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1 **ANALYSIS OF NON SCALAR CONTROL PROBLEMS FOR**
2 **PARABOLIC SYSTEMS BY THE BLOCK MOMENT METHOD**

3 FRANCK BOYER* AND MORGAN MORANCEY†

4 **Abstract.** This article deals with abstract linear time invariant controlled systems of parabolic
5 type. In [*Annales Henri Lebesgue*, **3** (2020), pp. 717–793], with A. Benabdallah, we introduced the
6 block moment method for scalar control operators. The principal aim of this method is to compute
7 the minimal time needed to drive an initial condition (or a space of initial conditions) to zero, in
8 particular in the case when spectral condensation occurs. The purpose of the present article is to
9 push forward the analysis to deal with any admissible control operator. The considered setting leads
10 to applications to one dimensional parabolic-type equations or coupled systems of such equations.

11 With such admissible control operator, the characterization of the minimal null control time is
12 obtained thanks to the resolution of an auxiliary vectorial block moment problem (*i.e.* set in the
13 control space) followed by a constrained optimization procedure of the cost of this resolution. This
14 leads to essentially sharp estimates on the resolution of the block moment problems which are uniform
15 with respect to the spectrum of the evolution operator in a certain class. This uniformity allows the
16 study of uniform controllability for various parameter dependent problems. We also deduce estimates
17 on the cost of controllability when the final time goes to the minimal null control time.

18 We illustrate how the method works on a few examples of such abstract controlled systems and
19 then we deal with actual coupled systems of one dimensional parabolic partial differential equations.
20 Our strategy enables us to tackle controllability issues that seem out of reach by existing techniques.

21 **Key words.** Control theory, parabolic partial differential equations, minimal null control time,
22 block moment method

23 **AMS subject classifications.** 93B05, 93C20, 93C25, 30E05, 35K90, 35P10

24 **1. Introduction.**

25 **1.1. Problem under study and state of the art.**

26 In this paper we study the controllability properties of the following linear control
27 system

28 (1.1)
$$\begin{cases} y'(t) + \mathcal{A}y(t) = \mathcal{B}u(t), \\ y(0) = y_0. \end{cases}$$

29 The assumptions on the operator \mathcal{A} (see Section 1.4) will lead to applications to
30 linear parabolic-type equations or coupled systems of such equations mostly in the
31 one dimensional setting. In all this article the Hilbert space of control will be denoted
32 by U and the operator \mathcal{B} will be a general admissible operator.

33 The question we address is the characterization of the minimal null control time
34 (possibly zero or infinite) from y_0 that is: for a given initial condition y_0 , what
35 is the minimal time $T_0(y_0)$ such that, for any $T > T_0(y_0)$, there exists a control
36 $u \in L^2(0, T; U)$ such that the associated solution of (1.1) satisfies $y(T) = 0$. A
37 more precise definition of the minimal null control time is given in Definition 1.1 in
38 Section 1.4.1.

39 For a presentation of null controllability of parabolic control problems as well as
40 the possible existence of a positive minimal null control time for such equations we

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41 refer to [4] or [9, Section 1.1] and the references therein. Such a positive minimal
 42 null control time is due either to insufficient observation of eigenvectors, or to con-
 43 densation of eigenvalues or to the geometry of generalized eigenspaces, or even to a
 44 combination of all those phenomena. Let us underline that this phenomenon is com-
 45 pletely unrelated to the minimal control time arising from constraints on the state
 46 or on the control as studied for instance in [31], or to the one arising in hyperbolic
 47 problems due to intrinsic finite speed of propagation in the equation.

48 Under the considered assumptions on \mathcal{A} , the problem of characterizing the mini-
 49 mal null control time has been solved for scalar controls ($\dim U = 1$) in [9] where the
 50 *block moment method* has been introduced in that purpose. The aim of the present
 51 article is to push forward the analysis of [9] to extend it to any admissible control
 52 operator. The new difficulties come from the interplay between spectral condensation
 53 phenomena and the particular geometry of the control operator.

54 To present the general ideas, let us assume for simplicity that the operator \mathcal{A}^*
 55 has a sequence of real and positive eigenvalues Λ and that the associated eigenvectors
 56 ϕ_λ , for $\lambda \in \Lambda$, form a complete family of the state space (the precise functional setting
 57 is detailed in Section 1.4). Then, the solution of system (1.1) satisfies $y(T) = 0$ if and
 58 only if the control $u \in L^2(0, T; U)$ solves the following moment problem

$$59 \quad (1.2) \quad \int_0^T e^{-\lambda t} \langle u(T-t), \mathcal{B}^* \phi_\lambda \rangle_U dt = -e^{-\lambda T} \langle y_0, \phi_\lambda \rangle, \quad \forall \lambda \in \Lambda.$$

60 • *Solving moment problems associated with a scalar control operator.*

61 In the scalar case ($U = \mathbb{R}$), provided that $\mathcal{B}^* \phi_\lambda \neq 0$, the moment problem reduces
 62 to

$$63 \quad (1.3) \quad \int_0^T e^{-\lambda t} u(T-t) dt = -e^{-\lambda T} \left\langle y_0, \frac{\phi_\lambda}{\mathcal{B}^* \phi_\lambda} \right\rangle, \quad \forall \lambda \in \Lambda.$$

64 This problem is usually solved by the construction of a biorthogonal family $(q_\lambda)_{\lambda \in \Lambda}$
 65 to the exponentials

$$66 \quad \{t \in (0, T) \mapsto e^{-\lambda t}; \lambda \in \Lambda\}$$

67 in $L^2(0, T; U)$, *i.e.*, a family $(q_\lambda)_{\lambda \in \Lambda}$ such that

$$68 \quad \int_0^T q_\lambda(t) e^{-\mu t} dt = \delta_{\lambda, \mu}, \quad \forall \lambda, \mu \in \Lambda.$$

69 From [36], the existence of such biorthogonal family is equivalent to the summability
 70 condition

$$71 \quad (1.4) \quad \sum_{\lambda \in \Lambda} \frac{1}{\lambda} < +\infty.$$

72 **REMARK 1.1.** *This condition (which will be assumed in the present article) is the*
 73 *main restriction to apply the moment method. Indeed, due to Weyl's law it imposes*
 74 *on many examples of partial differential equations of parabolic-type a restriction to the*
 75 *one dimensional setting. However, in some particular multi-dimensional geometries,*
 76 *the controllability problem can be transformed into a family of parameter dependent*
 77 *moment problems, each of them satisfying such assumption (see for instance [8, 3, 15]*
 78 *among others).*

79 With such a biorthogonal family, a formal solution of the moment problem (1.3)
 80 is given by

$$81 \quad u(T-t) = - \sum_{\lambda \in \Lambda} e^{-\lambda T} \left\langle y_0, \frac{\phi_\lambda}{\mathcal{B}^* \phi_\lambda} \right\rangle q_\lambda(t), \quad t \in (0, T).$$

82 Thus if, for any y_0 , the series defining u converges in $L^2(0, T; U)$ one obtains null
 83 controllability of system (1.1) in time T . To do so, it is crucial to prove upper bounds
 84 on $\|q_\lambda\|_{L^2(0, T)}$.

85 Suitable bounds on such biorthogonal families were provided in the pioneering
 86 work of Fattorini and Russell [21] in the case where the eigenvalues of \mathcal{A}^* are well
 87 separated *i.e.* satisfy the classical gap condition: $\inf \{|\lambda - \mu|; \lambda, \mu \in \Lambda, \lambda \neq \mu\} > 0$.
 88 When the eigenvalues are allowed to condensate we refer to the work [5] for almost
 89 sharp estimates implying the condensation index of the sequence Λ . A discussion on
 90 other references providing estimates on biorthogonal families is detailed below. These
 91 results have provided an optimal characterization of the minimal null control time
 92 when the eigenvectors of \mathcal{A}^* form a Riesz basis of the state space (and thus do not
 93 condensate).

94 However, as analyzed in [9], there are situations in which the eigenvectors also
 95 condensate and for which providing estimates on biorthogonal families is not sufficient
 96 to characterize the minimal null control time. In [9], it is assumed that the spectrum
 97 Λ can be decomposed as a union \mathcal{G} of well separated groups of bounded cardinality.
 98 Then, the control u is sought in the form

$$99 \quad u(T-t) = \sum_{G \in \mathcal{G}} v_G(t),$$

100 where, for any $G \in \mathcal{G}$, the function $v_G \in L^2(0, T; U)$ solves the block moment problem

$$101 \quad (1.5) \quad \begin{cases} \int_0^T e^{-\lambda t} v_G(t) dt = e^{-\lambda T} \left\langle y_0, \frac{\phi_\lambda}{\mathcal{B}^* \phi_\lambda} \right\rangle, & \forall \lambda \in G, \\ \int_0^T e^{-\lambda t} v_G(t) dt = 0, & \forall \lambda \notin G. \end{cases}$$

102 This enables to deal with the condensation of eigenvectors: the eigenvectors $(\phi_\lambda)_{\lambda \in \Lambda}$
 103 are only assumed to form a complete family of the state space.

104 • *Solving moment problems associated with a non scalar control operator.*

105 When the control is not scalar there are less available results in the literature. Here
 106 again, these results rely on the existence of a biorthogonal family to the exponentials
 107 with suitable bounds. For instance, in [6], null controllability in optimal time is proved
 108 using a subtle decomposition of the moment problem into two families of moment
 109 problems. In a more systematic way, one can take advantage of the biorthogonality
 110 in the time variable to seek for a solution u of the moment problem (1.2) in the form

$$111 \quad u(T-t) = - \sum_{\lambda \in \Lambda} e^{-\lambda T} \langle y_0, \phi_\lambda \rangle \frac{\mathcal{B}^* \phi_\lambda}{\|\mathcal{B}^* \phi_\lambda\|_U^2}.$$

112 This strategy was introduced by Lagnese in [25] for a one dimensional wave equation
 113 and used in the parabolic context for instance in [17, 2, 18, 3].

114 In the present article we deal with such general admissible control operators.
 115 As the eigenvectors will only be assumed to form a complete family, for each initial

116 condition y_0 , we study its null control time for system (1.1) by solving block moment
 117 problems of the following form

$$118 \quad (1.6) \quad \begin{cases} \int_0^T \langle V_G(t), e^{-\lambda t} \mathcal{B}^* \phi_\lambda \rangle_U dt = \langle y_0, e^{-\lambda T} \phi_\lambda \rangle, & \forall \lambda \in G, \\ \int_0^T \langle V_G(t), e^{-\lambda t} \mathcal{B}^* \phi_\lambda \rangle_U dt = 0, & \forall \lambda \notin G. \end{cases}$$

119 Let us recall that, for pedagogical purposes, we have restricted this first introductory
 120 subsection to the case of real simple eigenvalues. The general form of block moment
 121 problems under study in this article is detailed in Section 2.1.

122 The strategy to solve such block moment problem and estimate its solution is
 123 presented on an example in Section 1.3. Let us already notice that the geometry of
 124 the finite dimensional space $\text{Span}\{\mathcal{B}^* \phi_\lambda; \lambda \in G\}$ is crucial.

125 For instance, if this space is one dimensional, say generated by some $b \in U$, the
 126 strategy of Lagnese can be adapted if one seeks for V_G solution of the block moment
 127 problem (1.6) in the form

$$128 \quad V_G(t) = v_G(t)b,$$

129 where $v_G \in L^2(0, T; \mathbb{R})$ solves a scalar block moment problem of the same form
 130 as (1.5).

131 If, instead, the family $(\mathcal{B}^* \phi_\lambda)_{\lambda \in G}$ is composed of linearly independent vectors
 132 then it admits a biorthogonal family in U denoted by $(b_\lambda^*)_{\lambda \in G}$. Then, one can for
 133 instance seek for V_G solution of the block moment problem (1.6) in the form

$$134 \quad V_G(t) = v_G(t) \left(\sum_{\lambda \in G} b_\lambda^* \right).$$

135 where v_G solves a scalar block moment problem of the form (1.5). An upper bound
 136 of the minimal control time can then be obtained thanks to an estimate of the family
 137 $(b_\lambda^*)_{\lambda \in G}$, but without guarantee of optimality.

138 In the general setting, taking into account the geometry of the observations of
 139 eigenvectors to solve block moment problems of the form (1.6) is a more intricate
 140 question that we solve in this article, still under the summability condition (1.4).

141 Let us mention that we not only solve block moment problems of the form (1.6)
 142 but we also provide estimates on their solutions to ensure that the series defining the
 143 control converges. These estimates will lead to an optimal characterization of the
 144 minimal null control time for each given problem.

145 We pay particular attention to these estimates so that they do not directly depend
 146 on the sequence Λ but are uniform for classes of such sequences. This is an important
 147 step to tackle uniform controllability for parameter dependent control problems. Esti-
 148 mates of this kind have already proved their efficiency in various contexts such as: nu-
 149 merical analysis of semi-discrete control problems [2], oscillating coefficients [32], anal-
 150 ysis of degenerate control problems with respect to the degeneracy parameter [17, 18],
 151 analysis of higher dimensional controllability problems by reduction to families of one
 152 dimensional control problems [8, 1, 3, 15] or analysis of convergence of Robin-type
 153 controls to Dirichlet controls [11].

154 Another important feature of the estimates we obtain is to track the dependency
 155 with respect to the final time T when T goes to the minimal null control time. As pre-
 156 sented in Remark 2.5, this allows applications in higher dimensions (with a cylindrical
 157 geometry) or applications to nonlinear control problems.

- *An overview of some estimates on biorthogonal families.*

Finally, let us recall some classical results providing estimates for biorthogonal families to a sequence of exponentials.

Under the classical gap condition, uniform estimates for biorthogonal families were already obtained in [22] and sharp short-time estimates were obtained in [8]. In this setting, bounds with a detailed dependency with respect to parameters were given in [19]. In this work, the obtained bounds take into account the fact that the gap property between eigenvalues may be better in high frequencies. Similar results were also obtained in [26].

Under a weak-gap condition of the form (1.23), that is when the eigenvalues can be gathered in blocks of bounded cardinality with a gap between blocks (which is the setting of the present article), uniform estimates on biorthogonal sequences follow from the uniform estimates for the resolution of block moment problems proved in [9]. Similar estimates, but where the sharp dependency with respect to T of the different constants is tracked, were obtained in [24]. Using the strategy detailed in [12], the estimates of [9] can also be supplemented with such dependency with respect to T (see Theorem A.1). Let us mention that similar results were also obtained in [16] with stronger assumptions, namely with a weak-gap assumption on the square roots of the eigenvalues.

In the absence of any gap-type condition, estimates on biorthogonal families were first proved in [5] involving the condensation index and then later in [3] involving a local measure of the gap.

1.2. Structure of the article.

To ease the reading, let us give here the detailed outline of this article.

In Section 1.3 we detail, for a simple example, the obtained results as well as our strategy of proof. This allows to explain the contents of this article without introducing too many notations. Then, in Section 1.4, we detail the framework, assumptions and notations that will be used throughout this article.

The main results concerning the resolution of block moment problems with a non scalar control are stated in Section 2.1. The application of these results to the characterization of the minimal null control time is stated in Section 2.2. We provide in Section 2.3 more explicit formulas to compute the minimal null control time. We also deduce from our study some estimates on the cost of controllability that are given in Section 2.4.

The results concerning the resolution of block moment problems are proved in Section 3. The application of these results to the characterization of the minimal null control time and the study of the cost of null controllability are then proved in Section 4. More explicit formulas for the computation of the minimal null control time are proved in Section 5.

Finally we apply these results to different examples. First we deal in Section 6 with academic examples. For these examples the computations are rather simple and this allows to highlight the different phenomena at stake in this minimal null control time study. We end this article with the analysis of null controllability for systems of coupled linear partial differential equations of parabolic type in Section 7.

1.3. Strategy of proof.

To highlight the ideas we develop in this article (without drowning them in technicalities or notations), let us present our strategy of analysis of null controllability on an abstract simple example.

206 We consider $X = L^2(0, 1; \mathbb{R})^2$ and $\omega \subset (0, 1)$ a non empty open set. For a given
 207 $a > 0$ we define

$$208 \quad \Lambda = \left\{ \lambda_{k,1} := k^2, \lambda_{k,2} := k^2 + e^{-ak^2}; k \geq 1 \right\},$$

209 and take $(\varphi_k)_{k \geq 1}$ an Hilbert basis of X such that

$$210 \quad \inf_{k \geq 1} \|\varphi_k\|_{L^2(\omega)} > 0.$$

211 Let $\phi_{k,1} := \begin{pmatrix} \varphi_k \\ \varphi_k \end{pmatrix}$ and $\phi_{k,2} := \begin{pmatrix} 0 \\ \varphi_k \end{pmatrix}$. We define the operator \mathcal{A}^* in X by

$$212 \quad \mathcal{A}^* \phi_{k,1} = \lambda_{k,1} \phi_{k,1}, \quad \mathcal{A}^* \phi_{k,2} = \lambda_{k,2} \phi_{k,2},$$

213 with

$$214 \quad \mathcal{D}(\mathcal{A}^*) = \left\{ \sum_{k \geq 1} a_{k,1} \phi_{k,1} + a_{k,2} \phi_{k,2}; \sum_{k \geq 1} \lambda_{k,1}^2 a_{k,1}^2 + \lambda_{k,2}^2 a_{k,2}^2 < +\infty \right\}.$$

215 The control operator \mathcal{B} is defined by $U = L^2(0, 1; \mathbb{R})$ and

$$216 \quad \mathcal{B} : u \in U \mapsto \begin{pmatrix} 0 \\ \mathbf{1}_\omega u \end{pmatrix} \in X.$$

217 The condition $\inf_{k \geq 1} \|\varphi_k\|_{L^2(\omega)} > 0$ yields

$$218 \quad (1.7) \quad \mathcal{B}^* \phi_{k,1} = \mathcal{B}^* \phi_{k,2} = \mathbf{1}_\omega \varphi_k \neq 0, \quad \forall k \geq 1.$$

219 This ensures approximate controllability of system (1.1).

220 We insist on the fact that the goal of this article is not to deal with this particular
 221 example but to develop a general methodology to analyze the null controllability of
 222 system (1.1). The general assumptions that will be considered in this article are
 223 detailed in Section 1.4.

224 • Let $y_0 \in X$. From Proposition 1.1 and the fact that $\{\phi_{k,1}, \phi_{k,2}; k \geq 1\}$ forms a
 225 complete family of X , system (1.1) is null controllable from y_0 at time T if and only
 226 if there exists $u \in L^2(0, T; U)$ such that for any $k \geq 1$ and any $j \in \{1, 2\}$,

$$227 \quad \int_0^T e^{-\lambda_{k,j}t} \langle u(T-t), \mathcal{B}^* \phi_{k,j} \rangle_U dt = -e^{-\lambda_{k,j}T} \langle y_0, \phi_{k,j} \rangle_X.$$

228 Following the idea developed in [9], we seek for a control u of the form

$$229 \quad (1.8) \quad u(t) = - \sum_{k \geq 1} v_k(T-t)$$

230 where, for each $k \geq 1$, $v_k \in L^2(0, T; U)$ solves the block moment problem

$$231 \quad (1.9) \quad \begin{cases} \int_0^T e^{-\lambda_{k,j}t} \langle v_k(t), \mathcal{B}^* \phi_{k,j} \rangle_U dt = e^{-\lambda_{k,j}T} \langle y_0, \phi_{k,j} \rangle_X, & \forall j \in \{1, 2\}, \\ \int_0^T e^{-\lambda_{k',j}t} \langle v_k(t), \mathcal{B}^* \phi_{k',j} \rangle_U dt = 0, & \forall k' \neq k, \forall j \in \{1, 2\}. \end{cases}$$

232 • To solve (1.9), for a fixed k , we consider the following auxiliary block moment
 233 problem in the space U

$$234 \quad (1.10) \quad \begin{cases} \int_0^T e^{-\lambda_{k,j}t} v_k(t) dt = \Omega_{k,j}, & \forall j \in \{1, 2\}, \\ \int_0^T e^{-\lambda_{k',j}t} v_k(t) dt = 0, & \forall k' \neq k, \forall j \in \{1, 2\}, \end{cases}$$

235 where $\Omega_{k,j} \in U$ have to be precised. If we impose that $\Omega_{k,1}$ and $\Omega_{k,2}$ satisfy the
 236 constraints

$$237 \quad (1.11) \quad \langle \Omega_{k,j}, \mathcal{B}^* \phi_{k,j} \rangle_U = e^{-\lambda_{k,j}T} \langle y_0, \phi_{k,j} \rangle_X, \quad \forall j \in \{1, 2\},$$

238 we obtain that the solutions of (1.10) also solve (1.9). The existence of $\Omega_{k,1}$ and
 239 $\Omega_{k,2}$ satisfying the constraints (1.11) is ensured by the approximate controllabil-
 240 ity condition (1.7); however there exist infinitely many choices. A crucial point
 241 is that, by orthogonal projection, there exists $\Omega_{k,1}$ and $\Omega_{k,2}$ in the space $U_k =$
 242 $\text{Span}\{\mathcal{B}^* \phi_{k,1}, \mathcal{B}^* \phi_{k,2}\}$ satisfying the constraints (1.11).

243 Then, for any $\Omega_{k,1}, \Omega_{k,2} \in U_k$, since the space U_k is of finite dimension, applying
 244 the scalar results of [9] component by component leads to the existence of $v_k \in$
 245 $L^2(0, T; U)$ satisfying (1.10). It also gives the following estimate

$$246 \quad (1.12) \quad \|v_k\|_{L^2(0,T;U)}^2 \leq C_{T,\varepsilon} e^{\varepsilon \lambda_{k,1}} F(\Omega_{k,1}, \Omega_{k,2}),$$

247 with

$$248 \quad F : (\Omega_{k,1}, \Omega_{k,2}) \in U^2 \mapsto \|\Omega_{k,1}\|_U^2 + \left\| \frac{\Omega_{k,2} - \Omega_{k,1}}{\lambda_{k,2} - \lambda_{k,1}} \right\|_U^2.$$

249 Using (1.12) and minimizing the function F under the constraints (1.11) we obtain
 250 that there exists $v_k \in L^2(0, T; U)$ solution of the block moment problem (1.9) such
 251 that

$$252 \quad (1.13) \quad \|v_k\|_{L^2(0,T;U)}^2 \leq C_{T,\varepsilon} e^{\varepsilon \lambda_{k,1}} \inf \{F(\Omega_{k,1}, \Omega_{k,2}) ; \Omega_{k,1}, \Omega_{k,2} \text{ satisfy (1.11)}\}.$$

253 The corresponding general statements of the resolution of block moment problems
 254 are detailed in Section 2.1 (see Theorem 2.2) and proved in Section 3. Actually using
 255 a refined version of the results in [9] (see Theorem A.1) we obtain sharper results
 256 including dependency with respect to T .

257 • Now that we can solve the block moment problems (1.9), a way to characterize the
 258 minimal null control time is to estimate for which values of T the series (1.8) defining
 259 the control u converges in $L^2(0, T; U)$.

260 To achieve this goal, we isolate in the estimate (1.13) the dependency with respect
 261 to T . Notice that the function F does not depend on T but that the constraints (1.11)
 262 do.

263 For any $k \geq 1$ and any $\Omega_{k,1}, \Omega_{k,2} \in U_k$ we set

$$264 \quad \tilde{\Omega}_{k,j} := e^{\lambda_{k,j}T} \Omega_{k,j}, \quad \forall j \in \{1, 2\}.$$

265 Then, there is equivalence between the constraints (1.11) and the new constraints

$$266 \quad (1.14) \quad \left\langle \tilde{\Omega}_{k,j}, \mathcal{B}^* \phi_{k,j} \right\rangle_U = \langle y_0, \phi_{k,j} \rangle_X, \quad \forall j \in \{1, 2\}.$$

267 Now these constraints are independent of the variable T . From the mean value theo-
 268 rem we obtain

$$\begin{aligned}
 269 \quad F(\Omega_{k,1}, \Omega_{k,2}) &= \left\| e^{-\lambda_{k,1}T} \tilde{\Omega}_{k,1} \right\|_U^2 + \left\| \frac{e^{-\lambda_{k,2}T} \tilde{\Omega}_{k,2} - e^{-\lambda_{k,1}T} \tilde{\Omega}_{k,1}}{\lambda_{k,2} - \lambda_{k,1}} \right\|_U^2 \\
 270 \quad &\leq e^{-2\lambda_{k,1}T} \left\| \tilde{\Omega}_{k,1} \right\|_U^2 + 2e^{-2\lambda_{k,2}T} \left\| \frac{\tilde{\Omega}_{k,2} - \tilde{\Omega}_{k,1}}{\lambda_{k,2} - \lambda_{k,1}} \right\|_U^2 \\
 271 \quad &\quad + 2 \left(\frac{e^{-\lambda_{k,2}T} - e^{-\lambda_{k,1}T}}{\lambda_{k,2} - \lambda_{k,1}} \right)^2 \left\| \tilde{\Omega}_{k,1} \right\|_U^2 \\
 272 \quad &\leq 2(1 + T^2) e^{-2\lambda_{k,1}T} F(\tilde{\Omega}_{k,1}, \tilde{\Omega}_{k,2}).
 \end{aligned}$$

274 The general statement of this estimate is given in Lemma 4.1.

275 Plugging this estimate into (1.12) and optimizing the function F under the con-
 276 straints (1.14) yields

$$277 \quad (1.15) \quad \|v_k\|_{L^2(0,T;U)}^2 \leq C_{T,\varepsilon} e^{\varepsilon\lambda_{k,1}} e^{-2\lambda_{k,1}T} \mathcal{C}_k(y_0)$$

278 where $\mathcal{C}_k(y_0)$ is the quantity, independent of T , given by

$$\begin{aligned}
 280 \quad (1.16) \quad \mathcal{C}_k(y_0) &:= \inf \left\{ \left\| \tilde{\Omega}_1 \right\|_U^2 + \left\| \frac{\tilde{\Omega}_2 - \tilde{\Omega}_1}{\lambda_{k,2} - \lambda_{k,1}} \right\|_U^2 ; \tilde{\Omega}_1, \tilde{\Omega}_2 \in U_k \text{ satisfy} \right. \\
 281 \quad &\quad \left. \left\langle \tilde{\Omega}_j, \mathcal{B}^* \phi_{k,j} \right\rangle_U = \langle y_0, \phi_{k,j} \rangle_X, \forall j \in \{1, 2\} \right\}.
 \end{aligned}$$

283 Estimate (1.15) proves that for any time $T > 0$ such that

$$284 \quad T > \limsup_{k \rightarrow +\infty} \frac{\ln \mathcal{C}_k(y_0)}{2\lambda_{k,1}}$$

285 the series (1.8) defining the control u converges in $L^2(0, T; U)$. Thus, null controlla-
 286 bility of (1.1) from y_0 holds for such T .

287 We also prove that the obtained estimate (1.15) is sufficiently sharp so that it
 288 characterizes the minimal null control time from y_0 as

$$289 \quad (1.17) \quad T_0(y_0) = \limsup_{k \rightarrow +\infty} \frac{\ln \mathcal{C}_k(y_0)}{2\lambda_{k,1}}.$$

290 The corresponding general statements regarding the minimal null control time
 291 together with bounds on the cost of controllability are detailed in Section 2.1 (see
 292 Theorem 2.3) and proved in Section 4.

293 • At this stage we have characterized the minimal null control time as stated in (1.17).
 294 However to be able to estimate the actual value of $T_0(y_0)$ one should be able to
 295 estimate the quantity $\mathcal{C}_k(y_0)$ as defined in (1.16). This formula is not very explicit
 296 and it does not get better in the general setting.

297 However, we notice that (1.16) is a finite dimensional optimization problem that
 298 we explicitly solve in terms of the eigenelements of \mathcal{A}^* and their observations through
 299 \mathcal{B}^* .

300 Indeed the minimization problem (1.16) has a unique solution characterized by
 301 the existence of multipliers $m_1, m_2 \in \mathbb{R}$ such that for any $H_1, H_2 \in U_k$ we have
 (1.18)

$$302 \quad \left\langle H_1, \tilde{\Omega}_1 \right\rangle_U + \left\langle \frac{\tilde{\Omega}_2 - \tilde{\Omega}_1}{\lambda_{k,2} - \lambda_{k,1}}, \frac{H_2 - H_1}{\lambda_{k,2} - \lambda_{k,1}} \right\rangle_U = m_1 \langle H_1, \mathcal{B}^* \phi_{k,1} \rangle_U + m_2 \langle H_2, \mathcal{B}^* \phi_{k,2} \rangle_U.$$

303 Setting $H_1 = H_2 = H$ for any $H \in U_k$ implies

$$304 \quad \tilde{\Omega}_1 = m_1 \mathcal{B}^* \phi_{k,1} + m_2 \mathcal{B}^* \phi_{k,2}.$$

305 Setting $H_1 = 0$ and $H_2 = (\lambda_{k,2} - \lambda_{k,1})H$ for any $H \in U_k$ implies

$$306 \quad \tilde{\Omega}_2 = m_1 \mathcal{B}^* \phi_{k,1} + m_2 \mathcal{B}^* \phi_{k,2} + m_2 (\lambda_{k,2} - \lambda_{k,1})^2 \mathcal{B}^* \phi_{k,2}.$$

307 Getting back to the constraints (1.14) we obtain

$$308 \quad (1.19) \quad \begin{pmatrix} \langle y_0, \phi_{k,1} \rangle_X \\ \langle y_0, \phi_{k,2} \rangle_X \end{pmatrix} = M \begin{pmatrix} m_1 \\ m_2 \end{pmatrix},$$

309 where the 2×2 matrix M is defined by

$$310 \quad M = \text{Gram}_U(\mathcal{B}^* \phi_{k,1}, \mathcal{B}^* \phi_{k,2}) + \text{Gram}_U(0, (\lambda_{k,2} - \lambda_{k,1}) \mathcal{B}^* \phi_{k,2}).$$

311 Setting $H_1 = \tilde{\Omega}_1$ and $H_2 = \tilde{\Omega}_2$ in (1.18) and using (1.19) imply

$$312 \quad \mathcal{C}_k(y_0) = \left\| \tilde{\Omega}_1 \right\|_U^2 + \left\| \frac{\tilde{\Omega}_2 - \tilde{\Omega}_1}{\lambda_{k,2} - \lambda_{k,1}} \right\|_U^2 = \left\langle \begin{pmatrix} \langle y_0, \phi_{k,1} \rangle_X \\ \langle y_0, \phi_{k,2} \rangle_X \end{pmatrix}, \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \right\rangle \\ 313 \quad = \left\langle \begin{pmatrix} \langle y_0, \phi_{k,1} \rangle_X \\ \langle y_0, \phi_{k,2} \rangle_X \end{pmatrix}, M^{-1} \begin{pmatrix} \langle y_0, \phi_{k,1} \rangle_X \\ \langle y_0, \phi_{k,2} \rangle_X \end{pmatrix} \right\rangle. \\ 314$$

315 Thus, after computations, for the particular example we are considering here, the
 316 obtained formula reads

$$317 \quad \mathcal{C}_k(y_0) = \frac{1}{\|\varphi_k\|_{L^2(\omega)}^2} \left\langle y_0, \begin{pmatrix} \varphi_k \\ \varphi_k \end{pmatrix} \right\rangle_X^2 + \frac{e^{2ak^2}}{\|\varphi_k\|_{L^2(\omega)}^2} \left\langle y_0, \begin{pmatrix} \varphi_k \\ 0 \end{pmatrix} \right\rangle_X^2.$$

318 Then, from (1.17), it comes that the minimal null control time from X of this example
 319 is given by

$$320 \quad T_0(X) = a.$$

321 Notice, for instance, that this expression also gives that for a given y_0 if the set

$$322 \quad \left\{ k \in \mathbb{N}^* ; \left\langle y_0, \begin{pmatrix} \varphi_k \\ 0 \end{pmatrix} \right\rangle_X \neq 0 \right\}$$

323 is finite, then null controllability from y_0 holds in any positive time, *i.e.* $T_0(y_0) = 0$.

