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ANALYSIS OF NON SCALAR CONTROL PROBLEMS FOR PARABOLIC SYSTEMS BY THE BLOCK MOMENT METHOD

FRANCK BOYER* AND MORGAN MORANCEY†

Abstract. This article deals with abstract linear time invariant controlled systems. In [*Annales Henri Lebesgue*, **3** (2020), pp. 717–793], with A. Benabdallah, we introduced the block moment method for scalar control operators. The principal aim of this method is to answer the question of computing the minimal time needed to drive an initial condition (or a space of initial conditions) to zero. The purpose of the present article is to push forward the analysis to deal with any admissible control operator. The considered setting leads to applications to one dimensional parabolic-type equations or coupled systems of such equations.

With such admissible control operator, the characterization of the minimal null control time is obtained thanks to the resolution of an auxiliary vectorial block moment problem (*i.e.* set in the control space) followed by a constrained optimization procedure of the cost of this resolution. This leads to essentially sharp estimates on the resolution of the block moment problems which are uniform with respect to the spectrum of the evolution operator in a certain class. This uniformity allow the study of uniform controllability for various parameter dependent problems. We also deduce estimates on the cost of controllability when the final time goes to the minimal null control time.

We provide applications on abstract controlled system to illustrate how the method works and then deal with actual coupled systems of one dimensional parabolic partial differential equations. Our strategy enables us to gather previous results obtained by different methods but to also tackle controllability issues that seem out of reach by existing techniques.

Key words. Control theory, parabolic partial differential equations, minimal null control time, block moment method

AMS subject classifications. 93B05, 93C20, 93C25, 30E05, 35K90, 35P10

1. Introduction.

1.1. Problem under study and state of the art.

In this paper we study the controllability properties of the following linear control system

$$\begin{cases} y'(t) + \mathcal{A}y(t) = \mathcal{B}u(t), \\ y(0) = y_0. \end{cases} \tag{1.1}$$

The assumptions on the operator $\mathcal{A}$ (see Section 1.3) will lead to applications to linear parabolic-type equations or coupled systems of such equations mostly in the one dimensional setting. In all this article the Hilbert space of control will be denoted by U and the operator $\mathcal{B}$ will be a general admissible operator.

The question we address is the characterization of the minimal null control time (possibly zero or infinite) from y_0 that is: for a given initial condition y_0, what is the minimal time $T_0(y_0)$ such that, for any $T > T_0(y_0)$, there exists a control $u \in L^2(0,T;U)$ such that the associated solution of (1.1) satisfies $y(T) = 0$.

For a presentation of null controllability of parabolic control problems as well as the possible existence of a positive minimal null control time for such equations we refer to [4] or [10, Section 1.1] and the references therein. Such a positive minimal null

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control time is due either to insufficient observation of eigenvectors, or to condensation of eigenvalues or to the geometry of generalized eigenspaces, or even to a combination of all those phenomena.

Under the considered assumptions on $\mathcal{A}$, the problem of characterizing the minimal null control time has been solved for scalar controls ($\dim U = 1$) in [10] where the *block moment method* has been introduced in that purpose. The aim of the present article is to push forward the analysis of [10] to extend it to any admissible control operator.

To present the general ideas, let us assume for simplicity that the operator $\mathcal{A}^*$ has a sequence of positive eigenvalues Λ and that the associated eigenvectors ϕ_λ for $\lambda \in \Lambda$ form a complete family of the state space (the precise functional setting is detailed in Section 1.3). Then, the solution of system (1.1) satisfies $y(T) = 0$ if and only if the control $u \in L^2(0,T;U)$ solves the following moment problem

$$\int_0^T e^{-\lambda t} \langle u(T-t), \mathcal{B}^*\phi_\lambda \rangle_U \, \mathrm{d}t = -e^{-\lambda T} \langle y_0, \phi_\lambda \rangle, \qquad \forall \lambda \in \Lambda. \tag{1.2}$$

$\star$ In the scalar case ($\dim U = 1$), provided that $\mathcal{B}^*\phi_\lambda \neq 0$, this moment problem reduces to

$$\int_0^T e^{-\lambda t} u(T-t) \mathrm{d}t = -e^{-\lambda T} \left\langle y_0, \frac{\phi_\lambda}{\mathcal{B}^*\phi_\lambda} \right\rangle, \qquad \forall \lambda \in \Lambda. \tag{1.3}$$

This problem is usually solved by the construction of a biorthogonal family $(q_\lambda)_{\lambda\in\Lambda}$ to the exponentials

$$\left\{ t \in (0,T) \mapsto e^{-\lambda t} \, ; \, \lambda \in \Lambda \right\}$$

in $L^2(0,T;U)$, *i.e.*, a family $(q_\lambda)_{\lambda\in\Lambda}$ such that

$$\int_0^T q_\lambda(t) e^{-\mu t} \mathrm{d}t = \delta_{\lambda,\mu}, \qquad \forall \lambda, \mu \in \Lambda.$$

From [35], the existence of such biorthogonal family is equivalent to the summability condition

$$\sum_{\lambda\in\Lambda} \frac{1}{\lambda} < +\infty. \tag{1.4}$$

Remark 1.1. *This condition (which will be assumed in the present article) is the main restriction to apply the moment method. Indeed, due to Weyl's law it imposes on many examples of partial differential equations of parabolic-type a restriction to the one dimensional setting. However, in some particular multi-dimensional geometries, the controllability problem can be transformed into a family of parameter dependent moment problems, each of them satisfying such assumption (see for instance [9, 3, 18] among others).*

With such a biorthogonal family, a formal solution of the moment problem (1.3) is given by

$$u(T-t) = -\sum_{\lambda\in\Lambda} e^{-\lambda T} \left\langle y_0, \frac{\phi_\lambda}{\mathcal{B}^*\phi_\lambda} \right\rangle q_\lambda(t), \quad t \in (0,T).$$

Thus if, for any y_0, the series defining u converges in $L^2(0,T;U)$ one obtains null controllability of system (1.1) in time T. To do so, it is crucial to prove upper bounds on $\|q_\lambda\|_{L^2(0,T)}$.

Suitable bounds on such biorthogonal families were provided in the pioneering work of Fattorini and Russell [23] in the case where the eigenvalues of $\mathcal{A}^*$ are well separated *i.e.* satisfy the classical gap condition: $\inf\{|\lambda-\mu|\,;\,\lambda,\mu\in\Lambda,\,\lambda\neq\mu\}>0$. When the eigenvalues are allowed to condensate we refer to the work [5] for almost sharp estimates implying the condensation index of the sequence Λ. A discussion on other references providing estimates on biorthogonal families is detailed below. These results have provided an optimal characterization of the minimal null control time when the eigenvectors of $\mathcal{A}^*$ form a Riesz basis of the state space (and thus do not condensate).

However, as analyzed in [10], there are situations in which the eigenvectors also condensate and for which providing estimates on biorthogonal families is not sufficient to characterize the minimal null control time. In [10], it is assumed that the spectrum Λ can be decomposed as a union of well separated groups $(G_k)_{k\geq 1}$ of bounded cardinality. Then, the control u is seeked in the form

$$u(T-t)=\sum_{k\geq 1} v_k(t),$$

where, for any $k\geq 1$, the function $v_k\in L^2(0,T;U)$ solves the block moment problem

$$\begin{cases} \displaystyle\int_0^T e^{-\lambda t}v_k(t)\mathrm{d}t = e^{-\lambda T}\left\langle y_0, \frac{\phi_\lambda}{\mathcal{B}^*\phi_\lambda}\right\rangle, & \forall\lambda\in G_k,\\ \displaystyle\int_0^T e^{-\lambda t}v_k(t)\mathrm{d}t = 0, & \forall\lambda\notin G_k. \end{cases} \tag{1.5}$$

This enables to deal with the condensation of eigenvectors: the eigenvectors $(\phi_\lambda)_{\lambda\in\Lambda}$ are only assumed to form a complete family of the state space.

$\star$ When the control is not scalar there are less available results in the literature. Here again, these results rely on the existence of a biorthogonal family to the exponentials with suitable bounds. For instance, in [7], null controllability in optimal time is proved using a subtle decomposition of the moment problem into two families of moment problems. In a more systematic way, one can take advantage of the biorthogonality in the time variable to seek for a solution u of the moment problem (1.2) in the form

$$u(T-t)=-\sum_{\lambda\in\Lambda} e^{-\lambda T}\left\langle y_0,\phi_\lambda\right\rangle \frac{\mathcal{B}^*\phi_\lambda}{\|\mathcal{B}^*\phi_\lambda\|_U^2}.$$

This strategy was introduced by Lagnese in [28] for a one dimensional wave equation and used in the parabolic context for instance in [19, 2, 20, 3].

In the present article we deal with such general admissible control operators. As the eigenvectors will only be assumed to form a complete family, for each initial condition y_0, we study its null control time for system (1.1) by solving block moment problems of the following form

$$\begin{cases} \displaystyle\int_0^T \left\langle V_k(t), e^{-\lambda t}\mathcal{B}^*\phi_\lambda\right\rangle_U \mathrm{d}t = \left\langle y_0, e^{-\lambda T}\phi_\lambda\right\rangle, & \forall\lambda\in G_k,\\ \displaystyle\int_0^T \left\langle V_k(t), e^{-\lambda t}\mathcal{B}^*\phi_\lambda\right\rangle_U \mathrm{d}t = 0, & \forall\lambda\notin G_k. \end{cases} \tag{1.6}$$

Let us recall that, for pedagogical purposes, we have restricted this first introductory subsection to the case of simple eigenvalues. The general form of block moment problems under study in this article is detailed in Section 1.4.

The strategy to solve such block moment problems and estimate its solution is presented on an example in Section 1.2 together with the structure of the article. Let us already notice that the geometry of the finite dimensional space $\mathrm{Span}\{\mathcal{B}^*\phi_\lambda\ ;\ \lambda \in G_k\}$ is crucial.

For instance, if this space is one dimensional, say generated by some $b \in U$, the strategy of Lagnese can be adapted if one seeks for V_k solution of the block moment problem (1.6) in the form

$$V_k(t) = v_k(t)b,$$

where $v_k \in L^2(0,T;\mathbb{R})$ solves a scalar block moment problem of the same form as (1.5).

If, instead, the family $(\mathcal{B}^*\phi_\lambda)_{\lambda\in G_k}$ is composed of linearly independent vectors then it admits a biorthogonal family in U denoted by $(b^*_\lambda)_{\lambda\in G_k}$. Then, one can for instance seek for V_k solution of the block moment problem (1.6) in the form

$$V_k(t) = v_k(t)\left(\sum_{\lambda\in G_k} b^*_\lambda\right).$$

where v_k solves a scalar block moment problem of the form (1.5).

In the general setting, taking into account the geometry of the observations of eigenvectors to solve block moment problems of the form (1.6) is a more intricate question that we solve in this article, still under the summability condition (1.4).

Finally, let us mention that we not only solve block moment problems of the form (1.6) but we also provide estimates on their solutions to ensure that the series defining the control converges. These estimates will provide an optimal characterization of the minimal null control time for each given problem.

We add also an extra care on these estimates so that they do not directly depend on the sequence Λ but are uniform for classes of such sequences. It is an important step to tackle uniform controllability for parameter dependent control problems. Estimates of this kind have already proved their efficiency in various contexts such as: numerical analysis of semi-discrete control problems [2], oscillating coefficients [32], analysis of degenerate control problems with respect to the degeneracy parameter [19, 20], analysis of higher dimensional controllability problems by reduction of families of one dimensional control problems [9, 1, 3, 18] or analysis of convergence of Robin-type controls to Dirichlet controls [12].

Another important feature of the estimates we obtain is to track the dependency with respect to the final time T when T goes to the minimal null control time. As presented in Remark 1.7, this allows applications in higher dimensions (with a cylindrical geometry) or applications to nonlinear control problems.

Finally, let us recall some classical results providing estimates for biorthogonal families to a sequence of exponentials. Under the classical gap condition, uniform estimates for biorthogonal families were already obtained in [24] and sharp short-time estimates were obtained in [9]. In this setting, bounds with a detailed dependency with respect to parameters were given in [21]. In this work, the obtained bounds take into account the fact that the gap property between eigenvalues may be better in high frequencies.

Under a weak-gap condition of the form (1.21), that is when the eigenvalues can be gathered in blocks of bounded cardinality with a gap between blocks (which

is the setting of the present article), uniform estimates on biorthogonal sequences follow from the uniform estimates for the resolution of block moment problems proved in [10]. These estimates on biorthogonal family are improved with the dependency with respect to T in [26]. Let us mention that the estimates of [10] can also be supplemented with such dependency (see Theorem A.1) but only when the considered eigenvalues are assumed to be real (unlike the setting studied in [26]).

In the absence of any gap-type condition, estimates on biorthogonal families are proved in [5, 3].

1.2. Structure of the article and strategy of proof.

To highlight the ideas we develop in this article (without drowning them in technicalities or notations), let us present our strategy of analysis of null controllability on an abstract simple example.

We consider $X = L^2(0,1;\mathbb{R})^2$ and $\omega \subset (0,1)$ a non empty open set. For a given $a > 0$ we define

$$\Lambda = \left\{ \lambda_{k,1} := k^2,\ \lambda_{k,2} := k^2 + e^{-ak^2}\ ;\ k \geq 1 \right\},$$

and take $(\varphi_k)_{k\geq 1}$ an Hilbert basis of X such that

$$\inf_{k\geq 1} \|\varphi_k\|_{L^2(\omega)} > 0.$$

Let $\phi_{k,1} := \begin{pmatrix} \varphi_k \\ \varphi_k \end{pmatrix}$ and $\phi_{k,2} := \begin{pmatrix} 0 \\ \varphi_k \end{pmatrix}$. We define the operator $\mathcal{A}^*$ in X by

$$\mathcal{A}^*\phi_{k,1} = \lambda_{k,1}\phi_{k,1}, \qquad \mathcal{A}^*\phi_{k,2} = \lambda_{k,2}\phi_{k,2},$$

with

$$\mathcal{D}(\mathcal{A}^*) = \left\{ \sum_{k\geq 1} a_{k,1}\phi_{k,1} + a_{k,2}\phi_{k,2}\ ;\ \sum_{k\geq 1} \lambda_{k,1}^2 a_{k,1}^2 + \lambda_{k,2}^2 a_{k,2}^2 < +\infty \right\}.$$

The control operator $\mathcal{B}$ is defined by $U = L^2(0,1;\mathbb{R})$ and

$$\mathcal{B} : u \in U \mapsto \begin{pmatrix} 0 \\ \mathbf{1}_\omega u \end{pmatrix} \in X.$$

The condition $\inf_{k\geq 1} \|\varphi_k\|_{L^2(\omega)} > 0$ yields

$$\mathcal{B}^*\phi_{k,1} = \mathcal{B}^*\phi_{k,2} \neq 0, \qquad \forall k \geq 1. \tag{1.7}$$

This ensures approximate controllability of system (1.1).

We insist on the fact that the goal of this article is not to deal with this particular example but to develop a general methodology to analyze the null controllability of system (1.1).

• Let $y_0 \in X$. From Proposition 1.1 and the fact that $\{\phi_{k,1},\ \phi_{k,2}\ ;\ k \geq 1\}$ forms a complete family of X, system (1.1) is null controllable from y_0 at time T if and only if there exists $u \in L^2(0,T;U)$ such that for any $k \geq 1$ and any $j \in \{1,2\}$,

$$\int_0^T e^{-\lambda_{k,j}t} \left\langle u(T-t), \mathcal{B}^*\phi_{k,j} \right\rangle_U \mathrm{d}t = -e^{-\lambda_{k,j}T} \left\langle y_0, \phi_{k,j} \right\rangle_X.$$

Following the idea developed in [10], we seek for a control u of the form

$$u(t) = -\sum_{k\geq 1} v_k(T-t) \tag{1.8}$$

where, for each $k \geq 1$, v_k solves the block moment problem

$$\begin{cases} \displaystyle\int_0^T e^{-\lambda_{k,j}t} \left\langle v_k(t), \mathcal{B}^*\phi_{k,j}\right\rangle_U \mathrm{d}t = e^{-\lambda_{k,j}T} \left\langle y_0, \phi_{k,j}\right\rangle_X, \quad \forall j \in \{1,2\}, \\ \displaystyle\int_0^T e^{-\lambda_{k',j}t} \left\langle v_k(t), \mathcal{B}^*\phi_{k',j}\right\rangle_U \mathrm{d}t = 0, \quad \forall k' \neq k,\ \forall j \in \{1,2\}. \end{cases} \tag{1.9}$$

• To solve (1.9), for a fixed k, we consider the following auxiliary block moment problem in the space U

$$\begin{cases} \displaystyle\int_0^T e^{-\lambda_{k,j}t} v_k(t)\mathrm{d}t = \Omega_{k,j}, \quad \forall j \in \{1,2\}, \\ \displaystyle\int_0^T e^{-\lambda_{k',j}t} v_k(t)\mathrm{d}t = 0, \quad \forall k' \neq k,\ \forall j \in \{1,2\}, \end{cases} \tag{1.10}$$

where $\Omega_{k,j} \in U$ have to be precised. If we impose that $\Omega_{k,1}$ and $\Omega_{k,2}$ satisfy the constraints

$$\left\langle \Omega_{k,j}, \mathcal{B}^*\phi_{k,j}\right\rangle_U = e^{-\lambda_{k,j}T} \left\langle y_0, \phi_{k,j}\right\rangle_X, \qquad \forall j \in \{1,2\}, \tag{1.11}$$

we obtain that the solutions of (1.10) also solve (1.9). The existence of $\Omega_{k,1}$ and $\Omega_{k,2}$ satisfying the constraints (1.11) is ensured by (1.7), however there exist infinitely many choices.

