Gram}_X(e_1, \dots, e_n)T^*\omega, \omega \rangle = \langle \text{Gram}_X(e_1, \dots, e_n)(T^*\omega), (T^*\omega) \rangle$$

$$1927 \quad (D.4) \quad = \left\| \sum_{i=1}^n (T^*\omega)_i e_i \right\|^2$$

$$1928 \quad (D.5) \quad = \left\| \sum_{i=1}^n \sum_{j=1}^n \overline{T_{j,i}} \omega_j e_i \right\|^2$$

$$1929 \quad (D.6) \quad = \left\| \sum_{j=1}^n \omega_j (\overline{T}e)_j \right\|^2$$

$$1930 \quad (D.7) \quad = \langle \text{Gram}_X((\overline{T}e)_1, \dots, (\overline{T}e)_n)\omega, \omega \rangle. \quad \square$$

1932 Depending on the phenomenon at stake on actual examples, with a suitable choice of  
 1933  $\tilde{\xi}$  (i.e. of  $T$ ), the quantity  $\langle \widetilde{M}^{-1}\tilde{\xi}, \tilde{\xi} \rangle$  can be easier to estimate than  $\langle M^{-1}\xi, \xi \rangle$ .

#### 1934 D.1. Dilatations.

1935 Notice that

$$1936 \quad \langle \widetilde{M}^{-1}\tilde{\xi}, \tilde{\xi} \rangle \leq \|\widetilde{M}^{-1}\| \|\tilde{\xi}\|^2.$$

1937 When the minimal null control time can be estimated with rough estimates (this can  
 1938 only characterize the minimal time when  $T_0 = 0$ ), it can simplify the computations  
 1939 to have a bounded  $\|\tilde{\xi}\|$ . To do so, it is convenient to consider dilatations of  $\xi$ .

1940 Let  $X$  be a Hilbert space. Let  $n \in \mathbb{N}^*$  and  $e_1, \dots, e_n \in X$ . Let  $\xi \in \mathbb{C}^n$  and  $\beta \in \mathbb{C}^n$   
 1941 with non-zero entries. Let

$$1942 \quad T = D_\beta := \text{diag}(\beta) \in \text{GL}_n(\mathbb{C}), \quad \text{and} \quad \tilde{\xi} = T\xi.$$

1943 Then, from Lemma D.1, it comes that

$$1944 \quad T\text{Gram}_X(e_1, \dots, e_n)T^* = \text{Gram}_X(\overline{\beta}_1 e_1, \dots, \overline{\beta}_n e_n).$$

1945 **D.2. Invariance by scale change.**

1946 In our assumptions there is no normalization condition on the eigenvectors (see  
 1947 Remark 2.4). This allows to have simpler expressions for these eigenvectors. Actually,  
 1948 the computation of  $\langle M^{-1}\xi, \xi \rangle$  can be done with a different scale change on every  
 1949 generalized eigenvector as detailed in the following proposition.

1950 PROPOSITION D.2. *Let  $M$  and  $\xi$  be as defined in Theorem 2.8. Let  $\beta \in \mathbb{C}^{|\alpha|}$  be  
 1951 such that  $\beta_j^0 \neq 0$  for all  $j \in \llbracket 1, g \rrbracket$ . Set*

$$1952 \quad \tilde{\xi} = \begin{pmatrix} \langle y_0, (\beta\phi) [\lambda^{\mu^1}] \rangle_{-\diamond, \diamond} \\ \vdots \\ \langle y_0, (\beta\phi) [\lambda^{\mu^{|\alpha|}}] \rangle_{-\diamond, \diamond} \end{pmatrix}$$

1953 Then,

$$1954 \quad \langle M^{-1}\xi, \xi \rangle = \langle \widetilde{M}^{-1}\tilde{\xi}, \tilde{\xi} \rangle$$

1955 where

$$1956 \quad (\text{D.8}) \quad \widetilde{M} := \sum_{l=1}^{|\alpha|} \text{Gram}_U \left( \underbrace{0, \dots, 0}_{l-1}, (\beta b) [\lambda^{\mu^l - \mu^{l-1}}], \dots, (\beta b) [\lambda^{\mu^{|\alpha|} - \mu^{l-1}}] \right).$$