324 We obtain different explicit formula depending on the configuration for the multi-
 325 plicity of the eigenvalues of the considered block. The general statements of an explicit
 326 solution of the corresponding optimization problem are detailed in Section 2.3 (see
 327 Theorem 2.6 and Theorem 2.8) and proved in Section 5.

328 **1.4. Framework, spectral assumptions and notations.**

329 To state the main results of this article, we now detail the functional setting and
 330 assumptions we use.

331 **1.4.1. Functional setting.**

332 The functional setting for the study of system (1.1) is the same as in [9]. For the
 333 sake of completeness, let us briefly detail it. Unless explicitly stated, all the spaces
 334 are assumed to be complex vector spaces.

335 We consider X an Hilbert space, whose inner product and norm are denoted
 336 by $\langle \bullet, \bullet \rangle_X$ and $\|\bullet\|_X$ respectively. The space X is identified to its anti-dual through
 337 the Riesz theorem. Let $(\mathcal{A}, D(\mathcal{A}))$ be an unbounded operator in X such that $-\mathcal{A}$
 338 generates a C^0 -semigroup in X . Its adjoint in X is denoted by $(\mathcal{A}^*, D(\mathcal{A}^*))$. Up to
 339 a suitable translation, we can assume that 0 is in the resolvent set of \mathcal{A} .

340 We denote by X_1 (resp. X_1^*) the Hilbert space $D(\mathcal{A})$ (resp. $D(\mathcal{A}^*)$) equipped
 341 with the norm $\|x\|_1 := \|\mathcal{A}x\|_X$ (resp. $\|x\|_{1^*} := \|\mathcal{A}^*x\|_X$) and we define X_{-1} as the
 342 completion of X with respect to the norm

$$343 \quad \|y\|_{-1} := \sup_{z \in X_1^*} \frac{\langle y, z \rangle_X}{\|z\|_{1^*}}.$$

344 Notice that X_{-1} is isometrical to the topological anti-dual of X_1^* using X as a pivot
 345 space (see for instance [38, Proposition 2.10.2]). The corresponding duality bracket
 346 will be denoted by $\langle \bullet, \bullet \rangle_{-1,1^*}$ and satisfies

$$347 \quad \langle y, cz \rangle_{-1,1^*} = \bar{c} \langle y, z \rangle_{-1,1^*}, \quad \forall y \in X_{-1}, \forall z \in X_1^*, \forall c \in \mathbb{C}.$$

348 The control space U is an Hilbert space (that we will identify to its anti-dual). Its inner
 349 product and norm are denoted by $\langle \bullet, \bullet \rangle_U$ and $\|\bullet\|_U$ respectively. Let $\mathcal{B} : U \rightarrow X_{-1}$ be
 350 a linear continuous control operator and denote by $\mathcal{B}^* : X_1^* \rightarrow U$ its adjoint in the
 351 duality described above.

352 Let $(X_\diamond^*, \|\cdot\|_{\diamond^*})$ be an Hilbert space such that $X_1^* \subset X_\diamond^* \subset X$ with dense and
 353 continuous embeddings. We assume that X_\diamond^* is stable by the semigroup generated by
 354 $-\mathcal{A}^*$. We also define $X_{-\diamond}$ as the subspace of X_{-1} defined by

$$355 \quad X_{-\diamond} := \left\{ y \in X_{-1}; \|y\|_{-\diamond} := \sup_{z \in X_1^*} \frac{\langle y, z \rangle_{-1,1^*}}{\|z\|_{\diamond^*}} < +\infty \right\},$$

356 which is also isometrical to the anti-dual of X_\diamond^* with X as a pivot space. The cor-
 357 responding duality bracket will be denoted by $\langle \bullet, \bullet \rangle_{-\diamond, \diamond}$. Thus, we end up with the
 358 following five functional spaces

$$359 \quad X_1^* \subset X_\diamond^* \subset X \subset X_{-\diamond} \subset X_{-1}.$$

360 We say that the control operator \mathcal{B} is an admissible control operator for (1.1) with
 361 respect to the space $X_{-\diamond}$ if for any $T > 0$ there exists $C_T > 0$ such that

$$362 \quad (1.20) \quad \int_0^T \left\| \mathcal{B}^* e^{-(T-t)\mathcal{A}^*} z \right\|_U^2 dt \leq C_T \|z\|_{\diamond^*}^2, \quad \forall z \in X_1^*.$$

363 Notice that if (1.20) holds for some $T > 0$ it holds for any $T > 0$. The admissibility
 364 condition (1.20) implies that, by density, we can give a meaning to the map

$$365 \quad \left(t \mapsto \mathcal{B}^* e^{-(T-t)\mathcal{A}^*} z \right) \in L^2(0, T; U),$$

366 for any $z \in X_\diamond^*$. Then, we end up with the following well-posedness result (see [9,
367 Proposition 1.2]).

368 PROPOSITION 1.1. *Assume that (1.20) holds. Then, for any $T > 0$, any $y_0 \in X_{-\diamond}$,
369 and any $u \in L^2(0, T; U)$, there exists a unique $y \in C^0([0, T]; X_{-\diamond})$ solution to (1.1)
370 in the sense that it satisfies for any $t \in [0, T]$ and any $z_t \in X_\diamond^*$,*

$$371 \quad \langle y(t), z_t \rangle_{-\diamond, \diamond} - \langle y_0, e^{-t\mathcal{A}^*} z_t \rangle_{-\diamond, \diamond} = \int_0^t \langle u(s), \mathcal{B}^* e^{-(t-s)\mathcal{A}^*} z_t \rangle_U ds.$$

372 Moreover there exists $C_T > 0$ such that

$$373 \quad \sup_{t \in [0, T]} \|y(t)\|_{-\diamond} \leq C_T (\|y_0\|_{-\diamond} + \|u\|_{L^2(0, T; U)}).$$

374 REMARK 1.2. *By analogy with the semigroup notation, when $u = 0$, we set for
375 any $t \in [0, T]$, $e^{-t\mathcal{A}} y_0 := y(t)$. This extends the semigroup $e^{-\bullet\mathcal{A}}$ defined on X to $X_{-\diamond}$
376 and implies that for any $z \in X_{-\diamond}$,*

$$377 \quad (1.21) \quad \langle e^{-T\mathcal{A}} z, \phi \rangle_{-\diamond, \diamond} = \langle z, e^{-T\mathcal{A}^*} \phi \rangle_{-\diamond, \diamond}, \quad \forall \phi \in X_\diamond^*.$$

378 With this notion of solution at hand, we finally define the minimal null control
379 time from a subspace of initial conditions Y_0 .

380 DEFINITION 1.1. *Let Y_0 be a closed subspace of $X_{-\diamond}$ and let $T > 0$. The sys-
381 tem (1.1) is said to be null controllable from Y_0 at time T if for any $y_0 \in Y_0$, there
382 exists a control $u \in L^2(0, T; U)$ such that the associated solution of (1.1) satisfies
383 $y(T) = 0$.*

384 *The minimal null control time $T_0(Y_0) \in [0, +\infty]$ is defined by*

- 385 • *for any $T > T_0(Y_0)$, system (1.1) is null controllable from Y_0 at time T ;*
- 386 • *for any $T < T_0(Y_0)$, system (1.1) is not null controllable from Y_0 at time T .*

387 To simplify the notations, for any $y_0 \in X_{-\diamond}$, we define $T_0(y_0) := T_0(\text{Span}\{y_0\})$.

388 1.4.2. Spectral assumptions.

389 In all this article we assume that the operators \mathcal{A} and \mathcal{B} satisfy the assumptions
390 of Section 1.4.1. Moreover to solve the control problem we will need some additional
391 spectral assumptions.

392 ★ *Behavior of eigenvalues.*

393 We assume that the spectrum of \mathcal{A}^* , denoted by Λ , is only composed of (countably
394 many) eigenvalues. Moreover, we assume that the eigenvalues lie in a suitable sector
395 of the complex plane, i.e., there exists $\tau > 0$ such that

$$396 \quad (1.22) \quad \Lambda \subset S_\tau$$

397 where

$$398 \quad S_\tau := \{z \in \mathbb{C}; \Re z > 0 \text{ and } |\Im z| < (\sinh \tau) \Re z\}.$$

399 REMARK 1.3. *In [9], the assumption on Λ was stronger. Namely, in that article
400 it was assumed that $\Lambda \subset (1, +\infty)$. The fact that $\min \Lambda \geq 1$ was only used in the
401 lower bound on the solution of scalar block moment problems (see estimate (A.3)).
402 The extension to complex eigenvalues satisfying (1.22) was done in [12] and is stated
403 in Appendix A.*

404 *If necessary, one can replace the operator \mathcal{A} by $\mathcal{A} + \sigma$ without modifying the
405 controllability properties. Then, in the different estimates, the behavior with respect
406 to σ can be carefully tracked if needed.*

407 As in the case of a scalar control (see [9]) we assume that this spectrum satisfies
 408 a weak-gap condition. Namely, there exists $p \in \mathbb{N}^*$ and $\varrho > 0$ such that

$$409 \quad (1.23) \quad \#(\Lambda \cap D(\mu, \varrho/2)) \leq p, \quad \forall \mu \in \mathbb{C},$$

410 where $D(\mu, \varrho/2)$ denotes the open disk in the complex plane with center μ and radius
 411 $\varrho/2$. This means that the eigenvalues are allowed to condensate by groups but the
 412 cardinality of these groups should be bounded. To precise this, let us recall the notion
 413 of groupings used in [9, Definition 1.6, Proposition 7.1] and extended to the complex
 414 setting in [12, Proposition V.5.26].

415 **PROPOSITION 1.2.** *Let $p \in \mathbb{N}^*$ and $\varrho > 0$. Let $\Lambda \subset \mathbb{C}$ be such that the weak-gap
 416 condition (1.23) holds. Then, there exists a countable family \mathcal{G} of disjoint subsets of
 417 Λ satisfying*

$$418 \quad (1.24) \quad \Lambda = \bigcup_{G \in \mathcal{G}} G$$

419 and each $G \in \mathcal{G}$ satisfies

$$420 \quad (1.25) \quad \text{diam } G \leq \varrho,$$

421

$$422 \quad (1.26) \quad \#G \leq p,$$

423 and

$$424 \quad (1.27) \quad \text{dist}(\text{Conv } G, \Lambda \setminus G) \geq \frac{\varrho}{2 \times 4^{p-1}}.$$

425 Let us mention that the results do not depend on the particular construction done
 426 in [12, Proposition V.5.26] and remain valid for any grouping \mathcal{G} satisfying (1.24)-
 427 (1.27).

428 Concerning the asymptotic behavior of the spectrum we will use the counting
 429 function associated to Λ defined by

$$430 \quad N_\Lambda : r > 0 \mapsto \#\{\lambda \in \Lambda ; |\lambda| \leq r\}.$$

431 We assume that there exists $\kappa > 0$ and $\theta \in (0, 1)$ such that

$$432 \quad (1.28) \quad N_\Lambda(r) \leq \kappa r^\theta, \quad \forall r > 0$$

433 and

$$434 \quad (1.29) \quad |N_\Lambda(r) - N_\Lambda(s)| \leq \kappa \times (1 + |r - s|^\theta), \quad \forall r, s > 0.$$

435 Notice that this condition is slightly stronger than the classical summability condi-
 436 tion (1.4) used for instance in [22, 5, 9] and many other works.

437 **REMARK 1.4.** *Let us underline that if we do not assume (1.29) to hold all the
 438 results of the present article still hold with a slight change in the estimates. To lighten
 439 the writing we only detail this change for Theorem A.1 concerning the resolution of
 440 block moment problems with a scalar control (see Remark A.1). However, as proved
 441 in Section 7, the assumption (1.29) holds for many examples.*

442 Notice also that (1.28), with $r = \inf |\Lambda|$, implies the following lower bound on the
 443 bottom of the spectrum

$$444 \quad \inf |\Lambda| \geq \kappa^{-\theta}.$$

445 Our goal is not only to study the controllability properties of our system but also to
 446 obtain estimates that are uniform in a way to be precised. To do so, we define the
 447 following class of sequences: let $p \in \mathbb{N}^*$, $\varrho, \tau, \kappa > 0$, $\theta \in (0, 1)$ and consider the class

$$448 \quad (1.30) \quad \mathcal{L}_w(p, \varrho, \tau, \theta, \kappa) := \{\Lambda \subset S_\tau ; \Lambda \text{ satisfies (1.23), (1.28) and (1.29)}\}.$$

449 *★ Multiplicity of eigenvalues.*

450 In our study we allow both algebraic and geometric multiplicities for the eigenval-
 451 ues. We assume that these multiplicities are finite and that the algebraic multiplicity
 452 is globally bounded. More precisely, we assume that

$$453 \quad (1.31) \quad \gamma_\lambda := \dim \text{Ker}(\mathcal{A}^* - \lambda) < +\infty, \quad \forall \lambda \in \Lambda,$$

454 and that there exists $\eta \in \mathbb{N}^*$ such that

$$455 \quad (1.32) \quad \text{Ker}(\mathcal{A}^* - \lambda)^\eta = \text{Ker}(\mathcal{A}^* - \lambda)^{\eta+1}, \quad \forall \lambda \in \Lambda.$$

456 For any $\lambda \in \Lambda$ we denote by α_λ the smallest integer such that

$$457 \quad \text{Ker}(\mathcal{A}^* - \lambda)^{\alpha_\lambda} = \text{Ker}(\mathcal{A}^* - \lambda)^{\alpha_\lambda+1}$$

458 and set

$$459 \quad E_\lambda := \text{Ker}(\mathcal{A}^* - \lambda)^{\alpha_\lambda}.$$

460 *★ (Generalized) eigenvectors.*

461 To study null-controllability, we assume that the Fattorini-Hautus criterion is
 462 satisfied

$$463 \quad (1.33) \quad \text{Ker}(\mathcal{A}^* - \lambda) \cap \text{Ker } \mathcal{B}^* = \{0\}, \quad \forall \lambda \in \Lambda.$$

464 It is a necessary condition for approximate controllability. Note that, under additional
 465 assumptions on \mathcal{A} and \mathcal{B} it is also a sufficient condition for approximate controllability
 466 (see for instance [20, 34]). However, when studying null controllability of system (1.1)
 467 for initial conditions in a closed strict subspace Y_0 of $X_{-\diamond}$ the condition (1.33) can be
 468 too strong, see for instance Sections 7.1.2 and 7.1.3.

469 We assume that the family of generalized eigenvectors of \mathcal{A}^*

$$470 \quad \Phi = \{\phi \in E_\lambda ; \lambda \in \Lambda\} = \bigcup_{\lambda \in \Lambda} E_\lambda$$

471 is complete in X_{\diamond}^* i.e. for any $y \in X_{-\diamond}$,

$$472 \quad (1.34) \quad \left(\langle y, \phi \rangle_{-\diamond, \diamond} = 0, \quad \forall \phi \in \Phi \right) \implies y = 0.$$

473 In the following, to simplify the writing, we gather these assumptions and say
 474 that the operators \mathcal{A} and \mathcal{B} satisfy (H) if there exists $p \in \mathbb{N}^*$, $\varrho, \tau, \kappa > 0$ and $\theta \in (0, 1)$
 475 such that

$$476 \quad (\text{H}) \quad \begin{cases} \mathcal{A} \text{ and } \mathcal{B} \text{ satisfy the assumptions of Section 1.4.1;} \\ \Lambda = \text{Sp}(\mathcal{A}^*) \text{ belongs to } \mathcal{L}_w(p, \varrho, \tau, \theta, \kappa) \text{ and satisfies (1.31) and (1.32);} \\ \text{the associated (generalized) eigenvectors satisfy (1.33) and (1.34).} \end{cases}$$

477 **1.4.3. Notation.**

478 We give here some notation that will be used throughout this article.

- 479 • For any $a, b \in \mathbb{R}$, we define the following subsets of \mathbb{N} :

480
$$\llbracket a, b \rrbracket := [a, b] \cap \mathbb{N}, \quad \llbracket a, b[:= [a, b) \cap \mathbb{N}.$$

- 481 • In all the present paper, $\langle \bullet, \bullet \rangle$ denotes the usual inner product in finite di-
482 mension i.e.

483
$$\langle f, g \rangle = {}^t f \bar{g}.$$

- 484 • For any $t \in \mathbb{R}$ we denote by e_t the exponential function

485
$$e_t : \mathbb{C} \rightarrow \mathbb{C}$$

$$z \mapsto e^{-tz}.$$

- 486 • We shall denote by $C_{\nu_1, \dots, \nu_l} > 0$ a constant possibly varying from one line to
487 another but depending only on the parameters ν_1, \dots, ν_l .
488 • For any non empty subset $\Gamma \subset \Lambda$, we set

489 (1.35)
$$r_\Gamma := \inf_{\lambda \in \Gamma} \Re \lambda.$$

490 Notice that assumptions (1.22) and (1.23) imply that $r_\Gamma > 0$ for any $\Gamma \subset \Lambda$.

- 491 • For any multi-index $\alpha \in \mathbb{N}^n$, we denote its length by $|\alpha| = \sum_{j=1}^n \alpha_j$ and its
492 maximum by $|\alpha|_\infty = \max_{j \in \llbracket 1, n \rrbracket} \alpha_j$.

493 For $\alpha, \mu \in \mathbb{N}^n$, we say that $\mu \leq \alpha$ if and only if $\mu_j \leq \alpha_j$ for any $j \in \llbracket 1, n \rrbracket$.

- 494 • In all in this article the notation $f[\dots]$ stands for (generalized) divided dif-
495 ferences of a set of values (x_j, f_j) . Let us recall that, for pairwise distinct
496 $x_1, \dots, x_n \in \mathbb{C}$ and f_1, \dots, f_n in any vector space, the divided differences are
497 defined by

498
$$f[x_j] = f_j, \quad f[x_1, \dots, x_j] = \frac{f[x_2, \dots, x_j] - f[x_1, \dots, x_{j-1}]}{x_j - x_1}.$$

499 The two results that will be the most used in this article concerning divided
500 differences are the Leibniz formula

501
$$(gf)[x_1, \dots, x_j] = \sum_{k=1}^j g[x_1, \dots, x_k] f[x_k, \dots, x_j],$$

502 and Jensen inequality stating that, when $f_j = f(x_j)$ for an holomorphic
503 function f , we have

504
$$|f[x_1, \dots, x_j]| \leq \frac{|f^{(j-1)}(z)|}{(j-1)!},$$

505 with $z \in \text{Conv}\{x_1, \dots, x_j\}$. For more detailed statements and other useful
506 properties as well as their generalizations when x_1, \dots, x_n are not assumed to
507 be pairwise distinct we refer the reader to [12, Appendix A.2] This general-
508 ization is used in the present article whenever there are algebraically multiple
509 eigenvalues.

- 510 • For any closed subspace Y of $X_{-\diamond}$ we denote by P_Y the orthogonal projection
511 in $X_{-\diamond}$ onto Y . We denote by $P_Y^* \in L(X_\diamond^*)$ its adjoint in the duality $X_{-\diamond}$,
512 X_\diamond^* .

513 **2. Main results.**

514 We state in this section the main results of this article concerning the resolution
 515 of block moment problems and the application to the characterization of the minimal
 516 null control time.

517 **2.1. Resolution of block moment problems.**

518 *★ Definition of block moment problems.*

519 Using the notion of solution given in Proposition 1.1 and the assumption (1.34),
 520 null controllability from y_0 in time T reduces to the resolution of the following problem:
 521 find $u \in L^2(0, T; U)$ such that

522 (2.1)
$$\int_0^T \langle u(t), \mathcal{B}^* e^{-(T-t)\mathcal{A}^*} \phi \rangle_U dt = - \langle y_0, e^{-T\mathcal{A}^*} \phi \rangle_{-\diamond, \diamond}, \quad \forall \phi \in E_\lambda, \forall \lambda \in \Lambda.$$

523 Following the strategy initiated in [9] for scalar controls, we decompose this problem
 524 into block moment problems. Hence we look for a control of the form

525 (2.2)
$$u = - \sum_{G \in \mathcal{G}} v_G(T - \bullet)$$

526 where \mathcal{G} is a grouping (as stated in Proposition 1.2) and, for every $G \in \mathcal{G}$, $v_G \in$
 527 $L^2(0, T; U)$ solves the moment problem in the group G *i.e.*

528 (2.3a)
$$\int_0^T \langle v_G(t), \mathcal{B}^* e^{-t\mathcal{A}^*} \phi \rangle_U dt = \langle y_0, e^{-T\mathcal{A}^*} \phi \rangle_{-\diamond, \diamond}, \quad \forall \phi \in E_\lambda, \forall \lambda \in G,$$

529 (2.3b)
$$\int_0^T \langle v_G(t), \mathcal{B}^* e^{-t\mathcal{A}^*} \phi \rangle_U dt = 0, \quad \forall \phi \in E_\lambda, \forall \lambda \in \Lambda \setminus G.$$

 530

531 In fact it is sufficient to solve the following block moment problem

532 (2.4a)
$$\int_0^T \langle v_G(t), \mathcal{B}^* e^{-t\mathcal{A}^*} \phi \rangle_U dt = \langle e^{-T\mathcal{A}^*} y_0, \phi \rangle_{-\diamond, \diamond}, \quad \forall \phi \in E_\lambda, \forall \lambda \in G,$$

533 (2.4b)
$$\int_0^T v_G(t) t^l e^{-\bar{\lambda} t} dt = 0, \quad \forall \lambda \in \Lambda \setminus G, \forall l \in \llbracket 0, \eta \llbracket$$

 534

535 where $e^{-T\mathcal{A}^*} y_0$ is defined in (1.21).

536 Indeed, for any $\phi \in E_\lambda$, from [9, (1.22)], it comes that

537 (2.5)
$$e^{-t\mathcal{A}^*} \phi = e^{-\lambda t} \sum_{r \geq 0} \frac{(-t)^r}{r!} (\mathcal{A}^* - \lambda)^r \phi = \sum_{r \geq 0} e_t \left[\lambda^{(r+1)} \right] (\mathcal{A}^* - \lambda)^r \phi,$$

538 where the sums are finite (and contains at most the first α_λ terms). Thus, every
 539 solution of (2.4) solves (2.3). The orthogonality condition (2.4b) is more restrictive
 540 than (2.3b) but leads to negligible terms in the estimates.

541 *★ Resolution of block moment problems.*

542 In our setting, the block moment problem (2.4) is proved to be solvable for any
 543 $T > 0$. The resolution will follow from the scalar study done in [9] and refined in [12]
 544 (see Theorem A.1).

545 Due to (2.2), the main issue to prove null controllability of (1.1) is thus to sum
 546 those contributions to obtain a solution of (2.1). This is justified thanks to a precise
 547 estimate of the cost of the resolution of (2.4) for each group G which is the quantity

548
$$\inf \{ \|v_G\|_{L^2(0, T; U)} ; v_G \text{ solution of (2.4)} \}.$$

549 To state this result, we introduce some additional notation.

550 To solve the moment problem (2.4) we propose to lift it into a ‘vectorial block
551 moment problem’ of the following form (see (3.1))

$$552 \quad \begin{cases} \int_0^T v_G(t) \frac{(-t)^l}{l!} e^{-\bar{\lambda}t} dt = \Omega_{\bar{\lambda}}^l, & \forall \lambda \in G, \forall l \in \llbracket 0, \alpha_\lambda \rrbracket, \\ \int_0^T v_G(t) t^l e^{-\bar{\lambda}t} dt = 0, & \forall \lambda \in \Lambda \setminus G, \forall l \in \llbracket 0, \eta \rrbracket, \end{cases}$$

553 where $\Omega_{\bar{\lambda}}^l$ belongs to U . Following (2.5), to recover a solution of (2.4), we need
554 to impose some constraints on the right-hand side that are given in the following
555 definition.

556 DEFINITION 2.1. For any $\lambda \in \Lambda$ and any $z \in X_{-\diamond}$, we set

$$558 \quad (2.6) \quad \mathcal{O}(\lambda, z) = \left\{ (\Omega^0, \dots, \Omega^{\alpha_\lambda-1}) \in U^{\alpha_\lambda}; \right. \\ 559 \quad \left. \sum_{l=0}^{\alpha_\lambda-1} \langle \Omega^l, \mathcal{B}^*(\mathcal{A}^* - \lambda)^l \phi \rangle_U = \langle z, \phi \rangle_{-\diamond, \diamond}, \quad \forall \phi \in E_\lambda \right\}.$$

560 For a given group G , we set

$$562 \quad (2.7) \quad \mathcal{O}(G, z) = \prod_{\lambda \in G} \mathcal{O}(\lambda, z) \subset U^{|\alpha|}$$

563 where α is the multi-index of the algebraic multiplicities of the eigenvalues.

564 Consider any sequence of multi-indices $(\mu^l)_{l \in \llbracket 0, |\alpha| \rrbracket}$ such that

$$565 \quad (2.8) \quad \begin{cases} \mu^{l-1} \leq \mu^l, & \forall l \in \llbracket 1, |\alpha| \rrbracket, \\ |\mu^l| = l, & \forall l \in \llbracket 0, |\alpha| \rrbracket, \\ \mu^{|\alpha|} = \alpha. \end{cases}$$

566 To measure the cost associated to the group $G = \{\lambda_1, \dots, \lambda_g\}$ let us define the fol-
567 lowing functional

$$568 \quad (2.9) \quad F : \Omega = (\Omega_1^0, \dots, \Omega_1^{\alpha_1-1}, \dots, \Omega_g^0, \dots, \Omega_g^{\alpha_g-1}) \in U^{|\alpha|} \mapsto \sum_{l=1}^{|\alpha|} \left\| \Omega \left[\bar{\lambda}^{(\mu^l)} \right] \right\|_U^2$$

569 with the convention

$$570 \quad \Omega \left[\bar{\lambda}_j^{(l+1)} \right] = \Omega_{\bar{\lambda}_j}^l, \quad \forall j \in \llbracket 1, g \rrbracket, \forall l \in \llbracket 0, \alpha_j \rrbracket.$$

571 The use of such functional to measure the cost comes from the analysis conducted for
572 scalar controls in [9] (see Proposition 3.3). It appears in the following lower bound
573 for solutions of block moment problems.

574 PROPOSITION 2.1. Assume that the operators \mathcal{A} and \mathcal{B} satisfy the assumption (H)
575 (see Section 1.4.2). Let $T \in (0, +\infty)$, and $G \subset \Lambda$ be a group satisfying (1.26).

576 There exists $C_{p, \eta, r_\Lambda} > 0$ such that, for any $z \in X_{-\diamond}$, any $v_G \in L^2(0, T; U)$ solving

$$577 \quad \int_0^T \langle v_G(t), \mathcal{B}^* e^{-t\mathcal{A}^*} \phi \rangle_U dt = \langle z, \phi \rangle_{-\diamond, \diamond}, \quad \forall \phi \in E_\lambda, \forall \lambda \in G$$

578 *satisfies*

$$579 \quad (2.10) \quad \|v_G\|_{L^2(0,T;U)}^2 \geq C_{p,\eta,r,\Lambda} \mathcal{C}(G, z)$$

580 *where*

$$581 \quad (2.11) \quad \mathcal{C}(G, z) := \inf \{F(\Omega) ; \Omega \in \mathcal{O}(G, z)\}$$

582 *with F defined in (2.9) and $\mathcal{O}(G, z)$ defined in Definition 2.1.*

583 The first main result of this article concerns the resolution of block moment
584 problems of the form (2.4). It roughly states that, up to terms that turns out to be
585 negligible, the lower bound obtained in Proposition 2.1 is optimal.