For any $\Omega_{k,1}$, $\Omega_{k,2} \in U$, applying the results of [10] component by component in the finite dimensional subspace of U defined by $\mathrm{Span}\{\Omega_{k,1}, \Omega_{k,2}\}$ leads to the existence of $v_k \in L^2(0,T;U)$ satisfying (1.10). It also gives the following estimate

$$\|v_k\|^2_{L^2(0,T;U)} \leq C_{T,\varepsilon} e^{\varepsilon\lambda_{k,1}} F(\Omega_{k,1}, \Omega_{k,2}), \tag{1.12}$$

with

$$F : (\Omega_{k,1}, \Omega_{k,2}) \in U^2 \mapsto \|\Omega_{k,1}\|^2_U + \left\| \frac{\Omega_{k,2} - \Omega_{k,1}}{\lambda_{k,2} - \lambda_{k,1}} \right\|^2_U.$$

Using (1.12) and optimizing the function F under the constraints (1.11) we obtain that there exists $v_k \in L^2(0,T;U)$ solution of the block moment problem (1.9) such that

$$\|v_k\|^2_{L^2(0,T;U)} \leq C_{T,\varepsilon} e^{\varepsilon\lambda_{k,1}} \inf\left\{F(\Omega_{k,1}, \Omega_{k,2})\,;\ \Omega_{k,1}, \Omega_{k,2} \text{ satisfy (1.11)}\right\}. \tag{1.13}$$

The corresponding general statements of the resolution of block moment problems are detailed in Section 1.4 (see Theorem 1.2) and proved in Section 2.

• Now that we can solve the block moment problems (1.9), a way to characterize the minimal null control time is to estimate for which values of T the series (1.8) defining the control u converges in $L^2(0,T;U)$.

To achieve this goal, we isolate in the estimate (1.13) the dependency with respect to T. Notice that the function F does not depend on T but that the constraints (1.11) does.

For any $k \geq 1$ and any $\Omega_{k,1}$, $\Omega_{k,2} \in U$ we set

$$\widetilde{\Omega}_{k,j} := e^{\lambda_{k,j} T} \Omega_{k,j}, \qquad \forall j \in \{1,2\}.$$

Then, there is equivalence between the constraints (1.11) and the new constraints

$$\left\langle \widetilde{\Omega}_{k,j}, \mathcal{B}^* \phi_{k,j} \right\rangle_U = \langle y_0, \phi_{k,j} \rangle_X , \qquad \forall j \in \{1,2\}. \tag{1.14}$$

Now these constraints are independent of the variable T. From the mean value theorem we obtain

$$\begin{aligned} F(\Omega_{k,1}, \Omega_{k,2}) &= \left\| e^{-\lambda_{k,1} T} \widetilde{\Omega}_{k,1} \right\|_U^2 + \left\| \frac{e^{-\lambda_{k,2} T} \widetilde{\Omega}_{k,2} - e^{-\lambda_{k,1} T} \widetilde{\Omega}_{k,1}}{\lambda_{k,2} - \lambda_{k,1}} \right\|_U^2 . \\ &\leq e^{-2\lambda_{k,1} T} \left\| \widetilde{\Omega}_{k,1} \right\|_U^2 + 2 e^{-2\lambda_{k,2} T} \left\| \frac{\widetilde{\Omega}_{k,2} - \widetilde{\Omega}_{k,1}}{\lambda_{k,2} - \lambda_{k,1}} \right\|_U^2 \\ &\quad + 2 \left(\frac{e^{-\lambda_{k,2} T} - e^{-\lambda_{k,1} T}}{\lambda_{k,2} - \lambda_{k,1}} \right)^2 \left\| \widetilde{\Omega}_{k,1} \right\|_U^2 \\ &\leq 2(1+T^2) e^{-2\lambda_{k,1} T} F(\widetilde{\Omega}_{k,1}, \widetilde{\Omega}_{k,2}). \end{aligned}$$

The general statement of this estimate is given in Lemma 3.1.

Plugging this estimate into (1.12) and optimizing the function F under the constraints (1.14) yields

$$\|v_k\|^2_{L^2(0,T;U)} \leq C_{T,\varepsilon} e^{\varepsilon \lambda_{k,1}} e^{-2\lambda_{k,1} T} \mathcal{C}_k(y_0) \tag{1.15}$$

where

$$\mathcal{C}_k(y_0) := \inf \left\{ \left\| \widetilde{\Omega}_1 \right\|_U^2 + \left\| \frac{\widetilde{\Omega}_2 - \widetilde{\Omega}_1}{\lambda_{k,2} - \lambda_{k,1}} \right\|_U^2 ; \left\langle \widetilde{\Omega}_j, \mathcal{B}^* \phi_{k,j} \right\rangle_U = \langle y_0, \phi_{k,j} \rangle_X , \ \forall j \in \{1,2\} \right\}. \tag{1.16}$$

Estimate (1.15) proves that for any time T such that

$$T > \limsup_{k \to +\infty} \frac{\ln \mathcal{C}_k(y_0)}{2\lambda_{k,1}}$$

the series (1.8) defining the control u converges in $L^2(0,T;U)$. Thus, null controllability of (1.1) from y_0 holds for such T.

We also prove that the obtained estimate (1.15) is sufficiently sharp so that it characterizes the minimal null control time from y_0 as

$$T_0(y_0) = \limsup_{k \to +\infty} \frac{\ln \mathcal{C}_k(y_0)}{2\lambda_{k,1}}. \tag{1.17}$$

The corresponding general statements regarding the minimal null control time together with bounds on the cost of controllability are detailed in Section 1.4 (see Theorem 1.3) and proved in Section 3.

• At this stage we have characterized the minimal null control time as stated in (1.17). However to be able to estimate the actual value of $T_0(y_0)$ one should be able to estimate the quantity $\mathcal{C}_k(y_0)$ as defined in (1.16). This formula is not very explicit and it does not get better in the general setting.

To do so, we remark that (1.16) is a finite dimensional optimization problem that we can explicitly solve in terms of eigenelements of $\mathcal{A}^*$ and their observations through $\mathcal{B}^*$.

We obtain different results depending on the assumptions on the multiplicity of the eigenvalues of the considered blocks. The general statements of an explicit solution of the corresponding optimization problem are detailed in Section 1.5 (see Theorems 1.8 and 1.10) and proved in Section 4.

For the particular example we are considering here, the obtained formula reads

$$\mathcal{C}_k(y_0) = \frac{1}{\|\varphi_k\|^2_{L^2(\omega)}} \left\langle y_0, \begin{pmatrix} 0 \\ \varphi_k \end{pmatrix} \right\rangle_X^2 + \frac{e^{2ak^2}}{\|\varphi_k\|^2_{L^2(\omega)}} \left\langle y_0, \begin{pmatrix} \varphi_k \\ 0 \end{pmatrix} \right\rangle_X^2.$$

Then, the minimal null control time from X of this example is given by

$$T_0(X) = a.$$

Notice, for instance, that this expression also gives that for a given y_0 if the set

$$\left\{ k \in \mathbb{N}^* \, ; \, \left\langle y_0, \begin{pmatrix} \varphi_k \\ 0 \end{pmatrix} \right\rangle_X \neq 0 \right\}$$

is finite, then null controllability from y_0 holds in any positive time, *i.e.* $T_0(y_0) = 0$.

• Finally, we provide various examples of application of the results developed in this article. To highlight the ideas and phenomena we start with rather academic examples in Section 5. We then consider systems of coupled one dimensional linear parabolic equations with boundary or distributed controls in Section 6.

1.3. Framework, spectral assumptions and notations.

To state the main results of this article, we now detail the functional setting and assumptions we use.

1.3.1. Functional setting.

The functional setting for the study of system (1.1) is the same as in [10]. For the sake of completeness, let us detail it.

We consider X an Hilbert space, whose inner product and norm are denoted by $\langle \bullet, \bullet \rangle_X$ and $\|\bullet\|_X$ respectively. The space X is identified to its dual through the Riesz theorem. Let $(\mathcal{A}, D(\mathcal{A}))$ be an unbounded operator in X such that $-\mathcal{A}$ generates a C^0-semigroup in X. Its adjoint in X is denoted by $(\mathcal{A}^*, D(\mathcal{A}^*))$. Up to a suitable translation, we can assume that 0 is in the resolvent set of $\mathcal{A}$.

We denote by X_1 (resp. X_1^*) the Hilbert space $D(\mathcal{A})$ (resp. $D(\mathcal{A}^*)$) equipped with the norm $\|x\|_1 := \|\mathcal{A}x\|_X$ (resp. $\|x\|_{1^*} := \|\mathcal{A}^* x\|_X$) and we define X_{-1} as the completion of X with respect to the norm

$$\|y\|_{-1} := \sup_{z \in X_1^*} \frac{\langle y, z \rangle_X}{\|z\|_{1^*}}.$$

Notice that X_{-1} is isometrical to the topological dual of X_1^* using X as a pivot space (see for instance [37, Proposition 2.10.2]); the corresponding duality bracket will be denoted by $\langle \bullet, \bullet \rangle_{-1,1^*}$.

The control space U is an Hilbert space (that we will identify to its dual). Its inner product and norm are denoted by $\langle \bullet, \bullet \rangle_U$ and $\|\bullet\|_U$ respectively. Let $\mathcal{B} : U \to X_{-1}$ be a linear continuous control operator and denote by $\mathcal{B}^* : X_1^* \to U$ its adjoint in the duality described above.

Let $(X_\diamond^*, \|.\|_{\diamond^*})$ be an Hilbert space such that $X_1^* \subset X_\diamond^* \subset X$ with dense and continuous embeddings. We assume that $X_\diamond^*$ is stable by the semigroup generated by $-\mathcal{A}^*$. We also define $X_{-\diamond}$ as the subspace of X_{-1} defined by

$$X_{-\diamond} := \left\{ y \in X_{-1} \,;\, \|y\|_{-\diamond} := \sup_{z \in X_1^*} \frac{\langle y, z \rangle_{-1,1^*}}{\|z\|_{\diamond^*}} < +\infty \right\},$$

which is also isometrical to the dual of $X_\diamond^*$ with X as a pivot space. The corresponding duality bracket will be denoted by $\langle \bullet, \bullet \rangle_{-\diamond,\diamond}$. Thus, we end up with the following five functional spaces

$$X_1^* \subset X_\diamond^* \subset X \subset X_{-\diamond} \subset X_{-1}.$$

We say that the control operator $\mathcal{B}$ is an admissible control operator for (1.1) with respect to the space $X_{-\diamond}$ if for any $T > 0$ there exists $C_T > 0$ such that

$$\int_0^T \left\| \mathcal{B}^* e^{-(T-t)\mathcal{A}^*} z \right\|_U^2 \mathrm{d}t \leq C_T \|z\|_{\diamond^*}^2, \quad \forall z \in X_1^*. \tag{1.18}$$

Notice that if (1.18) holds for some $T > 0$ it holds for any $T > 0$. The admissibility condition (1.18) implies that, by density, we can give a meaning to the map

$$\left(t \mapsto \mathcal{B}^* e^{-(T-t)\mathcal{A}^*} z \right) \in L^2(0, T; U),$$

for any $z \in X_\diamond^*$. Then, we end up with the following well-posedness result (see [10, Proposition 1.2]).

PROPOSITION 1.1. *Assume that (1.18) holds. Then, for any $T > 0$, any $y_0 \in X_{-\diamond}$, and any $u \in L^2(0, T; U)$, there exists a unique $y \in C^0([0, T]; X_{-\diamond})$ solution to (1.1) in the sense that it satisfies for any $t \in [0, T]$ and any $z_t \in X_\diamond^*$,*

$$\langle y(t), z_t \rangle_{-\diamond,\diamond} - \left\langle y_0, e^{-t\mathcal{A}^*} z_t \right\rangle_{-\diamond,\diamond} = \int_0^t \left\langle u(s), \mathcal{B}^* e^{-(t-s)\mathcal{A}^*} z_t \right\rangle_U \mathrm{d}s.$$

Moreover there exists $C_T > 0$ such that

$$\sup_{t \in [0,T]} \|y(t)\|_{-\diamond} \leq C_T \big(\|y_0\|_{-\diamond} + \|u\|_{L^2(0,T;U)} \big).$$

REMARK 1.2. *By analogy with the semigroup notation, when $u = 0$, we set for any $t \in [0, T]$, $e^{-t\mathcal{A}} y_0 := y(t)$. This extends the semigroup $e^{-\bullet \mathcal{A}}$ defined on X to $X_{-\diamond}$ and implies that for any $z \in X_{-\diamond}$,*

$$\left\langle e^{-T\mathcal{A}} z, \phi \right\rangle_{-\diamond,\diamond} = \left\langle z, e^{-T\mathcal{A}^*} \phi \right\rangle_{-\diamond,\diamond}, \qquad \forall \phi \in X_\diamond^*. \tag{1.19}$$

With this notion of solution at hand, we finally define the minimal null control time from a subspace of initial conditions Y_0.

DEFINITION 1.1. *Let Y_0 be a closed subspace of $X_{-\diamond}$ and let $T > 0$. The system (1.1) is said to be null controllable from Y_0 at time T if for any $y_0 \in Y_0$, there exists a control $u \in L^2(0,T;U)$ such that the associated solution of (1.1) satisfies $y(T) = 0$.*

The minimal null control time $T_0(Y_0) \in [0, +\infty]$ is defined by

- *for any $T > T_0(Y_0)$, system (1.1) is null controllable from Y_0 at time T;*
- *for any $T < T_0(Y_0)$, system (1.1) is not null controllable from Y_0 at time T.*

To simplify the notations, for any $y_0 \in X_{-\diamond}$, we define $T_0(y_0) := T_0(\mathrm{Span}\{y_0\})$. In the formulas given in this article, it can happen that $T_0(Y_0) < 0$. In this case, one should replace $T_0(Y_0)$ by 0.

1.3.2. Spectral assumptions.

In all this article we assume that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy the assumptions of Proposition 1.1. Moreover to solve the control problem we will need some additional spectral assumptions.

⋆ *Behavior of eigenvalues.*

We assume that the spectrum of $\mathcal{A}^*$, denoted by Λ, is only composed of (countably many) eigenvalues.

In what follows we assume that

$$\Lambda \in (0, +\infty)^{\mathbb{N}}. \tag{1.20}$$

REMARK 1.3. *In [10], the assumption on Λ was slightly stronger. Namely, in that article it was assumed that $\Lambda \in (1, +\infty)^{\mathbb{N}}$. This stronger assumption was only used in the lower bound on the solution of scalar block moment problems (see estimate (A.4)). Thus the results of the article [10] that will be used in the present article remain valid under the assumption (1.20).*

If necessary, one can replace the operator $\mathcal{A}$ by $\mathcal{A} + \tau$ without modifying the controllability properties. Then, in the different estimates, the behavior with respect to τ can be carefully tracked if needed.

Most of the results of this article (but not all) also holds when the eigenvalues in Λ are complex valued (yet with a dominant real part). To avoid confusion we stick with the assumption (1.20) and we only discuss in Section 7.1 which results hold in the complex setting and what are the necessary adjustments.

As in the case of a scalar control (see [10]) we assume that this spectrum satisfies a weak-gap condition. Namely, there exists $p \in \mathbb{N}^*$ and $\varrho > 0$ such that

$$\sharp\Big(\Lambda \cap [\mu, \mu + \varrho]\Big) \leq p, \qquad \forall \mu \in [0, +\infty). \tag{1.21}$$

This means that the eigenvalues are allowed to condensate by groups but the cardinality of these groups should be bounded. To precise this, let us recall the notion of groupings introduced in [10, Definition 1.6].