1957 *Proof.* From Leibniz formula [9, Proposition 7.13], it comes that for any  $p \in$   
 1958  $\llbracket 1, |\alpha| \rrbracket$ ,

$$1959 \quad (\beta\phi) [\lambda^{\mu^p}] = \sum_{q=1}^{\mu^p} \beta [\lambda^{\mu^p - \mu^{q-1}}] \phi [\lambda^{\mu^q}].$$

1960 Thus,  $\tilde{\xi} = T\xi$  where  $T$  is the following lower triangular matrix

$$1961 \quad T = \left( \mathbf{1}_{q \leq p} \overline{\beta} [\lambda^{\mu^p - \mu^{q-1}}] \right)_{p, q \in \llbracket 1, |\alpha| \rrbracket}.$$

1962 The diagonal entries of this lower triangular matrix are  $\overline{\beta}_j^0$  and thus  $T \in \text{GL}_{|\alpha|}(\mathbb{C})$ .  
 1963 From (D.2), the associated matrix is

$$1964 \quad \widetilde{M} := \sum_{l=1}^{|\alpha|} T \text{Gram}_U \left( \underbrace{0, \dots, 0}_{l-1}, b [\lambda^{\mu^l - \mu^{l-1}}], \dots, b [\lambda^{\mu^{|\alpha|} - \mu^{l-1}}] \right) T^*.$$

1965 Let  $l \in \llbracket 1, |\alpha| \rrbracket$  and

$$1966 \quad e_1 = \dots = e_{l-1} = 0,$$

$$1967 \quad e_p = b [\lambda^{\mu^p - \mu^{l-1}}], \quad \forall p \in \llbracket l, |\alpha| \rrbracket.$$

1968

1969 Then, for any  $p \in \llbracket 1, |\alpha| \rrbracket$ ,

$$1970 \quad (\overline{T}e)_p = \sum_{q=1}^{|\alpha|} \mathbf{1}_{q \leq p} \beta [\lambda^{\mu^p - \mu^{q-1}}] e_q.$$

1971 Thus,  $(\bar{T}e)_1 = \dots = (\bar{T}e)_{l-1} = 0$  and, for any  $p \in \llbracket l, |\alpha| \rrbracket$ ,

$$1972 \quad (\bar{T}e)_p = \sum_{q=1}^{|\alpha|} \mathbf{1}_{q \leq p} \beta \left[ \lambda^{(\mu^p - \mu^{q-1})} \right] e_q = \sum_{q=l}^p \beta \left[ \lambda^{(\mu^p - \mu^{q-1})} \right] b \left[ \lambda^{(\mu^q - \mu^{l-1})} \right].$$

1973 Then, using again Leibniz formula [9, Proposition 7.13], we obtain

$$1974 \quad (\bar{T}e)_p = (\beta b) \left[ \lambda^{(\mu^p - \mu^{l-1})} \right].$$

1975 Finally, applying (D.1) and Lemma D.1 ends the proof of Proposition D.2.  $\square$

1976 **REMARK D.1.** *As there is no normalization condition on the eigenvectors a sim-*  
 1977 *ilar statement automatically holds with  $M$  and  $\xi$  defined in Theorem 2.10.*

### 1978 **D.3. An equivalent formula for simple eigenvalues.**

1979 In this section, we consider the case of a group of simple eigenvalues *i.e.*  $\alpha_\lambda =$   
 1980  $\gamma_\lambda = 1$  for every  $\lambda \in G$ . In that case, the cost of the group  $G$  can be computed either  
 1981 using the formula of Theorem 2.8 for geometrically simple eigenvalues or the formula  
 1982 of Theorem 2.10 for semi-simple eigenvalues. Even though these theorems imply that  
 1983 those two formulas coincide (as they are both the cost of the group) we give a direct  
 1984 proof of this statement.