586 **THEOREM 2.2.** *Assume that the operators \mathcal{A} and \mathcal{B} satisfy the assumption (H)*
587 *(see Section 1.4.2). Let $T \in (0, +\infty)$, and $G \subset \Lambda$ be a group satisfying (1.25)–(1.27).*
588 *For any $z \in X_{-\diamond}$, there exists $v_G \in L^2(0, T; U)$ solution of*

$$589 \quad (2.12a) \quad \int_0^T \left\langle v_G(t), \mathcal{B}^* e^{-t\mathcal{A}^*} \phi \right\rangle_U dt = \langle z, \phi \rangle_{-\diamond, \diamond}, \quad \forall \phi \in E_\lambda, \forall \lambda \in G,$$

$$590 \quad (2.12b) \quad \int_0^T v_G(t) t^l e^{-\bar{\lambda}t} dt = 0, \quad \forall \lambda \in \Lambda \setminus G, \forall l \in \llbracket 0, \eta \rrbracket,$$

591

592 *satisfying the following estimate*

$$593 \quad (2.13) \quad \|v_G\|_{L^2(0,T;U)}^2 \leq C \exp\left(\frac{C}{T^{1-\theta}}\right) \exp(r_G T) \exp(Cr_G^\theta) \mathcal{C}(G, z).$$

594 *In this estimate, $\mathcal{C}(G, z)$ is defined in (2.11) and r_G is defined in (1.35). The constant*
595 *$C > 0$ appearing in the estimate only depends on the parameters $\tau, p, \varrho, \eta, \theta$ and κ .*

596 Before giving the application of this resolution of block moment problems to the
597 null controllability of our initial system (1.1), let us give some comments.

598 • As it was the case in [9], the considered setting allows for a wide variety of
599 applications. In (1.34) the generalized eigenvectors are only assumed to form
600 a complete family (and not a Riesz basis as in many previous works) which
601 is the minimal assumption to use a moment method-like strategy. The weak
602 gap condition (1.23) is also well adapted to study systems of coupled one
603 dimensional parabolic equations (see Section 7).

604 • The main restriction is the assumption (1.28). As detailed in Section 1.1,
605 this assumption is common to most of the results based on a moment-like
606 method.

607 Though restrictive, let us underline that the moment method is, to the best
608 of our knowledge, the most suitable method to capture very sensitive features
609 such as a minimal null control time for parabolic control problems without
610 constraints.

611 • The main novelty of this theorem is to ensure solvability of block moment
612 problems coming from control problems with control operators that are only
613 assumed to be admissible. In particular, the space U can be of infinite di-
614 mension. Results concerning block moment problems with more general right-
615 hand sides, that is not necessarily coming from a controllability problem, are
616 stated in Appendix C

- 617 • The estimate (2.13) does not explicitly depend on the sequence of eigenval-
618 ues Λ but rather on some parameters such as the weak-gap parameters and
619 the asymptotic of the counting function. As presented in Section 1.1, the
620 uniformity of such bounds can be used to deal with parameter dependent
621 problems.
- 622 • Let us also underline that the obtained estimate (2.13) tracks the depen-
623 dency of the constants with respect to the controllability time T . This will
624 be crucial to estimate the cost of controllability in Proposition 2.9. We refer
625 to Remark 2.5 for possible applications of such estimates of the cost of
626 controllability.
- 627 • Though quite general and useful for the theoretical characterization of the
628 minimal null control time, the obtained estimate (2.13) still requires to be able
629 to evaluate quantities of the form $\mathcal{C}(G, z)$, which can be intricate. We provide
630 in Section 2.3 some explicit formulas that makes this estimation possible in
631 many actual examples.

632 2.2. Determination of the minimal null control time.

633 The resolution of block moment problems stated in Theorem 2.2 allows to obtain
634 the following characterization of the minimal null control time of our abstract control
635 problem from a given initial condition.

636 **THEOREM 2.3.** *Assume that the operators \mathcal{A} and \mathcal{B} satisfy the assumption (H)
637 (see Section 1.4.2) and let \mathcal{G} be an associated grouping as stated in Proposition 1.2.
638 Then, for any $y_0 \in X_{-\diamond}$, the minimal null control time of (1.1) from y_0 is given by*

$$639 \quad (2.14) \quad T_0(y_0) = \limsup_{G \in \mathcal{G}} \frac{\ln^+ \mathcal{C}(G, y_0)}{2r_G}$$

640 where $\mathcal{C}(G, y_0)$ is defined in (2.11).

641 In this statement we have used the notation $\ln^+ s = \max(0, \ln s)$, for any $s \geq 0$.

642 If one considers a space of initial conditions (instead of a single initial condition),
643 the characterization of the minimal null control time is given in the following corollary.

644 **COROLLARY 2.4.** *Let Y_0 be a closed subspace of $X_{-\diamond}$. Then, under the assump-
645 tions of Theorem 2.3, the minimal null control time from Y_0 is given by*

$$646 \quad T_0(Y_0) = \limsup_{G \in \mathcal{G}} \frac{\ln^+ \mathcal{C}(G, Y_0)}{2r_G}$$

647 with

$$648 \quad \mathcal{C}(G, Y_0) := \sup_{\substack{y_0 \in Y_0 \\ \|y_0\|_{-\diamond} = 1}} \mathcal{C}(G, y_0).$$

649 2.3. More explicit formulas.

650 Assume that the operators \mathcal{A} and \mathcal{B} satisfy the assumption (H). Let $G \subset \Lambda$ be
651 such that $\sharp G \leq p$ and $\text{diam } G \leq \varrho$. We have seen in Theorem 2.3 that the key quantity
652 to compute the minimal null control time from y_0 is

$$653 \quad \mathcal{C}(G, y_0) = \inf \{F(\Omega) ; \Omega \in \mathcal{O}(G, y_0)\}.$$

654 where the function F is defined in (2.9) and the constraints $\mathcal{O}(G, y_0)$ are defined
655 in (2.7). Let us give more explicit formulas to compute such costs.

656 Notice that, for any $z \in X_{-\circ}$, the quantity $\mathcal{C}(G, z)$ can be expressed as a finite
 657 dimensional constrained problem. Indeed, for a given group G we consider the finite
 658 dimensional subspace

$$659 \quad (2.15) \quad U_G = \mathcal{B}^* \text{Span} \{ \phi \in E_\lambda ; \lambda \in G \}$$

660 and P_{U_G} the orthogonal projection in U onto U_G . Then, for any $\Omega \in \mathcal{O}(G, z)$ it
 661 comes that $P_{U_G} \Omega \in \mathcal{O}(G, z)$ and $F(P_{U_G} \Omega) \leq F(\Omega)$. Thus, the optimization problem
 662 defining $\mathcal{C}(G, z)$ reduces to

$$663 \quad \mathcal{C}(G, z) = \inf \left\{ F(\Omega) ; \Omega \in \mathcal{O}(G, z) \cap U_G^{|\alpha|} \right\},$$

664 which is a finite dimensional optimization problem. From [9, Proposition 7.15], the
 665 function F is coercive which implies that the infimum is actually attained:

$$666 \quad (2.16) \quad \mathcal{C}(G, z) = \min \left\{ F(\Omega) ; \Omega \in \mathcal{O}(G, z) \cap U_G^{|\alpha|} \right\}.$$

667 In this section, solving the optimization problem (2.16), we provide more explicit
 668 formulas for this cost for some particular configurations for the multiplicities of the
 669 eigenvalues in the group G (and only in that particular group).

670 *★ A group G of geometrically simple eigenvalues.*

671 First, assume that the eigenvalues in $G = \{ \lambda_1, \dots, \lambda_g \}$ are all geometrically simple
 672 *i.e.* $\gamma_\lambda = 1$ for every $\lambda \in G$ where γ_λ is defined in (1.31).

673 For any $j \in \llbracket 1, g \rrbracket$ we denote by ϕ_j^0 an eigenvector of \mathcal{A}^* associated to the eigen-
 674 value λ_j and by $(\phi_j^l)_{l \in \llbracket 0, \alpha_j \rrbracket}$ an associated Jordan chain *i.e.*

$$675 \quad (\mathcal{A}^* - \lambda_j) \phi_j^l = \phi_j^{l-1}, \quad \forall l \in \llbracket 1, \alpha_j \rrbracket.$$

676 To simplify the writing, we set

$$677 \quad b_j^l := \mathcal{B}^* \phi_j^l \in U, \quad \forall l \in \llbracket 0, \alpha_j \rrbracket, \forall j \in \llbracket 1, g \rrbracket.$$

678 Recall that the sequence of multi-index $(\mu^l)_{l \in \llbracket 0, |\alpha| \rrbracket}$ satisfy (2.8) and let

$$679 \quad (2.17) \quad M := \sum_{l=1}^{|\alpha|} \Gamma_\mu^l$$

680 with

$$681 \quad \Gamma_\mu^l := \text{Gram}_U \left(\underbrace{0, \dots, 0}_{l-1}, b \left[\lambda_\bullet^{(\mu^l - \mu^{l-1})} \right], \dots, b \left[\lambda_\bullet^{(\mu^{|\alpha|} - \mu^{l-1})} \right] \right)$$

682 where for every $u_1, \dots, u_n \in U$, $\text{Gram}_U(u_1, \dots, u_n)$ denotes the Gram matrix whose
 683 entry on the i -th row and j -th column is $\langle u_j, u_i \rangle_U$. To explicit the cost $\mathcal{C}(G, y_0)$, we
 684 will use the inverse of this matrix. Its invertibility is guaranteed by the following
 685 proposition which is proved in Section 5.2.

686 PROPOSITION 2.5. *Under condition (1.33), the matrix M defined in (2.17) is in-*
 687 *vertible.*

688 The matrix M plays a crucial role in the computation of the cost $\mathcal{C}(G, y_0)$. Let us
689 give some comments. It is a sum of Gram matrices whose construction is summarized
690 in Figure 1 on an example with $G = \{\lambda_1, \lambda_2\}$ with $\alpha_1 = 3$ and $\alpha_2 = 2$. Each of these
691 matrices is of size $|\alpha|$ which is the number of eigenvalues (counted with their algebraic
692 multiplicities) that belong to the group G . Thus, on actual examples (see Section 7),
693 the size of these matrices is usually reasonably small.

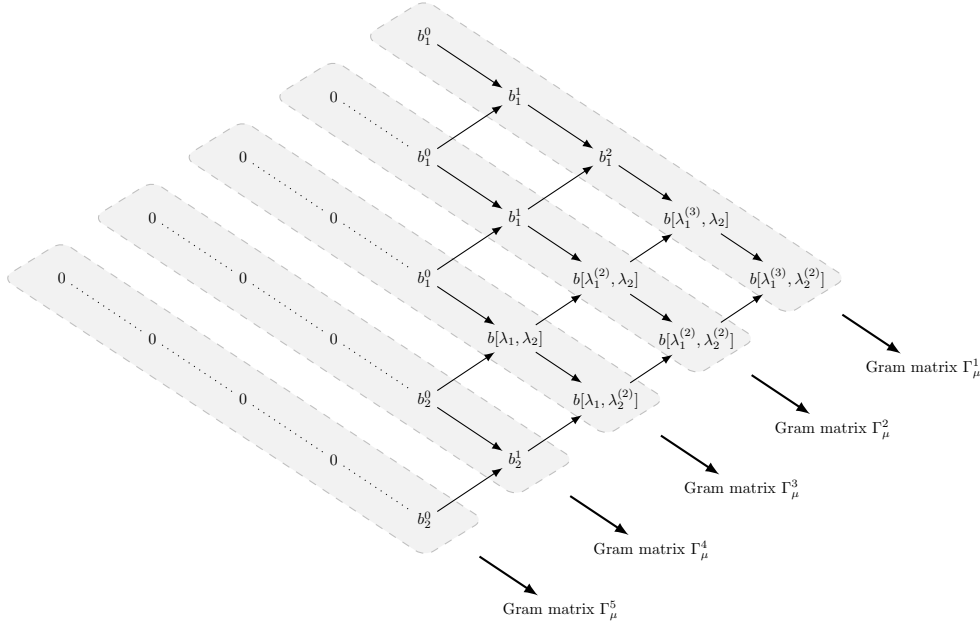


FIGURE 1. Construction of the Gram matrices Γ_μ^l in the case of a group $G = \{\lambda_1, \lambda_2\}$ with multiplicities $\alpha = (3, 2)$ and the sequence of multi-indices $\mu = ((0, 0), (1, 0), (2, 0), (3, 0), (3, 1), (3, 2))$

694 Then, we obtain the following formula for the cost of a group of geometrically
695 simple eigenvalues.

696 **THEOREM 2.6.** Assume that the operators \mathcal{A} and \mathcal{B} satisfy the assumption (H)
697 (see Section 1.4.2). Let $G = \{\lambda_1, \dots, \lambda_g\} \subset \Lambda$ be such that $\sharp G \leq p$ and $\text{diam } G \leq \rho$
698 and assume that $\gamma_\lambda = 1$ for every $\lambda \in G$. Then, for any $y_0 \in X_{-\diamond}$, we have

$$699 \quad \mathcal{C}(G, y_0) = \langle M^{-1}\xi, \xi \rangle, \quad \text{where } \xi = \begin{pmatrix} \langle y_0, \phi \left[\lambda_{\bullet}^{(\mu^1)} \right] \rangle_{-\diamond, \diamond} \\ \vdots \\ \langle y_0, \phi \left[\lambda_{\bullet}^{(\mu^{|\alpha|})} \right] \rangle_{-\diamond, \diamond} \end{pmatrix} \in \mathbb{C}^{|\alpha|}$$

700 and M is defined in (2.17).

701 Moreover, if Y_0 is a closed subspace of $X_{-\diamond}$,

$$702 \quad (2.18) \quad \mathcal{C}(G, Y_0) = \rho \left(\text{Gram}_{X_\diamond^*}(\psi_1, \dots, \psi_{|\alpha|}) M^{-1} \right)$$

703 where $\psi_j := P_{Y_0}^* \phi \left[\lambda_{\bullet}^{(\mu^j)} \right]$ and, for any matrix M , the notation $\rho(M)$ denotes the

704 spectral radius of the matrix M .

705 **REMARK 2.1.** Notice that we do not choose any particular eigenvector or Jordan
 706 chain. To compute explicitly the cost $\mathcal{C}(G, y_0)$ on actual examples, we will often choose
 707 them to satisfy

$$708 \quad \|b_j^0\|_U = 1, \quad \langle b_j^0, b_j^l \rangle_U = 0, \quad \forall l \in \llbracket 1, \alpha_j \rrbracket,$$

709 to simplify the Gram matrices. Obviously, as the quantity $\mathcal{C}(G, y_0)$ is independent of
 710 this choice, we can choose any other specific Jordan chains or eigenvectors that are
 711 more suitable to each problem.

712 **REMARK 2.2.** In the case where the eigenvalues of the considered group G are
 713 also algebraically simple, then the expression of M given in (2.17) reduces to

$$714 \quad (2.19) \quad M = \sum_{l=1}^g \Gamma^l \quad \text{with} \quad \Gamma^l = \text{Gram}_U \left(\underbrace{0, \dots, 0}_{l-1}, b[\lambda_l], \dots, b[\lambda_l, \dots, \lambda_g] \right)$$

715 and the expression of ξ reduces to

$$716 \quad \xi = \begin{pmatrix} \langle y_0, \phi[\lambda_1] \rangle_{-\diamond, \diamond} \\ \vdots \\ \langle y_0, \phi[\lambda_1, \dots, \lambda_g] \rangle_{-\diamond, \diamond} \end{pmatrix}.$$

717 \star A group G of semi-simple eigenvalues.

718 We now assume that all the eigenvalues in G are semi-simple *i.e.* for any $\lambda \in G$
 719 we have $\alpha_\lambda = 1$ where α_λ is defined in (1.32).

720 For any $j \in \llbracket 1, g \rrbracket$, we denote by $(\phi_{j,i})_{i \in \llbracket 1, \gamma_j \rrbracket}$ a basis of $\text{Ker}(\mathcal{A}^* - \lambda_j)$. To simplify
 721 the writing, we set

$$722 \quad b_{j,i} := \mathcal{B}^* \phi_{j,i}, \quad \forall j \in \llbracket 1, g \rrbracket, \forall i \in \llbracket 1, \gamma_j \rrbracket$$

723 and $\gamma_G := \gamma_1 + \dots + \gamma_g$.

724 For any $i \in \llbracket 1, g \rrbracket$, we set $\delta_1^i := 1$ and

$$725 \quad (2.20) \quad \delta_j^i := \prod_{k=1}^{j-1} (\lambda_i - \lambda_k), \quad \forall j \in \llbracket 2, g \rrbracket.$$

726 Notice that $\delta_j^i = 0$ as soon as $j > i$.

727 Let

$$728 \quad (2.21) \quad M = \sum_{l=1}^g \Gamma^l \quad \text{with} \quad \Gamma^l = \text{Gram}_U (\delta_l^1 b_{1,1}, \dots, \delta_l^1 b_{1,\gamma_1}, \dots, \delta_l^g b_{g,1}, \dots, \delta_l^g b_{g,\gamma_g}).$$

729 Here again, to explicit the cost $\mathcal{C}(G, y_0)$ we will use the inverse of this matrix. Its
 730 invertibility is guaranteed by the following proposition which is proved in Section 5.3.

731 **PROPOSITION 2.7.** Under condition (1.33), the matrix M defined in (2.21) is in-
 732 vertible.

733 Notice that the square matrix Γ^l is of size γ_G and can be seen as a block matrix where
 734 the block (i, j) with γ_i rows and γ_j columns is

$$735 \begin{pmatrix} \langle \delta_l^j b_{j,1}, \delta_l^i b_{i,1} \rangle_U & \cdots & \langle \delta_l^j b_{j,\gamma_j}, \delta_l^i b_{i,1} \rangle_U \\ \vdots & & \vdots \\ \langle \delta_l^j b_{j,1}, \delta_l^i b_{i,\gamma_i} \rangle_U & \cdots & \langle \delta_l^j b_{j,\gamma_j}, \delta_l^i b_{i,\gamma_i} \rangle_U \end{pmatrix}.$$

736 Thus, the block (i, j) of Γ^l is identically 0 for $i, j \in \llbracket 1, l \rrbracket$.

737 Then, we obtain the following formula for the cost of a group made of semi-simple
 738 eigenvalues.

739 **THEOREM 2.8.** *Assume that the operators \mathcal{A} and \mathcal{B} satisfy the assumption (H)
 740 (see Section 1.4.2). Let $G = \{\lambda_1, \dots, \lambda_g\} \subset \Lambda$ be such that $\sharp G \leq p$ and $\text{diam } G \leq p$
 741 and assume that $\alpha_\lambda = 1$ for every $\lambda \in G$. Then, for any $y_0 \in X_{-\diamond}$, we have*

$$742 \mathcal{C}(G, y_0) = \langle M^{-1}\xi, \xi \rangle$$

743 where

$$744 \xi = \begin{pmatrix} \langle y_0, \phi_{1,1} \rangle_{-\diamond, \diamond} \\ \vdots \\ \langle y_0, \phi_{1,\gamma_1} \rangle_{-\diamond, \diamond} \\ \vdots \\ \langle y_0, \phi_{g,1} \rangle_{-\diamond, \diamond} \\ \vdots \\ \langle y_0, \phi_{g,\gamma_g} \rangle_{-\diamond, \diamond} \end{pmatrix}$$

745 and M is defined in (2.21).

746 Moreover, if Y_0 is a closed subspace of $X_{-\diamond}$,

$$747 (2.22) \quad \mathcal{C}(G, Y_0) = \rho(\text{Gram}_{X_\diamond^*}(\psi_{1,1}, \dots, \psi_{1,\gamma_1}, \dots, \psi_{g,1}, \dots, \psi_{g,\gamma_g})M^{-1})$$

748 where $\psi_{j,i} := P_{Y_0}^* \phi_{j,i}$ and, for any matrix M , the notation $\rho(M)$ denotes its spectral
 749 radius.

750 **REMARK 2.3.** *When the eigenvalues of the group G are geometrically and alge-
 751 braically simple, Theorem 2.8 gives a characterization of the cost of the block $\mathcal{C}(G, y_0)$
 752 which is different from the one coming from Theorem 2.6 and detailed in Remark 2.2.
 753 A direct proof of this equivalence (stated in Proposition D.3) using algebraic manipu-
 754 lations is given in Appendix D.*

755 *★ Dealing simultaneously with geometric and algebraic multiplicity.*

756 Combining Theorems 2.6 and 2.8, we can deal with operators \mathcal{A}^* which have
 757 both groups of geometrically simple eigenvalues and groups of semi-simple eigenval-
 758 ues. However, for technical reasons, in the case where both algebraic and geometric
 759 multiplicities need to be taken into account into a group G we do obtain a general
 760 formula for the cost of this group $\mathcal{C}(G, y_0)$. Nevertheless, if this situation occurs in
 761 actual examples, computing this cost is a finite dimensional constrained optimization
 762 problem which can be solved ‘by hand’. We present in Section 5.4 an example of such
 763 resolution for a group G that does not satisfies the assumptions of Theorem 2.6 nor
 764 of Theorem 2.8.

765 **2.4. Estimate of the cost of null controllability.**

766 When system (1.1) is null controllable, we obtain the following bound on the cost
767 of controllability.

768 PROPOSITION 2.9. Assume that the operators \mathcal{A} and \mathcal{B} satisfy the assumption (H)
769 (see Section 1.4.2) and let \mathcal{G} be an associated grouping as stated in Proposition 1.2.

770 Let $y_0 \in X_{-\diamond}$ and let $T > T_0(y_0)$. There exists a control $u \in L^2(0, T; U)$ such
771 that the associated solution of (1.1) initiated from y_0 satisfies $y(T) = 0$ and

$$772 \|u\|_{L^2(0, T; U)}^2 \leq C \exp\left(\frac{C}{(T - T_0(y_0))^{\frac{\theta}{1-\theta}}}\right) (1+T)^{2\eta} \sum_{G \in \mathcal{G}} e^{-\frac{T-T_0(y_0)}{2} r_G} e^{-2r_G T_0(y_0)} \mathcal{C}(G, Y_0).$$

773 The constant $C > 0$ appearing in the estimate only depends on the parameters τ , p ,
774 ϱ , η , θ and κ .

775 Though quite general the above formula is not very explicit. More importantly,
776 it is proved in [29, Theorem 1.1] that, with a suitable choice of \mathcal{A} and \mathcal{B} satisfying
777 our assumptions, any blow-up of the cost of controllability can occur. We give below
778 a setting (inspired from [29, Theorem 1.2]) in which this upper bound on the cost of
779 controllability is simpler and can have some applications (see Remark 2.5).

780 COROLLARY 2.10. Assume that the operators \mathcal{A} and \mathcal{B} satisfy the assumption (H)
781 (see Section 1.4.2) and let \mathcal{G} be an associated grouping as stated in Proposition 1.2.
782 Let $\beta > 0$. For any $y_0 \in X_{-\diamond}$ satisfying,

$$783 (2.23) \quad \mathcal{C}(G, y_0) \leq \beta e^{2r_G T_0(y_0)} \|y_0\|_{-\diamond}^2, \quad \forall G \in \mathcal{G},$$

784 for any $T > T_0(y_0)$ close enough to $T_0(y_0)$, there exists a control $u \in L^2(0, T; U)$ such
785 that the associated solution of (1.1) satisfies $y(T) = 0$ and

$$786 \|u\|_{L^2(0, T; U)} \leq C \exp\left(\frac{C}{(T - T_0(y_0))^{\frac{\theta}{1-\theta}}}\right) \|y_0\|_{-\diamond},$$

787 where the constant $C > 0$ only depends on the parameters β , τ , p , ϱ , η , θ and κ .

788 REMARK 2.4. In the setting of Corollary 2.10, replacing the assumption (2.23)
789 by

$$790 \mathcal{C}(G, y_0) \leq \beta e^{\beta r_G^\sigma} e^{2r_G T_0(y_0)} \|y_0\|_{-\diamond}^2, \quad \forall G \in \mathcal{G},$$

791 with $\sigma \in (0, 1)$ leads to the following estimate

$$792 \|u\|_{L^2(0, T; U)} \leq C \exp\left(\frac{C}{(T - T_0(y_0))^{\frac{\max(\theta, \sigma)}{1-\max(\theta, \sigma)}}}\right) \|y_0\|_{-\diamond}.$$

793 REMARK 2.5. Giving the best possible estimate on the cost of small time null
794 controllability is a question that has drawn a lot of interest in the past years.

795 In classical cases, for instance for heat-like equations, null controllability holds
796 in any positive time and the cost of controllability in small time behaves like $\exp\left(\frac{C}{T}\right)$
797 (see for instance [37]). There are two mains applications of such estimate.

- 798 • Controllability in cylindrical domains.

799 It is proved in [8] that null controllability of parabolic problems in cylindrical
800 geometries (with operators compatible with this geometry) with a boundary
801 control located on the top of the cylinder can be proved thanks to null con-
802 trollability of the associated problem in the transverse variable together with

803 suitable estimates of the cost of controllability. Their proof relies on an adap-
 804 tation of the classical strategy of Lebeau and Robbiano [28] and thus uses
 805 an estimate of the cost of controllability in small time of the form $\exp(\frac{C}{T})$.
 806 These ideas were already present in [10] and later generalized in an abstract
 807 setting in [1].

808 • **Nonlinear control problems.**

809 The source term method has been introduced in [30] to prove controllability
 810 of a nonlinear fluid-structure system (see also [7, Section 2] for a general
 811 presentation of this strategy). Roughly speaking it amounts to prove null con-
 812 trollability with a source term in suitable weighted spaces and then use a fixed
 813 point argument. The null controllability with a source term is here proved by
 814 an iterative process which strongly uses that the cost of controllability of the
 815 linearized system behaves like $\exp(\frac{C}{T})$.

816 Notice that from the upper bound given in Corollary 2.10, the cost of controllability
 817 in small time can explode faster than $\exp(\frac{C}{T})$. Yet, as studied in [33] and in [35,
 818 Chapter 4], the arguments of the two previous applications can be adapted with an
 819 explosion of the cost of the form $\exp\left(\frac{C}{T^{1-\theta}}\right)$ with $\theta \in (0, 1)$.

820 However, these two applications uses a decomposition of the time interval $[0, T]$
 821 into an infinite number of sub-intervals (which explains the use of the asymptotic of
 822 the cost of controllability when the time goes to zero). Thus their extension in the
 823 case of a minimal null control time is an open problem.

824 **3. Resolution of block moment problems.**

825 In this section we prove Theorem 2.2 that is we solve the block moment prob-
 826 lem (2.12). To do so, we first consider a vectorial block moment problem (see (3.1)
 827 below) which is proved to be equivalent to the block moment problem (2.12) in Propo-
 828 sition 3.1. This equivalence strongly relies on the constraints (2.6). Then we prove
 829 the lower bound for solutions of block moment problems stated in Proposition 2.1.

830 Finally, in Section 3.2, we solve the vectorial block moment problem (3.1) which
 831 will conclude the proof of Theorem 2.2.

832 **3.1. An auxiliary equivalent vectorial block moment problem.**

833 Let $\Lambda \subset S_\tau$, $G = \{\lambda_1, \dots, \lambda_g\} \subset \Lambda$, $\eta \in \mathbb{N}^*$ and $\alpha = (\alpha_1, \dots, \alpha_g) \in \mathbb{N}^g$ with
 834 $|\alpha|_\infty \leq \eta$. For any

835
$$\Omega = (\Omega_1^0, \dots, \Omega_1^{\alpha_1-1}, \dots, \Omega_g^0, \dots, \Omega_g^{\alpha_g-1}) \in U^{|\alpha|},$$

836 we consider the following auxiliary vectorial block moment problem : find $v_G \in$
 837 $L^2(0, T; U)$ such that

838 (3.1a)
$$\int_0^T v_G(t) \frac{(-t)^l}{l!} e^{-\bar{\lambda}_j t} dt = \Omega_j^l, \quad \forall j \in \llbracket 1, g \rrbracket, \forall l \in \llbracket 0, \alpha_j \rrbracket,$$

839 (3.1b)
$$\int_0^T v_G(t) t^l e^{-\bar{\lambda} t} dt = 0, \quad \forall \lambda \in \Lambda \setminus G, \forall l \in \llbracket 0, \eta \rrbracket.$$

 840

841 This block moment problem is said to be *vectorial*: the right-hand side Ω belongs
 842 to $U^{|\alpha|}$ and its solution $v_G(t)$ belongs to the control space U for almost every t . Its
 843 resolution with (almost) sharp estimates is given in Proposition 3.3.

844 Through (2.5), when the right-hand side Ω of (3.1) satisfy the constraints (2.7),
 845 solving this vectorial block moment problem provides a solution of the original block
 846 moment problem (2.12). More precisely we have the following proposition

847 PROPOSITION 3.1. Let $T > 0$ and $z \in X_{-\diamond}$. The following two statements are
 848 equivalent:

- 849 i. there exists $\Omega \in \mathcal{O}(G, z)$ such that the function $v_G \in L^2(0, T; U)$ solves (3.1);
- 850 ii. the function $v_G \in L^2(0, T; U)$ solves (2.12).

851 *Proof.* Assume first that there exists $\Omega \in \mathcal{O}(G, z)$ and let $v \in L^2(0, T; U)$ be such
 852 that (3.1) holds.

853 Then, using (2.5), for any $j \in \llbracket 1, g \rrbracket$ and any $\phi \in E_{\lambda_j}$ we have

$$\begin{aligned}
 854 \quad \int_0^T \left\langle v(t), \mathcal{B}^* e^{-t\mathcal{A}^*} \phi \right\rangle_U dt &= \int_0^T \left\langle v(t), e^{-\lambda_j t} \sum_{l=0}^{\alpha_j-1} \frac{(-t)^l}{l!} \mathcal{B}^* (\mathcal{A}^* - \lambda_j)^l \phi \right\rangle_U dt \\
 855 &= \sum_{l=0}^{\alpha_j-1} \left\langle \int_0^T v(t) \frac{(-t)^l}{l!} e^{-\bar{\lambda}_j t} dt, \mathcal{B}^* (\mathcal{A}^* - \lambda_j)^l \phi \right\rangle_U \\
 856 &= \sum_{l=0}^{\alpha_j-1} \langle \Omega_j^l, \mathcal{B}^* (\mathcal{A}^* - \lambda_j)^l \phi \rangle_U. \\
 857
 \end{aligned}$$

858 Since $(\Omega_j^0, \dots, \Omega_j^{\alpha_j-1}) \in \mathcal{O}(\lambda_j, z)$ (see (2.6)), this leads to

$$859 \quad \int_0^T \left\langle v(t), \mathcal{B}^* e^{-t\mathcal{A}^*} \phi \right\rangle_U dt = \langle z, \phi \rangle_{-\diamond, \diamond}, \quad \forall j \in \llbracket 1, g \rrbracket, \forall \phi \in E_{\lambda_j},$$

860 which proves that v solves (2.12).