DEFINITION 1.2. *Let $p \in \mathbb{N}^*$ and $r, \varrho > 0$. A sequence of sets $(G_k)_{k\geq 1} \subset \mathcal{P}(\Lambda)$ is said to be a grouping for Λ with parameters p, r, ϱ (which we denote by $(G_k)_k \in \mathcal{G}(\Lambda, p, r, \varrho)$) if it is a covering of Λ*

$$\Lambda = \bigcup_{k\geq 1} G_k,$$

with the additional properties that, for every $k \geq 1$,

$$\sharp G_k \leq p, \qquad \sup(G_k) < \inf(G_{k+1}), \qquad \operatorname{dist}(G_k, G_{k+1}) \geq r$$

and

$$\operatorname{diam}(G_k) < \varrho.$$

As proved in [10, Proposition 7.1], the weak-gap condition (1.21) implies that

$$\mathcal{G}\left(\Lambda, p, \frac{\varrho}{p}, \varrho\right) \neq \varnothing.$$

REMARK 1.4. *For convenience, in the following we label the eigenvalues of a given group* G *in increasing order i.e.* $G = \{\lambda_1, \dots, \lambda_g\}$ *with* $\lambda_k < \lambda_{k+1}$ *but this is not mandatory.*

Concerning the asymptotic behavior of the spectrum we will use the counting function associated to Λ defined by

$$N_\Lambda : r > 0 \mapsto \sharp\{\lambda \in \Lambda\,;\ \lambda \leq r\}.$$

When there is no ambiguity we drop the subscript Λ. We assume that there exists $\bar{N} > 0$ and $a \in (0,1)$ such that

$$N_\Lambda(r) \leq \bar{N} r^a, \qquad \forall r > 0. \tag{1.22}$$

Notice that this condition is slightly stronger than the classical summability condition (1.4) used for instance in [24, 5, 10] and many other works.

Notice also that (1.22), with $r = \min\Lambda$, implies the following lower bound on the bottom of the spectrum

$$\min\Lambda \geq \bar{N}^{-a}.$$

Our goal is not only to study the controllability properties of our system but also to obtain estimates that are uniform in a way to be precised. To do so, we define the following class of sequences: let $p \in \mathbb{N}^*$, $\varrho, \bar{N} > 0$, $a \in (0,1)$ and consider the class

$$\mathcal{L}(p, \varrho, a, \bar{N}) := \left\{\Lambda \in (0,+\infty)^{\mathbb{N}}\,;\ \Lambda \text{ satisfies (1.21) and (1.22)}\right\}. \tag{1.23}$$

In this work, we obtain sharper estimates when replacing (1.22) by the stronger assumption

$$\left|N_\Lambda(r) - \bar{N} r^a\right| \leq \widetilde{N} r^{a'}, \qquad \forall r > 0, \tag{1.24}$$

with $\widetilde{N} > 0$ and $a' \in [0, a)$. This motivates the definition of the class

$$\mathcal{L}(p, \varrho, a, \bar{N}, a', \widetilde{N}) := \left\{\Lambda \in (0,+\infty)^{\mathbb{N}}\,;\ \Lambda \text{ satisfies (1.21) and (1.24)}\right\}. \tag{1.25}$$

Finally, we can also deal with the slightly larger class $\mathcal{L}(p, \varrho, \mathcal{N})$ used in [10] (see (A.1)) but this will not lead to explicit estimates with respect to the control time.

⋆ *Multiplicity of eigenvalues.*

In our study we allow both algebraic and geometric multiplicities for the eigenvalues. We assume that these multiplicities are finite and that the algebraic multiplicity is globally bounded. More precisely, we assume that

$$\gamma_\lambda := \dim \mathrm{Ker}(\mathcal{A}^* - \lambda) < +\infty, \qquad \forall \lambda \in \Lambda, \tag{1.26}$$

and that there exists $\eta \in \mathbb{N}^*$ such that

$$\mathrm{Ker}(\mathcal{A}^* - \lambda)^\eta = \mathrm{Ker}(\mathcal{A}^* - \lambda)^{\eta+1}, \qquad \forall \lambda \in \Lambda. \tag{1.27}$$

For any $\lambda \in \Lambda$ we denote by α_λ the smallest integer such that

$$\mathrm{Ker}(\mathcal{A}^* - \lambda)^{\alpha_\lambda} = \mathrm{Ker}(\mathcal{A}^* - \lambda)^{\alpha_\lambda + 1}$$

and set

$$E_\lambda := \mathrm{Ker}(\mathcal{A}^* - \lambda)^{\alpha_\lambda}.$$

For a given group $G = \{\lambda_1, \dots, \lambda_g\}$ we denote by $\alpha = (\alpha_1, \dots, \alpha_g)$ the multi-index of corresponding algebraic multiplicities.

⋆ *(Generalized) eigenvectors.*

To study null-controllability, we assume that the Fattorini-Hautus criterion is satisfied

$$\mathrm{Ker}(\mathcal{A}^* - \lambda) \cap \mathrm{Ker}\, \mathcal{B}^* = \{0\}, \qquad \forall \lambda \in \Lambda. \tag{1.28}$$

It is a necessary condition for approximate controllability. Note that, under additional assumptions on $\mathcal{A}$ and $\mathcal{B}$ it is also a sufficient condition for approximate controllability (see for instance [22, 33]). However, when studying null controllability of system (1.1) for initial conditions in a closed strict subspace Y_0 of $X_{-\diamond}$ the condition (1.28) can be too strong. This is discussed in Section 7.2.

We assume that the family of generalized eigenvectors of $\mathcal{A}^*$

$$\Phi = \{\phi \in E_\lambda \,;\, \lambda \in \Lambda\} = \bigcup_{\lambda \in \Lambda} E_\lambda$$

is complete in $X_\diamond^*$ *i.e.* for any $y \in X_{-\diamond}$,

$$\Big(\langle y, \phi \rangle_{-\diamond, \diamond} = 0, \quad \forall \phi \in \Phi \Big) \quad \Longrightarrow \quad y = 0. \tag{1.29}$$

In the following, to simplify the writing, we gather these assumptions and say that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy (H) if there exists $p \in \mathbb{N}^*$, $r, \varrho, \bar{N}, \widetilde{N} > 0$, $a \in (0,1)$, $a' \in [0, a)$ and $\mathcal{N} : (0, +\infty) \to \mathbb{R}$ such that

$$\text{(H)} \quad \begin{cases} \mathcal{A} \text{ and } \mathcal{B} \text{ satisfy the assumptions of Proposition 1.1 ;} \\ \Lambda = \mathrm{Sp}(\mathcal{A}^*) \text{ satisfies (1.20), (1.26), (1.27) and} \\ \Lambda \in \mathcal{L}(p, \varrho, \mathcal{N}) \cup \mathcal{L}(p, \varrho, a, \bar{N}) \cup \mathcal{L}(p, \varrho, a, \bar{N}, a', \widetilde{N}) \,; \\ \text{the associated (generalized) eigenvectors satisfy (1.28) and (1.29).} \end{cases}$$

1.3.3. Notation.

We give here some notation that will be used throughout this article.

- For any integers $a, b \in \mathbb{N}$, we define the following subset of $\mathbb{N}$:

$$[\![a, b]\!] := [a, b] \cap \mathbb{N}.$$

- For any $s \in \mathbb{R}$ we denote by e_s the exponential function

$$\begin{aligned} e_s : (0, +\infty) &\to \mathbb{R} \\ x &\mapsto e^{-sx}. \end{aligned}$$

- We shall denote by $C_{\theta_1,\dots,\theta_l} > 0$ a constant possibly varying from one line to another but depending only on the parameters $\theta_1, \dots, \theta_l$.
- For any multi-index $\alpha \in \mathbb{N}^n$, we denote its length by $|\alpha| = \sum_{j=1}^n \alpha_j$ and its maximum by $|\alpha|_\infty = \max_{j\in[\![1,n]\!]} \alpha_j$.
  For $\alpha, \mu \in \mathbb{N}^n$, we say that $\mu \leq \alpha$ if and only if $\mu_j \leq \alpha_j$ for any $j \in [\![1, n]\!]$.
- In all in this article the notation $f[\cdots]$ stands for (generalized) divided differences of a set of values (x_j, f_j). Let us recall that, for pairwise distinct $x_1, \dots, x_n \in \mathbb{R}$ and $f_1, \dots, f_n$ in any vector space, the divided differences are defined by

$$f[x_j] = f_j, \qquad f[x_1, \dots, x_j] = \frac{f[x_2, \dots, x_j] - f[x_1, \dots, x_{j-1}]}{x_j - x_1}.$$

  The two results that will the most used in this article concerning divided differences are the Leibniz formula

$$(gf)[x_1, \dots, x_j] = \sum_{k=1}^{j} g[x_1, \dots, x_k] f[x_k, \dots, x_j],$$

  and the Lagrange theorem stating that, when $f_j = f(x_j)$ for a sufficiently regular function f, we have

$$f[x_1, \dots, x_j] = \frac{f^{(j-1)}(z)}{(j-1)!},$$

  with $z \in \mathrm{Conv}\{x_1, \dots, x_j\}$. For more detailed statements and other useful properties as well as their generalizations when $x_1, \dots, x_n$ are not assumed to be pairwise distinct we refer the reader to [10, Section 7.3]. This generalization is used in the present article whenever there are algebraically multiple eigenvalues.
- For any closed subspace Y of $X_{-\diamond}$ we denote by P_Y the orthogonal projection in $X_{-\diamond}$ onto Y. We denote by $P_Y^* \in L(X_\diamond^*)$ its adjoint in the duality $X_{-\diamond}$, $X_\diamond^*$.

1.4. Block moment problems and minimal time for null-controllability.

⋆ *Definition of block moment problems.*

Using the notion of solution given in Proposition 1.1 and the assumption (1.29), null controllability from y_0 in time T reduces to the resolution of the following problem: find $u \in L^2(0, T; U)$ such that

$$\int_0^T \left\langle u(t), \mathcal{B}^* e^{-(T-t)\mathcal{A}^*} \phi \right\rangle_U \mathrm{d}t = -\left\langle y_0, e^{-T\mathcal{A}^*} \phi \right\rangle_{-\diamond,\diamond}, \quad \forall \phi \in E_\lambda,\ \forall \lambda \in \Lambda. \tag{1.30}$$

Following the strategy initiated in [10] for scalar controls, we decompose this problem into block moments problem. Hence we look for a control of the form

$$u = -\sum_{k\geq 1} v_k(T-\bullet) \tag{1.31}$$

where for every $k \in \mathbb{N}^*$, $v_k \in L^2(0,T;U)$ solves the moment problem in the group G_k *i.e.*

$$\int_0^T \left\langle v_k(t), \mathcal{B}^* e^{-t\mathcal{A}^*}\phi \right\rangle_U \mathrm{d}t = \left\langle y_0, e^{-T\mathcal{A}^*}\phi \right\rangle_{-\diamond,\diamond}, \quad \forall \phi \in E_\lambda,\ \forall \lambda \in G_k, \tag{1.32a}$$

$$\int_0^T \left\langle v_k(t), \mathcal{B}^* e^{-t\mathcal{A}^*}\phi \right\rangle_U \mathrm{d}t = 0, \quad \forall \phi \in E_\lambda,\ \forall \lambda \in \Lambda\backslash G_k. \tag{1.32b}$$

Let us rewrite the orthogonality condition between groups (1.32b) in a more convenient way. For any $\phi \in E_\lambda$, from [10, (1.22)], it comes that

$$e^{-t\mathcal{A}^*}\phi = e^{-\lambda t}\sum_{r\geq 0}\frac{(-t)^r}{r!}(\mathcal{A}^*-\lambda)^r\phi = \sum_{r\geq 0} e_t\left[\lambda^{(r+1)}\right](\mathcal{A}^*-\lambda)^r\phi, \tag{1.33}$$

where the sums are finite (and contains at most the first α_λ terms).

From (1.32) and (1.33), we study in this article the following block moment problems for a given group G

$$\int_0^T \left\langle v(t), \mathcal{B}^* e^{-t\mathcal{A}^*}\phi \right\rangle_U \mathrm{d}t = \left\langle e^{-T\mathcal{A}}y_0, \phi \right\rangle_{-\diamond,\diamond}, \quad \forall \phi \in E_\lambda,\ \forall \lambda \in G, \tag{1.34a}$$

$$\int_0^T v(t) t^l e^{-\lambda t}\mathrm{d}t = 0, \qquad \forall \lambda \in \Lambda\backslash G,\ \forall l \in [\![0, \eta-1]\!]. \tag{1.34b}$$

where $e^{-T\mathcal{A}}y_0$ is defined in (1.19).

Remark 1.5. *Thanks to* (1.33)*, every solution of* (1.34) *solves* (1.32)*. Yet, the orthogonality condition between groups* (1.34b) *is more restrictive than* (1.32b)*: it is stated directly in U and each eigenvalue outside the group G is considered as if it has maximal algebraic multiplicity η. Those two choices allow a unification of the writing when the eigenvalues in different groups have different spectral behaviors and have no influence on the obtained results.*

⋆ *Resolution of block moment problems.*

In our setting, the block moment problem (1.34) is proved to be solvable for any $T > 0$. The resolution will follow from the scalar study done in [10] (see Theorem 1.2).

Due to (1.31), the main issue to prove null controllability of (1.1) is thus to sum these solutions to obtain a solution of (1.30). This is justified thanks to a precise estimate of the cost of the resolution of (1.34) for each group G that is the quantity

$$\inf\left\{\|v\|_{L^2(0,T;U)}\ ;\ v \text{ solution of } (1.34)\right\}.$$

To state this result, we introduce some additional notation.

To solve the moment problem (1.34) we lift it into a '*vectorial block moment*

problem' of the following form (see (2.1))

$$\begin{cases} \displaystyle\int_0^T v(t)\frac{(-t)^l}{l!}e^{-\lambda t}\mathrm{d}t = \Omega_\lambda^l, \quad \forall \lambda \in G,\ \forall l \in [\![0, \alpha_\lambda - 1]\!], \\ \displaystyle\int_0^T v(t) t^l e^{-\lambda t}\mathrm{d}t = 0, \quad \forall \lambda \in \Lambda\backslash G,\ \forall l \in [\![0, \eta - 1]\!], \end{cases}$$

where Ω_λ^l belongs to U. Following (1.33), to recover a solution of (1.34), we need to impose some constraints on the right-hand side. Thus, for any $\lambda \in \Lambda$ and any $z \in X_{-\diamond}$, we set

$$\mathcal{O}(\lambda, z) = \Big\{ (\Omega^0, \dots, \Omega^{\alpha_\lambda - 1}) \in U^{\alpha_\lambda}\ ;\ \sum_{l=0}^{\alpha_\lambda - 1} \left\langle \Omega^l, \mathcal{B}^*(\mathcal{A}^* - \lambda)^l \phi \right\rangle_U = \langle z, \phi \rangle_{-\diamond,\diamond}\,, \quad \forall \phi \in E_\lambda \Big\}. \tag{1.35}$$

For a given group $G = \{\lambda_1, \dots, \lambda_g\}$ we set

$$\mathcal{O}(G, z) = \mathcal{O}(\lambda_1, z) \times \cdots \times \mathcal{O}(\lambda_g, z) \subset U^{|\alpha|}. \tag{1.36}$$

Recall that $\alpha = (\alpha_1, \dots, \alpha_g)$ is the multi-index of algebraic multiplicities. Consider any sequence of multi-indices $(\mu^l)_{l \in [\![0, |\alpha|]\!]}$ such that

$$\begin{cases} \mu^{l-1} \leq \mu^l, \quad \forall l \in [\![1, |\alpha|]\!], \\ \left|\mu^l\right| = l, \quad \forall l \in [\![0, |\alpha|]\!], \\ \mu^{|\alpha|} = \alpha. \end{cases} \tag{1.37}$$

To measure the cost associated to the group G let us define the following functional

$$F : \Omega = \left(\Omega_1^0, \dots, \Omega_1^{\alpha_1 - 1}, \dots, \Omega_g^0, \dots, \Omega_g^{\alpha_g - 1}\right) \in U^{|\alpha|} \mapsto \sum_{l=1}^{|\alpha|} \left\| \Omega \left[\lambda_\bullet^{(\mu^l)} \right] \right\|_U^2. \tag{1.38}$$

The use of such functional to measure the cost comes from the analysis conducted in [10] (see Proposition 2.1).

The first main result of this article concerns the resolution of block moment problems of the form (1.34).

THEOREM 1.2. *Assume that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy the assumption (H) (see page 12) and let $(G_k)_{k \geq 1} \in \mathcal{G}(\Lambda, p, r, \varrho)$ be an associated grouping. Let $T \in (0, +\infty)$.*

For any $G = \{\lambda_1, \dots, \lambda_g\} \in (G_k)_k$ and any $z \in X_{-\diamond}$, there exists $v \in L^2(0, T; U)$ solution of

$$\int_0^T \left\langle v(t), \mathcal{B}^* e^{-t\mathcal{A}^*} \phi \right\rangle_U \mathrm{d}t = \langle z, \phi \rangle_{-\diamond,\diamond}\,, \qquad \forall \phi \in E_{\lambda_j},\ \forall j \in [\![1, g]\!], \tag{1.39a}$$

$$\int_0^T v(t) t^l e^{-\lambda t}\mathrm{d}t = 0, \qquad \forall \lambda \in \Lambda\backslash G,\ \forall l \in [\![0, \eta - 1]\!]. \tag{1.39b}$$

Moreover, we have the following estimate

$$\|v\|^2_{L^2(0,T;U)} \leq \mathcal{E}(\lambda_1)\, \mathcal{K}(T)\, \mathcal{C}(G, z), \tag{1.40}$$

where

$$\mathcal{C}(G,z) := \inf\{F(\Omega)\,;\,\Omega \in \mathcal{O}(G,z)\} \tag{1.41}$$

with F defined in (1.38), $\mathcal{O}(G,z)$ defined in (1.36) and the functions $\mathcal{E}$ and $\mathcal{K}$ satisfy the bounds given in Theorem A.1.