1985 **PROPOSITION D.3.** *Let  $M$  and  $\xi$  be the matrix and the vector given in Theorem 2.8*  
 1986 *i.e.*

$$1987 \quad M := \sum_{l=1}^g \text{Gram}_U \left( \underbrace{0, \dots, 0}_{l-1}, b[\lambda_l], \dots, b[\lambda_l, \dots, \lambda_g] \right)$$

1988 *and*

$$1989 \quad \xi = \begin{pmatrix} \langle y_0, \phi[\lambda_1] \rangle_{-\diamond, \diamond} \\ \vdots \\ \langle y_0, \phi[\lambda_1, \dots, \lambda_g] \rangle_{-\diamond, \diamond} \end{pmatrix}.$$

1990 *Let  $\widetilde{M}$  and  $\widetilde{\xi}$  be the matrix and the vector given in Theorem 2.10 i.e.*

$$1991 \quad (\text{D.9}) \quad \widetilde{M} := \sum_{l=1}^g \text{Gram}_U (\delta_l^1 b[\lambda_1], \dots, \delta_l^g b[\lambda_g]) \quad \text{and} \quad \widetilde{\xi} := \begin{pmatrix} \langle y_0, \phi[\lambda_1] \rangle_{-\diamond, \diamond} \\ \vdots \\ \langle y_0, \phi[\lambda_g] \rangle_{-\diamond, \diamond} \end{pmatrix}.$$

1992 *Then,*

$$1993 \quad \langle M^{-1} \xi, \xi \rangle = \langle \widetilde{M}^{-1} \widetilde{\xi}, \widetilde{\xi} \rangle$$

1994 *Proof.* The usual interpolation formula [9, Proposition 7.6] gives

$$1995 \quad (\text{D.10}) \quad \phi[\lambda_i] = \sum_{j=1}^i \left( \prod_{k=1}^{j-1} (\lambda_i - \lambda_k) \right) \phi[\lambda_1, \dots, \lambda_j].$$

1996 Recall that the notation  $\delta_j^i$  has been introduced in (2.36). With these notations,  
 1997  $\tilde{\xi} = T\xi$  where  $T$  is the following lower triangular matrix

1998 
$$T = \left( \overline{\delta_j^i} \right)_{i,j \in \llbracket 1, g \rrbracket} \in \text{GL}_g(\mathbb{C}).$$

1999 From (D.2), we define

2000 
$$\widehat{M} := \sum_{l=1}^g T \text{Gram}_U \left( \underbrace{0, \dots, 0}_{l-1}, b[\lambda_l], \dots, b[\lambda_l, \dots, \lambda_g] \right) T^*,$$

2001 so that we have  $\langle M^{-1}\xi, \xi \rangle = \langle \widehat{M}^{-1}\tilde{\xi}, \tilde{\xi} \rangle$ . We will now prove that  $\widehat{M} = \widetilde{M}$ .

2002 Let  $l \in \llbracket 1, g \rrbracket$  and

2003 
$$e_1 = \dots = e_{l-1} = 0,$$
  
 2004 
$$e_j = b[\lambda_l, \dots, \lambda_j], \quad \forall j \in \llbracket l, g \rrbracket.$$

2006 Then,  $(\overline{T}e)_1 = \dots = (\overline{T}e)_{l-1} = 0$  and for  $i \in \llbracket l, g \rrbracket$ , using again the interpolation  
 2007 property [9, Proposition 7.6], we obtain

2008 
$$\begin{aligned} (\overline{T}e)_i &= \sum_{j=l}^g \delta_j^i b[\lambda_l, \dots, \lambda_j] \\ 2009 &= \sum_{j=l}^i \delta_j^i b[\lambda_l, \dots, \lambda_j] \\ 2010 &= \delta_l^i \sum_{j=l}^i \left( \prod_{k=l}^{j-1} (\lambda_i - \lambda_k) \right) b[\lambda_l, \dots, \lambda_j] \\ 2011 &= \delta_l^i b[\lambda_i]. \end{aligned}$$

2013 Recalling that  $\delta_l^1 = \dots = \delta_l^{l-1} = 0$ , we thus obtain

2014 
$$(\overline{T}e)_i = \delta_l^i b[\lambda_i], \quad \forall i \in \llbracket 1, g \rrbracket.$$

2015 Finally, from Lemma D.1, we deduce that  $\widehat{M} = \widetilde{M}$  which ends the proof of Proposi-  
 2016 tion D.3.  $\square$

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