861 Assume now that $v \in L^2(0, T; U)$ solves (2.12). Setting

$$862 \quad \Omega_j^l := \int_0^T v(t) \frac{(-t)^l}{l!} e^{-\bar{\lambda}_j t} dt$$

863 we obtain that v solves (3.1). As in the previous step, the identity (2.5) implies that
 864 $\Omega \in \mathcal{O}(G, z)$. \square

865 Using this vectorial block moment problem allows to prove the lower bound stated
 866 in Proposition 2.1.

867 *Proof (of Proposition 2.1).* Let $v_G \in L^2(0, T; U)$ be any solution of (2.12a). Let

$$868 \quad \Omega_j^l := \int_0^T v_G(t) \frac{(-t)^l}{l!} e^{-\bar{\lambda}_j t} dt = \int_0^T v_G(t) e_t \left[\bar{\lambda}_j^{(l+1)} \right] dt, \quad \forall j \in \llbracket 1, g \rrbracket, \forall l \in \llbracket 0, \alpha_j \rrbracket.$$

869 As in the proof of Proposition 3.1, the use of (2.5) implies that

$$870 \quad \Omega = (\Omega_1^0, \dots, \Omega_1^{\alpha_1-1}, \dots, \Omega_g^0, \dots, \Omega_g^{\alpha_g-1}) \in \mathcal{O}(G, z).$$

871 Thus,

$$872 \quad (3.2) \quad \mathcal{C}(G, z) \leq F(\Omega) = \sum_{l=1}^{|\alpha|} \left\| \Omega \left[\bar{\lambda}^{(\mu^l)} \right] \right\|_U^2.$$

873 Notice that

$$874 \quad \Omega \left[\bar{\lambda}_{\bullet}^{(\mu^l)} \right] = \int_0^T v_G(t) e_t \left[\bar{\lambda}_{\bullet}^{(\mu^l)} \right] dt, \quad \forall l \in \llbracket 0, |\alpha| \rrbracket.$$

875 Using Jensen inequality [9, Proposition 6.1] yields,

$$876 \quad \left| e_t \left[\bar{\lambda}_{\bullet}^{(\mu^l)} \right] \right| = \left| e_t \left[\lambda_{\bullet}^{(\mu^l)} \right] \right| \leq \frac{t^{l-1} e^{-r_G t}}{(l-1)!}.$$

877 Together with Cauchy-Schwarz inequality this implies

$$878 \quad \left\| \Omega \left[\bar{\lambda}_{\bullet}^{(\mu^l)} \right] \right\|_U \leq \left(\int_0^{+\infty} \frac{t^{l-1} e^{-r_G t}}{(l-1)!} dt \right)^{\frac{1}{2}} \|v_G\|_{L^2(0,T;U)}.$$

879 Then, as $r_G \geq r_\Lambda$ and $|\alpha| \leq p\eta$, estimate (3.2) ends the proof of Theorem 2.2. \square

880 3.2. Solving the original moment problem.

881 In view of Proposition 3.1, to solve (2.12), we prove that there exists at least one
882 Ω satisfying the constraints (2.6).

883 PROPOSITION 3.2. *Let $\lambda \in \Lambda$ and $z \in X_{-\diamond}$. Then, under assumption (1.33), we
884 have*

$$885 \quad \mathcal{O}(\lambda, z) \neq \emptyset.$$

886 *Proof.* Let $T > 0$. The finite dimensional space E_λ is stable by the semi-
887 group $e^{-\bullet \mathcal{A}^*}$ (see for instance (2.5)). Using the approximate controllability assump-
888 tion (1.33) we have that

$$889 \quad \phi \in E_\lambda \mapsto \left\| \mathcal{B}^* e^{-\bullet \mathcal{A}^*} \phi \right\|_{L^2(0,T,U)}$$

890 is a norm on E_λ . Then, the equivalence of norms in finite dimension implies that the
891 following HUM-type functional

$$892 \quad J : \phi \in E_\lambda \mapsto \frac{1}{2} \left\| \mathcal{B}^* e^{-\bullet \mathcal{A}^*} \phi \right\|_{L^2(0,T,U)}^2 - \Re \langle z, \phi \rangle_{-\diamond, \diamond}$$

893 is coercive. Let $\tilde{\phi} \in E_\lambda$ be such that

$$894 \quad J(\tilde{\phi}) = \inf_{\phi \in E_\lambda} J(\phi)$$

895 and $v := \mathcal{B}^* e^{-\bullet \mathcal{A}^*} \tilde{\phi}$. The optimality condition gives (paying attention to the fact that
896 E_λ is a complex vector space)

$$897 \quad (3.3) \quad \int_0^T \left\langle v(t), \mathcal{B}^* e^{-t \mathcal{A}^*} \phi \right\rangle_U dt = \langle z, \phi \rangle_{-\diamond, \diamond}, \quad \forall \phi \in E_\lambda.$$

898 Finally, we set $\Omega := (\Omega^0, \dots, \Omega^{\alpha_\lambda - 1})$ with

$$899 \quad \Omega^l := \int_0^T v(t) \frac{(-t)^l}{l!} e^{-\bar{\lambda} t} dt, \quad \forall l \in \llbracket 0, \alpha_\lambda \rrbracket.$$

900 Using (3.3) and following the computations of Proposition 3.1 we obtain that $\Omega \in$
901 $\mathcal{O}(\lambda, z)$. \square

902 We now turn to the resolution of the vectorial block moment problem (3.1).

903 PROPOSITION 3.3. Let $p \in \mathbb{N}^*$, $\varrho, \tau, \kappa > 0$ and $\theta \in (0, 1)$. Assume that

904
$$\Lambda \in \mathcal{L}_w(p, \varrho, \tau, \theta, \kappa).$$

905 Let $G = \{\lambda_1, \dots, \lambda_g\} \subset \Lambda$ be a group satisfying (1.25)–(1.27). Let $T \in (0, +\infty)$ and
 906 $\eta \in \mathbb{N}^*$. For any multi-index $\alpha \in \mathbb{N}^g$ with $|\alpha|_\infty \leq \eta$ and any

907
$$\Omega = (\Omega_1^0, \dots, \Omega_1^{\alpha_1-1}, \dots, \Omega_g^0, \dots, \Omega_g^{\alpha_g-1}) \in U^{|\alpha|},$$

908 there exists $v_G \in L^2(0, T; U)$ solution of (3.1) such that

909
$$\|v_G\|_{L^2(0, T; U)}^2 \leq C \exp\left(\frac{C}{T^{1-\theta}}\right) \exp(r_G T) \exp(Cr_G^\theta) F(\Omega),$$

910 where F is defined in (2.9) and r_G is defined in (1.35). The constant $C > 0$ appearing
 911 in the estimate only depends on the parameters $\tau, p, \varrho, \eta, \theta$ and κ .

912 *Proof.* Let $(e_j)_{j \in \llbracket 1, d \rrbracket}$ be an orthonormal basis of the finite dimensional subspace
 913 of U given by

914
$$\text{Span} \left\{ \Omega_j^l; j \in \llbracket 1, g \rrbracket, l \in \llbracket 0, \alpha_j \rrbracket \right\}.$$

915 Then, for any $j \in \llbracket 1, g \rrbracket$ and $l \in \llbracket 0, \alpha_j \rrbracket$, there exists $\left(a_i \left[\overline{\lambda_j}^{(l+1)} \right] \right)_{i \in \llbracket 1, d \rrbracket} \in \mathbb{C}^{|\alpha|}$ such
 916 that the decomposition of Ω_j^l reads

917
$$\Omega_j^l = \sum_{i=1}^d a_i \left[\overline{\lambda_j}^{(l+1)} \right] e_i.$$

918 From Theorem A.1, for any $i \in \llbracket 1, d \rrbracket$, there exists $v_i \in L^2(0, T; \mathbb{C})$ such that

919
$$\begin{cases} \int_0^T v_i(t) \frac{(-t)^l}{l!} e^{-\overline{\lambda_j} t} dt = a_i \left[\overline{\lambda_j}^{(l+1)} \right], & \forall j \in \llbracket 1, g \rrbracket, \forall l \in \llbracket 0, \alpha_j \rrbracket, \\ \int_0^T v_i(t) t^l e^{-\overline{\lambda} t} dt = 0, & \forall \lambda \in \Lambda \setminus G, \forall l \in \llbracket 0, \eta \rrbracket, \end{cases}$$

920 and

921
$$\|v_i\|_{L^2(0, T; \mathbb{C})}^2 \leq C e^{CT^{-\frac{\theta}{1-\theta}}} e^{Cr_G^\theta} \max_{\substack{\mu \in \mathbb{N}^g \\ \mu \leq \alpha}} \left| a_i \left[\overline{\lambda_1}^{(\mu_1)}, \dots, \overline{\lambda_g}^{(\mu_g)} \right] \right|^2.$$

922 Setting

923
$$v := \sum_{i=1}^d v_i e_i,$$

924 we get that v solves (3.1) and using [9, Proposition 7.15]

$$\begin{aligned}
925 \quad \|v\|_{L^2(0,T;U)}^2 &= \sum_{i=1}^d \|v_i\|_{L^2(0,T;\mathbb{C})}^2 \\
926 \quad &\leq C e^{CT^{-\frac{\theta}{1-\theta}}} e^{r_G T} e^{Cr_G^\theta} \sum_{i=1}^d \max_{\substack{\mu \in \mathbb{N}^g \\ \mu \leq \alpha}} |a_i [\bar{\lambda}_1^{(\mu_1)}, \dots, \bar{\lambda}_g^{(\mu_g)}]|^2 \\
927 \quad &\leq C_{p,\varrho,\eta} C e^{CT^{-\frac{\theta}{1-\theta}}} e^{r_G T} e^{Cr_G^\theta} \sum_{p=1}^{|\alpha|} \left(\sum_{i=1}^d |a_i [\bar{\lambda}_\bullet^{(\mu^p)}]|^2 \right) \\
928 \quad &= C e^{CT^{-\frac{\theta}{1-\theta}}} e^{r_G T} e^{Cr_G^\theta} \sum_{p=1}^{|\alpha|} \left\| \Omega [\bar{\lambda}_\bullet^{(\mu^p)}] \right\|^2. \\
929
\end{aligned}$$

930 This ends the proof of Proposition 3.3. \square

931 We now have all the ingredients to prove Theorem 2.2.

932 *Proof (of Theorem 2.2).* From Proposition 3.2, we have $\mathcal{O}(G, z) \neq \emptyset$. Recall
933 that, from (2.16), the optimization problem defining $\mathcal{C}(G, z)$ can be reduced to a
934 finite dimensional optimization problem for which the infimum is attained. Thus, let
935 $\Omega \in \mathcal{O}(G, z)$ be such that

$$936 \quad F(\Omega) = \mathcal{C}(G, z).$$

937 Let $v_G \in L^2(0, T; U)$ be the solution of (3.1) given by Proposition 3.3 with Ω as right-
938 hand side. As $\Omega \in \mathcal{O}(G, z)$, from Proposition 3.1 we deduce that v_G solves (2.12).
939 The upper bound (2.13) on $\|v_G\|_{L^2(0,T;U)}$ is given by Proposition 3.3. \square

940 4. Application to the determination of the minimal null control time.

941 This section is dedicated to the consequences of Theorem 2.2 on the null control-
942 lability properties of system (1.1).

943 From Theorem 2.2, the resolution of block moment problems (2.4) associated with
944 null controllability of (1.1) will involve the quantity $\mathcal{C}(G, e^{-T\mathcal{A}}y_0)$. To formulate the
945 minimal null control time we isolate the dependency with respect to the variable T
946 leading to quantities involving $\mathcal{C}(G, y_0)$. The comparison between these two costs is
947 detailed in Section 4.1.

948 Then, this leads to the formulation of the minimal null control time stated in
949 Theorem 2.3. We then prove the estimates on the cost of null controllability stated
950 in Proposition 2.9 and Corollary 2.10. This is detailed in Section 4.2.

951 4.1. Relating the different costs.

952 Let us prove that the cost $\mathcal{C}(G, e^{-T\mathcal{A}}z)$ appearing in Theorem 2.2 roughly behaves
953 like $e^{-2r_G T} \mathcal{C}(G, z)$. More precisely, we have the following estimates.

954 LEMMA 4.1. *Assume that the operators \mathcal{A} and \mathcal{B} satisfy the assumption (H) (see*
955 *Section 1.4.2). There exists $C_{p,\varrho,\eta} > 0$ such that for any $G \subset \Lambda$ with $\sharp G \leq p$ and*
956 *diam $G \leq \varrho$, for any $T > 0$ and any $z \in X_{-\varrho}$,*

$$957 \quad (4.1) \quad \mathcal{C}(G, e^{-T\mathcal{A}}z) \leq C_{p,\varrho,\eta} (1+T)^{2|\alpha|} e^{-2r_G T} \mathcal{C}(G, z)$$

958 and

$$959 \quad (4.2) \quad e^{-2r_G T} \mathcal{C}(G, z) \leq C_{p,\varrho,\eta} (1+T)^{2|\alpha|} e^{2\varrho T} \mathcal{C}(G, e^{-T\mathcal{A}}z).$$

960 *Proof.* Recall that from (1.21) we have

$$961 \quad \left\langle y_0, e^{-T\mathcal{A}^*} \phi \right\rangle_{-\diamond, \diamond} = \left\langle e^{-T\mathcal{A}} y_0, \phi \right\rangle_{-\diamond, \diamond}, \quad \forall \phi \in X_\diamond^*.$$

962 We set $G = \{\lambda_1, \dots, \lambda_g\}$.

963 • We start with the proof of (4.1).

964 From (2.16), let $\tilde{\Omega} \in \mathcal{O}(G, z)$ be such that $F(\tilde{\Omega}) = \mathcal{C}(G, z)$. We define Ω by

$$965 \quad \Omega_j^l := (e_T \tilde{\Omega}) \left[\overline{\lambda_j^{(l+1)}} \right], \quad \forall j \in \llbracket 1, g \rrbracket, \forall l \in \llbracket 0, \alpha_j \rrbracket,$$

966 with the convention

$$967 \quad \tilde{\Omega} \left[\overline{\lambda_j^{(l+1)}} \right] = \tilde{\Omega}_j^l, \quad \forall j \in \llbracket 1, g \rrbracket, \forall l \in \llbracket 0, \alpha_j \rrbracket.$$

968 Let us prove that $\Omega \in \mathcal{O}(G, e^{-T\mathcal{A}}z)$. For any $j \in \llbracket 1, g \rrbracket$ and any $\phi \in E_{\lambda_j}$, using [9,
969 Definition 7.12] we obtain

$$\begin{aligned} 970 \quad \sum_{l \geq 0} \langle \Omega_j^l, \mathcal{B}^*(\mathcal{A}^* - \lambda_j)^l \phi \rangle_U &= \sum_{l \geq 0} \sum_{r=0}^l e_T \left[\overline{\lambda_j^{(r+1)}} \right] \langle \tilde{\Omega}_j^{l-r}, \mathcal{B}^*(\mathcal{A}^* - \lambda_j)^l \phi \rangle_U \\ 971 &= \sum_{r \geq 0} e_T \left[\overline{\lambda_j^{(r+1)}} \right] \sum_{l \geq r} \langle \tilde{\Omega}_j^{l-r}, \mathcal{B}^*(\mathcal{A}^* - \lambda_j)^l \phi \rangle_U \\ 972 &= \sum_{r \geq 0} e_T \left[\overline{\lambda_j^{(r+1)}} \right] \sum_{l \geq 0} \langle \tilde{\Omega}_j^l, \mathcal{B}^*(\mathcal{A}^* - \lambda_j)^{l+r} \phi \rangle_U. \end{aligned}$$

974 Since $\tilde{\Omega} \in \mathcal{O}(G, z)$ and $e_T \left[\overline{\lambda_j^{(r+1)}} \right] = \overline{e_T \left[\lambda_j^{(r+1)} \right]}$ for any $r \geq 0$, using (2.5) this
975 yields

$$\begin{aligned} 976 \quad \sum_{l \geq 0} \langle \Omega_j^l, \mathcal{B}^*(\mathcal{A}^* - \lambda_j)^l \phi \rangle_U &= \sum_{r \geq 0} \overline{e_T \left[\lambda_j^{(r+1)} \right]} \langle z, ((\mathcal{A}^* - \lambda_j)^r \phi) \rangle_{-\diamond, \diamond} \\ 977 \quad (4.3) \quad &= \left\langle z, e^{-T\mathcal{A}^*} \phi \right\rangle_{-\diamond, \diamond} = \left\langle e^{-T\mathcal{A}} z, \phi \right\rangle_{-\diamond, \diamond}. \end{aligned}$$

979 This proves the claim.

980 Applying Leibniz formula [9, Proposition 7.13] and Jensen inequality [9, Propo-
981 sition 6.1] we obtain,

$$\begin{aligned} 982 \quad \left\| \Omega \left[\overline{\lambda_\bullet^{(\mu^l)}} \right] \right\|_U &= \left\| \sum_{q=1}^l e_T \left[\overline{\lambda_\bullet^{(\mu^l - \mu^{q-1})}} \right] \tilde{\Omega} \left[\overline{\lambda_\bullet^{(\mu^q)}} \right] \right\| \\ 983 &\leq C_{p, \varrho, \eta} (1+T)^{|\alpha|} e^{-r_G T} \left(\sum_{q=1}^l \left\| \tilde{\Omega} \left[\overline{\lambda_\bullet^{(\mu^q)}} \right] \right\|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

984 Thus,

$$986 \quad F(\Omega) \leq C_{p, \varrho, \eta} (1+T)^{2|\alpha|} e^{-2r_G T} F(\tilde{\Omega}) = C_{p, \varrho, \eta} (1+T)^{2|\alpha|} e^{-2r_G T} \mathcal{C}(G, z).$$

987 As $\Omega \in \mathcal{O}(G, e^{-T\mathcal{A}}z)$, this proves (4.1).

988 • The proof of (4.2) uses the same ingredients.
 989 From (2.16), let $\Omega \in \mathcal{O}(G, e^{-T\mathcal{A}}z)$ be such that $F(\Omega) = \mathcal{C}(G, e^{-T\mathcal{A}}z)$. For any
 990 $j \in \llbracket 1, g \rrbracket$ and any $l \in \llbracket 0, \alpha_j \rrbracket$, let

$$991 \quad \tilde{\Omega}_j^l := (e_{-T}\Omega) \left[\bar{\lambda}_j^{(l+1)} \right]$$

992 where

$$993 \quad \Omega \left[\bar{\lambda}_j^{(l+1)} \right] := \Omega_j^l.$$

994 As previously, applying Leibniz formula [9, Proposition 7.13] and Jensen inequality [9,
 995 Proposition 6.1], since λ_j satisfies $\Re \lambda_j \leq r_G + \varrho$ for any $j \in \llbracket 1, g \rrbracket$, we obtain

$$996 \quad \left\| \tilde{\Omega} \left[\bar{\lambda}_{\bullet}^{(\mu^l)} \right] \right\|_U \leq C_{p,\varrho,\eta} (1+T)^{2|\alpha|} e^{(r_G+\varrho)T} \left(\sum_{q=1}^l \left\| \Omega \left[\bar{\lambda}_{\bullet}^{(\mu^q)} \right] \right\|^2 \right)^{\frac{1}{2}}.$$

997 The same computations as in (4.3) give that $\Omega \in \mathcal{O}(G, z)$. Thus

$$998 \quad \begin{aligned} \mathcal{C}(G, z) &\leq F(\tilde{\Omega}) \leq C_{p,\varrho,\eta} (1+T)^{2|\alpha|} e^{2(r_G+\varrho)T} F(\Omega) \\ 1000 &= C_{p,\varrho,\eta} (1+T)^{2|\alpha|} e^{2(r_G+\varrho)T} \mathcal{C}(G, e^{-T\mathcal{A}}z) \end{aligned}$$

1001 and (4.2) is proved. □

1002 4.2. The minimal null control time.

1003 This section is dedicated to the proof of Theorem 2.3 and Corollary 2.4 concerning
 1004 the minimal null control time. Proposition 2.9 and Corollary 2.10 concerning the
 1005 cost of null controllability will follow from the estimates obtained in the proof of
 1006 Theorem 2.3. This is discussed at the end of the current section.

1007 *Proof (of Theorem 2.3).*

1008 • We start with the proof of null controllability in time $T > T_0(y_0)$.

1009 We set $\varepsilon = T - T_0(y_0) > 0$. Let $G \in \mathcal{G}$ and let $v_G \in L^2(0, \varepsilon; U)$ be the solution
 1010 of the block moment problem (2.12) in time ε associated with $z = e^{-T\mathcal{A}}y_0$ given by
 1011 Theorem 2.2 *i.e.*

$$1012 \quad \int_0^\varepsilon \left\langle v_G(t), \mathcal{B}^* e^{-t\mathcal{A}^*} \phi \right\rangle_U dt = \langle e^{-T\mathcal{A}}y_0, \phi \rangle_{-\diamond, \diamond}, \quad \forall \phi \in E_\lambda, \forall \lambda \in G,$$

$$1013 \quad \int_0^\varepsilon v_G(t) t^l e^{-\bar{\lambda}t} dt = 0, \quad \forall \lambda \in \Lambda \setminus G, \forall l \in \llbracket 0, \eta \rrbracket.$$

1015 We still denote by $v_G \in L^2(0, T; U)$ the extension of v_G by 0. Thus, v_G satisfies

$$1016 \quad \int_0^T \left\langle v_G(t), \mathcal{B}^* e^{-t\mathcal{A}^*} \phi \right\rangle_U dt = \langle e^{-T\mathcal{A}}y_0, \phi \rangle_{-\diamond, \diamond}, \quad \forall \phi \in E_\lambda, \forall \lambda \in G,$$

$$1017 \quad \int_0^T v_G(t) t^l e^{-\bar{\lambda}t} dt = 0, \quad \forall \lambda \in \Lambda \setminus G, \forall l \in \llbracket 0, \eta \rrbracket.$$

1019 From (2.5), this implies that v_G solves (2.3). Thus, the only point left is to prove that
 1020 the series (2.2) defining the control u converges in $L^2(0, T; U)$.

1021 From Theorem 2.2 we have that

$$1022 \quad \|v_G\|_{L^2(0, T; U)}^2 = \|v_G\|_{L^2(0, \varepsilon; U)}^2 \leq C e^{C\varepsilon^{-\frac{\theta}{1-\theta}}} e^{\varepsilon r_G} e^{Cr_G^\theta} \mathcal{C}(G, e^{-T\mathcal{A}}y_0).$$

1023 Studying the maximum of the function $x \in [0, +\infty) \mapsto Cx^\theta - \frac{\varepsilon}{2}x$ it comes that

$$1024 \quad e^{Cr_G^\theta} \leq e^{C\varepsilon^{-\frac{\theta}{1-\theta}}} e^{\frac{\varepsilon}{2}r_G}$$

1025 where in the right-hand side the constant $C > 0$ has changed but still depend on the
1026 same parameters. Thus,

$$1027 \quad \|v_G\|_{L^2(0,T;U)}^2 \leq C e^{C\varepsilon^{-\frac{\theta}{1-\theta}}} e^{\frac{3}{2}\varepsilon r_G} \mathcal{C}(G, e^{-T\mathcal{A}}y_0).$$

1028 Using (4.1) we obtain

$$1029 \quad \|v_G\|_{L^2(0,T;U)}^2 \leq C e^{C\varepsilon^{-\frac{\theta}{1-\theta}}} (1+T)^{2|\alpha|} e^{-\frac{\varepsilon}{2}r_G} e^{-2r_G(T-\varepsilon)} \mathcal{C}(G, y_0).$$

1030 Recalling that $\varepsilon = T - T_0(y_0)$ this gives
1031 (4.4)

$$1031 \quad \|v_G\|_{L^2(0,T;U)}^2 \leq C \exp\left(\frac{C}{(T - T_0(y_0))^{\frac{\theta}{1-\theta}}}\right) (1+T)^{2|\alpha|} e^{-\frac{T-T_0(y_0)}{2}r_G} e^{-2r_G T_0(y_0)} \mathcal{C}(G, y_0).$$

1032 Recall that in (2.14) we have defined $T_0(y_0)$ by

$$1033 \quad T_0(y_0) = \limsup_{G \in \mathcal{G}} \frac{\ln^+ \mathcal{C}(G, y_0)}{2r_G}.$$

1034 Thus, when r_G is sufficiently large, we have

$$1035 \quad e^{-2r_G T_0(y_0)} \mathcal{C}(G, y_0) \leq \exp\left(\frac{T - T_0(y_0)}{4} r_G\right).$$

1036 Together with (4.4) this implies, for r_G sufficiently large,

$$1037 \quad \|v_G\|_{L^2(0,T;U)}^2 \leq C \exp\left(\frac{C}{(T - T_0(y_0))^{\frac{\theta}{1-\theta}}}\right) (1+T)^{2|\alpha|} \exp\left(-\frac{T - T_0(y_0)}{4} r_G\right)$$

1038 and proves that the series

$$1039 \quad (4.5) \quad u = \sum_{G \in \mathcal{G}} v_G(T - \bullet)$$

1040 converges in $L^2(0, T; U)$. This proves null controllability of (1.1) from y_0 in any time
1041 $T > T_0(y_0)$.

1042 • We now end the proof of Theorem 2.3 by proving that null controllability does not
1043 hold in time $T < T_0(y_0)$. The proof mainly relies on the optimality of the resolution
1044 of the block moment problems given in Proposition 2.1 (see (2.10)).

1045 Let $T > 0$. Assume that problem (1.1) is null controllable from y_0 in time T .
1046 Thus there exists $u \in L^2(0, T; U)$ such that $y(T) = 0$ and

$$1047 \quad \|u\|_{L^2(0,T;U)} \leq C_T \|y_0\|_{-\diamond}.$$

1048 Let $v := -u(T - \bullet)$. Then, for any $G \in \mathcal{G}$, v satisfies (2.12a) with $z = e^{-T\mathcal{A}}y_0$.
1049 From (2.10), this implies

$$1050 \quad (4.6) \quad C_T^2 \|y_0\|_{-\diamond}^2 \geq \|u\|_{L^2(0,T;U)}^2 = \|v\|_{L^2(0,T;U)}^2 \geq C_{p,\eta,r_\Lambda} \mathcal{C}(G, e^{-T\mathcal{A}}y_0).$$

1051 Applying (4.2) we obtain

$$1052 \quad \mathcal{C}(G, y_0) \leq C_{T,p,\varrho,\eta} e^{2r_G T} \mathcal{C}(G, e^{-TA} y_0).$$

1053 Together with (4.6) this implies

$$1054 \quad (4.7) \quad \mathcal{C}(G, y_0) \leq C_{T,p,\varrho,\eta,r_\Lambda} \|y_0\|_{-\infty}^2 e^{2r_G T}.$$

1055 Getting back to the definition of $T_0(y_0)$ given in (2.14), this implies that $T \geq T_0(y_0)$
 1056 and ends the proof of Theorem 2.3. \square

1057 **REMARK 4.1.** *It is worth noticing that the control v_G constructed is only active on*
 1058 *the time interval $(0, T - T_0(y_0))$. Thus, whenever $T_0(y_0) > 0$, the series (4.5) defining*
 1059 *the control u proves that it is possible to control y_0 to 0 in any time $T > T_0(y_0)$ with*
 1060 *a control that is identically vanishing on the time interval $(0, T_0(y_0))$.*

1061 We now turn to the proof of Corollary 2.4.

1062 *Proof (of Corollary 2.4).* By definition, we have $T_0(Y_0) = \sup_{y_0 \in Y_0} T_0(y_0)$. Using
 1063 the definition of $\mathcal{C}(G, Y_0)$ and Theorem 2.3, it directly comes that

$$1064 \quad T_0(Y_0) \leq \limsup_{G \in \mathcal{G}} \frac{\ln^+ \mathcal{C}(G, Y_0)}{2r_G}.$$

1065 We now focus on the converse inequality. Let $T > 0$ such that

$$1066 \quad T < \limsup_{G \in \mathcal{G}} \frac{\ln^+ \mathcal{C}(G, Y_0)}{2r_G}$$

1067 and let us prove that $T \leq T_0(Y_0)$.

1068 There exists $\varepsilon > 0$ and a sequence of groups $(G_k)_{k \in \mathbb{N}} \in \mathcal{G}^{\mathbb{N}}$ such that for any
 1069 $k \in \mathbb{N}^*$, there exists $y_{0,k} \in Y_0$ with $\|y_{0,k}\|_{-\infty} = 1$ satisfying

$$1070 \quad (4.8) \quad T + \varepsilon < \frac{\ln \mathcal{C}(G_k, y_{0,k})}{2r_{G_k}}.$$

1071 By contradiction, assume that for any $y_0 \in Y_0$, we have $T > T_0(y_0)$. Thus,
 1072 from (4.7), there exists $C_{T,p,\varrho,\eta,r_\Lambda} > 0$ such that for any $k \in \mathbb{N}^*$

$$1073 \quad \frac{\ln \mathcal{C}(G_k, y_{0,k})}{2r_{G_k}} \leq \frac{\ln C_{T,p,\varrho,\eta,r_\Lambda}}{2r_{G_k}} + T.$$

1074 Taking k sufficiently large, this is in contradiction with (4.8). \square

1075 We end this section with the proof of Proposition 2.9 and Corollary 2.10 concern-
 1076 ing the cost of null controllability.