Moreover, there exists $C_{p,\eta,\min\Lambda} > 0$ such that any $v \in L^2(0,T;U)$ solving (1.39a) satisfies

$$\|v\|^2_{L^2(0,T;U)} \geq C_{p,\eta,\min\Lambda}\mathcal{C}(G,z). \tag{1.42}$$

Before giving the application of this resolution of block moment problems to null controllability of problem (1.1), let us give some comments.

- As it was the case in [10], the considered setting allows for a wide variety of applications. In (1.29) the generalized eigenvectors are only assumed to form a complete family (and not a Riesz basis as in many previous works) which is the minimal assumption to use a moment method-like strategy. The weak gap condition (1.21) is also well adapted to study systems of coupled one dimensional parabolic equations (see Section 6).
- The main restriction is the assumption (1.22) or (1.24) (or (A.1)). As detailed in Section 1.1, this assumption is common to most of the results based on a moment-like method.
  Though restrictive, let us underline that the moment method is, to the best of our knowledge, the most suitable method to capture very sensitive features such as a minimal null control time for parabolic control problems without constraints.
- The main novelty of this theorem is to ensure solvability of block moment problems coming from control problems with control operators that are only assumed to be admissible. In particular, the space U can be of infinite dimension.
- As proved by (1.42), for any fixed $T > 0$, up to the factor $\mathcal{E}(\lambda_1)$, the obtained estimate (1.40) is optimal in the asymptotic $\min G \to +\infty$. This will be crucial to completely characterize the minimal null control time in Theorem 1.3. In the applications to control theory, this term $\mathcal{E}(\lambda_1)$ which accounts for the orthogonality condition (1.39b), will always be negligible (see the bounds given in Theorem A.1).
- The estimate (1.40) does not explicitly depend on the sequence of eigenvalues Λ but rather on some parameters such as the weak-gap parameters and the asymptotic of the counting function. As presented in Section 1.1, the uniformity of such bounds can be used to deal with parameter dependent problems.
- Let us also underline that the obtained estimate (1.40) tracks the dependency of the constants with respect to the controllability time T when

$$\Lambda \in \mathcal{L}(p,\varrho,a,\bar{N}) \cup \mathcal{L}(p,\varrho,a,\bar{N},a',\widetilde{N}).$$

  This will be crucial to estimate the cost of controllability in Proposition 1.5. We refer to Remark 1.7 for possible applications of such estimates of the cost of controllability.
- Though quite general and useful for the theoretical characterization of the minimal null control time, the obtained estimate (1.40) is not very easy to

deal with on actual examples. With slightly stronger assumptions on the eigenvalues of the group G we provide in Section 1.5 more explicit formulas.

- The formulation of the right-hand side of (1.39a) is not standard. Usually, to set a moment problem, a specific basis of the generalized eigenspace is exhibited. Here, in our study, we do not exhibit any particular generalized eigenvector. This enables to choose different normalization condition on different examples and then simplify the computations on actual examples (see Sections 5 and 6). Our study also leads to the resolution of block moment problems with 'standard' right-hand sides. The obtained results are detailed in Appendix C.

⋆ *Application to null controllability.*

The resolution of block moment problems stated in Theorem 1.2 allows to obtain the following characterization of the minimal null control time from a given initial condition.

THEOREM 1.3. *Assume that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy the assumption (H) (see page 12) and let $(G_k)_{k\geq 1} \in \mathcal{G}(\Lambda, p, r, \varrho)$ be an associated grouping. Then, for any $y_0 \in X_{-\diamond}$, the minimal null control time of (1.1) from y_0 is given by*

$$T_0(y_0) = \limsup_{k\to+\infty} \frac{\ln \mathcal{C}(G_k, y_0)}{2\min G_k} \tag{1.43}$$

where $\mathcal{C}(G_k, y_0)$ is defined in (1.41).

If one considers a space of initial conditions (instead of a single initial condition), the characterization of the minimal null control time is given in the following corollary.

COROLLARY 1.4. *Let Y_0 be a closed subspace of $X_{-\diamond}$. Then, under the assumptions of Theorem 1.3, the minimal null control time from Y_0 is given by*

$$T_0(Y_0) = \limsup_{k\to+\infty} \frac{\ln \mathcal{C}(G_k, Y_0)}{2\min G_k}$$

with

$$\mathcal{C}(G, Y_0) := \sup_{\substack{y_0\in Y_0 \\ \|y_0\|_{-\diamond}=1}} \mathcal{C}(G, y_0).$$

The remarks on the assumptions and their benefits and restrictions stated after Theorem 1.2 remain valid.

When system (1.1) is null controllable, we obtain the following bound on the cost of controllability.

PROPOSITION 1.5. *Assume that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy the assumption (H) (see page 12) and let $(G_k)_{k\geq 1} \in \mathcal{G}(\Lambda, p, r, \varrho)$ be an associated grouping. Let Y_0 be a closed subspace of $X_{-\diamond}$ and let $T > T_0(Y_0)$.*

For any $y_0 \in Y_0$ with $\|y_0\|_{-\diamond} = 1$, there exists a control $u \in L^2(0, T; U)$ such that the associated solution of (1.1) satisfies $y(T) = 0$ and

$$\|u\|^2_{L^2(0,T;U)} \leq \mathcal{K}(T) \sum_{k\geq 1} (1+T)^{2|\alpha_{G_k}|} \mathcal{E}(\min G_k) e^{-2(\min G_k)T} \mathcal{C}(G_k, Y_0),$$

where the functions $\mathcal{E}$ and $\mathcal{K}$ satisfy the bounds given in Theorem A.1.

Though quite general the above formula is not very explicit. More importantly, it is proved in [30, Theorem 1.1] that, with a suitable choice of $\mathcal{A}$ and $\mathcal{B}$ satisfying our assumptions, any blow-up of the cost of controllability can occur. We give below a setting (inspired from [30, Theorem 1.2]) in which this upper bound on the cost of controllability is simpler and can have some applications (see Remark 1.7).

COROLLARY 1.6. *Assume that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy the assumption (H) (see page 12) with*

$$\Lambda \in \mathcal{L}(p, \varrho, a, \bar{N}, a', \widetilde{N})$$

as defined in (1.25). Let $\kappa > 0$. There exists $C > 0$ depending only on κ, p, ϱ, η, a, $\bar{N}$, a' and $\widetilde{N}$ such that for any $y_0 \in X_{-\diamond}$ satisfying

$$\mathcal{C}(G_k, y_0) \leq \kappa e^{2(\min G_k)T_0(y_0)} \|y_0\|^2_{-\diamond}, \qquad \forall k \geq 1, \tag{1.44}$$

for any $T > T_0(y_0)$ close enough to $T_0(y_0)$, there exists a control $u \in L^2(0,T;U)$ such that the associated solution of (1.1) satisfies $y(T) = 0$ and

$$\|u\|_{L^2(0,T;U)} \leq C \exp\left(\frac{C}{(T - T_0(y_0))^{\frac{a}{1-a}}}\right) \|y_0\|_{-\diamond}.$$

REMARK 1.6. *In the setting of Corollary 1.6, replacing the assumption (1.44) by*

$$\mathcal{C}(G_k, y_0) \leq \kappa e^{C(\min G_k)^b} e^{2(\min G_k)T_0(y_0)} \|y_0\|^2_{-\diamond}, \qquad \forall k \geq 1,$$

with $b \in (0,1)$ leads to the following estimate

$$\|u\|_{L^2(0,T;U)} \leq C \exp\left(\frac{C}{T^{\frac{a}{1-a}}} + \frac{C}{(T - T_0(y_0))^{\frac{\max(a,b)}{1-\max(a,b)}}}\right) \|y_0\|_{-\diamond}.$$

REMARK 1.7. *Giving the best possible estimate on the cost of small time null controllability is a question that has drawn a lot of interest in the past years.*

In classical cases, for instance for heat-like equations, null controllability holds in any positive time and the cost of controllability in small time behaves like $\exp\left(\frac{C}{T}\right)$ (see for instance [36]). There are two mains applications of such estimate.

- *Controllability in cylindrical domains.*
  It is proved in [9] that null controllability of parabolic problems in cylindrical geometries (with operators compatible with this geometry) with a boundary control located on the top of the cylinder can be proved thanks to null controllability of the associated problem in the transverse variable together with suitable estimates of the cost of controllability. Their proof relies on an adaptation of the classical strategy of Lebeau and Robbiano [29] and thus uses an estimate of the cost of controllability in small time of the form $\exp\left(\frac{C}{T}\right)$. These ideas were already present in [11] and later generalized in an abstract setting in [1].
- *Nonlinear control problems.*
  The source term method has been introduced in [31] to prove controllability of a nonlinear fluid-structure system (see also [8, Section 2] for a general presentation of this strategy). Roughly speaking it amounts to prove null controllability with a source term in suitable weighted spaces and then use a fixed point argument. The null controllability with a source term is here proved by an iterative process which strongly uses that the cost of controllability of the linearized system behaves like $\exp\left(\frac{C}{T}\right)$.

Notice that from the upper bound given in Corollary 1.6, the cost of controllability in small time can explode faster than $\exp\left(\frac{C}{T}\right)$. *Yet, as studied in [34, Chapter 4], the arguments of the two previous applications can be adapted with an explosion of the cost of the form* $\exp\left(\frac{C}{T^{\frac{a}{1-a}}}\right)$ *with* $a \in (0,1)$.

However, these two applications uses a decomposition of the time interval $[0,T]$ *into an infinite number of sub-intervals (which explains the use of the asymptotic of the cost of controllability when the time goes to zero). Thus their extension in the case of a minimal null control time is an open problem.*

1.5. A more explicit formula.

Assume that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy the assumption (H) (see page 12). Let $G = \{\lambda_1, \dots, \lambda_g\} \subset \Lambda$ be such that $\sharp G \leq p$ and $\operatorname{diam} G \leq \varrho$. We have seen in Theorem 1.3 that the key quantity to compute the minimal null control time from y_0 is

$$\mathcal{C}(G, y_0) = \inf\left\{F(\Omega)\,;\, \Omega \in \mathcal{O}(G, y_0)\right\}.$$

where the function F is defined in (1.38) and the constraints $\mathcal{O}(G, y_0)$ are defined in (1.36).

Notice that, for any $z \in X_{-\diamond}$, the quantity $\mathcal{C}(G, z)$ can be expressed as a finite dimensional constrained problem. Indeed, for a given group G we consider the finite dimensional subspace

$$U_G = \mathcal{B}^* \operatorname{Span}\left\{\phi \in E_\lambda\,;\, \lambda \in G\right\} \tag{1.45}$$

and P_{U_G} the orthogonal projection in U onto U_G. Then, for any $\Omega \in \mathcal{O}(G, z)$ it comes that $P_{U_G}\Omega \in \mathcal{O}(G, z)$ and $F(P_{U_G}\Omega) \leq F(\Omega)$. Thus, the optimization problem defining $\mathcal{C}(G, z)$ reduces to

$$\mathcal{C}(G, z) = \inf\left\{F(\Omega)\,;\, \Omega \in \mathcal{O}(G, z) \cap U_G^{|\alpha|}\right\},$$

which is a finite dimensional optimization problem. From [10, Proposition 7.15], the function F is coercive which implies that the infimum is attained:

$$\mathcal{C}(G, z) = \min\left\{F(\Omega)\,;\, \Omega \in \mathcal{O}(G, z) \cap U_G^{|\alpha|}\right\}. \tag{1.46}$$

In this section, solving the optimization problem (1.46), we provide more explicit formulas for this cost depending on stronger assumptions on the multiplicity of the eigenvalues in the group G (and only in the group G).

Remark 1.8. *All the results in this section only concern the group* G. *Then, the assumption* (H) *is stronger than needed. For instance, one does not need the weak gap condition* (1.21) *on the whole spectrum* Λ *but only that* $\sharp G \leq p$ *and* $\operatorname{diam} G \leq \varrho$. *However, to simplify the reading we stick with assumption* (H).

⋆ *A group* G *of geometrically simple eigenvalues.*

First, assume that the eigenvalues in G are all geometrically simple *i.e.* $\gamma_\lambda = 1$ for every $\lambda \in G$ where γ_λ is defined in (1.26).

For any $j \in [\![1, g]\!]$ we denote by ϕ_j^0 an eigenvector of $\mathcal{A}^*$ associated to the eigenvalue λ_j and by $(\phi_j^l)_{l \in [\![0, \alpha_j - 1]\!]}$ an associated Jordan chain *i.e.*

$$(\mathcal{A}^* - \lambda_j)\phi_j^l = \phi_j^{l-1}, \qquad \forall l \in [\![1, \alpha_j - 1]\!].$$

To simplify the writing, we set

$$b_j^l := \mathcal{B}^* \phi_j^l \in U, \qquad \forall l \in [\![0, \alpha_j - 1]\!],\ \forall j \in [\![1, g]\!].$$

Recall that the sequence of multi-index $(\mu^l)_{l \in [\![0,|\alpha|]\!]}$ satisfy (1.37) and let

$$M := \sum_{l=1}^{|\alpha|} \Gamma_\mu^l \tag{1.47}$$

with

$$\Gamma_\mu^l := \mathrm{Gram}_U \left(\underbrace{0, \dots, 0}_{l-1}, b\left[\lambda_\bullet^{(\mu^l - \mu^{l-1})}\right], \dots, b\left[\lambda_\bullet^{(\mu^{|\alpha|} - \mu^{l-1})}\right] \right)$$

where $\mathrm{Gram}_U(\cdots)$ denotes the Gram matrix of the arguments with respect to the scalar product in U. To explicit the cost $\mathcal{C}(G, y_0)$, we will use the inverse of this matrix. Its invertibility is guaranteed by the following proposition which is proved in Section 4.2.

PROPOSITION 1.7. *Under condition (1.28), the matrix M defined in (1.47) is invertible.*

The matrix M plays a crucial role in the computation of the cost $\mathcal{C}(G, y_0)$. Let us give some comments. It is a sum of Gram matrices whose construction is summarized in Figure 1 on an example with $G = \{\lambda_1, \lambda_2\}$ with $\alpha_1 = 3$ and $\alpha_2 = 2$. Each of these matrices is of size $|\alpha|$ which is the number of eigenvalues (counted with their algebraic multiplicities) that belong to the group G. Thus, on actual examples (see Section 6), the size of these matrices is usually reasonably small.

Then, we obtain the following formula for the cost of a group of geometrically simple eigenvalues.

THEOREM 1.8. *Assume that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy the assumption (H) (see page 12). Let $G = \{\lambda_1, \dots, \lambda_g\} \subset \Lambda$ be such that $\sharp G \le p$ and $\operatorname{diam} G \le \varrho$ and assume that $\gamma_\lambda = 1$ for every $\lambda \in G$. Then, for any $y_0 \in X_{-\diamond}$, we have*

$$\mathcal{C}(G, y_0) = \left\langle M^{-1}\xi, \xi \right\rangle, \qquad \text{where } \xi = \begin{pmatrix} \left\langle y_0, \phi\left[\lambda_\bullet^{(\mu^1)}\right]\right\rangle_{-\diamond,\diamond} \\ \vdots \\ \left\langle y_0, \phi\left[\lambda_\bullet^{(\mu^{|\alpha|})}\right]\right\rangle_{-\diamond,\diamond} \end{pmatrix}$$

and M is defined in (1.47).

Moreover, if Y_0 is a closed subspace of $X_{-\diamond}$,

$$\mathcal{C}(G, Y_0) = \rho\left(\mathrm{Gram}_{X_\diamond^*}(\psi_1, \dots, \psi_{|\alpha|}) M^{-1}\right) \tag{1.48}$$

where $\psi_j := P_{Y_0}^* \phi\left[\lambda_\bullet^{(\mu^j)}\right]$ and, for any matrix M, the notation $\rho(\mathsf{M})$ denotes the spectral radius of the matrix M.