1077 A careful inspection of the proof of null controllability in time $T > T_0(y_0)$ detailed
 1078 in Section 4.2 allows to give a bound on the cost of controllability.

1079 *Proof (of Proposition 2.9 and Corollary 2.10).* The proof of Proposition 2.9 fol-
 1080 lows directly from (2.2) and (4.4).

1081 The proof of Corollary 2.10 then follows directly from Proposition 2.9, assump-
 1082 tion (2.23) and the estimate

$$1083 \quad \sum_{G \in \mathcal{G}} e^{-r_G x} \leq \frac{C_{\theta,\kappa}}{x^\theta}, \quad \forall x > 0,$$

1084 proved in [12, Proposition A.5.32]. \square

1085 **5. Computation of the cost of a block.**

1086 In this section we prove more explicit formulas to estimate the cost $\mathcal{C}(G, y_0)$
 1087 of the resolution of a block moment problem depending on the assumptions on the
 1088 eigenvalues in the group G . More precisely, we prove here Theorems 2.6 and 2.8.
 1089 For pedagogical purpose, we start in Section 5.1 with Theorem 2.6 for algebraically
 1090 (and geometrically) simple eigenvalues *i.e.* when $\alpha_\lambda = \gamma_\lambda = 1$ for any $\lambda \in G$. Then,
 1091 in Section 5.2, we prove the general statement of Theorem 2.6 that is when all the
 1092 eigenvalues in the group are geometrically simple *i.e.* $\gamma_\lambda = 1$ for any $\lambda \in G$.

1093 The formula for the cost $\mathcal{C}(G, y_0)$ when all the eigenvalues in the group G are
 1094 semi-simple (*i.e.* $\alpha_\lambda = 1$ for any $\lambda \in G$) stated in Theorem 2.8 is then proved in
 1095 Section 5.3. The extension to spaces of initial conditions (2.18) and (2.22) does not
 1096 depend on the matrix M and follows directly from Lemma B.1. Thus, their proofs
 1097 are not detailed here.

1098 When both algebraic and geometric multiplicities appear in the same group we
 1099 do not get a general formula but describe the procedure on an example in Section 5.4.

1100 Recall that from (2.16), computing $\mathcal{C}(G, y_0)$ is a finite dimensional optimization
 1101 problem given by

$$1102 \quad \mathcal{C}(G, y_0) = \min \left\{ F(\Omega) ; \Omega \in \mathcal{O}(G, y_0) \cap U_G^{|\alpha|} \right\}$$

1103 where the function F is defined in (2.9), the constraints associated with $\mathcal{O}(G, y_0)$ are
 1104 defined in (2.7) and U_G is defined in (2.15).

1105 **5.1. The case of simple eigenvalues.**

1106 In all this section, we consider the simpler case where $\alpha_\lambda = \gamma_\lambda = 1$ for every
 1107 $\lambda \in G$. Thus, in the rest of this section, we drop the superscript 0 associated to
 1108 eigenvectors.

1109 We start with the proof of the invertibility of the matrix M stated in Proposi-
 1110 tion 2.5.

1111 *Proof.* Recall that, as $\alpha_\lambda = \gamma_\lambda = 1$, the positive semi-definite matrix M is defined
 1112 in (2.19). Let $\tau \in \mathbb{C}^g$ be such that $\langle M\tau, \tau \rangle = 0$. Then, for each $l \in \llbracket 1, g \rrbracket$, we have

$$1113 \quad \langle \Gamma^l \tau, \tau \rangle = 0.$$

1114 We prove that $\tau = 0$. By contradiction let

$$1115 \quad l = \max\{j \in \llbracket 1, g \rrbracket ; \tau_j \neq 0\}.$$

1116 Then from (2.19) this leads to $\langle \Gamma^l \tau, \tau \rangle = \|b[\lambda_l]\|_U^2 |\tau_l|^2$. Using (1.33) implies $\tau_l = 0$.
 1117 This is in contradiction with the definition of l which proves the invertibility of M . \square

1118 We now prove Theorem 2.6.

1119 *Proof.* First of all, notice that the function F to minimize reduces to

$$1120 \quad F(\Omega) = \sum_{j=1}^g \|\Omega[\bar{\lambda}_1, \dots, \bar{\lambda}_j]\|^2$$

1121 and, as $\gamma_\lambda = \alpha_\lambda = 1$, the constraints defining the set $\mathcal{O}(\lambda_j, y_0)$ reduce to

$$1122 \quad \langle \Omega_j, b_j \rangle_U = \langle y_0, \phi_j \rangle_{-\infty, \infty}.$$

1123 Thus, the minimization problem reduces to

1124

1125 (5.1) $\mathcal{C}(G, y_0) = \min \left\{ F(\Omega) ; \Omega = (\Omega_1, \dots, \Omega_g) \in U_G^g \text{ such that} \right.$

1126
1127

$$\left. \langle \Omega_j, b_j \rangle_U = \langle y_0, \phi_j \rangle_{-\diamond, \diamond}, \forall j \in \llbracket 1, g \rrbracket \right\}.$$

1128 For the sake of generality, let us consider for this proof any $\omega_1, \dots, \omega_g \in \mathbb{C}$ and the
1129 more general constraints

1130 (5.2)
$$\langle \Omega_j, b_j \rangle_U = \omega_j, \quad \forall j \in \llbracket 1, g \rrbracket.$$

1131 Using the formalism of divided differences, this is equivalent to the family of con-
1132 straints

1133 (5.3)
$$\langle \Omega, b \rangle_U [\bar{\lambda}_1, \dots, \bar{\lambda}_j] = \omega [\bar{\lambda}_1, \dots, \bar{\lambda}_j], \quad \forall j \in \llbracket 1, g \rrbracket.$$

1134 We consider the constrained complex minimization problem

1135
$$\min \{ F(\Omega) ; \Omega = (\Omega_1, \dots, \Omega_g) \in U_G^g \text{ such that (5.3) holds} \}.$$

1136 It has a unique solution, which is characterised by the existence of multipliers $(m_j)_{j \in \llbracket 1, g \rrbracket} \subset$
1137 \mathbb{C} such that

1138 (5.4)
$$\sum_{j=1}^g \langle H[\bar{\lambda}_1, \dots, \bar{\lambda}_j], \Omega[\bar{\lambda}_1, \dots, \bar{\lambda}_j] \rangle_U = \sum_{j=1}^g \bar{m}_j \langle H, b \rangle_U [\bar{\lambda}_1, \dots, \bar{\lambda}_j],$$

1139 for any $H_1, \dots, H_g \in U_G$.

1140 Then, for a given $q \in \llbracket 1, g \rrbracket$, using Leibniz formula [12, Proposition A.2.11], the
1141 constraints (5.3) can be rewritten as

1142 (5.5)
$$\omega [\bar{\lambda}_1, \dots, \bar{\lambda}_q] = \langle \Omega, b \rangle_U [\bar{\lambda}_1, \dots, \bar{\lambda}_q] = \sum_{j=1}^q \langle \Omega[\bar{\lambda}_1, \dots, \bar{\lambda}_j], b[\lambda_j, \dots, \lambda_q] \rangle_U$$

1143 To relate (5.5) and (5.4), we look for $H_1, \dots, H_g \in U_G$ such that, for a given $q \in \llbracket 1, g \rrbracket$
1144 we have

1145
$$H[\bar{\lambda}_1, \dots, \bar{\lambda}_j] = \begin{cases} b[\lambda_j, \dots, \lambda_q], & \text{for } j \leq q, \\ 0, & \text{for } j > q. \end{cases}$$

1146 This can be done by setting $H_1 = b[\lambda_1, \dots, \lambda_q]$ and, from the interpolation formula [9,
1147 Proposition 7.6], by defining H_j by the formula

1148
$$H_j = \sum_{i=1}^j \left(\prod_{k=1}^{j-1} (\bar{\lambda}_i - \bar{\lambda}_k) \right) H[\bar{\lambda}_1, \dots, \bar{\lambda}_i], \quad \forall j \in \llbracket 2, g \rrbracket.$$

1149 Then, from (5.5) we obtain

1150
$$\omega [\bar{\lambda}_1, \dots, \bar{\lambda}_q] = \sum_{j=1}^g \langle \Omega[\bar{\lambda}_1, \dots, \bar{\lambda}_j], H[\bar{\lambda}_1, \dots, \bar{\lambda}_j] \rangle_U.$$

1151 Now relation (5.4) leads, after conjugation, to

$$1152 \quad \overline{\omega[\bar{\lambda}_1, \dots, \bar{\lambda}_g]} = \sum_{j=1}^g \bar{m}_j \langle H, b \rangle_U [\bar{\lambda}_1, \dots, \bar{\lambda}_j].$$

1153 The application of Leibniz formula [12, Proposition A.2.11] yields

$$1154 \quad \overline{\omega[\bar{\lambda}_1, \dots, \bar{\lambda}_g]} = \sum_{j=1}^g \bar{m}_j \left(\sum_{l=1}^j \langle H[\bar{\lambda}_1, \dots, \bar{\lambda}_l], b[\lambda_l, \dots, \lambda_j] \rangle_U \right)$$

$$1155 \quad = \sum_{j=1}^g \bar{m}_j \left(\sum_{l=1}^{\min(j, g)} \langle b[\lambda_l, \dots, \lambda_g], b[\lambda_l, \dots, \lambda_j] \rangle_U \right).$$

1156

1157 Conjugating this relation leads to

$$1158 \quad \omega[\bar{\lambda}_1, \dots, \bar{\lambda}_g] = \sum_{j=1}^g m_j \left(\sum_{l=1}^{\min(j, g)} \langle b[\lambda_l, \dots, \lambda_j], b[\lambda_l, \dots, \lambda_g] \rangle_U \right)$$

$$1159 \quad = \sum_{l=1}^g \sum_{j=1}^g m_j \Gamma_{q, j}^l = (Mm)_q,$$

1160

1161 where Γ^l and M are defined in (2.19).

1162 Let

$$1163 \quad \xi := \begin{pmatrix} \omega[\bar{\lambda}_1] \\ \vdots \\ \omega[\bar{\lambda}_1, \dots, \bar{\lambda}_g] \end{pmatrix} \in \mathbb{C}^g.$$

1164 We have just proved that $m = M^{-1}\xi$. Getting back to (5.4) with $H = \Omega$ together
1165 with the constraints (5.3), we obtain

$$1166 \quad F(\Omega) = \sum_{j=1}^g \bar{m}_j \langle \Omega, b \rangle_U [\bar{\lambda}_1, \dots, \bar{\lambda}_j] = \langle M^{-1}\xi, \xi \rangle.$$

1167 With the specific choice, $\omega_j = \langle y_0, \phi_j \rangle_{-\diamond, \diamond}$, this ends the proof of Theorem 2.6 with
1168 the extra assumption that $\alpha_\lambda = 1$ for all $\lambda \in G$. Indeed, by anti-linearity we have

$$1169 \quad \omega[\bar{\lambda}_1, \dots, \bar{\lambda}_j] = \langle y_0, \phi[\lambda_1, \dots, \lambda_j] \rangle_{-\diamond, \diamond}, \quad \forall j \in \llbracket 1, g \rrbracket. \quad \square$$

1170 **REMARK 5.1.** As mentioned in Remark 2.1, estimate (5.1) implies that the cost
1171 of the block G (i.e. the quantity $\langle M^{-1}\xi, \xi \rangle$) can be estimated using any eigenvectors:
1172 there is no normalization condition.

1173 **REMARK 5.2.** Rewriting the constraints in the form (5.3) is not mandatory but,
1174 as the function to minimize involves divided differences, it leads to more exploitable
1175 formulas and will ease the writing when dealing with algebraic multiplicity of eigen-
1176 values. Dealing directly with (5.2) would lead to the expression (D.9) for the cost of
1177 the block G as it will appear in the proof of Theorem 2.8.

1178 **5.2. The case of geometrically simple eigenvalues.**

1179 The proof of Proposition 2.5 and Theorem 2.6 under the sole assumption $\gamma_\lambda = 1$
 1180 for any $\lambda \in G$ follows closely the proof done in Section 5.1. The main difference is the
 1181 use of generalized divided differences (see [9, Section 7.3]) instead of classical divided
 1182 differences as detailed below.

1183 *Proof (of Proposition 2.5).* Due to (2.8), for any $l \in \llbracket 1, |\alpha| \rrbracket$ the multi-index $\mu^l -$
 1184 μ^{l-1} is composed of only one 1 and $g - 1$ zeros. Thus,

$$1185 \quad b \left[\lambda_{\bullet}^{(\mu^l - \mu^{l-1})} \right] = b_j^0$$

1186 for a certain $j \in \llbracket 1, g \rrbracket$. From (1.33) it comes that

$$1187 \quad b \left[\lambda_{\bullet}^{(\mu^l - \mu^{l-1})} \right] \neq 0, \quad \forall l \in \llbracket 1, |\alpha| \rrbracket.$$

1188 The rest of the proof follows as in Section 5.1. □

1189 *Proof (of Theorem 2.6).* As $\gamma_\lambda = 1$, the constraints defining the set $\mathcal{O}(\lambda_j, y_0)$
 1190 reduce to

$$1191 \quad \sum_{r=0}^l \langle \Omega_j^r, b_j^{l-r} \rangle_U = \sum_{r=0}^l \langle \Omega_j^r, \mathcal{B}^*(\mathcal{A}^* - \lambda_j)^r \phi_j^l \rangle_U$$

$$1192 \quad = \langle y_0, \phi_j^l \rangle_{-\diamond, \diamond}, \quad \forall l \in \llbracket 0, \alpha_j \rrbracket.$$

1194 By definition of $\langle \Omega, b \rangle_U \left[\overline{\lambda}_j^{(l+1)} \right]$, this is equivalent to

$$1195 \quad \langle \Omega, b \rangle_U \left[\overline{\lambda}_j^{(l+1)} \right] = \langle y_0, \phi_j^l \rangle_{-\diamond, \diamond}, \quad \forall l \in \llbracket 0, \alpha_j \rrbracket.$$

1196 Thus,

$$1197 \quad (5.6) \quad \mathcal{C}(G, y_0) = \min \left\{ F(\Omega) ; \Omega = (\Omega_1^0, \dots, \Omega_1^{\alpha_1-1}, \dots, \Omega_g^0, \dots, \Omega_g^{\alpha_g-1}) \in U_G^{|\alpha|} \right.$$

$$1199 \quad \left. \text{such that } \langle \Omega, b \rangle_U \left[\overline{\lambda}_j^{(l+1)} \right] = \langle y_0, \phi_j^l \rangle_{-\diamond, \diamond}, \quad \forall j \in \llbracket 1, g \rrbracket, \forall l \in \llbracket 0, \alpha_j \rrbracket \right\}.$$

1201 For the sake of generality, let us consider for this proof any

$$1202 \quad (\omega_1^0, \dots, \omega_1^{\alpha_1-1}, \dots, \omega_g^0, \dots, \omega_g^{\alpha_g-1}) \in \mathbb{C}^{|\alpha|}$$

1203 and the more general constraints

$$1204 \quad \langle \Omega, b \rangle_U \left[\overline{\lambda}_j^{(l+1)} \right] = \omega_j^l, \quad \forall j \in \llbracket 1, g \rrbracket, \forall l \in \llbracket 0, \alpha_j \rrbracket.$$

1205 From (2.8), this is equivalent to the family of constraints

$$1206 \quad \langle \Omega, b \rangle_U \left[\overline{\lambda}_{\bullet}^{(\mu^p)} \right] = \omega \left[\overline{\lambda}_{\bullet}^{(\mu^p)} \right], \quad \forall p \in \llbracket 1, |\alpha| \rrbracket,$$

1207 and we proceed as in Section 5.1. The only difference is the use of generalized divided
 1208 differences. For instance, the equation (5.4) now reads

$$1209 \quad \sum_{l=1}^{|\alpha|} \left\langle H \left[\overline{\lambda}_{\bullet}^{(\mu^l)} \right], \Omega \left[\overline{\lambda}_{\bullet}^{(\mu^l)} \right] \right\rangle_U = \sum_{l=1}^{|\alpha|} \overline{m}_l \langle H, b \rangle_U \left[\overline{\lambda}_{\bullet}^{(\mu^l)} \right], \quad \forall H = (H_j^l) \in U_G^{|\alpha|}.$$

1210 The rest of the proof remains unchanged. □

1211 REMARK 5.3. As mentioned in Remark 2.1, estimate (5.6) implies that the cost
 1212 of the block G (i.e. the quantity $\langle M^{-1}\xi, \xi \rangle$) can be estimated using any eigenvectors
 1213 and any associated Jordan chains.

1214 **5.3. The case of semi-simple eigenvalues.**

1215 We start with the proof of Proposition 2.7.

1216 *Proof (of Proposition 2.7).* Recall that the positive semi-definite matrix M is de-
 1217 fined in (2.21). Let $\tau \in \mathbb{C}^{\gamma_G}$ be such that $\langle M\tau, \tau \rangle = 0$. Then, for any $l \in \llbracket 1, g \rrbracket$,
 1218 $\langle \Gamma^l \tau, \tau \rangle = 0$. We prove that $\tau = 0$. By contradiction let

$$1219 \quad \tilde{l} = \max\{j \in \llbracket 1, \gamma_G \rrbracket ; \tau_j \neq 0\}$$

1220 and $l \in \llbracket 1, g \rrbracket$ be such that

$$1221 \quad \gamma_1 + \dots + \gamma_{l-1} < \tilde{l} \leq \gamma_1 + \dots + \gamma_l$$

1222 with the convention that $l = 1$ when $\tilde{l} \leq \gamma_1$. We denote by $\sigma \in \mathbb{C}^{\gamma_l}$ the l^{th} block of τ
 1223 i.e.

$$1224 \quad \sigma = \begin{pmatrix} \tau_{\gamma_1 + \dots + \gamma_{l-1} + 1} \\ \vdots \\ \tau_{\gamma_1 + \dots + \gamma_l} \end{pmatrix}.$$

1225 From (2.20) we have $\delta_i^i = 0$ when $i < l$. Thus all the blocks (i, j) of Γ^l are equal to 0
 1226 when $i, j \in \llbracket 1, l \rrbracket$. This leads to

$$1227 \quad \langle \Gamma^l \tau, \tau \rangle = |\delta_l^l|^2 \langle \text{Gram}_U(b_{l,1}, \dots, b_{l,\gamma_l}) \sigma, \sigma \rangle.$$

1228 As the eigenvalues $\lambda_1, \dots, \lambda_g$ are distinct it comes that $\delta_l^l \neq 0$ (see (2.20)) which
 1229 implies

$$1230 \quad \langle \text{Gram}_U(b_{l,1}, \dots, b_{l,\gamma_l}) \sigma, \sigma \rangle = 0.$$

1231 From (1.33), we have that $b_{l,1}, \dots, b_{l,\gamma_l}$ are linearly independent. This proves the
 1232 invertibility of $\text{Gram}_U(b_{l,1}, \dots, b_{l,\gamma_l})$ and gives $\sigma = 0$. This is in contradiction with
 1233 the definition of \tilde{l} which proves the invertibility of M . \square

1234 We now turn to the proof of Theorem 2.8.

1235 *Proof (of Theorem 2.8).* First of all, notice that the function F to minimize re-
 1236 duces to

$$1237 \quad F(\Omega) = \sum_{j=1}^g \|\Omega[\bar{\lambda}_1, \dots, \bar{\lambda}_j]\|^2$$

1238 and, as $\alpha_\lambda = 1$, the constraints defining the set $\mathcal{O}(\lambda_j, y_0)$ reduce to

$$1239 \quad \langle \Omega_j, \mathcal{B}^* \phi \rangle_U = \langle y_0, \phi \rangle_{-\diamond, \diamond}, \quad \forall \phi \in \text{Ker}(\mathcal{A}^* - \lambda_j).$$

1240 To simplify the writing, let us consider the linear maps

$$1241 \quad B_j := \begin{pmatrix} \langle \bullet, \mathcal{B}^* \phi_{j,1} \rangle_U \\ \vdots \\ \langle \bullet, \mathcal{B}^* \phi_{j,\gamma_j} \rangle_U \end{pmatrix} \in \mathcal{L}(U, \mathbb{C}^{\gamma_j}).$$

1242 Then the constraints defining $\mathcal{O}(\lambda_j, y_0)$ can be rewritten as the equality

$$1243 \quad (5.7) \quad B_j \Omega_j = \begin{pmatrix} \langle y_0, \phi_{j,1} \rangle_{-\diamond, \diamond} \\ \vdots \\ \langle y_0, \phi_{j,\gamma_j} \rangle_{-\diamond, \diamond} \end{pmatrix}.$$

1244 Thus,

1245

$$1246 \quad (5.8) \quad \mathcal{C}(G, y_0) = \min \left\{ F(\Omega) ; \Omega = (\Omega_1, \dots, \Omega_g) \in U_G^g \right.$$

1247
1248

such that (5.7) holds for any $j \in \llbracket 1, g \rrbracket$ }.

1249 For the sake of generality, let us consider for this proof, for any $j \in \llbracket 1, g \rrbracket$, any $\omega_j \in \mathbb{C}^{\gamma_j}$
1250 and the more general constraints

$$1251 \quad (5.9) \quad B_j \Omega_j = \omega_j, \quad \forall j \in \llbracket 1, g \rrbracket.$$

1252 As the ω_j 's have different sizes we avoid in this proof the use of divided differences
1253 to rewrite the constraints. This is why we end up with the formula (2.21) rather than
1254 an adaptation of (2.19) (see also the discussion in Remark 5.2).

1255 Arguing as before, the solution of our optimisation problem satisfies

$$1256 \quad (5.10) \quad \sum_{j=1}^g \langle H[\bar{\lambda}_1, \dots, \bar{\lambda}_j], \Omega[\bar{\lambda}_1, \dots, \bar{\lambda}_j] \rangle_U = \sum_{j=1}^g \langle B_j H_j, m_j \rangle, \quad \forall H_1, \dots, H_g \in U_G,$$

1257 for some $m_j \in \mathbb{C}^{\gamma_j}, j = 1, \dots, g$.

1258 Recall that in (2.20) we defined the numbers

$$1259 \quad \delta_j^i = \prod_{k \in \llbracket 1, j \llbracket} (\lambda_i - \lambda_k), \quad \forall j \in \llbracket 2, g \rrbracket.$$

1260 Then, from the interpolation formula [9, Proposition 7.6], we obtain that

$$1261 \quad (5.11) \quad \Omega_i = \sum_{l=1}^i \delta_l^i \Omega[\bar{\lambda}_1, \dots, \bar{\lambda}_l].$$

1262 For any $H \in U_G$ and $i \in \llbracket 1, g \rrbracket$, let us design $H_1^{(i)}, \dots, H_g^{(i)} \in U_G$ such that

$$1263 \quad (5.12) \quad H^{(i)}[\bar{\lambda}_1, \dots, \bar{\lambda}_l] = \delta_l^i H, \quad \forall l \in \llbracket 1, i \rrbracket.$$

1264 To do so, we set $H_1^{(i)} = H$ then, using the interpolation formula [9, Proposition 7.6],
1265 we define recursively

$$1266 \quad H_j^{(i)} = \sum_{l=1}^j \delta_l^j H^{(i)}[\bar{\lambda}_1, \dots, \bar{\lambda}_l] = \left(\sum_{l=1}^j \delta_l^i \delta_l^j \right) H = a_j^{(i)} H$$

1267 with

$$1268 \quad (5.13) \quad a_j^{(i)} := \sum_{l=1}^g \delta_l^i \delta_l^j = \sum_{l=1}^{\min(i,j)} \delta_l^i \delta_l^j.$$

1269 This ensures (5.12). Plugging this set of values $H_j^{(i)}, j = 1, \dots, g$ in (5.10) and taking
 1270 into account (5.11), leads to

$$\begin{aligned}
 1271 \quad \sum_{j=1}^g a_j^{(i)} \langle B_j H, m_j \rangle &= \sum_{j=1}^g \langle B_j H_j^{(i)}, m_j \rangle \\
 1272 &= \sum_{j=1}^g \delta_j^i \langle H, \Omega[\bar{\lambda}_1, \dots, \bar{\lambda}_j] \rangle_U \\
 1273 &= \left\langle H, \sum_{j=1}^g \bar{\delta}_j^i \Omega[\bar{\lambda}_1, \dots, \bar{\lambda}_j] \right\rangle_U \\
 1274 \quad &= \langle H, \Omega_i \rangle_U.
 \end{aligned}$$

1276 This being true for any $H \in U_G$, we end up with

$$1277 \quad (5.14) \quad \Omega_i = \sum_{j=1}^g \overline{a_j^{(i)}} B_j^* m_j.$$

1278 Together with (5.9), using (5.13), we obtain that

$$\begin{aligned}
 1279 \quad \omega_i &= \sum_{j=1}^g \overline{a_j^{(i)}} B_i B_j^* m_j \\
 1280 &= \sum_{l=1}^g \sum_{j=1}^g (\bar{\delta}_l^i B_i) (\bar{\delta}_l^j B_j)^* m_j \\
 1281 &= (Mm)_i
 \end{aligned}$$

1283 where M is defined in (2.21) and $(Mm)_i \in \mathbb{C}^{\gamma_i}$ denotes the i^{th} block of $Mm \in \mathbb{C}^{\gamma_G}$.

1284 Finally, if we set

$$1285 \quad \xi := \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_g \end{pmatrix} \in \mathbb{C}^{\gamma_G},$$

1286 we have proved that the multiplier is given by $m = M^{-1}\xi$. Applying (5.10) with
 1287 $H_j = \Omega_j$ and using the constraints (5.7) leads to

$$1288 \quad F(\Omega) = \sum_{j=1}^g \|\Omega[\bar{\lambda}_1, \dots, \bar{\lambda}_j]\|^2 = \langle M^{-1}\xi, \xi \rangle,$$

1289 which proves the claim. \square

1290 **REMARK 5.4.** *As mentioned in Remark 2.1, estimate (5.8) implies that the cost*
 1291 *of the block G (i.e. the quantity $\langle M^{-1}\xi, \xi \rangle$) can be estimated using any basis of*
 1292 *eigenvectors.*

1293 **5.4. Dealing simultaneously with algebraic and geometric multiplici-** 1294 **ties.**

1295 The proof of Theorem 2.6 strongly relies on the use of divided differences to rewrite
 1296 the constraints whereas the proof of Theorem 2.8 is based on the vectorial writing

1297 of the constraints through the operators $B_j \in \mathcal{L}(U; \mathbb{C}^{\gamma_j})$. As the target spaces of
 1298 these operators do not have the same dimension, one cannot directly compute divided
 1299 differences. Thus, the setting we developed to compute the cost of a given block does
 1300 not lead to a general formula when both kind of multiplicities need to be taken into
 1301 account in the same group. However, for actual problems, the computation of this
 1302 cost is a finite dimensional constrained optimization problem which can be explicitly
 1303 solved.

1304 Let us give an example of such a group that does not fit into Theorem 2.6 nor
 1305 into Theorem 2.8 but for which we manage to compute the cost *by hand*. To simplify
 1306 a little the presentation, we give this example in the case of real Hilbert spaces and
 1307 real eigenvalues.

1308 We consider a group $G = \{\lambda_1, \lambda_2\}$ of two distinct eigenvalues such that $\gamma_{\lambda_1} =$
 1309 $\alpha_{\lambda_1} = 2$ and $\gamma_{\lambda_2} = \alpha_{\lambda_2} = 1$. Let $(\phi_{1,1}^0, \phi_{1,2}^0)$ be a basis of $\text{Ker}(\mathcal{A}^* - \lambda_1)$ and $\phi_{2,1}^0$ be
 1310 an eigenvector of \mathcal{A}^* associated to the eigenvalue λ_2 . Assume that the generalized
 1311 eigenvector $\phi_{1,1}^1$ is such that

$$1312 \quad (\mathcal{A}^* - \lambda_1)\phi_{1,1}^1 = \phi_{1,1}^0,$$

1313 and that $\{\phi_{1,1}^0, \phi_{1,1}^1, \phi_{1,2}^0\}$ forms a basis of $\text{Ker}(\mathcal{A}^* - \lambda_1)^2$.

1314 For this group, in the same spirit as in Theorems 2.6 and 2.8, we obtain the
 1315 following result.

1316 PROPOSITION 5.1. *For any $y_0 \in X_{-\diamond}$, we have*

$$1317 \quad \mathcal{C}(G, y_0) = \langle M^{-1}\xi, \xi \rangle \quad \text{where} \quad \xi = \begin{pmatrix} \langle y_0, \phi_{1,1}^0 \rangle_{-\diamond, \diamond} \\ \langle y_0, \phi_{1,2}^0 \rangle_{-\diamond, \diamond} \\ \langle y_0, \phi_{1,1}^1 \rangle_{-\diamond, \diamond} \\ \langle y_0, \phi_{2,1}^0 \rangle_{-\diamond, \diamond} \end{pmatrix}$$

1318 and M is the invertible matrix defined by

$$1319 \quad M = \text{Gram}_U(b_{1,1}^0, b_{1,2}^0, b_{1,1}^1, b_{2,1}^0) \\
 1320 \quad + \text{Gram}_U(0, 0, b_{1,1}^0, \delta b_{2,1}^0) \\
 1321 \quad + \text{Gram}_U(0, 0, 0, \delta^2 b_{2,1}^0) \\
 1322$$

1323 with $\delta = \lambda_2 - \lambda_1$.

1324 *Proof.* Let

$$1325 \quad (\omega_{1,1}^0, \omega_{1,2}^0, \omega_{1,1}^1, \omega_{2,1}^0)^t \in \mathbb{R}^4.$$

1326 As in the proofs of Theorems 2.6 and 2.8, the goal is to compute the minimum of the
 1327 function

$$1328 \quad F : (\Omega_1^0, \Omega_1^1, \Omega_2^0) \in U_G^3 \mapsto \|\Omega_1^0\|^2 + \|\Omega_1^1\|^2 + \|\Omega[\lambda_1^{(2)}, \lambda_2]\|^2,$$

1329 under the 4 constraints

$$1330 \quad \langle \Omega_j^0, b_{j,i}^0 \rangle_U = \omega_{j,i}^0, \quad \forall i \in \llbracket 1, \gamma_j \rrbracket, \forall j \in \llbracket 1, 2 \rrbracket, \\
 1331 \quad \langle \Omega_1^0, b_{1,1}^1 \rangle_U + \langle \Omega_1^1, b_{1,1}^0 \rangle_U = \omega_{1,1}^1.$$

1333 Then, the Lagrange multipliers $m_{1,1}^0, m_{1,2}^0, m_{1,1}^1$ and $m_{2,1}^0$ satisfy the equations

1334

$$1335 \quad (5.15) \quad \langle \Omega_1^0, H_1^0 \rangle_U + \langle \Omega_1^1, H_1^1 \rangle_U + \langle \Omega[\lambda_1^{(2)}, \lambda_2], H[\lambda_1^{(2)}, \lambda_2] \rangle_U = m_{1,1}^0 \langle H_1^0, b_{1,1}^0 \rangle_U \\
 1336 \quad + m_{1,2}^0 \langle H_1^0, b_{1,2}^0 \rangle_U + m_{1,1}^1 \left(\langle H_1^0, b_{1,1}^1 \rangle_U + \langle H_1^1, b_{1,1}^0 \rangle_U \right) + m_{2,1}^0 \langle H_2^0, b_{2,1}^0 \rangle_U,$$

1337

1338 for every $H_1^0, H_1^1, H_2^0 \in U_G$. Considering successively

$$1339 \quad H_1^0 = b_{1,1}^0, \quad H_1^1 = 0, \quad H_2^0 = b_{1,1}^0,$$

1340

$$1341 \quad H_1^0 = b_{1,2}^0, \quad H_1^1 = 0, \quad H_2^0 = b_{1,2}^0,$$

1342

$$1343 \quad H_1^0 = b_{1,1}^1, \quad H_1^1 = b_{1,1}^0, \quad H_2^0 = b_{1,1}^1 + (\lambda_2 - \lambda_1)b_{1,1}^0,$$

1344 and

$$1345 \quad H_1^0 = b_{2,1}^0, \quad H_1^1 = (\lambda_2 - \lambda_1)b_{2,1}^0, \quad H_2^0 = (1 + (\lambda_2 - \lambda_1)^2 + (\lambda_2 - \lambda_1)^4)b_{2,1}^0,$$

1346 and plugging it into (5.15), we obtain that $\begin{pmatrix} \omega_{1,1}^0 \\ \omega_{1,2}^0 \\ \omega_{1,1}^1 \\ \omega_{2,1}^0 \end{pmatrix} = M \begin{pmatrix} m_{1,1}^0 \\ m_{1,2}^0 \\ m_{1,1}^1 \\ m_{2,1}^0 \end{pmatrix}$. Then, the same

1347 argument as in the proofs of Theorems 2.6 and 2.8 ends the proof. \square

1348 6. Application to the study of null controllability of academic exam- 1349 ples.

1350 In this section we provide examples to illustrate how to use the formulas obtained
1351 in Theorems 2.3, 2.6 and 2.8 in order to compute the minimal null control time.

1352 We start with academic examples for which computations are simpler. Then, in
1353 Section 7, we study coupled systems of actual partial differential equations of parabolic
1354 type.

1355 6.1. Setting and notations.

1356 Let A be the unbounded Sturm-Liouville operator defined in $L^2(0, 1; \mathbb{R})$ by

$$1357 \quad (6.1) \quad D(A) = H^2(0, 1; \mathbb{R}) \cap H_0^1(0, 1; \mathbb{R}), \quad A = -\partial_x(\gamma \partial_x \bullet) + c\bullet,$$

1358 with $c \in L^\infty(0, 1; \mathbb{R})$ satisfying $c \geq 0$ and $\gamma \in C^1([0, 1]; \mathbb{R})$ satisfying $\inf_{[0,1]} \gamma > 0$.

1359 The operator A admits an increasing sequence of eigenvalues denoted by $(\nu_k)_{k \in \mathbb{N}^*}$.
1360 The associated normalized eigenvectors $(\varphi_k)_{k \in \mathbb{N}^*}$ form an Hilbert basis of $L^2(0, 1; \mathbb{R})$.

1361 **REMARK 6.1.** *The assumption $c \geq 0$ ensures that for any $k \geq 1$, the eigenvalues*
1362 *satisfies $\nu_k > 0$. From Remark 1.3, the controllability results proved in the present*
1363 *article still hold when the function c is bounded from below.*

1364 To lighten the notations, for any $I \subset (0, 1)$ we set $\|\bullet\|_I = \|\bullet\|_{L^2(I)}$.

1365 Let $f : \text{Sp}(A) \rightarrow \mathbb{R}$ be a bounded function. Associated to this function we consider
1366 the operator $f(A)$ defined on $D(A)$ by the spectral theorem by

$$1367 \quad (6.2) \quad f(A) = \sum_{k \geq 1} f(\nu_k) \langle \bullet, \varphi_k \rangle_{L^2(0,1;\mathbb{R})} \varphi_k.$$

1368 **6.2. Spectral properties of Sturm-Liouville operators.**

1369 Let A be the Sturm-Liouville operator defined in (6.1). All the examples studied
 1370 in this article are based on this operator. We recall here some spectral properties that
 1371 will be used in our study.

1372 From [2, Theorem 1.1 and Remark 2.1], there exist $\varrho > 0$ and $C > 0$ such that

1373 (6.3)
$$\varrho < \nu_{k+1} - \nu_k, \quad \forall k \geq 1,$$

1374
 1375 (6.4)
$$\frac{1}{C}\sqrt{\nu_k} \leq |\varphi'_k(x)| \leq C\sqrt{\nu_k}, \quad \forall x \in \{0, 1\}, \forall k \geq 1,$$

1376 and, for any non-empty open set $\omega \subset (0, 1)$,

1377 (6.5)
$$\inf_{k \geq 1} \|\varphi_k\|_\omega > 0.$$

1378 Moreover, using [12, Theorem IV.1.3], the associated counting function satisfies

1379 (6.6)
$$N_{(\nu_k)_k}(r) \leq C\sqrt{r}, \quad \forall r > 0,$$

1380 and

1381 (6.7)
$$|N_{(\nu_k)_k}(r) - N_{(\nu_k)_k}(s)| \leq C \left(1 + \sqrt{|r - s|}\right), \quad \forall r, s > 0.$$

1382 We also recall the classical Lebeau-Robbiano spectral inequality

1383 (6.8)
$$\left\| \sum_{k \leq K} a_k \varphi_k \right\|_\Omega \leq C e^{C\sqrt{\nu_K}} \left\| \sum_{k \leq K} a_k \varphi_k \right\|_\omega, \quad \forall K \geq 1, \forall (a_k)_k \subset \mathbb{R}.$$

1384 Indeed, as detailed for instance in [12, Theorem IV.2.19], the proof of this spectral
 1385 inequality given in [27] directly extends to the low regularity coefficients considered
 1386 here.

1387 **6.3. Perturbation of a 2x2 Jordan block.**

1388 Let $\omega \subset (0, 1)$ be a non-empty open set and $U = L^2(\omega)$. Let A be the Sturm-
 1389 Liouville operator defined in (6.1) and $f(A)$ be the operator defined in (6.2) with
 1390 $f : \text{Sp}(A) \rightarrow \mathbb{R}$ satisfying

1391
$$|f(\nu_k)| < \frac{\varrho}{2}, \quad \forall k \geq 1.$$

1392 We consider the operator \mathcal{A} on $X = L^2(0, 1; \mathbb{R})^2$ defined by

1393 (6.9)
$$\mathcal{A} = \begin{pmatrix} A & I \\ 0 & A + f(A) \end{pmatrix}, \quad D(\mathcal{A}) = D(A) \times D(A),$$

1394 and

1395 (6.10)
$$\mathcal{B} : u \in U \mapsto \begin{pmatrix} 0 \\ \mathbf{1}_\omega u \end{pmatrix}.$$

1396 Then,

1397
$$\mathcal{B}^* : \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \in X \mapsto \mathbf{1}_\omega \varphi_2.$$

1398 It is easy to see that $(-\mathcal{A}, D(\mathcal{A}))$ generates a \mathcal{C}_0 -semigroup on X and that $\mathcal{B} : U \rightarrow X$
 1399 is bounded. Thus we consider for this example that $X_\diamond^* = X = X_{-\diamond}$.

1400 PROPOSITION 6.1. *Let us consider the control system (1.1) with \mathcal{A} and \mathcal{B} given*
 1401 *by (6.9)-(6.10). Then, null-controllability from $X_{-\diamond}$ holds in any time i.e. $T_0(X_{-\diamond}) =$*
 1402 *0.*

1403 *Proof.* The spectrum of $(\mathcal{A}^*, D(\mathcal{A}))$ is given by

$$1404 \quad \Lambda = \{\nu_k ; k \geq 1\} \cup \{\nu_k + f(\nu_k) ; k \geq 1\}.$$

1405 Recall that $(\nu_k)_{k \geq 1}$ satisfies (6.3), (6.6) and (6.7). From [12, Lemma V.4.20] it comes
 1406 that there exists $\kappa > 0$ such that $\Lambda \in \mathcal{L}_w(2, \frac{\kappa}{2}, \frac{1}{2}, \kappa)$.

1407 An associated grouping is given by

$$1408 \quad \begin{cases} G_k := \{\lambda_{k,1} := \nu_k, \lambda_{k,2} := \nu_k + f(\nu_k)\}, & \text{if } f(\nu_k) \neq 0, \\ G_k := \{\lambda_{k,1} := \nu_k\}, & \text{if } f(\nu_k) = 0. \end{cases}$$

1409 If $f(\nu_k) \neq 0$ the eigenvalues $\lambda_{k,1}$ and $\lambda_{k,2}$ are simple and we consider the associated
 1410 eigenvectors

$$1411 \quad \phi_{k,1}^0 = \begin{pmatrix} -f(\nu_k) \\ 1 \end{pmatrix} \varphi_k, \quad \phi_{k,2}^0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varphi_k.$$

1412 If $f(\nu_k) = 0$ the eigenvalue $\lambda_{k,1}$ is algebraically double and we consider the associated
 1413 Jordan chain

$$1414 \quad \phi_{k,1}^0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varphi_k, \quad \phi_{k,1}^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \varphi_k.$$

1415 From (6.5) it comes that (1.33) and (1.34) are satisfied. Thus, from Theorem 2.3, we
 1416 obtain that for any $y_0 \in X_{-\diamond}$,

$$1417 \quad T_0(y_0) = \limsup_{k \rightarrow +\infty} \frac{\ln^+ \mathcal{C}(G_k, y_0)}{2 \min G_k}.$$

1418 Let us now conclude by estimating $\mathcal{C}(G_k, y_0)$.

1419 • Consider first that $f(\nu_k) \neq 0$. Then, $\phi[\lambda_{k,1}, \lambda_{k,2}] = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \varphi_k$ and

$$1420 \quad b[\lambda_{k,1}, \lambda_{k,2}] = \mathcal{B}^* \phi[\lambda_{k,1}, \lambda_{k,2}] = \frac{\mathbf{1}_\omega \varphi_k - \mathbf{1}_\omega \varphi_k}{f(\nu_k)} = 0.$$

1421 From Theorem 2.6 it comes that

$$1422 \quad \mathcal{C}(G_k, y_0) = \langle M^{-1} \xi, \xi \rangle$$

1423 with

$$1424 \quad M = \text{Gram}(b[\lambda_{k,1}], b[\lambda_{k,1}, \lambda_{k,2}]) + \text{Gram}(0, b[\lambda_{k,2}]) = \begin{pmatrix} \|\varphi_k\|_\omega^2 & 0 \\ 0 & \|\varphi_k\|_\omega^2 \end{pmatrix}$$

1425 and

$$1426 \quad \xi = \begin{pmatrix} \langle y_0, \phi[\lambda_{k,1}] \rangle_{-\diamond, \diamond} \\ \langle y_0, \phi[\lambda_{k,1}, \lambda_{k,2}] \rangle_{-\diamond, \diamond} \end{pmatrix} = \begin{pmatrix} \left\langle y_0, \begin{pmatrix} -f(\nu_k) \\ 1 \end{pmatrix} \varphi_k \right\rangle_{-\diamond, \diamond} \\ \left\langle y_0, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \varphi_k \right\rangle_{-\diamond, \diamond} \end{pmatrix}.$$

1427 Thus,

$$1428 \quad \mathcal{C}(G_k, y_0) = \left\langle y_0, \begin{pmatrix} -f(\nu_k) \\ 1 \end{pmatrix} \frac{\varphi_k}{\|\varphi_k\|_\omega} \right\rangle_{-\diamond, \diamond}^2 + \left\langle y_0, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{\varphi_k}{\|\varphi_k\|_\omega} \right\rangle_{-\diamond, \diamond}^2.$$

1430 • Consider now that $f(\nu_k) = 0$. Then, $b[\lambda_{k,1}^{(2)}] = 0$. From Theorem 2.6 it comes
 1431 that

$$1432 \quad \mathcal{C}(G_k, y_0) = \langle M^{-1}\xi, \xi \rangle$$

1433 with

$$1434 \quad M_k = \text{Gram}(b[\lambda_{k,1}], b[\lambda_{k,1}^{(2)}]) + \text{Gram}(0, b[\lambda_{k,1}]) = \begin{pmatrix} \|\varphi_k\|_\omega^2 & 0 \\ 0 & \|\varphi_k\|_\omega^2 \end{pmatrix}. \\ 1435$$

1436 and

$$1437 \quad \xi = \begin{pmatrix} \langle y_0, \phi[\lambda_{k,1}] \rangle_{-\diamond, \diamond} \\ \langle y_0, \phi[\lambda_{k,1}^{(2)}] \rangle_{-\diamond, \diamond} \end{pmatrix} = \begin{pmatrix} \left\langle y_0, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varphi_k \right\rangle_{-\diamond, \diamond} \\ \left\langle y_0, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \varphi_k \right\rangle_{-\diamond, \diamond} \end{pmatrix}.$$

1438 As previously,

$$1439 \quad \mathcal{C}(G_k, y_0) = \left\langle y_0, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \frac{\varphi_k}{\|\varphi_k\|_\omega} \right\rangle_{-\diamond, \diamond}^2 + \left\langle y_0, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{\varphi_k}{\|\varphi_k\|_\omega} \right\rangle_{-\diamond, \diamond}^2.$$

1440 Gathering both cases and using estimate (6.5) we obtain, for any $y_0 \in X_{-\diamond}$,

$$1441 \quad \mathcal{C}(G_k, y_0) \leq C \|y_0\|_{-\diamond}^2, \quad \forall k \geq 1.$$

1442 Thus,

$$1443 \quad T_0(y_0) = \limsup_{k \rightarrow +\infty} \frac{\ln^+ \mathcal{C}(G_k, y_0)}{2 \min G_k} = 0. \quad \square$$

1444 **6.4. Competition between different perturbations.**

1445 Let $\omega_1, \omega_2 \subset (0, 1)$ be two open sets with $\omega_1 \neq \emptyset$ and $U = L^2(\Omega)^2$. Let $B_1, B_2 \in$
 1446 \mathbb{R}^3 . To simplify the computations, we assume that

$$1447 \quad B_i = \begin{pmatrix} 0 \\ B_{i,2} \\ B_{i,3} \end{pmatrix}.$$

1448 Let $\alpha, \beta > 0$ with $\alpha \neq \beta$ and $f, g : \text{Sp}(A) \rightarrow \mathbb{R}$ be defined by

$$1449 \quad f(\nu_k) = \frac{\varrho}{2} e^{-\alpha\nu_k}, \quad g(\nu_k) = \frac{\varrho}{2} e^{-\beta\nu_k}.$$

1450 As previously, we consider the associated operators $f(A)$ and $g(A)$ defined by the
 1451 spectral theorem and we define the evolution operator \mathcal{A} on $X = L^2(0, 1; \mathbb{R})^3$ by

$$1452 \quad (6.11) \quad \mathcal{A} = \begin{pmatrix} A & I & 0 \\ 0 & A + f(A) & 0 \\ 0 & 0 & A + g(A) \end{pmatrix}, \quad D(\mathcal{A}) = D(A)^3,$$

1453 and the control operator by

$$1454 \quad (6.12) \quad \mathcal{B} : \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in U \mapsto \mathbf{1}_{\omega_1} u_1 B_1 + \mathbf{1}_{\omega_2} u_2 B_2.$$

1455 Then, the observation operator reads

$$1456 \quad \mathcal{B}^* : \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} \in X \mapsto \begin{pmatrix} \mathbf{1}_{\omega_1} (B_{1,2}\varphi_2 + B_{1,3}\varphi_3) \\ \mathbf{1}_{\omega_2} (B_{2,2}\varphi_2 + B_{2,3}\varphi_3) \end{pmatrix}.$$

1457

1458 PROPOSITION 6.2. *Let us consider the control system (1.1) with \mathcal{A} and \mathcal{B} given*
 1459 *by (6.11)-(6.12).*

1460 *i. If $\omega_2 = \emptyset$, we assume that*

$$1461 \quad (6.13) \quad B_{1,2}B_{1,3} \neq 0.$$

1462 *Then,*

$$1463 \quad T_0(X_{-\diamond}) = \beta + \min\{\alpha, \beta\}.$$

1464 *ii. If $\omega_2 \neq \emptyset$, we assume that*

$$1465 \quad (6.14) \quad (B_{1,2}^2 + B_{2,2}^2) (B_{1,3}^2 + B_{2,3}^2) \neq 0.$$

1466 *(a) If B_1 and B_2 are linearly independent, then,*

$$1467 \quad T_0(X_{-\diamond}) = 0.$$

1468 *(b) If B_1 and B_2 are not linearly independent, then,*

$$1469 \quad T_0(X_{-\diamond}) = \beta + \min\{\alpha, \beta\}.$$

1470 *Proof.* It is easy to see that $(-\mathcal{A}, D(\mathcal{A}))$ generates a \mathcal{C}_0 -semigroup on X and that
 1471 $\mathcal{B} : U \rightarrow X$ is bounded. Thus we consider for this example that $X_\diamond^* = X = X_{-\diamond}$ and
 1472 $Y_0 = X_{-\diamond}$.

1473 The spectrum of $(\mathcal{A}^*, D(\mathcal{A}))$ is given by $\Lambda = \bigcup_{k \geq 1} G_k$ where

$$1474 \quad G_k := \{\lambda_{k,1} := \nu_k, \lambda_{k,2} := \nu_k + f(\nu_k), \lambda_{k,3} := \nu_k + g(\nu_k)\}.$$

1475 Again, since $(\nu_k)_{k \geq 1}$ satisfies (6.3), (6.6) and (6.7), the application of [12, Lemma
 1476 V.4.20] yields the existence of $\kappa > 0$ such that $\Lambda \in \mathcal{L}_w(3, \frac{\kappa}{2}, \frac{1}{2}, \kappa)$. The sequence
 1477 $(G_k)_{k \geq 1}$ is an associated grouping.

1478 The eigenvalues are simple and the corresponding eigenvectors are given by

$$1479 \quad \phi_{k,1}^0 = \begin{pmatrix} -f(\nu_k) \\ 1 \\ 0 \end{pmatrix} \varphi_k, \quad \phi_{k,2}^0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \varphi_k, \quad \phi_{k,3}^0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \varphi_k.$$

1480 Thus, the assumption (1.34) hold. Moreover,

$$1481 \quad (6.15) \quad b_1 = b_2 = \begin{pmatrix} \mathbf{1}_{\omega_1} \varphi_k B_{1,2} \\ \mathbf{1}_{\omega_2} \varphi_k B_{2,2} \end{pmatrix}, \quad b_3 = \begin{pmatrix} \mathbf{1}_{\omega_1} \varphi_k B_{1,3} \\ \mathbf{1}_{\omega_2} \varphi_k B_{2,3} \end{pmatrix}$$

1482 From (6.5) and (6.13) or (6.14) (depending on the assumption on ω_2) it comes
 1483 that (1.33) is satisfied. Thus, from Theorem 2.3, it comes that for any $y_0 \in X_{-\diamond}$,

$$1484 \quad T_0(y_0) = \limsup_{k \rightarrow +\infty} \frac{\ln^+ \mathcal{C}(G_k, y_0)}{2 \min G_k}.$$

1485 Let us now estimate $\mathcal{C}(G_k, y_0)$. From Theorem 2.6 it comes that

1486
$$\mathcal{C}(G_k, y_0) = \langle M^{-1}\xi, \xi \rangle$$

1487 with

1488
$$M = \text{Gram}(b[\lambda_{k,1}], b[\lambda_{k,1}, \lambda_{k,2}], b[\lambda_{k,1}, \lambda_{k,2}, \lambda_{k,3}])$$

 1489
$$+ \text{Gram}(0, b[\lambda_{k,2}], b[\lambda_{k,2}, \lambda_{k,3}]) + \text{Gram}(0, 0, b[\lambda_{k,3}])$$

1491 and

1492
$$\xi = \begin{pmatrix} \langle y_0, \phi[\lambda_{k,1}] \rangle_{-\diamond, \diamond} \\ \langle y_0, \phi[\lambda_{k,1}, \lambda_{k,2}] \rangle_{-\diamond, \diamond} \\ \langle y_0, \phi[\lambda_{k,1}, \lambda_{k,2}, \lambda_{k,3}] \rangle_{-\diamond, \diamond} \end{pmatrix}.$$

1493 Explicit computations yield

1494
$$\phi[\lambda_{k,1}] = \begin{pmatrix} -f(\nu_k) \\ 1 \\ 0 \end{pmatrix} \varphi_k, \quad \phi[\lambda_{k,1}, \lambda_{k,2}] = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \varphi_k,$$

1495 and

1496
$$\phi[\lambda_{k,1}, \lambda_{k,2}, \lambda_{k,3}] = \frac{1}{g(\nu_k)(g(\nu_k) - f(\nu_k))} \begin{pmatrix} f(\nu_k) - g(\nu_k) \\ -1 \\ 1 \end{pmatrix} \varphi_k.$$

1497 i. Assume that $\omega_2 = \emptyset$.

1498 After the change of variables

1499
$$z = \text{diag} \left(\frac{1}{B_{1,2}}, \frac{1}{B_{1,2}}, \frac{1}{B_{1,3}} \right) y,$$

1500 the system under study reads

1501
$$\begin{cases} \partial_t z + \begin{pmatrix} A & I & 0 \\ 0 & A + f(A) & 0 \\ 0 & 0 & A + g(A) \end{pmatrix} z = \mathbf{1}_{\omega_1} u_1(t, x) \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \\ z(t, 0) = z(t, 1) = 0. \end{cases}$$

1502 This leads to

1503
$$b[\lambda_{k,1}] = b[\lambda_{k,2}] = b[\lambda_{k,3}] = \mathbf{1}_{\omega_1} \varphi_k.$$

1504 Thus, $M = \|\varphi_k\|_{\omega_1}^2 I_3$ and

1505
$$\mathcal{C}(G_k, y_0) = \left\langle y_0, \begin{pmatrix} -f(\nu_k) \\ 1 \\ 0 \end{pmatrix} \frac{\varphi_k}{\|\varphi_k\|_{\omega_1}} \right\rangle_{-\diamond, \diamond}^2 + \left\langle y_0, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \frac{\varphi_k}{\|\varphi_k\|_{\omega_1}} \right\rangle_{-\diamond, \diamond}^2$$

 1506
$$+ \left(\frac{1}{g(\nu_k)(g(\nu_k) - f(\nu_k))} \right)^2 \left\langle y_0, \begin{pmatrix} f(\nu_k) - g(\nu_k) \\ -1 \\ 1 \end{pmatrix} \frac{\varphi_k}{\|\varphi_k\|_{\omega_1}} \right\rangle_{-\diamond, \diamond}^2.$$

 1507
 1508

1509 From (6.5), we obtain for any $y_0 \in X_{-\diamond}$,

1510
$$\mathcal{C}(G_k, y_0) \leq C \|y_0\|_{-\diamond}^2 \left(1 + \left(\frac{1}{g(\nu_k)(g(\nu_k) - f(\nu_k))} \right)^2 \right).$$

1511 This leads to

$$1512 \quad T_0(X_{-\diamond}) \leq \limsup_{k \rightarrow +\infty} \frac{-\ln^+ |g(\nu_k)(g(\nu_k) - f(\nu_k))|}{\nu_k}.$$

1513 Conversely, with the particular choice

$$1514 \quad y_0 = \sum_{k \geq 1} \frac{1}{\nu_k} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \varphi_k,$$

1515 we have

$$1516 \quad \mathcal{C}(G_k, y_0) = \frac{1}{\nu_k^2 \|\varphi_k\|_{\omega_1}^2} \left(\frac{1}{g(\nu_k)(g(\nu_k) - f(\nu_k))} \right)^2.$$

1517 Thus, from (6.5), we obtain

$$1518 \quad T_0(X_{-\diamond}) \geq T_0(y_0) = \limsup_{k \rightarrow +\infty} \frac{-\ln |g(\nu_k)(g(\nu_k) - f(\nu_k))|}{\nu_k}$$

1519 which gives

$$1520 \quad T_0(X_{-\diamond}) = \limsup_{k \rightarrow +\infty} \frac{-\ln |g(\nu_k)(g(\nu_k) - f(\nu_k))|}{\nu_k}.$$

1521 Then, the same computations as [9, Section 5.1.3] yield

$$1522 \quad T_0(X_{-\diamond}) = \beta + \min\{\alpha, \beta\}.$$

1523 ii. We now consider the case $\omega_2 \neq \emptyset$.

1524 (a) Assume that B_1 and B_2 are linearly independent. If necessary, we con-
 1525 sider smaller control sets so that $\omega_1 \cap \omega_2 = \emptyset$. As we will prove that
 1526 $T_0(X_{-\diamond}) = 0$, this is not a restrictive assumption.

1527 To ease the reading we drop the index k in what follows. As previ-
 1528 ously, the vector ξ is not bounded. Let us consider the dilatation
 1529 $D_\epsilon = \text{diag}(1, 1, \epsilon)$ with

$$1530 \quad \epsilon = g(\nu)(g(\nu) - f(\nu))$$

1531 and $\tilde{\xi} = D_\epsilon \xi$. Then, from Section D.1, it comes that

$$1532 \quad \mathcal{C}(G, y_0) = \left\langle \tilde{M}^{-1} \tilde{\xi}, \tilde{\xi} \right\rangle$$

1533 with

$$1534 \quad \tilde{M} = \text{Gram}(b[\lambda_1], b[\lambda_1, \lambda_2], \epsilon b[\lambda_1, \lambda_2, \lambda_3]) \\ 1535 \quad + \text{Gram}(0, b[\lambda_2], \epsilon b[\lambda_2, \lambda_3]) + \text{Gram}(0, 0, \epsilon b[\lambda_3]).$$

1537 As $\|\tilde{\xi}\|$ is bounded, we simply give a lower bound on the smallest eigen-
 1538 value of \tilde{M} . Using (6.15), it comes that

$$1539 \quad b[\lambda_1, \lambda_2] = 0, \quad b[\lambda_2, \lambda_3] = \frac{b_3 - b_1}{g(\nu) - f(\nu)}, \quad b[\lambda_1, \lambda_2, \lambda_3] = \frac{1}{\epsilon}(b_3 - b_1).$$

1540

Thus,

1541

$$\begin{aligned} \widetilde{M} = & \text{Gram}(b_1, 0, b_3 - b_1) + \text{Gram}(0, b_1, g(\nu)(b_3 - b_1)) \\ & + \text{Gram}(0, 0, \epsilon b_3). \end{aligned}$$

1543

1544

This gives that, for any $\tau \in \mathbb{R}^3$, we have

1545

1546

$$(6.16) \quad \left\langle \widetilde{M}\tau, \tau \right\rangle = \|\tau_1 b_1 + \tau_3(b_3 - b_1)\|_U^2 + \|\tau_2 b_1 + g(\nu)\tau_3(b_3 - b_1)\|_U^2 \\ + \epsilon^2 \|\tau_3 b_3\|_U^2.$$

1548

1549

To obtain a lower bound on this quantity we use the following lemma.

1550

LEMMA 6.3. *There exists $C > 0$ (independent of k) such that for any $\theta_1, \theta_3 \in \mathbb{R}$,*

1551

1552

$$\|\theta_1 b_1 + \theta_3 b_3\|_U^2 \geq C(\theta_1^2 + \theta_3^2).$$

1553

Proof. As $\omega_1 \cap \omega_2 = \emptyset$,

1554

$$\begin{aligned} \|\theta_1 b_1 + \theta_3 b_3\|_U^2 = & (B_{1,2}\theta_1 + B_{1,3}\theta_3)^2 \|\varphi_k\|_{\omega_1}^2 \\ & + (B_{2,2}\theta_1 + B_{2,3}\theta_3)^2 \|\varphi_k\|_{\omega_2}^2. \end{aligned}$$

1556

1557

Using (6.5) it comes that

1558

$$\begin{aligned} \|\theta_1 b_1 + \theta_3 b_3\|_U^2 \geq & C \left((B_{1,2}\theta_1 + B_{1,3}\theta_3)^2 + (B_{2,2}\theta_1 + B_{2,3}\theta_3)^2 \right) \\ = & \left\| \begin{pmatrix} B_{1,2} & B_{1,3} \\ B_{2,2} & B_{2,3} \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_3 \end{pmatrix} \right\|^2. \end{aligned}$$

1559

1560

Since B_1 and B_2 are linearly independent, this ends the proof. \square

1561

1562

Applying this lemma twice to (6.16) yield

1563

$$\begin{aligned} \left\langle \widetilde{M}\tau, \tau \right\rangle \geq & C \left((\tau_1 - \tau_3)^2 + \tau_3^2 + (\tau_2 - g(\nu)\tau_3)^2 + g(\nu)^2 \tau_3^2 + \epsilon^2 \tau_3^2 \right) \\ \geq & C \left((\tau_1 - \tau_3)^2 + \tau_3^2 + (\tau_2 - g(\nu)\tau_3)^2 \right). \end{aligned}$$

1564

1566

Taking into account that $0 < g(\nu) < \frac{1}{2}$ for ν large enough, the study of this quadratic form in \mathbb{R}^3 leads to

1567

1568

$$\left\langle \widetilde{M}\tau, \tau \right\rangle \geq C(\tau_1^2 + \tau_2^2 + \tau_3^2).$$

1569

Thus the smallest eigenvalue of \widetilde{M} is bounded from below. This leads to the boundedness of $\left\langle \widetilde{M}^{-1}\widetilde{\xi}, \widetilde{\xi} \right\rangle$ which concludes the proof of case ii (a).