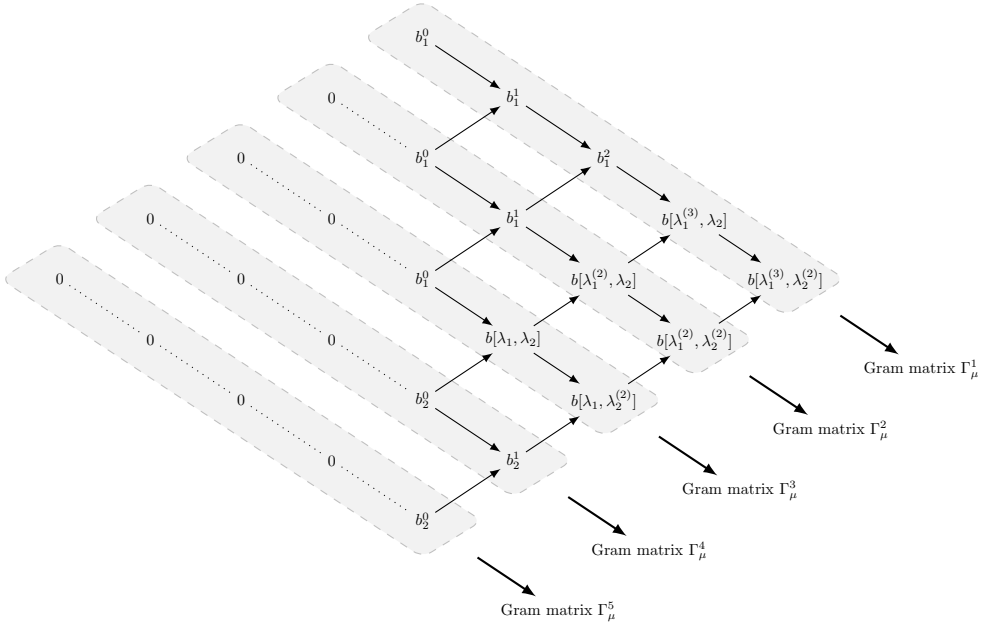

FIGURE 1. *Construction of the Gram matrices Γ_μ^l in the case of a group $G = \{\lambda_1, \lambda_2\}$ with multiplicities $\alpha = (3, 2)$ and the sequence of multi-indices $\mu = \big((0,0),(1,0),(2,0),(3,0),(3,1),(3,2)\big)$*

REMARK 1.9. *Notice that we do not choose any particular eigenvector or Jordan chain. To compute explicitly the cost $\mathcal{C}(G, y_0)$ on actual examples, we will often choose them to satisfy*

$$\|b_j^0\|_U = 1, \qquad \big\langle b_j^0, b_j^l \big\rangle_U = 0, \quad \forall l \in [\![1, \alpha_j - 1]\!],$$

to simplify the Gram matrices. Obviously, as the quantity $\mathcal{C}(G, y_0)$ is independent of this choice, we can choose any other specific Jordan chains or eigenvectors that are more suitable to each problem.

REMARK 1.10. *In the case where the eigenvalues of the considered group G are also algebraically simple, then the expression of M given in (1.47) reduces to*

$$M = \sum_{l=1}^{g} \Gamma^l \quad \text{with} \quad \Gamma^l = \mathrm{Gram}_U \left(\underbrace{0, \dots, 0}_{l-1}, b[\lambda_l], \dots, b[\lambda_l, \dots, \lambda_g] \right) \tag{1.49}$$

and the expression of ξ reduces to

$$\xi = \begin{pmatrix} \langle y_0, \phi[\lambda_1] \rangle_{-\diamond,\diamond} \\ \vdots \\ \langle y_0, \phi[\lambda_1, \dots, \lambda_g] \rangle_{-\diamond,\diamond} \end{pmatrix}.$$

⋆ *A group G of semi-simple eigenvalues.*

We now assume that all the eigenvalues in G are semi-simple *i.e.* for any $\lambda \in G$ we have $\alpha_\lambda = 1$ where α_λ is defined in (1.27).

For any $j \in [\![1, g]\!]$, we denote by $(\phi_{j,i})_{i\in[\![1,\gamma_j]\!]}$ a basis of $\mathrm{Ker}(\mathcal{A}^* - \lambda_j)$. To simplify the writing, we set

$$b_{j,i} := \mathcal{B}^*\phi_{j,i}, \qquad \forall j \in [\![1, g]\!],\ \forall i \in [\![1, \gamma_j]\!]$$

and $\gamma_G := \gamma_1 + \cdots + \gamma_g$.

For any $i \in [\![1, g]\!]$, we set $\delta_1^i := 1$ and

$$\delta_j^i := \prod_{k=1}^{j-1} \big(\lambda_i - \lambda_k\big), \qquad \forall j \in [\![2, g]\!]. \tag{1.50}$$

Notice that $\delta_j^i = 0$ as soon as $j > i$.

Let

$$M = \sum_{l=1}^{g} \Gamma^l \ \text{ with } \ \Gamma^l = \mathrm{Gram}_U\left(\delta_l^1 b_{1,1}, \ldots, \delta_l^1 b_{1,\gamma_1}, \ldots, \delta_l^g b_{g,1}, \ldots, \delta_l^g b_{g,\gamma_g}\right). \tag{1.51}$$

Here again, to explicit the cost $\mathcal{C}(G, y_0)$ we will use the inverse of this matrix. Its invertibility is guaranteed by the following proposition which is proved in Section 4.3.

PROPOSITION 1.9. *Under condition (1.28), the matrix M defined in (1.51) is invertible.*

Notice that the square matrix Γ^l is of size γ_G and can be seen as a block matrix where the block (i, j) is

$$\begin{pmatrix} \left\langle \delta_l^i b_{i,1}, \delta_l^j b_{j,1} \right\rangle_U & \cdots & \left\langle \delta_l^i b_{i,1}, \delta_l^j b_{j,\gamma_j} \right\rangle_U \\ \vdots & & \vdots \\ \left\langle \delta_l^i b_{i,\gamma_i}, \delta_l^j b_{j,1} \right\rangle_U & \cdots & \left\langle \delta_l^i b_{i,\gamma_i}, \delta_l^j b_{j,\gamma_j} \right\rangle_U \end{pmatrix}$$

Thus, the block (i, j) of Γ^l is identically 0 for $i, j \in [\![1, l-1]\!]$.

Then, we obtain the following formula for the cost of a group made of semi-simple eigenvalues.

THEOREM 1.10. *Assume that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy the assumption (H) (see page 12). Let $G = \{\lambda_1, \ldots, \lambda_g\} \subset \Lambda$ be such that $\sharp G \le p$ and $\mathrm{diam}\, G \le \varrho$ and assume that $\alpha_\lambda = 1$ for every $\lambda \in G$. Then, for any $y_0 \in X_{-\diamond}$, we have*

$$\mathcal{C}(G, y_0) = \left\langle M^{-1}\xi, \xi \right\rangle$$

where

$$\xi = \begin{pmatrix} \langle y_0, \phi_{1,1}\rangle_{-\diamond,\diamond} \\ \vdots \\ \langle y_0, \phi_{1,\gamma_1}\rangle_{-\diamond,\diamond} \\ \vdots \\ \langle y_0, \phi_{g,1}\rangle_{-\diamond,\diamond} \\ \vdots \\ \langle y_0, \phi_{g,\gamma_g}\rangle_{-\diamond,\diamond} \end{pmatrix}$$

and M is defined in (1.51).

Moreover, if Y_0 is a closed subspace of $X_{-\diamond}$,

$$\mathcal{C}(G, Y_0) = \rho\left(\mathrm{Gram}_{X_\diamond^*}(\psi_{1,1}, \dots, \psi_{1,\gamma_1}, \dots, \psi_{g,1}, \dots, \psi_{g,\gamma_g}) M^{-1}\right) \tag{1.52}$$

where $\psi_{j,i} := P_{Y_0}^* \phi_{j,i}$ and, for any matrix M, the notation $\rho(\mathsf{M})$ denotes the spectral radius of the matrix M.

REMARK 1.11. *When the eigenvalues of the group G are geometrically and algebraically simple, Theorem 1.10 gives a characterization of the cost of the block $\mathcal{C}(G, y_0)$ which is different from the one coming from Theorem 1.8 and detailed in Remark 1.10. A direct proof of this equivalence (stated in Proposition D.3) using algebraic manipulations is given in Appendix D.*

⋆ *Dealing simultaneously with geometric and algebraic multiplicity.*

Combining Theorems 1.8 and 1.10, we can deal with operators $\mathcal{A}^*$ which have both groups of geometrically simple eigenvalues and groups of semi-simple eigenvalues. However, for technical reasons, in the case where both algebraic and geometric multiplicities need to be taken into account into a group G we do obtain a general formula for the cost of this group $\mathcal{C}(G, y_0)$. Nevertheless, if this situation occurs in actual examples, computing this cost is a finite dimensional constrained optimization problem which can be solved '*by hand*'. We present in Section 4.4 an example of such resolution for a group G that does not satisfies the assumptions of Theorem 1.8 nor of Theorem 1.10.

2. Resolution of block moment problems.

In this section we prove Theorem 1.2 that is we solve the block moment problem (1.39). To do so, we first consider a vectorial block moment problem (see (2.1) below). We solve it in Section 2.1 with an estimate of the cost of this resolution. Then, using the constraints (1.36), we prove that this implies Theorem 1.2. This is detailed in Section 2.2.

2.1. An auxiliary vectorial block moment problem.

Let $\Lambda \in (0, +\infty)^{\mathbb{N}}$, $G = \{\lambda_1, \dots, \lambda_g\} \subset \Lambda$, $\eta \in \mathbb{N}^*$ and $\alpha = (\alpha_1, \dots, \alpha_g) \in \mathbb{N}^g$ with $|\alpha|_\infty \leq \eta$. For any

$$\Omega = \left(\Omega_1^0, \dots, \Omega_1^{\alpha_1 - 1}, \dots, \Omega_g^0, \dots, \Omega_g^{\alpha_g - 1}\right) \in U^{|\alpha|},$$

we consider the following auxiliary vectorial block moment problem

$$\int_0^T v(t) \frac{(-t)^l}{l!} e^{-\lambda_j t} \mathrm{d}t = \Omega_j^l, \quad \forall j \in [\![1, g]\!],\ \forall l \in [\![0, \alpha_j - 1]\!], \tag{2.1a}$$

$$\int_0^T v(t) t^l e^{-\lambda t} \mathrm{d}t = 0, \quad \forall \lambda \in \Lambda \backslash G,\ \forall l \in [\![0, \eta - 1]\!]. \tag{2.1b}$$

This block moment problem is said to be *vectorial* since the right-hand side Ω belongs to $U^{|\alpha|}$. Its resolution with (almost) sharp estimates is given in the following proposition.

PROPOSITION 2.1. *Let $p \in \mathbb{N}^*$, $r, \varrho, \bar{N}, \widetilde{N} > 0$, $a \in (0, 1)$, $a' \in [0, a)$ and $\mathcal{N} : (0, +\infty) \to \mathbb{R}$. Assume that*

$$\Lambda \in \mathcal{L}(p, \varrho, \mathcal{N}) \cup \mathcal{L}(p, \varrho, a, \bar{N}) \cup \mathcal{L}(p, \varrho, a, \bar{N}, a', \widetilde{N})$$

and let $(G_k)_{k\geq 1} \in \mathcal{G}(\Lambda, p, r, \varrho)$ *be an associated grouping. Recall that these classes are defined in* (A.1), (1.23) *and* (1.25).

Let $T \in (0, +\infty)$ *and* $\eta \in \mathbb{N}^*$. *For any* $G = \{\lambda_1, \dots, \lambda_g\} \in (G_k)_k$, *for any multi-index* $\alpha \in \mathbb{N}^g$ *with* $|\alpha|_\infty \leq \eta$ *and any*

$$\Omega = \left(\Omega_1^0, \dots, \Omega_1^{\alpha_1 - 1}, \dots, \Omega_g^0, \dots, \Omega_g^{\alpha_g - 1}\right) \in U^{|\alpha|},$$

there exists $v \in L^2(0, T; U)$ *solution of* (2.1) *such that*

$$\|v\|^2_{L^2(0,T;U)} \leq \mathcal{E}(\lambda_1)\, \mathcal{K}(T)\, F(\Omega),$$

where F *is defined in* (1.38) *and the functions* $\mathcal{E}$ *and* $\mathcal{K}$ *satisfy the bounds given in Theorem A.1.*

Proof. Let $(e_j)_{j \in [\![1,d]\!]}$ be an orthonormal basis of the finite dimensional subspace of U

$$\text{Span}\left\{\Omega_j^l \,;\, j \in [\![1, g]\!],\, l \in [\![0, \alpha_j - 1]\!]\right\}.$$

Then, we decompose Ω_j^l as

$$\Omega_j^l = \sum_{i=1}^d a_{j,i}^l e_i.$$

From Theorem A.1, for any $i \in [\![1, d]\!]$, there exists $v_i \in L^2(0, T; \mathbb{R})$ such that

$$\begin{cases} \displaystyle\int_0^T v_i(t) \frac{(-t)^l}{l!} e^{-\lambda_j t} \mathrm{d}t = a_{j,i}^l, \quad \forall j \in [\![1, g]\!],\, \forall l \in [\![0, \alpha_j - 1]\!], \\ \displaystyle\int_0^T v_i(t) t^l e^{-\lambda t} \mathrm{d}t = 0, \quad \forall \lambda \in \Lambda \backslash G,\, \forall l \in [\![0, \eta - 1]\!], \end{cases}$$

and

$$\|v_i\|^2_{L^2(0,T;\mathbb{R})} \leq \mathcal{E}(\lambda_1)\mathcal{K}(T) \max_{\substack{\mu \in \mathbb{N}^g \\ \mu \leq \alpha}} \left| a_{\bullet,i}^{\bullet} \left[\lambda_1^{(\mu_1)}, \dots, \lambda_g^{(\mu_g)}\right]\right|^2.$$

Setting

$$v := \sum_{i=1}^d v_i e_i,$$

we get that v solves (2.1) and using [10, Proposition 7.15]

$$\begin{aligned} \|v\|^2_{L^2(0,T;U)} &= \sum_{i=1}^d \|v_i\|^2_{L^2(0,T;\mathbb{R})} \\ &\leq \mathcal{E}(\lambda_1)\mathcal{K}(T) \sum_{i=1}^d \max_{\substack{\mu \in \mathbb{N}^g \\ \mu \leq \alpha}} \left| a_{\bullet,i}^{\bullet} \left[\lambda_1^{(\mu_1)}, \dots, \lambda_g^{(\mu_g)}\right]\right|^2 \\ &\leq C_{p,\varrho,\eta} \mathcal{E}(\lambda_1)\mathcal{K}(T) \sum_{p=1}^{|\alpha|} \left(\sum_{i=1}^d \left| a_{\bullet,i}^{\bullet} \left[\lambda_{\bullet}^{(\mu^p)}\right]\right|^2 \right) \\ &= C_{p,\varrho,\eta} \mathcal{E}(\lambda_1)\mathcal{K}(T) \sum_{p=1}^{|\alpha|} \left\| \Omega \left[\lambda_{\bullet}^{(\mu^p)}\right]\right\|^2. \end{aligned}$$

Modifying the constants appearing in $\mathcal{E}$ and $\mathcal{K}$ (still satisfying the bounds given in Theorem A.1) ends the proof of Proposition 2.1. ∎

2.2. Solving the original moment problem.

Through (1.33), when the right-hand side Ω of (2.1) satisfy the constraints (1.36), solving this vectorial block moment problem provides a solution of the original block moment problem (1.39). More precisely we have the following proposition

PROPOSITION 2.2. *Let $T > 0$ and $z \in X_{-\diamond}$. The following two statements are equivalent:*

i. there exists $\Omega \in \mathcal{O}(G, z)$ such that the function $v \in L^2(0, T; U)$ solves (2.1);

ii. the function $v \in L^2(0, T; U)$ solves (1.39).

Proof. Assume first that there exists $\Omega \in \mathcal{O}(G, z)$ and let $v \in L^2(0, T; U)$ be such that (2.1) holds.

Then, using (1.33), for any $j \in [\![1, g]\!]$ and any $\phi \in E_{\lambda_j}$ we have

$$\begin{aligned}\int_0^T \left\langle v(t), \mathcal{B}^* e^{-t\mathcal{A}^*}\phi \right\rangle_U \mathrm{d}t &= \int_0^T \left\langle v(t), e^{-\lambda t} \sum_{l=0}^{\alpha_j - 1} \frac{(-t)^l}{l!} (\mathcal{A}^* - \lambda_j)^l \phi \right\rangle_U \mathrm{d}t \\ &= \sum_{l=0}^{\alpha_j - 1} \left\langle \int_0^T v(t) \frac{(-t)^l}{l!} e^{-\lambda_j t} \mathrm{d}t, (\mathcal{A}^* - \lambda_j)^l \phi \right\rangle_U \\ &= \sum_{l=0}^{\alpha_j - 1} \left\langle \Omega_j^l, (\mathcal{A}^* - \lambda_j)^l \phi \right\rangle_U .\end{aligned}$$

Since $\left(\Omega_j^0, \dots, \Omega_j^{\alpha_j - 1}\right) \in \mathcal{O}(\lambda_j, z)$, this leads to

$$\int_0^T \left\langle v(t), \mathcal{B}^* e^{-t\mathcal{A}^*}\phi \right\rangle_U \mathrm{d}t = \langle z, \phi \rangle_{-\diamond, \diamond}, \qquad \forall j \in [\![1, g]\!], \forall \phi \in E_{\lambda_j},$$

which proves that v solves (1.39).

Assume now that $v \in L^2(0, T; U)$ solves (1.39). Setting

$$\Omega_j^l := \int_0^T v(t) \frac{(-t)^l}{l!} e^{-\lambda_j t} \mathrm{d}t$$

we obtain that v solves (2.1). As in the previous step, the identity (1.33) implies that $\Omega \in \mathcal{O}(G, z)$. ∎

Finally, to solve (1.39) we prove that there exists at least one Ω satisfying the constraints (1.36).