1570

1571

(b) Assume now that B_1 and B_2 are not linearly independent. Then, there exist $x_1, x_2 \in \mathbb{R}$ such that

1572

1573

$$\begin{cases} x_1 B_{1,2} + x_2 B_{1,3} = 0 \\ x_1 B_{2,2} + x_2 B_{2,3} = 0. \end{cases}$$

1574

Up to a change of normalization of the eigenvectors (independent of k) we obtain

1575

1576

$$b_1 = b_2 = b_3 = \begin{pmatrix} \mathbf{1}_{\omega_1} \varphi_k x_1 B_{1,2} \\ \mathbf{1}_{\omega_2} \varphi_k x_1 B_{2,2} \end{pmatrix}$$

1577

and this amounts to case i. \square

1578 **7. Analysis of controllability for systems of partial differential equa-**
 1579 **tions.**

1580 We now turn to the analysis of null controllability of actual partial differential
 1581 equations. We consider here coupled systems of two linear one dimensional parabolic
 1582 equations.

1583 **7.1. Coupled heat equations with different diffusion coefficients.**

1584 In this application, we consider the Sturm-Liouville operator A defined in (6.1)
 1585 and we define in $X = L^2(0, 1; \mathbb{R})^2$ the operator

1586
$$\mathcal{A} = \begin{pmatrix} A & I \\ 0 & dA \end{pmatrix}, \quad D(\mathcal{A}) = D(A)^2,$$

1587 with $d > 0$. We will assume $d \neq 1$, since the case $d = 1$ is much simpler and already
 1588 studied in the literature: see the computations of Section 6.3 in the case $f = 0$ or, for
 1589 instance, [23] for a more general study based on Carleman estimates.

1590 We will consider two cases : the case where two boundary controls are applied to
 1591 the system, and the case where we consider the same distributed control in the two
 1592 equations of the system.

1593 **7.1.1. Spectrum of \mathcal{A}^* .** Let $\Lambda_1 := \text{Sp}(A) = \{\nu_k ; k \geq 1\}$ and $\Lambda_2 := d\Lambda_1$.

1594 The spectrum of \mathcal{A}^* is given by $\Lambda = \Lambda_1 \cup \Lambda_2$ which belongs to $\mathcal{L}_w(2, \varrho, \frac{1}{2}, \kappa)$ for
 1595 some $\varrho, \kappa > 0$ (see [12, Lemma V.4.20]).

1596 For any $\lambda \in \Lambda$, there are two non mutually exclusive cases:

- 1597
 - If $\lambda = \nu_k \in \Lambda_1$, then we can associate an eigenvector given by

1598
$$\phi_{\lambda,1} = \begin{pmatrix} 1 \\ \varepsilon_k \end{pmatrix} \varphi_k,$$

1599 with $\varepsilon_k = \frac{1}{\nu_k(1-d)}$. Note that ε_k tends to zero when k goes to infinity.

- 1600
 - If $\lambda = d\nu_l \in \Lambda_2$, then we can associate an eigenvector given by

1601
$$\phi_{\lambda,2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varphi_l.$$

1602 It clearly appears that the elements in $\Lambda_1 \cap \Lambda_2$ (if this set is not empty) are ge-
 1603 ometrically double eigenvalues of \mathcal{A}^* , since in that case $\phi_{\lambda,1}$ and $\phi_{\lambda,2}$ are linearly
 1604 independent.

1605 Note that (1.34) holds for the choices of X_\diamond^* that we will make in the sequel, since
 1606 $(\varphi_k)_{k \geq 1}$ is an Hilbert basis of $L^2(0, 1; \mathbb{R})$.

1607 **7.1.2. Two boundary controls.** In this section, we study the following bound-
 1608 ary control system

1609 (7.1)
$$\begin{cases} \partial_t y + \mathcal{A}y = 0, & t \in (0, T), \\ y(t, 0) = B_0 u_0(t), \quad y(t, 1) = B_1 u_1(t), & t \in (0, T), \end{cases}$$

1610 with

1611 (7.2)
$$B_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad B_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

1612 The control operator \mathcal{B} is defined in a weak sense as in [38]. The expression of its
 1613 adjoint is given by

$$1614 \quad \mathcal{B}^* : \begin{pmatrix} f \\ g \end{pmatrix} \in X_1^* \mapsto \begin{pmatrix} -B_0^* \begin{pmatrix} f'(0) \\ g'(0) \end{pmatrix} \\ B_1^* \begin{pmatrix} f'(1) \\ g'(1) \end{pmatrix} \end{pmatrix} = \begin{pmatrix} -(f'(0) + g'(0)) \\ g'(1) \end{pmatrix}.$$

1615 Considering $X_\diamond^* = H_0^1(0, 1; \mathbb{R})^2$, we obtain that \mathcal{B} is admissible with respect to $X_{-\diamond} =$
 1616 $H^{-1}(0, 1; \mathbb{R})^2$.

1617 **PROPOSITION 7.1.** *For any $d \neq 1$, there exists Y_0 a closed subspace of $H^{-1}(0, 1; \mathbb{R})^2$*
 1618 *of finite codimension such that*

- 1619 • for any $y_0 \notin Y_0$, system (7.1) is not approximately controllable;
- 1620 • for any $y_0 \in Y_0$, system (7.1) is null controllable in any time $T > 0$.

1621 **REMARK 7.1.** *The situation with a single control is quite different. Indeed, con-*
 1622 *sidering $B_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $B_1 = 0$, it is proved in [5] that, when A is the Dirichlet*
 1623 *Laplace operator, approximate controllability holds if and only if $\sqrt{d} \notin \mathbb{Q}$ and in this*
 1624 *case that*

$$1625 \quad T_0(X_{-\diamond}) = \limsup_{\substack{\lambda \rightarrow \infty \\ \lambda \in \Lambda}} \frac{\ln^+ \left(\frac{1}{\text{dist}(\lambda, \Lambda \setminus \{\lambda\})} \right)}{\lambda}.$$

1626 *With this formula the authors prove that, for any $\tau \in [0, +\infty]$, there exists a diffusion*
 1627 *ratio $d > 0$ such that the minimal null control time of system (7.1) satisfies $T_0(X_{-\diamond}) =$*
 1628 *τ .*

1629 **REMARK 7.2.** *From the definition of Y_0 in the following proof, we directly obtain*
 1630 *that in the case where \mathcal{A} is the Dirichlet Laplace operator on the interval $(0, 1)$, then*
 1631 *$Y_0 = H^{-1}(0, 1; \mathbb{R})^2$.*

1632 **REMARK 7.3.** *The particular choice of B_0 and B_1 is done to simplify the com-*
 1633 *putations. Notice that with this choice, it is not possible to steer to zero the second*
 1634 *equation and then control the first equation. This would be the case with the simpler*
 1635 *choice*

$$1636 \quad B_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad B_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

1637 *Proof.* Let us compute the observations associated to the eigenvectors of \mathcal{A}^* .

1638 For any $k \geq 1$, we define $s_k \in \mathbb{R}$ be such that $\varphi'_k(1) = s_k \varphi'_k(0)$. From (6.4), there
 1639 exists $C > 0$ such that

$$1640 \quad (7.3) \quad \frac{1}{C} \leq |s_k| \leq C, \quad \forall k \geq 1.$$

- 1641 • For any $\lambda = \nu_k \in \Lambda_1$, we have

$$1642 \quad (7.4) \quad \mathcal{B}^* \phi_{\lambda,1} = -\varphi'_k(0) \begin{pmatrix} 1 + \varepsilon_k \\ -s_k \varepsilon_k \end{pmatrix}.$$

- 1643 • For any $\lambda = d\nu_l \in \Lambda_2$, we have

$$1644 \quad (7.5) \quad \mathcal{B}^* \phi_{\lambda,2} = -\varphi'_l(0) \begin{pmatrix} 1 \\ -s_l \end{pmatrix}.$$

1645 Due to (6.4) and (7.3), it comes that (1.33) holds for any simple eigenvalue $\lambda \in$
 1646 $(\Lambda_1 \setminus \Lambda_2) \cup (\Lambda_2 \setminus \Lambda_1)$.

1647 However, for a geometrically double eigenvalue $\lambda \in \Lambda_1 \cap \Lambda_2$, there can be non-
 1648 observable modes. Indeed, let k and l such that $\lambda = \nu_k = d\nu_l$. Then, the condition

$$1649 \quad \text{Ker}(\mathcal{A}^* - \lambda) \cap \text{Ker} \mathcal{B}^* \neq \{0\}$$

1650 is equivalent to the fact that $\mathcal{B}^* \phi_{\lambda,1}$ and $\mathcal{B}^* \phi_{\lambda,2}$ given by (7.4)-(7.5) are linearly
 1651 independent, which is itself equivalent to the condition

$$1652 \quad (7.6) \quad s_k \varepsilon_k = s_l (1 + \varepsilon_k).$$

1653 Due to the asymptotics $\varepsilon_k \xrightarrow[k \rightarrow +\infty]{} 0$ it turns out that the set

$$1654 \quad \Theta := \{ \lambda = \nu_k = d\nu_l \in \Lambda_1 \cap \Lambda_2 ; (7.6) \text{ holds} \},$$

1655 is finite.

1656 For any $\lambda \in \Theta$, we can find $\psi_\lambda \in \text{Span}(\phi_{\lambda,1}, \phi_{\lambda,2})$ such that $\mathcal{B}^* \psi_\lambda = 0$ and $\psi_\lambda \neq 0$,
 1657 that is a non observable mode.

1658 Finally, we introduce the set

$$1659 \quad Y_0 := \left\{ y_0 \in X_{-\diamond} ; \langle y_0, \psi_\lambda \rangle_{-\diamond, \diamond} = 0, \forall \lambda \in \Theta \right\}$$

1660 which is, by construction, of finite codimension. For $y_0 \in Y_0$, the associated moment
 1661 problem reduces to the one where the geometrically double eigenvalues $\lambda \in \Theta$ are now
 1662 considered as simple eigenvalues with associated eigenvector $\phi_{\lambda,2}$, since the moment
 1663 equation is automatically satisfied for the other eigenvector ψ_λ .

1664 We consider now a grouping \mathcal{G} as given by Proposition 1.2, with $p = 2$ and $\varrho > 0$
 1665 small enough such that for $i \in \{1, 2\}$ we have

$$1666 \quad (7.7) \quad |\lambda - \mu| > \varrho, \quad \forall \lambda, \mu \in \Lambda_i, \lambda \neq \mu.$$

1667 Hence, Theorem 2.3 gives the formula

$$1668 \quad T_0(y_0) = \limsup_{G \in \mathcal{G}} \frac{\ln^+ \mathcal{C}(G, y_0)}{2r_G}.$$

1669 We will prove in the sequel, analyzing the different possible blocks, that

$$1670 \quad (7.8) \quad \sup_{G \in \mathcal{G}} \mathcal{C}(G, y_0) < +\infty,$$

1671 which will let us conclude the claim, that is $T_0(y_0) = 0$.

1672 • Blocks of a simple eigenvalue.

We immediately obtain

$$\mathcal{C}(G, y_0) = \begin{cases} \frac{\left| \langle y_0, \phi_{\lambda,1} \rangle_{-\diamond, \diamond} \right|^2}{((1 + \varepsilon_k)^2 + s_k^2 \varepsilon_k^2) |\varphi'_k(0)|^2} & , \text{ if } \lambda = \nu_k, \\ \frac{\left| \langle y_0, \phi_{\lambda,2} \rangle_{-\diamond, \diamond} \right|^2}{(1 + s_l^2) |\varphi'_l(0)|^2} & , \text{ if } \lambda = d\nu_l. \end{cases}$$

1673 Using again (6.4) the estimate (7.3) and the fact that $(\varepsilon_k)_k$ goes to 0 as k goes
 1674 to infinity, we observe that the blocks consisting of a single simple eigenvalue
 1675 do not contribute to the minimal time: the quantity $\mathcal{C}(G, y_0)$ is bounded
 1676 independently of G .

1677 Moreover, by the discussion above, the blocks consisting of a single double
 1678 eigenvalue belonging to Θ do not contribute either.

- Blocks of two simple eigenvalues: $G = \{\lambda_1 := \nu_k\} \cup \{\lambda_2 := d\nu_l\}$.
 From Theorem 2.8 we obtain

$$1681 \quad \mathcal{C}(G, y_0) = \langle M^{-1}\xi, \xi \rangle$$

1682 with

$$1683 \quad M = \text{Gram}(b[\lambda_1], b[\lambda_2]) + \text{Gram}(0, (\lambda_2 - \lambda_1)b[\lambda_2])$$

1684 and

$$1685 \quad \xi = \begin{pmatrix} \langle y_0, \phi_{\lambda_1,1} \rangle_{-\diamond, \diamond} \\ \langle y_0, \phi_{\lambda_2,2} \rangle_{-\diamond, \diamond} \end{pmatrix}.$$

1686 To ease the reading, we use the following change of normalization for the
 1687 eigenvectors

$$1688 \quad \tilde{\phi}_{\lambda_1} := \frac{\phi_{\lambda_1,1}}{-\varphi'_k(0)}, \quad \tilde{\phi}_{\lambda_2} := \frac{\phi_{\lambda_2,2}}{-\varphi'_l(0)},$$

1689 and we denote by \tilde{M} and $\tilde{\xi}$ the associated quantities. Notice that, due to (6.4),
 1690 the quantity $\|\tilde{\xi}\|$ is bounded. Thus, to estimate $\mathcal{C}(G, y_0)$ we give a lower bound
 1691 on the smallest eigenvalue of \tilde{M} . We have

$$1692 \quad \begin{aligned} \tilde{M} &= \text{Gram}(\tilde{b}[\lambda_1], \tilde{b}[\lambda_2]) + \text{Gram}(0, (\lambda_2 - \lambda_1)\tilde{b}[\lambda_2]) \\ 1693 \quad &= \underbrace{\begin{pmatrix} \varepsilon_k^2 s_k^2 + (1 + \varepsilon_k)^2 & 1 + \varepsilon_k + \varepsilon_k s_k s_l \\ 1 + \varepsilon_k + \varepsilon_k s_k s_l & 1 + s_l^2 \end{pmatrix}}_{=\Gamma^1} + \begin{pmatrix} 0 & 0 \\ 0 & (\lambda_2 - \lambda_1)^2(1 + s_l^2) \end{pmatrix}. \end{aligned}$$

1694 For any $\tau \in \mathbb{R}^2$, $\langle \tilde{M}\tau, \tau \rangle \geq \langle \Gamma^1\tau, \tau \rangle$. Then,

$$1696 \quad \min \text{Sp}(\Gamma^1) \geq \frac{\det(\Gamma^1)}{\text{tr}(\Gamma^1)} = \frac{((1 + \varepsilon_k)s_l - \varepsilon_k s_k)^2}{1 + (1 + \varepsilon_k)^2 + \varepsilon_k^2 s_k^2 + s_l^2}$$

1697 From (7.3), it comes that, for k large enough, $\min \text{Sp}(\Gamma^1)$ is bounded from
 1698 below by a positive constant independent of G .

- Blocks made of a geometrically double eigenvalue which does not belong to Θ :

1700 Consider $G = \{\lambda\}$ with $\lambda = \nu_k = d\nu_l \in \Lambda_1 \cap \Lambda_2$. With the same notations as
 1701 previously, Theorem 2.8 implies that

$$1703 \quad \mathcal{C}(G, y_0) = \langle \tilde{M}^{-1}\tilde{\xi}, \tilde{\xi} \rangle$$

1704 where

$$1705 \quad \tilde{\xi} = \begin{pmatrix} \left\langle y_0, \frac{\phi_{\lambda,1}}{-\varphi'_k(0)} \right\rangle_{-\diamond, \diamond} \\ \left\langle y_0, \frac{\phi_{\lambda,2}}{-\varphi'_l(0)} \right\rangle_{-\diamond, \diamond} \end{pmatrix}$$

1706 and

$$1707 \quad \widetilde{M} = \text{Gram} \left(\begin{array}{c} \mathcal{B}^* \phi_{\lambda,1} \\ -\varphi'_k(0) \end{array}, \begin{array}{c} \mathcal{B}^* \phi_{\lambda,2} \\ -\varphi'_l(0) \end{array} \right) = \Gamma^1. \\ 1708$$

1709 Notice that since $\lambda \notin \Theta$, we have $\det(\Gamma^1) = ((1 + \varepsilon_k)s_l - \varepsilon_k s_k)^2 > 0$.

1710 Thus, the study of the previous item proves that, for λ large enough, $\min \text{Sp}(\Gamma^1)$
1711 is bounded from below by a positive constant independent of λ .

1712 Gathering all cases, we deduce (7.8) and the proof is complete. \square

1713 **7.1.3. Simultaneous distributed control.** Let us now consider the following
1714 control problem

$$1715 \quad (7.9) \quad \begin{cases} \partial_t y + \mathcal{A}y = \mathbf{1}_\omega \begin{pmatrix} 1 \\ 1 \end{pmatrix} u(t, x), & t \in (0, T), \\ y(t, 0) = y(t, 1) = 0, & t \in (0, T). \end{cases}$$

1716 In that case, the observation operator \mathcal{B}^* is given by

$$1717 \quad \mathcal{B}^* : \begin{pmatrix} f \\ g \end{pmatrix} \in X_1^* \mapsto \mathbf{1}_\omega(f + g),$$

1718 and is clearly admissible with respect to the pivot space X . Our result concerning
1719 this example is very similar to Proposition 7.1 and reads as follows.

1720 **PROPOSITION 7.2.** *For any $d \neq 1$, there exists Y_0 a closed subspace of $H^{-1}(0, 1; \mathbb{R})^2$
1721 of codimension less or equal than 1 such that*

- 1722 • for any $y_0 \notin Y_0$, system (7.9) is not approximately controllable;
- 1723 • for any $y_0 \in Y_0$, system (7.9) is null controllable in any time $T > 0$.

1724 **REMARK 7.4.** *During the proof it will appear that there exists a countable set
1725 $D \subset (1, +\infty)$ such that for any $d \notin D \cup \{1\}$, we have $Y_0 = H^{-1}(0, 1; \mathbb{R})^2$, which
1726 means that our system is null-controllable at any time $T > 0$ for any initial data. In
1727 particular, it is noticeable that this property holds for any $d < 1$, that is in the case
1728 where the diffusion coefficient is lower in the second equation (the one which does not
1729 contain coupling terms).*

1730 *Proof.* We start by computing the observations related to the eigenelements of
1731 \mathcal{A}^*

- 1732 • For any $\lambda = \nu_k \in \Lambda_1$, we have

$$1733 \quad (7.10) \quad \mathcal{B}^* \phi_{\lambda,1} = (1 + \varepsilon_k) \varphi_k \mathbf{1}_\omega.$$

- 1734 • For any $\lambda = d\nu_l \in \Lambda_2$, we have

$$1735 \quad (7.11) \quad \mathcal{B}^* \phi_{\lambda,2} = \varphi_l \mathbf{1}_\omega.$$

If for some k we have $1 + \varepsilon_k = 0$, then we clearly get that (1.33) does not hold. We
can thus introduce the set

$$\Theta := \{\lambda = \nu_k ; 1 + \varepsilon_k = 0\},$$

1736 which is of cardinal less or equal than 1 (by definition of the sequence $(\varepsilon_k)_k$, see
1737 Section 7.1.1). Note also that for $d < 1$, we always have $\varepsilon_k > 0$, so that $\Theta = \emptyset$, see
1738 Remark 7.4.

1739 We notice however that, for any $\lambda = d\nu_l$, we have $\mathcal{B}^*\phi_{\lambda,2} \neq 0$ and that if $\lambda =$
 1740 $\nu_k = d\nu_l \in \Lambda_1 \cap \Lambda_2$, with $\lambda \notin \Theta$, then $\mathcal{B}^*\phi_{\lambda,1}$ and $\mathcal{B}^*\phi_{\lambda,2}$ are linearly independent.

Let us introduce

$$Y_0 := \{y_0 \in X; \text{ s.t. } \langle y_0, \phi_{\lambda,1} \rangle_X = 0, \forall \lambda \in \Theta\}.$$

1741 By definition of this set, for any initial data in Y_0 , the moment equation (1.2) related
 1742 to the eigenvector $\phi_{\lambda,1}$ for $\lambda \in \Theta$ is automatically satisfied for any control since both
 1743 members are equal to zero.

1744 As in the proof of Proposition 7.1, we consider a grouping \mathcal{G} as given by Propo-
 1745 sition 1.2, with $p = 2$ and $\varrho > 0$ small enough such that for $i \in \{1, 2\}$ we have

$$1746 \quad |\lambda - \mu| > \varrho, \quad \forall \lambda, \mu \in \Lambda_i, \lambda \neq \mu.$$

1747 Hence, Theorem 2.3 gives the formula

$$1748 \quad T_0(y_0) = \limsup_{G \in \mathcal{G}} \frac{\ln^+ \mathcal{C}(G, y_0)}{2r_G}.$$

1749 Let us now evaluate the quantities $\mathcal{C}(G, y_0)$ for every possible block.

- 1750 • Blocks made of a simple eigenvalue that does not belong to Θ .
 We immediately obtain

$$\mathcal{C}(G, y_0) = \begin{cases} \frac{|\langle y_0, \phi_{\lambda,1} \rangle_X|^2}{(1 + \varepsilon_k)^2 \|\varphi_k\|_\omega^2}, & \text{if } \lambda = \nu_k, \\ \frac{|\langle y_0, \phi_{\lambda,2} \rangle_X|^2}{\|\varphi_l\|_\omega^2}, & \text{if } \lambda = d\nu_l, \end{cases}$$

1751 which is a bounded quantity thanks to (6.5) and the fact that $(\varepsilon_k)_k$ tends to
 1752 zero at infinity.

- 1753 • Blocks made of two eigenvalues: $G = \{\lambda_1 := \nu_k\} \cup \{\lambda_2 := d\nu_l\}$. Note that
 1754 the proof below works exactly the same in the case where $\lambda_1 \neq \lambda_2$, that is if
 1755 the two eigenvalues are simple, or in the case where $\lambda_1 = \lambda_2$, that is if there
 1756 is only a geometrically double eigenvalue.

1757 By the discussion above, we can assume that λ_1 does not belong to Θ (if
 1758 not, this block has to be considered as a block containing only the simple
 1759 eigenvalue λ_2).

Thanks to Theorem 2.8 we have $\mathcal{C}(G, y_0) \leq \langle \widetilde{M}^{-1}\xi, \xi \rangle$ where

$$\begin{aligned} \widetilde{M} &= \text{Gram}(\mathbf{1}_\omega \varphi_k, \mathbf{1}_\omega \varphi_l), \\ \xi &= \begin{pmatrix} \langle y_0, \phi_{\lambda_1,1} \rangle_X \\ 1 + \varepsilon_k \\ \langle y_0, \phi_{\lambda_2,2} \rangle_X \end{pmatrix} \end{aligned}$$

By using the Lebeau-Robbiano inequality (6.8), and the fact that $|\lambda_1 - \lambda_2| \leq$
 ϱ , we have that

$$\langle \widetilde{M}^{-1}\xi, \xi \rangle \leq C_1 e^{C_1 \sqrt{r_G}} \|\xi\|^2 \leq C_2 e^{C_1 \sqrt{r_G}} \|y_0\|_X^2,$$

1760 where C_1, C_2 only depends on ϱ, ω and on the operator \mathcal{A} .

All in all, we have obtained that

$$\ln^+ \mathcal{C}(G, y_0) \leq C (1 + \sqrt{r_G}).$$

1761 Gathering all cases, we conclude that $T_0(y_0) = 0$. □

1762 **7.2. Other applications.**

1763 Let us consider the following control system

(7.12)

$$\begin{cases} \partial_t y + \begin{pmatrix} -\partial_{xx} + c_1(x) & 1 \\ 0 & -\partial_{xx} + c_2(x) \end{pmatrix} y = \begin{pmatrix} 0 \\ \mathbf{1}_\omega u(t, x) \end{pmatrix}, & (t, x) \in (0, T) \times (0, 1), \\ y(t, 0) = y(t, 1) = 0, & t \in (0, T), \\ y(0, x) = y_0(x), \end{cases}$$

1764

1765 where $c_1, c_2 \in L^2(0, 1; \mathbb{R})$.

1766 With the technics developed in this article, one can prove the following control-
1767 lability result.

1768 **PROPOSITION 7.3.** *For any non-negative potentials c_1, c_2 , system (7.12) is null*
1769 *controllable in any time $T > 0$ from $L^2(0, 1; \mathbb{R})^2$.*

1770 The proof follows closely the computations done for the same system with a boundary
1771 control in [9, Section 5.2.1]. The only difference is that the contributions of terms of
1772 the form $\|\mathcal{B}^* \bullet\|_U = \|\bullet\|_\omega$ are estimated using (6.5).

1773 As the result stated in Proposition 7.3 is already known (it is for instance an
1774 application of [23] with a proof based on Carleman estimates), we do not detail the
1775 proof here to lighten this article.

1776 With the technics developed in this article we can also analyze null controllability
1777 for the following control system

(7.13)

$$\begin{cases} \partial_t y + \begin{pmatrix} A & q(x) \\ 0 & A \end{pmatrix} y = \begin{pmatrix} 0 \\ \mathbf{1}_\omega u(t, x) \end{pmatrix}, & (t, x) \in (0, T) \times (0, 1), \\ y(t, 0) = y(t, 1) = 0, & t \in (0, T), \\ y(0, x) = y_0(x), \end{cases}$$

1778

1779 where the coupling function q belongs to $L^\infty(0, 1; \mathbb{R})$ and $\omega \subset (0, 1)$ is a non empty
1780 open set. We manage to characterize the value of the minimal null-control time
1781 without any other assumption on q and ω .

1782 This analysis extends previous results of [14] where approximate controllability
1783 was studied and those of [6] where null controllability was studied in the particular
1784 case where \mathcal{A} is the Dirichlet Laplace operator and ω is an interval disjoint of $\text{Supp } q$.
1785 Our formalism also allows us to recover null controllability in any time when q has a
1786 strict sign on a subdomain of ω as proved in [23] by means of Carleman estimates.

1787 Since the analysis of this example makes use of refined spectral properties of
1788 the underlying operator whose proofs are rather intricate, we will develop it in the
1789 forthcoming paper [13].

1790 **Appendix A. Some refinements in the case of scalar controls.**

1791 In [9], the block moment method was introduced to solve null controllability
1792 problems with scalar controls ($U = \mathbb{R}$). With respect to block moment problems, the
1793 main result of this paper is [9, Theorem 4.1]. In this work there were no assumptions
1794 on the counting function. The spectrum Λ was only assumed to satisfy $\Lambda \subset [1, +\infty)$
1795 and

1796

$$\sum_{\lambda \in \Lambda} \frac{1}{\lambda} < +\infty.$$

1797 Using the slightly more restrictive condition (1.28) on the asymptotics of the count-
1798 ing function we allow the eigenvalues to be complex valued and we obtain sharper

1799 estimates together with the explicit dependency of the constants with respect to the
1800 final time T (see Remark 2.5 for possible applications of such estimates). This im-
1801 proved resolution of scalar block moment problems reads as follow and is proved in [12,
1802 Theorem V.4.25].