PROPOSITION 2.3. *Let $\lambda \in \Lambda$ and $z \in X_{-\diamond}$. Then, under assumption (1.28), we have*

$$\mathcal{O}(\lambda, z) \neq \varnothing.$$

Proof. Let $T > 0$. From (1.33) the finite dimensional space E_λ is stable by the semigroup $e^{-\bullet \mathcal{A}^*}$. Using the approximate controllability assumption (1.28) we have that

$$\phi \in E_\lambda \mapsto \left\| \mathcal{B}^* e^{-\bullet \mathcal{A}^*} \phi \right\|_{L^2(0,T,U)}$$

is a norm on E_λ. Then, the equivalence of norms in finite dimension implies that the following HUM-type functional

$$J : \phi \in E_\lambda \mapsto \frac{1}{2} \left\| \mathcal{B}^* e^{-\bullet \mathcal{A}^*} \phi \right\|^2_{L^2(0,T,U)} - \langle z, \phi \rangle_{-\diamond, \diamond}$$

is coercive. Let $\tilde{\phi} \in E_\lambda$ be such that

$$J(\tilde{\phi}) = \inf_{\phi \in E_\lambda} J(\phi)$$

and $v := \mathcal{B}^* e^{-\bullet \mathcal{A}^*} \tilde{\phi}$. Then it comes that

$$\int_0^T \left\langle v(t), \mathcal{B}^* e^{-t\mathcal{A}^*} \phi \right\rangle_U \mathrm{d}t = \langle z, \phi \rangle_{-\diamond,\diamond}, \quad \forall \phi \in E_\lambda. \tag{2.2}$$

Finally, we set $\Omega := \left(\Omega^0, \dots, \Omega^{\alpha_\lambda - 1}\right)$ with

$$\Omega^l := \int_0^T v(t) \frac{(-t)^l}{l!} e^{-\lambda t} \mathrm{d}t, \quad \forall l \in [\![0, \alpha_\lambda - 1]\!].$$

Using (2.2) and following the computations of Proposition 2.2 we obtain that $\Omega \in \mathcal{O}(\lambda, z)$. ∎

We now have all the ingredients to prove Theorem 1.2.

Proof (of Theorem 1.2).

• From Proposition 2.3, we have $\mathcal{O}(G, z) \neq \varnothing$. Recall that, from (1.46), the optimization problem defining $\mathcal{C}(G, z)$ can be reduced to a finite dimensional optimization for which the infimum is attained. Thus, let $\Omega \in \mathcal{O}(G, z)$ be such that

$$F(\Omega) = \mathcal{C}(G, z).$$

Let $v \in L^2(0, T; U)$ be the solution of (2.1) given by Proposition 2.1 with Ω as right-hand side. As $\Omega \in \mathcal{O}(G, z)$, from Proposition 2.2 we deduce that v solves (1.39). The upper bound (1.40) on $\|v\|_{L^2(0,T;U)}$ is given by Proposition 2.1.

• We now turn to the lower bound (1.42). Let $v \in L^2(0, T; U)$ be any solution of (1.39a). Let

$$\Omega_j^l := \int_0^T v(t) \frac{(-t)^l}{l!} e^{-\lambda_j t} \mathrm{d}t = \int_0^T v(t) e_t\left[\lambda_j^{(l+1)}\right] \mathrm{d}t, \quad \forall j \in [\![1, g]\!],\ \forall l \in [\![0, \alpha_j - 1]\!].$$

As in the proof of Proposition 2.2, the use of (1.33) implies that

$$\Omega = \left(\Omega_1^0, \dots, \Omega_1^{\alpha_1 - 1}, \dots, \Omega_g^0, \dots, \Omega_g^{\alpha_g - 1}\right) \in \mathcal{O}(G, z).$$

Thus,

$$\mathcal{C}(G, z) \leq F(\Omega) = \sum_{l=1}^{|\alpha|} \left\| \Omega\left[\lambda_\bullet^{(\mu^l)}\right] \right\|_U^2. \tag{2.3}$$

Notice that

$$\Omega\left[\lambda_\bullet^{(\mu^l)}\right] = \int_0^T v(t) e_t\left[\lambda_\bullet^{(\mu^l)}\right] \mathrm{d}t, \quad \forall l \in [\![0, |\alpha|]\!].$$

Using Lagrange theorem [10, Proposition 7.14] yields,

$$\left| e_t\left[\lambda_\bullet^{(\mu^l)}\right] \right| \leq \frac{t^{l-1} e^{-\lambda_1 t}}{(l-1)!}.$$

Together with Cauchy-Schwarz inequality this implies

$$\left\| \Omega\left[\lambda_\bullet^{(\mu^l)}\right] \right\|_U \leq \left(\int_0^{+\infty} \frac{t^{l-1} e^{-\lambda_1 t}}{(l-1)!} \mathrm{d}t \right)^{\frac{1}{2}} \|v\|_{L^2(0,T;U)}.$$

Then, as $\lambda_1 \geq \min \Lambda$ and $|\alpha| \leq p\eta$, estimate (2.3) ends the proof of Theorem 1.2. ∎

3. Application to the determination of the minimal null control time.

This section is dedicated to the consequences of Theorem 1.2 on the null controllability properties of system (1.1).

From Theorem 1.2, the resolution of block moment problems (1.34) associated with null controllability of (1.1) will involve the quantity $\mathcal{C}(G, e^{-T\mathcal{A}}y_0)$. To formulate the minimal null control time we isolate the dependency with respect to the variable T leading to quantities involving $\mathcal{C}(G, y_0)$. The comparison between these two costs is detailed in Section 3.1.

Then, this leads to the formulation of the minimal null control time stated in Theorem 1.3. This is detailed in Section 3.2.

We then prove, in Section 3.3, the estimates on the cost of null controllability stated in Proposition 1.5 and Corollary 1.6.

3.1. Relating the different costs.

The costs $\mathcal{C}(G, z)$ and $\mathcal{C}(G, e^{-T\mathcal{A}}z)$ satisfy the following estimates.

LEMMA 3.1. *Assume that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy the assumption (H) (see page 12). There exists $C_{p,\varrho,\eta} > 0$ such that for any $G \subset \Lambda$ with $\sharp G \leq p$ and $\operatorname{diam} G \leq \varrho$, for any $T > 0$ and any $z \in X_{-\diamond}$,*

$$\mathcal{C}(G, e^{-T\mathcal{A}}z) \leq C_{p,\varrho,\eta}(1+T)^{2|\alpha|}e^{-2(\min G)T}\mathcal{C}(G, z) \tag{3.1}$$

and

$$\mathcal{C}(G, z) \leq C_{p,\varrho,\eta}(1+T)^{2|\alpha|}e^{2(\max G)T}\mathcal{C}(G, e^{-T\mathcal{A}}z). \tag{3.2}$$

Proof. Recall that from (1.19) we have

$$\left\langle y_0, e^{-T\mathcal{A}^*}\phi \right\rangle_{-\diamond,\diamond} = \left\langle e^{-T\mathcal{A}}y_0, \phi \right\rangle_{-\diamond,\diamond}, \qquad \forall \phi \in X_\diamond^*.$$

We set $G = \{\lambda_1, \dots, \lambda_g\}$ with $\lambda_1 < \cdots < \lambda_g$.

• We start with the proof of (3.1).

From (1.46), let $\widetilde{\Omega} \in \mathcal{O}(G, z)$ be such that $F(\widetilde{\Omega}) = \mathcal{C}(G, z)$. We define Ω by

$$\Omega_j^l := (e_T\widetilde{\Omega})\left[\lambda_j^{(l+1)}\right], \qquad \forall j \in [\![1, g]\!],\ \forall l \in [\![0, \alpha_j - 1]\!].$$

Let us prove that $\Omega \in \mathcal{O}(G, e^{-T\mathcal{A}}z)$. For any $j \in [\![1, g]\!]$ and any $\phi \in E_{\lambda_j}$, using Leibniz formula [10, Proposition 7.13] and $\widetilde{\Omega} \in \mathcal{O}(G, z)$, we obtain

$$\begin{aligned}
\sum_{l\geq 0}\left\langle \Omega_j^l, \mathcal{B}^*(\mathcal{A}^* - \lambda_j)^l\phi\right\rangle_U &= \sum_{l\geq 0}\sum_{r=0}^{l} e_T\left[\lambda_j^{(r+1)}\right]\left\langle \widetilde{\Omega}_j^{l-r}, \mathcal{B}^*(\mathcal{A}^* - \lambda_j)^l\phi\right\rangle_U \\
&= \sum_{r\geq 0} e_T\left[\lambda_j^{(r+1)}\right]\sum_{l\geq r}\left\langle \widetilde{\Omega}_j^{l-r}, \mathcal{B}^*(\mathcal{A}^* - \lambda_j)^l\phi\right\rangle_U \\
&= \sum_{r\geq 0} e_T\left[\lambda_j^{(r+1)}\right]\sum_{l\geq 0}\left\langle \widetilde{\Omega}_j^{l}, \mathcal{B}^*(\mathcal{A}^* - \lambda_j)^{l+r}\phi\right\rangle_U \\
&= \sum_{r\geq 0} e_T\left[\lambda_j^{(r+1)}\right]\langle z, ((\mathcal{A}^* - \lambda_j)^r\phi)\rangle_{-\diamond,\diamond} \\
&= \left\langle z, e^{-T\mathcal{A}^*}\phi\right\rangle_{-\diamond,\diamond} = \left\langle e^{-T\mathcal{A}}z, \phi\right\rangle_{-\diamond,\diamond}.
\end{aligned} \tag{3.3}$$

This proves the claim.

Applying Leibniz formula [10, Proposition 7.13] and Lagrange theorem [10, Proposition 7.14] we obtain,

$$\left\|\Omega\left[\lambda_\bullet^{(\mu^l)}\right]\right\|_U = \left\|\sum_{q=1}^{l} e_T\left[\lambda_\bullet^{(\mu^l-\mu^{q-1})}\right]\ \widetilde{\Omega}\left[\lambda_\bullet^{(\mu^q)}\right]\right\|$$

$$\leq C_{p,\varrho,\eta}(1+T)^{|\alpha|}e^{-\lambda_1 T}\left(\sum_{q=1}^{l}\left\|\widetilde{\Omega}\left[\lambda_\bullet^{(\mu^q)}\right]\right\|^2\right)^{\frac{1}{2}}.$$

Thus,

$$F(\Omega) \leq C_{p,\varrho,\eta}(1+T)^{2|\alpha|}e^{-2\lambda_1 T}F(\widetilde{\Omega}) = C_{p,\varrho,\eta}(1+T)^{2|\alpha|}e^{-2\lambda_1 T}\mathcal{C}(G,z).$$

As $\Omega \in \mathcal{O}(G, e^{-T\mathcal{A}}z)$, this proves (3.1).

• The proof of (3.2) uses the same ingredients.

From (1.46), let $\Omega \in \mathcal{O}(G, e^{-T\mathcal{A}}z)$ be such that $F(\Omega) = \mathcal{C}(G, e^{-T\mathcal{A}}z)$. For any $j \in [\![1, g]\!]$ and any $l \in [\![0, \alpha_j - 1]\!]$, let

$$\widetilde{\Omega}_j^l := (e_{-T}\Omega)\left[\lambda_j^{(l+1)}\right].$$

As previously, applying Leibniz formula [10, Proposition 7.13] and Lagrange theorem [10, Proposition 7.14], we obtain

$$\left\|\widetilde{\Omega}\left[\lambda_\bullet^{(\mu^l)}\right]\right\|_U \leq C_{p,\varrho,\eta}(1+T)^{2|\alpha|}e^{\lambda_g T}\left(\sum_{q=1}^{l}\left\|\Omega\left[\lambda_\bullet^{(\mu^q)}\right]\right\|^2\right)^{\frac{1}{2}}.$$

The same computations as (3.3) give that $\Omega \in \mathcal{O}(G, z)$. Thus

$$\mathcal{C}(G,z) \leq F(\widetilde{\Omega}) \leq C_{p,\varrho,\eta}(1+T)^{2|\alpha|}e^{2\lambda_g T}F(\Omega) = C_{p,\varrho,\eta}(1+T)^{2|\alpha|}e^{2\lambda_g T}\mathcal{C}(G, e^{-T\mathcal{A}}z)$$

and (3.2) is proved. ∎

3.2. The minimal null control time.

This section is dedicated to the proof of Theorem 1.3 and Corollary 1.4.

Proof (of Theorem 1.3).

• We start with the proof of null controllability in time $T > T_0(y_0)$. Let $k \in \mathbb{N}^*$ and set $G_k := \{\lambda_1, \dots, \lambda_g\}$. Let $v_k \in L^2(0,T;U)$ be the solution of the block moment problem (1.39) associated with $z = e^{-T\mathcal{A}}y_0$ given by Theorem 1.2 *i.e.*

$$\int_0^T \left\langle v_k(t), \mathcal{B}^* e^{-t\mathcal{A}^*}\phi\right\rangle_U \mathrm{d}t = \left\langle e^{-T\mathcal{A}}y_0, \phi\right\rangle_{-\diamond,\diamond}, \qquad \forall \phi \in E_{\lambda_j},\ \forall j \in [\![1,g]\!],$$

$$\int_0^T v_k(t)t^l e^{-\lambda t}\mathrm{d}t = 0, \qquad \forall \lambda \in \Lambda\backslash G_k,\ \forall l \in [\![0, \eta-1]\!].$$

From (1.33), this implies that v_k solves (1.32). Thus, the only point left is to prove that the series (1.31) defining the control u converges in $L^2(0,T;U)$.

From Theorem 1.2 we have that

$$\|v_k\|^2_{L^2(0,T;U)} \leq \mathcal{E}(\lambda_1)\,\mathcal{K}(T)\,\mathcal{C}(G_k, e^{-T\mathcal{A}}y_0).$$

Using (3.1), up to a modification of the constants appearing in $\mathcal{E}$ and $\mathcal{K}$ (still depending on the same parameters), we obtain

$$\|v_k\|^2_{L^2(0,T;U)} \leq (1+T)^{2|\alpha|}\mathcal{E}(\lambda_1)\,\mathcal{K}(T)\,e^{-2(\min G_k)T}\,\mathcal{C}(G_k, y_0). \tag{3.4}$$

Using the definition of $T_0(y_0)$ given in (1.43) and the bound on $\mathcal{E}$ given in Theorem A.1, it comes that the series

$$\sum_{k\in\mathbb{N}^*} v_k(T-\bullet)$$

converges in $L^2(0,T;U)$ when $T > T_0(y_0)$ which proves null controllability of (1.1) from y_0 in any time $T > T_0(y_0)$.

• We now end the proof of Theorem 1.3 by proving that null controllability does not hold in time $T < T_0(y_0)$. The proof mainly relies on the optimality of the resolution of the block moment problems given in Theorem 1.2 (see (1.42)).

Let $T > 0$. Assume that problem (1.1) is null controllable from y_0 in time T. Thus there exists $u \in L^2(0,T;U)$ such that $y(T) = 0$ and

$$\|u\|_{L^2(0,T;U)} \leq C_T\,\|y_0\|_{-\diamond}\,.$$

Let $v := -u(T-\bullet)$. Then, for any $k \in \mathbb{N}^*$, v satisfies (1.39a) with $z = e^{-T\mathcal{A}}y_0$. From (1.42), this implies

$$C_T^2\,\|y_0\|^2_{-\diamond} \geq \|u\|^2_{L^2(0,T;U)} = \|v\|^2_{L^2(0,T;U)} \geq C_{p,\eta,\min\Lambda}\mathcal{C}(G_k, e^{-T\mathcal{A}}y_0). \tag{3.5}$$

Applying (3.2) and using that $e^{2(\max G_k)T} \leq e^{2\varrho T}e^{2(\min G_k)T}$ we obtain

$$\mathcal{C}(G_k, y_0) \leq C_{T,p,\varrho,\eta}e^{2(\min G_k)T}\mathcal{C}(G_k, e^{-T\mathcal{A}}y_0).$$

Together with (3.5) this implies

$$\mathcal{C}(G_k, y_0) \leq C_{T,p,\varrho,\eta,\min\Lambda}\,\|y_0\|^2_{-\diamond}\,e^{2(\min G_k)T}. \tag{3.6}$$

Getting back to the definition of $T_0(y_0)$ given in (1.43), this implies that $T \geq T_0(y_0)$ and ends the proof of Theorem 1.3. ∎

We end this subsection with the proof of Corollary 1.4.

Proof (of Corollary 1.4). By definition, we have $T_0(Y_0) = \sup_{y_0\in Y_0} T_0(y_0)$. Using the definition of $\mathcal{C}(G, Y_0)$ and Theorem 1.3, it directly comes that

$$T_0(Y_0) \leq \limsup_{k\to+\infty} \frac{\ln \mathcal{C}(G_k, Y_0)}{2\min G_k}.$$

We now focus on the converse inequality. Let

$$T < \limsup_{k\to+\infty} \frac{\ln \mathcal{C}(G_k, Y_0)}{2\min G_k}$$

and let us prove that $T \leq T_0(Y_0)$.

There exists $\varepsilon > 0$ and an increasing sequence of integers $(n_k)_{k\in\mathbb{N}^*}$ such that for any $k \in \mathbb{N}^*$, there exists $y_{0,k} \in Y_0$ with $\|y_{0,k}\|_{-\diamond} = 1$ satisfying

$$T + \varepsilon < \frac{\ln \mathcal{C}(G_{n_k}, y_{0,k})}{2 \min G_{n_k}}. \tag{3.7}$$

By contradiction, assume that for any $y_0 \in Y_0$, we have $T > T_0(y_0)$. Thus, from (3.6), there exists $C_{T,p,\varrho,\eta,\min\Lambda} > 0$ such that for any $k \in \mathbb{N}^*$

$$\frac{\ln \mathcal{C}(G_{n_k}, y_{0,k})}{2 \min G_{n_k}} \leq \frac{\ln C_{T,p,\varrho,\eta,\min\Lambda}}{2 \min G_{n_k}} + T.$$

Taking k sufficiently large, this is in contradiction with (3.7). ∎

3.3. On the cost of controllability.

A careful inspection of the proof of null controllability in time $T > T_0(y_0)$ detailed in Section 3.2 allows to give a bound on the cost of controllability. Indeed, the proof of Proposition 1.5 follows directly from (1.31) and (3.4) and will not be more detailed. Its consequences stated in Corollary 1.6 are proved below.

Proof (of Corollary 1.6). Let $y_0 \in X_{-\diamond}$ and $T > T_0(y_0)$. From Proposition 1.5 it comes that there exists $u \in L^2(0,T;U)$ such that $y(T) = 0$ and

$$\|u\|^2_{L^2(0,T;U)} \leq \mathcal{K}(T) \sum_{k\geq 1} (1+T)^{2|\alpha_{G_k}|} \mathcal{E}(\min G_k) e^{-2(\min G_k)T} \mathcal{C}(G_k, y_0),$$

The seeked estimate is only interesting when $T \to T_0(y_0)^+$ thus, as $|\alpha_{G_k}| \leq \eta p$, the term $(1+T)^{2|\alpha_{G_k}|}$ is bounded. To simplify the writing we set $\lambda_{k,1} := \min G_k$. Due to the assumption on Λ, it comes from Theorem A.1 that

$$\mathcal{E}(\lambda) \leq C \exp\left(C\lambda^a\right), \quad \forall \lambda \in \Lambda,$$

and

$$\mathcal{K}(T) \leq C \exp\left(\frac{C}{T^{\frac{a}{1-a}}}\right), \quad \forall T > 0,$$

where the constant C depends only on p, r, ϱ, η, a, $\bar{N}$, a' and $\widetilde{N}$. Thus, using the assumption (1.44), we obtain

$$\|u\|^2_{L^2(0,T;U)} \leq C \exp\left(\frac{C}{T^{\frac{a}{1-a}}}\right) \|y_0\|^2_{-\diamond} \sum_{k\geq 1} e^{-\lambda_{k,1}\left(T - T_0(y_0)\right)} e^{C\lambda_{k,1}^a - \lambda_{k,1}\left(T - T_0(y_0)\right)}. \tag{3.8}$$

The maximum of the function $x \mapsto Cx^a - x\big(T - T_0(y_0)\big)$ is given by

$$(Ca)^{\frac{1}{1-a}} \left(\frac{1}{a} - 1\right) \frac{1}{\big(T - T_0(y_0)\big)^{\frac{a}{1-a}}}.$$

Thus, up to a modification of C (still depending on the same parameters),

$$e^{C\lambda_{k,1}^a - \lambda_{k,1}\left(T - T_0(y_0)\right)} \leq \exp\left(\frac{C}{\big(T - T_0(y_0)\big)^{\frac{a}{1-a}}}\right), \quad \forall k \geq 1. \tag{3.9}$$

From [16, Propositions A.6.20 and A.6.21], it comes that

$$\sum_{k\geq 1} e^{-\lambda_{k,1}\left(T-T_0(y_0)\right)} \leq \frac{C_{a,\bar{N}}}{T-T_0(y_0)}. \tag{3.10}$$

Plugging (3.10) and (3.9) into (3.8) ends the proof of Corollary 1.6. ∎

4. Computation of the cost of a block.

In this section we prove more explicit formulas to estimate the cost $\mathcal{C}(G, y_0)$ of the resolution of a block moment problem depending on the assumptions on the eigenvalues in the group G. More precisely, we prove here Theorems 1.8 and 1.10. For pedagogical purpose, we start in Section 4.1 with Theorem 1.8 for algebraically (and geometrically) simple eigenvalues *i.e.* when $\alpha_\lambda = \gamma_\lambda = 1$ for any $\lambda \in G$. Then, in Section 4.2, we prove the general statement of Theorem 1.8 that is when all the eigenvalues in the group are geometrically simple *i.e.* $\gamma_\lambda = 1$ for any $\lambda \in G$.

The formula for the cost $\mathcal{C}(G, y_0)$ when all the eigenvalues in the group G are semi-simple (i.e. $\alpha_\lambda = 1$ for any $\lambda \in G$) stated in Theorem 1.10 is then proved in Section 4.3. The extension to spaces of initial conditions (1.48) and (1.52) does not depend on the matrix M and follows directly from Lemma B.1. Thus, their proofs are not detailed here.

When both algebraic and geometric multiplicities appear in the same group we do not get a general formula but describe the procedure on an example in Section 4.4.

Recall that from (1.46), computing $\mathcal{C}(G, y_0)$ is a finite dimensional optimization problem given by

$$\mathcal{C}(G, y_0) = \min\left\{F(\Omega)\,;\, \Omega \in \mathcal{O}(G, y_0) \cap U_G^{|\alpha|}\right\}$$

where the function F is defined in (1.38), the constraints associated with $\mathcal{O}(G, y_0)$ are defined in (1.36) and U_G is defined in (1.45).

4.1. The case of simple eigenvalues.

In all this section, we consider the simpler case where $\alpha_\lambda = \gamma_\lambda = 1$ for every $\lambda \in G$. Thus, in the rest of this section, we drop the superscript 0 associated to eigenvectors.

We start with the proof of the invertibility of the matrix M stated in Proposition 1.7.

Proof. Recall that, as $\alpha_\lambda = \gamma_\lambda = 1$, the positive semi-definite matrix M is defined in (1.49). Let $\tau \in \mathbb{R}^g$ be such that $\langle M\tau, \tau\rangle = 0$. Then, for each $l \in [\![1, g]\!]$, we have

$$\left\langle \Gamma^l \tau, \tau \right\rangle = 0.$$

We prove that $\tau = 0$. By contradiction let

$$l = \max\{j \in [\![1, g]\!]\,;\, \tau_j \neq 0\}.$$

Then from (1.49) this leads to $\left\langle \Gamma^l \tau, \tau \right\rangle = \|b[\lambda_l]\|_U^2\, \tau_l^2$. Using (1.28) implies $\tau_l = 0$. This is in contradiction with the definition of l which proves the invertibility of M. ∎

We now prove Theorem 1.8.

Proof. First of all, notice that the function F to minimize reduces to

$$F(\Omega) = \sum_{j=1}^{g} \|\Omega[\lambda_1, \dots, \lambda_j]\|^2$$

and, as $\gamma_\lambda = \alpha_\lambda = 1$, the constraints defining the set $\mathcal{O}(\lambda_j, y_0)$ reduce to

$$\langle \Omega_j, b_j \rangle_U = \langle y_0, \phi_j \rangle_{-\diamond,\diamond}.$$

Thus, the minimization problem reduces to

$$\mathcal{C}(G, y_0) = \min \Big\{ F(\Omega)\,;\, \Omega = (\Omega_1, \dots, \Omega_g) \in U_G^g \text{ such that } \langle \Omega_j, b_j \rangle_U = \langle y_0, \phi_j \rangle_{-\diamond,\diamond},\ \forall j \in [\![1, g]\!] \Big\}. \tag{4.1}$$

For the sake of generality, let us consider for this proof any $\omega_1, \dots, \omega_g \in \mathbb{R}$ and the more general constraints

$$\langle \Omega_j, b_j \rangle_U = \omega_j, \qquad \forall j \in [\![1, g]\!]. \tag{4.2}$$

Using the formalism of divided differences, this is equivalent to the family of constraints

$$\langle \Omega, b \rangle_U [\lambda_1, \dots, \lambda_j] = \omega[\lambda_1, \dots, \lambda_j], \qquad \forall j \in [\![1, g]\!]. \tag{4.3}$$

Denote by $(m_j)_{j \in [\![1,g]\!]}$ the Lagrange multipliers associated with the minimization problem

$$\min \{ F(\Omega)\,;\, \Omega = (\Omega_1, \dots, \Omega_g) \in U_G^g \text{ such that (4.3) holds} \}.$$

Then, we obtain that the minimum satisfies

$$\sum_{j=1}^{g} \langle \Omega[\lambda_1, \dots, \lambda_j], H[\lambda_1, \dots, \lambda_j] \rangle_U = \sum_{j=1}^{g} m_j \langle H, b \rangle_U [\lambda_1, \dots, \lambda_j], \tag{4.4}$$

for any $H_1, \dots, H_g \in U_G$.

Then, for a given $q \in [\![1, g]\!]$, using Leibniz formula [10, Proposition 7.7], the constraints (4.3) can be rewritten as

$$\omega[\lambda_1, \dots, \lambda_q] = \langle \Omega, b \rangle_U [\lambda_1, \dots, \lambda_q] = \sum_{j=1}^{q} \langle \Omega[\lambda_1, \dots, \lambda_j], b[\lambda_j, \dots, \lambda_q] \rangle_U \tag{4.5}$$

To relate (4.5) and (4.4), we set $H_1, \dots, H_g \in U_G$ such that

$$H[\lambda_1, \dots, \lambda_j] = \begin{cases} b[\lambda_j, \dots, \lambda_q], & \text{for } j \le q, \\ 0, & \text{for } j > q. \end{cases}$$

This can be done defining $H_1 = b[\lambda_1, \dots, \lambda_q]$ and, from the interpolation formula [10, Proposition 7.6], defining H_j by the formula

$$H_j = \sum_{i=1}^{j} \left(\prod_{k=1}^{j-1} (\lambda_i - \lambda_k) \right) H[\lambda_1, \dots, \lambda_i], \quad \forall j \in [\![2, g]\!].$$

Then, from (4.5) we obtain

$$\omega[\lambda_1,\dots,\lambda_q] = \sum_{j=1}^{g} \langle \Omega[\lambda_1,\dots,\lambda_j], H[\lambda_1,\dots,\lambda_j]\rangle_U .$$

Now relation (4.4) leads to

$$\omega[\lambda_1,\dots,\lambda_q] = \sum_{j=1}^{g} m_j \langle H, b\rangle_U [\lambda_1,\dots,\lambda_j].$$

The application of Leibniz formula [10, Proposition 7.7] yields

$$\begin{aligned}\omega[\lambda_1,\dots,\lambda_q] &= \sum_{j=1}^{g} m_j \left(\sum_{l=1}^{j} \langle H[\lambda_1,\dots,\lambda_l], b[\lambda_l,\dots,\lambda_j]\rangle_U \right) \\ &= \sum_{j=1}^{g} m_j \left(\sum_{l=1}^{\min(j,q)} \langle b[\lambda_l,\dots,\lambda_q], b[\lambda_l,\dots,\lambda_j]\rangle_U \right) \\ &= \sum_{l=1}^{g}\sum_{j=1}^{g} m_j \Gamma^l_{q,j} = (Mm)_q\end{aligned}$$

where Γ^l and M are defined in (1.49).

Let

$$\xi := \begin{pmatrix} \omega[\lambda_1] \\ \vdots \\ \omega[\lambda_1,\dots,\lambda_g] \end{pmatrix}.$$

As $m = M^{-1}\xi$, getting back to (4.4) with $H = \Omega$ together with the constraints (4.3), we obtain

$$F(\Omega) = \sum_{j=1}^{g} m_j \langle \Omega, b\rangle_U [\lambda_1,\dots,\lambda_j] = \sum_{j=1}^{g} (M^{-1}\xi)_j \xi_j = \langle M^{-1}\xi, \xi\rangle.$$

With the specific choice, $\omega_j = \langle y_0, \phi_j\rangle_{-\diamond,\diamond}$, this ends the proof of Theorem 1.8 with the extra assumption that $\alpha_\lambda = 1$ for all $\lambda \in G$. ∎

REMARK 4.1. *As mentioned in Remark 1.9, estimate (4.1) implies that the cost of the block G (i.e. the quantity $\langle M^{-1}\xi, \xi\rangle$) can be estimated using any eigenvectors: there is no normalization condition.*

REMARK 4.2. *Rewriting the constraints in the form (4.3) is not mandatory but, as the function to minimize involves divided differences, it leads to more exploitable formulas and will ease the writing when dealing with algebraic multiplicity of eigenvalues. Dealing directly with (4.2) would lead to the expression (D.9) for the cost of the block G as it will appear in the proof of Theorem 1.10.*

4.2. The case of geometrically simple eigenvalues.

The proof of Proposition 1.7 and Theorem 1.8 under the sole assumption $\gamma_\lambda = 1$ for any $\lambda \in G$ follows closely the proof done in Section 4.1. The main difference is the use of generalized divided differences (see [10, Section 7.3]) instead of classical divided differences as detailed below.

Proof (of Proposition 1.7). Due to (1.37), for any $l \in [\![1, |\alpha|]\!]$ the multi-index $\mu^l - \mu^{l-1}$ is composed of only one 1 and $g-1$ zeros. Thus,

$$b\left[\lambda_\bullet^{(\mu^l-\mu^{l-1})}\right] = b_j^0$$

for a certain $j \in [\![1, g]\!]$. From (1.28) it comes that

$$b\left[\lambda_\bullet^{(\mu^l-\mu^{l-1})}\right] \neq 0, \qquad \forall l \in [\![1, |\alpha|]\!].$$

The rest of the proof follows as in Section 4.1. ∎

Proof (of Theorem 1.8). As $\gamma_\lambda = 1$, the constraints defining the set $\mathcal{O}(\lambda_j, y_0)$ reduce to

$$\begin{aligned}\sum_{r=0}^{l} \left\langle \Omega_j^r, b_j^{l-r} \right\rangle_U &= \sum_{r=0}^{l} \left\langle \Omega_j^r, \mathcal{B}^*(\mathcal{A}^* - \lambda_j)^r \phi_j^l \right\rangle_U \\ &= \left\langle y_0, \phi_j^l \right\rangle_{-\diamond,\diamond}, \quad \forall l \in [\![0, \alpha_j - 1]\!].\end{aligned}$$

Using Leibniz formula [10, Proposition 7.13] this is equivalent to

$$\langle \Omega, b \rangle_U \left[\lambda_j^{(l+1)}\right] = \left\langle y_0, \phi_j^l \right\rangle_{-\diamond,\diamond}, \quad \forall l \in [\![0, \alpha_j - 1]\!].$$

Thus,

$$\begin{aligned}(4.6)\quad \mathcal{C}(G, y_0) = \min \Big\{ & F(\Omega)\,;\, \Omega = (\Omega_1^0, \dots, \Omega_1^{\alpha_1-1}, \dots, \Omega_g^0, \dots, \Omega_g^{\alpha_g-1}) \in U_G^{|\alpha|} \\ & \text{such that } \langle \Omega, b \rangle_U \left[\lambda_j^{(l+1)}\right] = \left\langle y_0, \phi_j^l \right\rangle_{-\diamond,\diamond}, \quad \forall j \in [\![1, g]\!],\ \forall l \in [\![0, \alpha_j - 1]\!] \Big\}.\end{aligned}$$

For the sake of generality, let us consider for this proof any

$$\left(\omega_1^0, \dots, \omega_1^{\alpha_1-1}, \dots, \omega_g^0, \dots, \omega_g^{\alpha_g-1}\right) \in \mathbb{R}^{|\alpha|}$$

and the more general constraints

$$\langle \Omega, b \rangle_U \left[\lambda_j^{(l+1)}\right] = \omega_j^l, \quad \forall j \in [\![1, g]\!],\ \forall l \in [\![0, \alpha_j - 1]\!].$$

From (1.37), this is equivalent to the family of constraints

$$\langle \Omega, b \rangle_U \left[\lambda_\bullet^{(\mu^p)}\right] = \omega \left[\lambda_\bullet^{(\mu^p)}\right], \quad \forall p \in [\![1, |\alpha|]\!],$$

and we proceed as in Section 4.1. The only difference is the use of generalized divided differences. For instance, the equation (4.4) defining the Lagrange multipliers now reads

$$\sum_{l=1}^{|\alpha|} \langle \Omega[\lambda^{(\mu^l)}], H[\lambda^{(\mu^l)}] \rangle = \sum_{l=1}^{|\alpha|} m_l \langle H, b \rangle [\lambda^{(\mu^l)}], \qquad \forall H = (H_j^l) \in U_G^{|\alpha|},$$

and the Leibniz formula [10, Proposition 7.7] is replaced by its generalization [10, Proposition 7.13]. The rest of the proof remains unchanged. ∎

REMARK 4.3. *As mentioned in Remark 1.9, estimate (4.6) implies that the cost of the block G (i.e. the quantity $\left\langle M^{-1}\xi, \xi \right\rangle$) can be estimated using any eigenvectors and any associated Jordan chains.*

4.3. The case of semi-simple eigenvalues.

We start with the proof of Proposition 1.9.