1803 THEOREM A.1. *Let $p \in \mathbb{N}^*$, $\varrho, \tau, \kappa > 0$ and $\theta \in (0, 1)$. Assume that*

$$1804 \quad \Lambda \in \mathcal{L}_w(p, \varrho, \tau, \theta, \kappa).$$

1805 *Let $G = \{\lambda_1, \dots, \lambda_g\} \subset \Lambda$ be a group satisfying (1.25)–(1.27). Let $T \in (0, +\infty)$ and*
1806 *$\eta \in \mathbb{N}^*$. For any multi-index $\alpha \in \mathbb{N}^g$ with $|\alpha|_\infty \leq \eta$ and any*

$$1807 \quad \omega = (\omega_1^0, \dots, \omega_1^{\alpha_1-1}, \dots, \omega_g^0, \dots, \omega_g^{\alpha_g-1}) \in \mathbb{C}^{|\alpha|},$$

1808 *there exists $v_G \in L^2(0, T; \mathbb{C})$ satisfying*

$$1809 \quad (\text{A.1a}) \quad \int_0^T v_G(t) \frac{(-t)^l}{l!} e^{-\bar{\lambda}_j t} dt = \omega_j^l, \quad \forall j \in \llbracket 1, g \rrbracket, \forall l \in \llbracket 0, \alpha_j \rrbracket,$$

$$1810 \quad (\text{A.1b}) \quad \int_0^T v_G(t) \frac{(-t)^l}{l!} e^{-\bar{\lambda} t} dt = 0, \quad \forall \lambda \in \Lambda \setminus G, \forall l \in \llbracket 0, \eta \rrbracket.$$

1812 *The solution v_G satisfies the following estimate*

$$1813 \quad (\text{A.2}) \quad \|v_G\|_{L^2(0, T; U)} \leq C \exp\left(\frac{C}{T^{\frac{\theta}{1-\theta}}}\right) \exp\left(r_G \frac{T}{2}\right) \exp(Cr_G^\theta) \max_{\substack{\mu \in \mathbb{N}^g \\ \mu \leq \alpha}} \left| \omega \left[\bar{\lambda}_\bullet^{(\mu)} \right] \right|,$$

1814 *where r_G is defined in (1.35) and with the convention*

$$1815 \quad \omega \left[\bar{\lambda}_j^{(l+1)} \right] = \omega_j^l, \quad \forall j \in \llbracket 1, g \rrbracket, \forall l \in \llbracket 0, \alpha_j \rrbracket.$$

1816 *The constant $C > 0$ appearing in the estimate only depends on the parameters $\tau, p,$
1817 ϱ, η, θ and κ .*

1818 *Moreover, there exists a constant $C_{p, \eta, r_\Lambda} > 0$ such that any $v_G \in L^2(0, T; U)$
1819 *solution of (A.1a) satisfy**

$$1820 \quad (\text{A.3}) \quad \|v_G\|_{L^2(0, T; \mathbb{C})} \geq C_{p, \eta, r_\Lambda} \max_{\substack{\mu \in \mathbb{N}^g \\ \mu \leq \alpha}} \left| \omega \left[\bar{\lambda}_\bullet^{(\mu)} \right] \right|.$$

1821 REMARK A.1. *If every assumption hold except (1.29) in the definition of the class*
1822 *$\mathcal{L}_w(p, \varrho, \tau, \theta, \kappa)$, Theorem A.1 remains valid replacing θ in estimate (A.2) by any*
1823 *$\theta' \in (\theta, 1)$ (see [12, Theorem V.4.25]).*

1824 *Since every estimate on the resolution of block moment problems proved in this*
1825 *paper follows from (A.2), this remark holds in the whole current paper. Notably it*
1826 *applies to Theorem 2.2 and to the estimates of the cost of controllability stated in*
1827 *Proposition 2.9 and Corollary 2.10.*

1828 The application of this theorem to the resolution of scalar block moment problems
1829 can be found in [12, Section V.5.3]. Notice that in the estimate (A.2) the term $e^{r_G T/2}$
1830 can seem to be annoying. In [12, Corollary V.5.29], it is dealt with solving scalar
1831 block moment problems in small time T . The same strategy is used in the present
1832 paper to prove Theorem 2.3 in Section 4.2.

1833 **Appendix B. An auxiliary optimization argument.**

1834 LEMMA B.1. Let Y be a closed subspace of $X_{-\diamond}$. Let $g \in \mathbb{N}^*$ and $\psi_1, \dots, \psi_g \in$
 1835 $P_Y^* X_{\diamond}^*$. For any $y \in Y$, let

$$1836 \quad \xi_y = \begin{pmatrix} \langle y, \psi_1 \rangle_{-\diamond, \diamond} \\ \vdots \\ \langle y, \psi_g \rangle_{-\diamond, \diamond} \end{pmatrix}.$$

1837 Then, for any positive semi-definite hermitian square matrix $M \in \mathcal{M}_g(\mathbb{C})$, we have

$$1838 \quad (\text{B.1}) \quad \sup_{\substack{y \in Y \\ \|y\|_{-\diamond} = 1}} \langle M \xi_y, \xi_y \rangle = \rho(\mathbf{G}_\psi M)$$

1839 with $\mathbf{G}_\psi = \text{Gram}_{X_{\diamond}^*}(\psi_1, \dots, \psi_g)$.

In the course of the proof we will use that there exists an isometric linear bijection $I : X_{-\diamond} \mapsto X_{\diamond}^*$ such that

$$\langle y, \varphi \rangle_{-\diamond, \diamond} = (Iy, \varphi)_{\diamond^*}, \quad \forall y \in X_{-\diamond}, \forall \varphi \in X_{\diamond}^*.$$

Note that it satisfies

$$(Iy, \varphi)_{\diamond^*} = (y, I^{-1}\varphi)_{-\diamond}, \quad \forall y \in X_{-\diamond}, \forall \varphi \in X_{\diamond}^*.$$

Proof. Let S be the value of the supremum in the left-hand side of (B.1). By assumption on the $(\psi_i)_i$, we first observe that the supremum can be taken on the whole space $X_{-\diamond}$ instead of Y without changing its value. Then, for any $1 \leq i \leq g$, we have

$$\langle y, \psi_i \rangle_{-\diamond, \diamond} = (y, I^{-1}\psi_i)_{-\diamond},$$

and therefore the value of S does not change if we take the supremum over the set

$$\tilde{\Psi} = \text{Span}(\tilde{\psi}_1, \dots, \tilde{\psi}_g) \subset X_{-\diamond},$$

1840 with

$$1841 \quad (\text{B.2}) \quad \tilde{\psi}_i = I^{-1}\psi_i.$$

We write any element $y \in \tilde{\Psi}$ as follows $y = \sum_{i=1}^g x_i \tilde{\psi}_i$, with $x = (x_j)_{j \in [1, g]} \in \mathbb{C}^g$ so that we can compute

$$\left(y, \tilde{\psi}_i \right)_{-\diamond} = \sum_{j=1}^g x_j \left(\tilde{\psi}_j, \tilde{\psi}_i \right)_{-\diamond} = (\mathbf{G}_{\tilde{\psi}} x)_i, \quad \forall i \in [1, g],$$

$$(y, y)_{-\diamond} = \sum_{i=1}^g \sum_{j=1}^g \bar{x}_i x_j \left(\tilde{\psi}_j, \tilde{\psi}_i \right)_{-\diamond} = \left\langle \mathbf{G}_{\tilde{\psi}} x, x \right\rangle,$$

where $\mathbf{G}_{\tilde{\psi}}$ is the Gram matrix in $X_{-\diamond}$ of the family $\{\tilde{\psi}_1, \dots, \tilde{\psi}_g\}$. Using that I is an isometry from $X_{-\diamond}$ onto X_{\diamond}^* it actually appears that

$$\mathbf{G}_{\tilde{\psi}} = \mathbf{G}_\psi.$$

Finally, we have proved that

$$\xi_y = \mathbf{G}_\psi x, \quad \text{and} \quad \|y\|_{-\diamond}^2 = \langle \mathbf{G}_\psi x, x \rangle.$$

The supremum we are looking for thus reads

$$S = \sup_{\substack{x \in \mathbb{C}^g \\ \langle \mathbf{G}_\psi x, x \rangle = 1}} \langle M \mathbf{G}_\psi x, \mathbf{G}_\psi x \rangle.$$

- By compactness, we know that this supremum is actually achieved at some point $x_0 \in \mathbb{C}^g$, that is

$$\langle MG_\psi x_0, G_\psi x_0 \rangle = S, \quad \text{and} \quad \langle G_\psi x_0, x_0 \rangle = 1.$$

1842 The Lagrange multiplier theorem gives that there exists $\lambda \in \mathbb{C}$ such that

$$1843 \quad (\text{B.3}) \quad \langle MG_\psi x_0, G_\psi h \rangle = \lambda \langle G_\psi x_0, h \rangle, \quad \forall h \in \mathbb{C}^g.$$

Taking $h = x_0$ in this equation, we get

$$\langle MG_\psi x_0, G_\psi x_0 \rangle = \lambda \langle G_\psi x_0, x_0 \rangle = \lambda,$$

1844 and thus $\lambda = S$, in particular λ is a non negative real number.

From (B.3), we deduce

$$G_\psi M G_\psi x_0 = \lambda G_\psi x_0.$$

and since $G_\psi x_0 \neq 0$ (we recall that $\langle G_\psi x_0, x_0 \rangle = 1$), we conclude that λ is an eigenvalue of $G_\psi M$ and therefore

$$S = \lambda \leq \rho(G_\psi M).$$

We have thus proved that

$$S \leq \rho(G_\psi M).$$

- If $\rho(G_\psi M) = 0$, the claim is proved. If not, we set

$$\lambda = \rho(G_\psi M) = \rho(MG_\psi) = \rho\left(G_\psi^{\frac{1}{2}} M G_\psi^{\frac{1}{2}}\right),$$

which is a positive number which is an eigenvalue of the three matrices above. In particular, there exists $x_0 \in \mathbb{C}^g \setminus \{0\}$ such that

$$M G_\psi x_0 = \lambda x_0.$$

Taking the inner product with $G_\psi x_0$ we obtain

$$\langle M G_\psi x_0, G_\psi x_0 \rangle = \lambda \langle x_0, G_\psi x_0 \rangle,$$

and since $\langle x_0, G_\psi x_0 \rangle = \left\| G_\psi^{\frac{1}{2}} x_0 \right\|^2$ cannot be equal to zero, we deduce that

$$\lambda \leq S,$$

1845 and the proof is complete. \square

1846 **Appendix C. Solving general block moment problems.**

1847 As this paper is oriented towards control theory we do not deal with the most
 1848 general block moment problems. Indeed, in Theorem 2.2, the considered block mo-
 1849 ment problems have a specific right-hand side which is a linear form. This formalism
 1850 is chosen in order to avoid exhibiting a particular basis of the generalized eigenspaces.
 1851 The price to pay is this restriction on the considered right-hand sides. However the
 1852 proofs detailed in Sections 3 and 5 directly lead to the following more general results.

1853 The study with a group composed of geometrically simple eigenvalues (see Sec-
 1854 tions 5.1 and 5.2) leads to the following theorem.

1855 THEOREM C.1. Let $p \in \mathbb{N}^*$, $\varrho, \tau, \kappa > 0$ and $\theta \in (0, 1)$. Assume that

1856
$$\Lambda \in \mathcal{L}_w(p, \varrho, \tau, \theta, \kappa).$$

1857 Recall that this class of sequences is defined in (1.30). Let $G = \{\lambda_1, \dots, \lambda_g\} \subset \Lambda$ be
 1858 a group satisfying (1.25)–(1.27). Let $T \in (0, +\infty)$ and $\eta \in \mathbb{N}^*$. For any multi-index
 1859 $\alpha \in \mathbb{N}^g$ with $|\alpha|_\infty \leq \eta$, any

1860
$$\omega = (\omega_1^0, \dots, \omega_1^{\alpha_1-1}, \dots, \omega_g^0, \dots, \omega_g^{\alpha_g-1}) \in \mathbb{C}^{|\alpha|},$$

1861 and any $b \in U^{|\alpha|}$ with

1862
$$b_j^0 \neq 0, \quad \forall j \in \llbracket 1, g \rrbracket,$$

1863 there exists $v_G \in L^2(0, T; U)$ satisfying

1864 (C.1a)
$$\int_0^T \left\langle v_G(t), (e_t b) \left[\bar{\lambda}_j^{(l+1)} \right] \right\rangle_U dt = \omega_j^l, \quad \forall j \in \llbracket 1, g \rrbracket, \forall l \in \llbracket 0, \alpha_j \rrbracket,$$

1865 (C.1b)
$$\int_0^T v_G(t) t^l e^{-\bar{\lambda} t} dt = 0, \quad \forall \lambda \in \Lambda \setminus G, \forall l \in \llbracket 0, \eta \rrbracket.$$

 1866

1867 The solution v_G satisfies the following estimate

1868
$$\|v_G\|_{L^2(0, T; U)} \leq C \exp\left(\frac{C}{T^{1-\theta}}\right) \exp\left(r_G \frac{T}{2}\right) \exp(Cr_G^\theta) \langle M^{-1} \xi, \xi \rangle,$$

1869 where

1870
$$\xi := \begin{pmatrix} \omega \left[\bar{\lambda}_\bullet^{(\mu^1)} \right] \\ \vdots \\ \omega \left[\bar{\lambda}_\bullet^{(\mu^{|\alpha|})} \right] \end{pmatrix},$$

1871 the sequence $(\mu^p)_{p \in \llbracket 0, |\alpha| \rrbracket}$ is defined in (2.8), the associated matrix M is defined
 1872 in (2.17), r_G is defined in (1.35) and with the convention

1873
$$\omega \left[\bar{\lambda}_j^{(l+1)} \right] = \omega_j^l, \quad \forall j \in \llbracket 1, g \rrbracket, \forall l \in \llbracket 0, \alpha_j \rrbracket.$$

1874 The constant $C > 0$ appearing in the estimate only depends on the parameters $\tau, p,$
 1875 ϱ, η, θ and κ .

1876 Moreover, there exists a constant $C_{p, \eta, r_\Lambda} > 0$ such that any $v_G \in L^2(0, T; U)$
 1877 solution of (C.1a) satisfy

1878
$$\|v_G\|_{L^2(0, T; U)} \geq C_{p, \eta, r_\Lambda} \langle M^{-1} \xi, \xi \rangle.$$

1879 REMARK C.1. As detailed in Remark 2.2, when the eigenvalues in G are also
 1880 algebraically simple, i.e. $\alpha_\lambda = \gamma_\lambda = 1$ for any $\lambda \in G$, the expression of ξ reduces to

1881
$$\xi := \begin{pmatrix} \omega \left[\bar{\lambda}_1 \right] \\ \vdots \\ \omega \left[\bar{\lambda}_1, \dots, \bar{\lambda}_g \right] \end{pmatrix},$$

1882 and the expression of M reduces to the one given in (2.19).

1883 The study with a group composed of semi-simple eigenvalues (see Section 5.3)
 1884 leads to the following theorem.

1885 THEOREM C.2. Let $p \in \mathbb{N}^*$, $\varrho, \tau, \kappa > 0$ and $\theta \in (0, 1)$. Assume that

1886
$$\Lambda \in \mathcal{L}_w(p, \varrho, \tau, \theta, \kappa).$$

1887 Recall that this class of sequences is defined in (1.30). Let $G = \{\lambda_1, \dots, \lambda_g\} \subset \Lambda$ be a
 1888 group satisfying (1.25)–(1.27). Let $\gamma_1, \dots, \gamma_g \in \mathbb{N}^*$ and $\gamma_G = \gamma_1 + \dots + \gamma_g$. Let $\eta \in \mathbb{N}^*$
 1889 and $T \in (0, +\infty)$.

1890 For any $(\omega_{j,i})_{j \in \llbracket 1, g \rrbracket, i \in \llbracket 1, \gamma_j \rrbracket} \in \mathbb{C}^{\gamma_G}$ and any $(b_{j,i})_{j \in \llbracket 1, g \rrbracket, i \in \llbracket 1, \gamma_j \rrbracket} \in U^{\gamma_G}$ such
 1891 that $b_{j,1}, \dots, b_{j,\gamma_j}$ are linearly independent for every $j \in \llbracket 1, g \rrbracket$, there exists $v_G \in$
 1892 $L^2(0, T; U)$ satisfying

1893 (C.2a)
$$\int_0^T \langle v_G(t), e^{-\bar{\lambda}_j t} b_{j,i} \rangle_U dt = \omega_{j,i}, \quad \forall j \in \llbracket 1, g \rrbracket, \forall i \in \llbracket 1, \gamma_j \rrbracket,$$

1894 (C.2b)
$$\int_0^T v_G(t) t^l e^{-\bar{\lambda} t} dt = 0, \quad \forall \lambda \in \Lambda \setminus G, \forall l \in \llbracket 0, \eta \rrbracket.$$

 1895

1896 The solution v_G satisfies the following estimate

1897
$$\|v_G\|_{L^2(0, T; U)} \leq C \exp\left(\frac{C}{T^{1-\theta}}\right) \exp\left(r_G \frac{T}{2}\right) \exp(Cr_G^\theta) \langle M^{-1} \xi, \xi \rangle,$$

1898 where $\xi \in \mathbb{C}^{\gamma_G}$ is defined by blocks with

1899
$$\xi_j := \begin{pmatrix} \omega_{j,1} \\ \vdots \\ \omega_{j,g} \end{pmatrix},$$

1900 the associated matrix M is defined in (2.21) and r_G is defined in (1.35). The constant
 1901 $C > 0$ appearing in the estimate only depends on the parameters $\tau, p, \varrho, \eta, \theta$ and κ .

1902 Moreover, there exists a constant $C_{p, \eta, r_\Lambda} > 0$ such that any $v_G \in L^2(0, T; U)$
 1903 solution of (C.2a) satisfy

1904
$$\|v_G\|_{L^2(0, T; U)} \geq C_{p, \eta, r_\Lambda} \langle M^{-1} \xi, \xi \rangle.$$

1905 Appendix D. Post-processing formulas.

1906 The minimal null control time given in Theorem 2.3, together with the compu-
 1907 tation of the contribution of each group given in Theorems 2.6 and 2.8, allow to
 1908 answer the question of minimal null control time for a wide variety of one dimensional
 1909 parabolic control problems. However, for a given problem, the precise estimate of the
 1910 quantity of interest $\langle M^{-1} \xi, \xi \rangle$ can remain a tricky question.

1911 There is no normalization condition on the eigenvectors and no uniqueness of
 1912 the considered Jordan chains. Thus, it happens that there are choices for which the
 1913 quantity of interest $\langle M^{-1} \xi, \xi \rangle$ is easier to compute (see for instance Remark 2.1). We
 1914 gather here some results that are used in Sections 6 and 7 to estimate such quantities.

1915 We will make an intensive use of the following reformulation. Let $n \in \mathbb{N}^*$ and let
 1916 $T, M \in \text{GL}_n(\mathbb{C})$. For any $\xi \in \mathbb{C}^n$, let $\tilde{\xi} := T\xi$. Then,

1917 (D.1)
$$\langle M^{-1} \xi, \xi \rangle = \langle M^{-1} T^{-1} \tilde{\xi}, T^{-1} \tilde{\xi} \rangle = \langle \widetilde{M}^{-1} \tilde{\xi}, \tilde{\xi} \rangle$$

1918 where

$$1919 \quad (D.2) \quad \widetilde{M} := TMT^*.$$

1920 As the matrix M is a sum of Gram matrices we will also use the following lemma.

1921 LEMMA D.1. *Let X be an Hilbert space. Let $n \in \mathbb{N}^*$ and $e = (e_1, \dots, e_n) \in X^n$.*
 1922 *Let $T \in \mathcal{M}_n(\mathbb{C})$. Then,*

$$1923 \quad T\text{Gram}_X(e_1, \dots, e_n)T^* = \text{Gram}_X((\overline{T}e)_1, \dots, (\overline{T}e)_n)$$

1924 where, for any $i \in \llbracket 1, n \rrbracket$, $(\overline{T}e)_i$ is defined by

$$1925 \quad (\overline{T}e)_i := \sum_{j=1}^n \overline{T_{i,j}} e_j.$$

1926 *Proof.* For any $\omega \in \mathbb{C}^n$, it comes that

$$1927 \quad (D.3) \quad \langle T\text{Gram}_X(e_1, \dots, e_n)T^*\omega, \omega \rangle = \langle \text{Gram}_X(e_1, \dots, e_n)(T^*\omega), (T^*\omega) \rangle$$

$$1928 \quad (D.4) \quad = \left\| \sum_{i=1}^n (T^*\omega)_i e_i \right\|^2$$

$$1929 \quad (D.5) \quad = \left\| \sum_{i=1}^n \sum_{j=1}^n \overline{T_{j,i}} \omega_j e_i \right\|^2$$

$$1930 \quad (D.6) \quad = \left\| \sum_{j=1}^n \omega_j (\overline{T}e)_j \right\|^2$$

$$1931 \quad (D.7) \quad = \langle \text{Gram}_X((\overline{T}e)_1, \dots, (\overline{T}e)_n)\omega, \omega \rangle. \quad \square$$

1933 Depending on the phenomenon at stake on actual examples, with a suitable choice of
 1934 $\tilde{\xi}$ (i.e. of T), the quantity $\langle \widetilde{M}^{-1}\tilde{\xi}, \tilde{\xi} \rangle$ can be easier to estimate than $\langle M^{-1}\xi, \xi \rangle$.

1935 D.1. Dilatations.

1936 Notice that

$$1937 \quad \langle \widetilde{M}^{-1}\tilde{\xi}, \tilde{\xi} \rangle \leq \|\widetilde{M}^{-1}\| \|\tilde{\xi}\|^2.$$

1938 When the minimal null control time can be estimated with rough estimates (this can
 1939 only characterize the minimal time when $T_0 = 0$), it can simplify the computations
 1940 to have a bounded $\|\tilde{\xi}\|$. To do so, it is convenient to consider dilatations of ξ .

1941 Let X be an Hilbert space. Let $n \in \mathbb{N}^*$ and $e_1, \dots, e_n \in X$. Let $\xi \in \mathbb{C}^n$ and
 1942 $\beta \in \mathbb{C}^n$ with non-zero entries. Let

$$1943 \quad T = D_\beta := \text{diag}(\beta) \in \text{GL}_n(\mathbb{C}), \quad \text{and} \quad \tilde{\xi} = T\xi.$$

1944 Then, from Lemma D.1, it comes that

$$1945 \quad T\text{Gram}_X(e_1, \dots, e_n)T^* = \text{Gram}_X(\overline{\beta}_1 e_1, \dots, \overline{\beta}_n e_n).$$

1946 **D.2. Invariance by scale change.**

1947 In our assumptions there is no normalization condition on the eigenvectors (see
 1948 Remark 2.1). This allows to have simpler expressions for these eigenvectors. Actually,
 1949 the computation of $\langle M^{-1}\xi, \xi \rangle$ can be done with a different scale change on every
 1950 generalized eigenvector as detailed in the following proposition.

1951 PROPOSITION D.2. *Let M and ξ be as defined in Theorem 2.6. Let $\beta \in \mathbb{C}^{|\alpha|}$ be
 1952 such that $\beta_j^0 \neq 0$ for all $j \in \llbracket 1, g \rrbracket$. Set*

$$1953 \quad \tilde{\xi} = \begin{pmatrix} \langle y_0, (\beta\phi) [\lambda^{\mu^1}] \rangle_{-\diamond, \diamond} \\ \vdots \\ \langle y_0, (\beta\phi) [\lambda^{\mu^{|\alpha|}}] \rangle_{-\diamond, \diamond} \end{pmatrix}$$

1954 Then,

$$1955 \quad \langle M^{-1}\xi, \xi \rangle = \langle \widetilde{M}^{-1}\tilde{\xi}, \tilde{\xi} \rangle$$

1956 where

$$1957 \quad (\text{D.8}) \quad \widetilde{M} := \sum_{l=1}^{|\alpha|} \text{Gram}_U \left(\underbrace{0, \dots, 0}_{l-1}, (\beta b) [\lambda^{\mu^l - \mu^{l-1}}], \dots, (\beta b) [\lambda^{\mu^{|\alpha|} - \mu^{l-1}}] \right).$$

1958 *Proof.* From Leibniz formula [9, Proposition 7.13], it comes that for any $p \in$
 1959 $\llbracket 1, |\alpha| \rrbracket$,

$$1960 \quad (\beta\phi) [\lambda^{\mu^p}] = \sum_{q=1}^{\mu^p} \beta [\lambda^{\mu^p - \mu^{q-1}}] \phi [\lambda^{\mu^q}].$$

1961 Thus, $\tilde{\xi} = T\xi$ where T is the following lower triangular matrix

$$1962 \quad T = \left(\mathbf{1}_{q \leq p} \overline{\beta} [\lambda^{\mu^p - \mu^{q-1}}] \right)_{p, q \in \llbracket 1, |\alpha| \rrbracket}.$$

1963 The diagonal entries of this lower triangular matrix are $\overline{\beta}_j^0$ and thus $T \in \text{GL}_{|\alpha|}(\mathbb{C})$.
 1964 From (D.2), the associated matrix is

$$1965 \quad \widetilde{M} := \sum_{l=1}^{|\alpha|} T \text{Gram}_U \left(\underbrace{0, \dots, 0}_{l-1}, b [\lambda^{\mu^l - \mu^{l-1}}], \dots, b [\lambda^{\mu^{|\alpha|} - \mu^{l-1}}] \right) T^*.$$

1966 Let $l \in \llbracket 1, |\alpha| \rrbracket$ and

$$1967 \quad e_1 = \dots = e_{l-1} = 0,$$

$$1968 \quad e_p = b [\lambda^{\mu^p - \mu^{l-1}}], \quad \forall p \in \llbracket l, |\alpha| \rrbracket.$$

1969

1970 Then, for any $p \in \llbracket 1, |\alpha| \rrbracket$,

$$1971 \quad (\overline{T}e)_p = \sum_{q=1}^{|\alpha|} \mathbf{1}_{q \leq p} \beta [\lambda^{\mu^p - \mu^{q-1}}] e_q.$$

1972 Thus, $(\bar{T}e)_1 = \dots = (\bar{T}e)_{l-1} = 0$ and, for any $p \in \llbracket l, |\alpha| \rrbracket$,

$$1973 \quad (\bar{T}e)_p = \sum_{q=1}^{|\alpha|} \mathbf{1}_{q \leq p} \beta \left[\lambda^{(\mu^p - \mu^{q-1})} \right] e_q = \sum_{q=l}^p \beta \left[\lambda^{(\mu^p - \mu^{q-1})} \right] b \left[\lambda^{(\mu^q - \mu^{l-1})} \right].$$

1974 Then, using again Leibniz formula [9, Proposition 7.13], we obtain

$$1975 \quad (\bar{T}e)_p = (\beta b) \left[\lambda^{(\mu^p - \mu^{l-1})} \right].$$

1976 Finally, applying (D.1) and Lemma D.1 ends the proof of Proposition D.2. \square

1977 **REMARK D.1.** *As there is no normalization condition on the eigenvectors a sim-*
 1978 *ilar statement automatically holds with M and ξ defined in Theorem 2.8.*

1979 **D.3. An equivalent formula for simple eigenvalues.**

1980 In this section, we consider the case of a group of simple eigenvalues *i.e.* $\alpha_\lambda =$
 1981 $\gamma_\lambda = 1$ for every $\lambda \in G$. In that case, the cost of the group G can be computed either
 1982 using the formula of Theorem 2.6 for geometrically simple eigenvalues or the formula
 1983 of Theorem 2.8 for semi-simple eigenvalues. Even though these theorems imply that
 1984 those two formulas coincide (as they are both the cost of the group) we give a direct
 1985 proof of this statement.

1986 **PROPOSITION D.3.** *Let M and ξ be the matrix and the vector given in Theorem 2.6*
 1987 *i.e.*

$$1988 \quad M := \sum_{l=1}^g \text{Gram}_U \left(\underbrace{0, \dots, 0}_{l-1}, b[\lambda_l], \dots, b[\lambda_l, \dots, \lambda_g] \right)$$

1989 *and*

$$1990 \quad \xi = \begin{pmatrix} \langle y_0, \phi[\lambda_1] \rangle_{-\diamond, \diamond} \\ \vdots \\ \langle y_0, \phi[\lambda_1, \dots, \lambda_g] \rangle_{-\diamond, \diamond} \end{pmatrix}.$$

1991 *Let \widetilde{M} and $\widetilde{\xi}$ be the matrix and the vector given in Theorem 2.8 i.e.*

$$1992 \quad (\text{D.9}) \quad \widetilde{M} := \sum_{l=1}^g \text{Gram}_U (\delta_l^1 b[\lambda_1], \dots, \delta_l^g b[\lambda_g]) \quad \text{and} \quad \widetilde{\xi} := \begin{pmatrix} \langle y_0, \phi[\lambda_1] \rangle_{-\diamond, \diamond} \\ \vdots \\ \langle y_0, \phi[\lambda_g] \rangle_{-\diamond, \diamond} \end{pmatrix}.$$

1993 *Then,*

$$1994 \quad \langle M^{-1} \xi, \xi \rangle = \langle \widetilde{M}^{-1} \widetilde{\xi}, \widetilde{\xi} \rangle$$

1995 *Proof.* The usual interpolation formula [9, Proposition 7.6] gives

$$1996 \quad (\text{D.10}) \quad \phi[\lambda_i] = \sum_{j=1}^i \left(\prod_{k=1}^{j-1} (\lambda_i - \lambda_k) \right) \phi[\lambda_1, \dots, \lambda_j].$$

1997 Recall that the notation δ_j^i has been introduced in (2.20). With these notations,
 1998 $\tilde{\xi} = T\xi$ where T is the following lower triangular matrix

1999
$$T = \left(\overline{\delta_j^i} \right)_{i,j \in \llbracket 1, g \rrbracket} \in \text{GL}_g(\mathbb{C}).$$

2000 From (D.2), we define

2001
$$\widehat{M} := \sum_{l=1}^g T \text{Gram}_U \left(\underbrace{0, \dots, 0}_{l-1}, b[\lambda_l], \dots, b[\lambda_l, \dots, \lambda_g] \right) T^*,$$

2002 so that we have $\langle M^{-1}\xi, \xi \rangle = \langle \widehat{M}^{-1}\tilde{\xi}, \tilde{\xi} \rangle$. We will now prove that $\widehat{M} = \widetilde{M}$.

2003 Let $l \in \llbracket 1, g \rrbracket$ and

2004
$$e_1 = \dots = e_{l-1} = 0,$$

 2005
$$e_j = b[\lambda_l, \dots, \lambda_j], \quad \forall j \in \llbracket l, g \rrbracket.$$

2007 Then, $(\overline{T}e)_1 = \dots = (\overline{T}e)_{l-1} = 0$ and for $i \in \llbracket l, g \rrbracket$, using again the interpolation
 2008 property [9, Proposition 7.6], we obtain

2009
$$(\overline{T}e)_i = \sum_{j=l}^g \delta_j^i b[\lambda_l, \dots, \lambda_j]$$

 2010
$$= \sum_{j=l}^i \delta_j^i b[\lambda_l, \dots, \lambda_j]$$

 2011
$$= \delta_l^i \sum_{j=l}^i \left(\prod_{k=l}^{j-1} (\lambda_i - \lambda_k) \right) b[\lambda_l, \dots, \lambda_j]$$

 2012
$$= \delta_l^i b[\lambda_i].$$

2014 Recalling that $\delta_l^1 = \dots = \delta_l^{l-1} = 0$, we thus obtain

2015
$$(\overline{T}e)_i = \delta_l^i b[\lambda_i], \quad \forall i \in \llbracket 1, g \rrbracket.$$

2016 Finally, from Lemma D.1, we deduce that $\widehat{M} = \widetilde{M}$ which ends the proof of Proposi-
 2017 tion D.3. \square

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