Proof (of Proposition 1.9). Recall that the positive semi-definite matrix M is defined in (1.51). Let $\tau \in \mathbb{R}^{\gamma_G}$ be such that $\langle M\tau, \tau\rangle = 0$. Then, for any $l \in [\![1, g]\!]$, $\langle \Gamma^l \tau, \tau\rangle = 0$. We prove that $\tau = 0$. By contradiction let

$$\tilde{l} = \max\{j \in [\![1, \gamma_G]\!] \, ; \, \tau_j \neq 0\}$$

and $l \in [\![1, g]\!]$ be such that

$$\gamma_1 + \cdots + \gamma_{l-1} < \tilde{l} \leq \gamma_1 + \cdots + \gamma_l$$

with the convention that $l = 1$ when $\tilde{l} \leq \gamma_1$. We denote by $\sigma \in \mathbb{R}^{\gamma_l}$ the l^{th} block of τ *i.e.*

$$\sigma = \begin{pmatrix} \tau_{\gamma_1 + \cdots + \gamma_{l-1} + 1} \\ \vdots \\ \tau_{\gamma_1 + \cdots + \gamma_l} \end{pmatrix}.$$

From (1.50) we have $\delta_l^i = 0$ when $i < l$. Thus all the blocks (i, j) of Γ^l are equal to 0 when $i, j \in [\![1, l-1]\!]$. This leads to

$$\langle \Gamma^l \tau, \tau\rangle = \left\langle \left(\delta_l^l\right)^2 \mathrm{Gram}_U\left(b_{l,1}, \ldots, b_{l,\gamma_l}\right) \sigma, \sigma \right\rangle.$$

As the eigenvalues $\lambda_1, \ldots, \lambda_g$ are distinct it comes that $\delta_l^l \neq 0$ (see (1.50)) which implies

$$\langle \mathrm{Gram}_U\left(b_{l,1}, \ldots, b_{l,\gamma_l}\right) \sigma, \sigma\rangle = 0.$$

From (1.28), we have that $b_{l,1}, \ldots, b_{l,\gamma_l}$ are linearly independent. This proves the invertibility of $\mathrm{Gram}_U\left(b_{l,1}, \ldots, b_{l,\gamma_l}\right)$ and gives $\sigma = 0$. This is in contradiction with the definition of $\tilde{l}$ which proves the invertibility of M. ∎

We now turn to the proof of Theorem 1.10

Proof (of Theorem 1.10). First of all, notice that the function F to minimize reduces to

$$F(\Omega) = \sum_{j=1}^{g} \|\Omega[\lambda_1, \ldots, \lambda_j]\|^2$$

and, as $\alpha_\lambda = 1$, the constraints defining the set $\mathcal{O}(\lambda_j, y_0)$ reduce to

$$\langle \Omega_j, \mathcal{B}^*\phi\rangle_U = \langle y_0, \phi\rangle_{-\diamond,\diamond}, \qquad \forall \phi \in \mathrm{Ker}(\mathcal{A}^* - \lambda_j).$$

To simplify the writing, let us consider the maps

$$B_j := \begin{pmatrix} \langle \mathcal{B}^*\phi_{j,1}, \bullet\rangle_U \\ \vdots \\ \langle \mathcal{B}^*\phi_{j,\gamma_j}, \bullet\rangle_U \end{pmatrix} \in \mathcal{L}(U, \mathbb{R}^{\gamma_j}).$$

Then the constraints defining $\mathcal{O}(\lambda_j, y_0)$ can be rewritten as the equality

$$B_j\Omega_j = \begin{pmatrix} \langle y_0, \phi_{j,1}\rangle_{-\diamond,\diamond} \\ \vdots \\ \langle y_0, \phi_{j,\gamma_j}\rangle_{-\diamond,\diamond} \end{pmatrix}. \tag{4.7}$$

Thus,

$$\mathcal{C}(G, y_0) = \min \Big\{ F(\Omega)\,;\ \Omega = (\Omega_1, \dots, \Omega_g) \in U_G^g \text{ such that (4.7) holds for any } j \in [\![1, g]\!] \Big\}. \tag{4.8}$$

For the sake of generality, let us consider for this proof any

$$(\omega_{1,1}, \dots, \omega_{1,\gamma_1}, \dots, \omega_{g,1}, \dots, \omega_{g,\gamma_g}) \in \mathbb{R}^{\gamma_G}$$

and the more general constraints

$$B_j \Omega_j = \omega_j, \qquad \forall j \in [\![1, g]\!],$$

where ω_j denotes $\begin{pmatrix} \omega_{j,1} \\ \vdots \\ \omega_{j,\gamma_j} \end{pmatrix} \in \mathbb{R}^{\gamma_j}$.

As the ω_j's have different sizes we avoid in this proof the use of divided differences to rewrite the constraints. This is why we end up with the formula (1.51) rather than an adaptation of (1.49) (see also the discussion in Remark 4.2).

Denoting by $m_j \in \mathbb{R}^{\gamma_j}$ the Lagrange multipliers we obtain that the minimum satisfies

$$\sum_{j=1}^{g} \langle \Omega[\lambda_1, \dots, \lambda_j], H[\lambda_1, \dots, \lambda_j] \rangle_U = \sum_{j=1}^{g} \langle m_j, B_j H_j \rangle\,, \quad \forall H_1, \dots, H_g \in U_G. \tag{4.9}$$

Recall that in (1.50) we defined

$$\delta_j^i = \prod_{l \in [\![1, j-1]\!]} (\lambda_i - \lambda_l), \qquad \forall j \in [\![2, g]\!].$$

Then, from the interpolation formula [10, Proposition 7.6], we obtain that

$$\Omega_i = \sum_{l=1}^{i} \delta_l^i \Omega[\lambda_1, \dots, \lambda_l]. \tag{4.10}$$

For any $H \in U_G$ and $i \in [\![1, g]\!]$, let us design $H_1^{(i)}, \dots, H_g^{(i)} \in U_G$ such that

$$H^{(i)}[\lambda_1, \dots, \lambda_l] = \delta_l^i H, \quad \forall l \in [\![1, i]\!]. \tag{4.11}$$

We set $H_1^{(i)} = H$ and, using the interpolation formula [10, Proposition 7.6], we define recursively

$$H_j^{(i)} = \sum_{l=1}^{j} \delta_l^j H^{(i)}[\lambda_1, \dots, \lambda_l] = \left(\sum_{l=1}^{j} \delta_l^i \delta_l^j \right) H = \theta_j^{(i)} H$$

with

$$\theta_j^{(i)} := \sum_{l=1}^{g} \delta_l^i \delta_l^j = \sum_{l=1}^{\min(i,j)} \delta_l^i \delta_l^j. \tag{4.12}$$

This ensures (4.11). Taking into account (4.10), plugging $H_j^{(i)}$ in (4.9) leads to

$$\Omega_i = \sum_{j=1}^{g} \theta_j^{(i)} B_j^* m_j. \tag{4.13}$$

Together with (4.7), using (4.12), we obtain that

$$\begin{aligned}
\omega_i &= \sum_{j=1}^{g} \theta_j^{(i)} B_i B_j^* m_j \\
&= \sum_{l=1}^{g} \sum_{j=1}^{g} \left(\delta_l^i B_i\right) \left(\delta_l^j B_j\right)^* m_j \\
&= (Mm)_i
\end{aligned}$$

where M is defined in (1.51) and $(Mm)_i \in \mathbb{R}^{\gamma_i}$ denotes the i^{th} block of $Mm \in \mathbb{R}^{\gamma_G}$.

Finally, if we set

$$\xi := \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_g \end{pmatrix} \in \mathbb{R}^{\gamma_G}.$$

we have proved that the Lagrange multipliers are given by $m = M^{-1}\xi$. Applying (4.9) with $H_j = \Omega_j$ and using the constraints (4.7) leads to

$$F(\Omega) = \sum_{j=1}^{g} \|\Omega[\lambda_1, \dots, \lambda_j]\|^2 = \sum_{j=1}^{g} \left\langle (M^{-1}\xi)_j, \xi_j \right\rangle = \left\langle M^{-1}\xi, \xi \right\rangle,$$

which proves the claim. ∎

REMARK 4.4. *From (4.13) and $m = M^{-1}\xi$ it comes that*

$$\mathcal{C}(G, y_0) = \min \Big\{ F(\Omega)\,;\, \Omega = (\Omega_1, \dots, \Omega_g) \in U_G^g \text{ such that (4.7) holds for any } j \in [\![1, g]\!] \Big\}$$

is attained for

$$\Omega_i = \sum_{j=1}^{g} \left(\sum_{l=1}^{g} \delta_l^i \delta_l^j \right) B_j^* (M^{-1}\xi)_j.$$

REMARK 4.5. *As mentioned in Remark 1.9, estimate (4.8) implies that the cost of the block G (i.e. the quantity $\left\langle M^{-1}\xi, \xi \right\rangle$) can be estimated using any basis of eigenvectors.*

4.4. Dealing simultaneously with algebraic and geometric multiplicities.

The proof of Theorem 1.8 strongly relies on the use of divided differences to rewrite the constraints whereas the proof of Theorem 1.10 is based on the vectorial writing of the constraints through the operators $B_j \in \mathcal{L}(U; \mathbb{R}^{\gamma_j})$. As the target spaces of these operators do not have the same dimension, one cannot directly compute divided

differences. Thus, the setting we developed to compute the cost of a given block does not lead to a general formula when both kind of multiplicities need to be taken into account in the same group. However, for actual problems, the computation of this cost is a finite dimensional constrained optimization problem which can be explicitly solved.

Let us give an example of such a group that does not fit into Theorem 1.8 nor into Theorem 1.10 but for which we manage to compute the cost '*by hand*'.

We consider a group $G = \{\lambda_1, \lambda_2\}$ such that $\gamma_{\lambda_1} = \alpha_{\lambda_1} = 2$ and $\gamma_{\lambda_2} = \alpha_{\lambda_2} = 1$. Let $(\phi^0_{1,1}, \phi^0_{1,2})$ be a basis of $\mathrm{Ker}(\mathcal{A}^* - \lambda_1)$ and $\phi^0_{2,1}$ be an eigenvector of $\mathcal{A}^*$ associated to the eigenvalue λ_2. Assume that the generalized eigenvector $\phi^1_{1,1}$ is such that

$$(\mathcal{A}^* - \lambda_1)\,\phi^1_{1,1} = \phi^0_{1,1},$$

and that $(\phi^0_{1,1}, \phi^1_{1,1}, \phi^0_{1,2})$ forms a basis of $\mathrm{Ker}(\mathcal{A}^* - \lambda_1)^2$.

For this group, in the same spirit as in Theorems 1.8 and 1.10, we obtain the following result.

PROPOSITION 4.1. *For any $y_0 \in X_{-\diamond}$, we have*

$$\mathcal{C}(G, y_0) = \left\langle M^{-1}\xi, \xi \right\rangle \qquad \textit{where} \qquad \xi = \begin{pmatrix} \left\langle y_0, \phi^0_{1,1} \right\rangle_{-\diamond,\diamond} \\ \left\langle y_0, \phi^0_{1,2} \right\rangle_{-\diamond,\diamond} \\ \left\langle y_0, \phi^1_{1,1} \right\rangle_{-\diamond,\diamond} \\ \left\langle y_0, \phi^0_{2,1} \right\rangle_{-\diamond,\diamond} \end{pmatrix}$$

and M is the invertible matrix defined by

$$\begin{aligned} M =& \mathrm{Gram}_U\big(b^0_{1,1},\, b^0_{1,2},\, b^1_{1,1},\, b^0_{2,1}\big) \\ &+ \mathrm{Gram}_U\big(0,\, 0,\, b^0_{1,1},\, \delta b^0_{2,1}\big) \\ &+ \mathrm{Gram}_U\big(0,\, 0,\, 0,\, \delta^2 b^0_{2,1}\big) \end{aligned}$$

with $\delta = \lambda_2 - \lambda_1$.

Proof. Let

$$\big(\omega^0_{1,1}, \omega^0_{1,2}, \omega^1_{1,1}, \omega^0_{2,1}\big)^t \in \mathbb{R}^4.$$

As in the proofs of Theorems 1.8 and 1.10, the goal is to compute the minimum of the function

$$F : \big(\Omega^0_1, \Omega^1_1, \Omega^0_2\big) \in U_G^3 \mapsto \|\Omega^0_1\|^2 + \|\Omega^1_1\|^2 + \|\Omega[\lambda_1, \lambda_1, \lambda_2]\|^2,$$

under the 4 constraints

$$\begin{aligned} &\left\langle \Omega^0_j, b^0_{j,i} \right\rangle_U = \omega^0_{j,i}, \quad \forall i \in [\![1, \gamma_j]\!],\ \forall j \in [\![1, 2]\!], \\ &\left\langle \Omega^0_1, b^1_{1,1} \right\rangle_U + \left\langle \Omega^1_1, b^0_{1,1} \right\rangle_U = \omega^1_{1,1}. \end{aligned}$$

Then, the Lagrange multipliers $m^0_{1,1}, m^0_{1,2}, m^1_{1,1}$ and $m^0_{2,1}$ satisfy the equations

$$\begin{aligned} &\left\langle \Omega^0_1, H^0_1 \right\rangle_U + \left\langle \Omega^1_1, H^1_1 \right\rangle_U + \langle \Omega[\lambda_1, \lambda_1, \lambda_2], H[\lambda_1, \lambda_1, \lambda_2] \rangle_U = m^0_{1,1} \left\langle H^0_1, b^0_{1,1} \right\rangle_U \\ &+ m^0_{1,2} \left\langle H^0_1, b^0_{1,2} \right\rangle_U + m^1_{1,1} \left(\left\langle H^0_1, b^1_{1,1} \right\rangle_U + \left\langle H^1_1, b^0_{1,1} \right\rangle_U \right) + m^0_{2,1} \left\langle H^0_2, b^0_{2,1} \right\rangle_U, \end{aligned} \tag{4.14}$$

for every H_1^0, H_1^1, $H_2^0 \in U_G$. Considering successively

$$H_1^0 = b_{1,1}^0, \quad H_1^1 = 0, \quad H_2^0 = b_{1,1}^0,$$

$$H_1^0 = b_{1,2}^0, \quad H_1^1 = 0, \quad H_2^0 = b_{1,2}^0,$$

$$H_1^0 = b_{1,1}^1, \quad H_1^1 = b_{1,1}^0, \quad H_2^0 = b_{1,1}^1 + (\lambda_2 - \lambda_1) b_{1,1}^0,$$

and

$$H_1^0 = b_{2,1}^0, \quad H_1^1 = (\lambda_2 - \lambda_1) b_{2,1}^0, \quad H_2^0 = \left(1 + (\lambda_2 - \lambda_1)^2 + (\lambda_2 - \lambda_1)^4\right) b_{2,1}^0,$$

and plugging it into (4.14), we obtain that $\begin{pmatrix} \omega_{1,1}^0 \\ \omega_{1,2}^0 \\ \omega_{1,1}^1 \\ \omega_{2,1}^0 \end{pmatrix} = M \begin{pmatrix} m_{1,1}^0 \\ m_{1,2}^0 \\ m_{1,1}^1 \\ m_{2,1}^0 \end{pmatrix}$ which ends the proof. ∎

5. Application to the study of null controllability of academic examples.

In this section we provide examples to illustrate how to use the formulas obtained in Theorems 1.3, 1.8 and 1.10 in order to compute the minimal null control time.

We start with academic examples for which computations are simpler. Then, in Section 6, we study coupled systems of actual partial differential equations of parabolic type.

5.1. Setting and notations.

Let A be the unbounded Sturm-Liouville operator defined in $L^2(0,1;\mathbb{R})$ by

$$D(A) = H^2(0,1;\mathbb{R}) \cap H_0^1(0,1;\mathbb{R}), \qquad A\bullet = -\partial_x\big(\gamma \partial_x \bullet\big) + c\bullet, \tag{5.1}$$

with $c \in L^\infty(0,1;\mathbb{R})$ satisfying $c \geq 0$ and $\gamma \in C^1([0,1];\mathbb{R})$ satisfying $\inf\limits_{[0,1]} \gamma > 0$.

The operator A admits an increasing sequence of eigenvalues denoted by $(\nu_k)_{k\in\mathbb{N}^*}$. The associated normalized eigenvectors $(\varphi_k)_{k\in\mathbb{N}^*}$ form an Hilbert basis of $L^2(0,1;\mathbb{R})$.

Remark 5.1. *The assumption $c \geq 0$ ensures that for any $k \geq 1$, the eigenvalues satisfies $\nu_k > 0$. From Remark 1.3, the controllability results proved in the present article still hold when the function c is bounded from below.*

To lighten the notations, for any $I \subset (0,1)$ we set $\| \bullet \|_I = \| \bullet \|_{L^2(I)}$.

Let $f : \mathrm{Sp}(A) \to \mathbb{R}$ be a bounded function. Associated to this function we consider the operator $f(A)$ defined on $D(A)$ by the spectral theorem by

$$f(A) = \sum_{k\geq 1} f(\nu_k) \left\langle \bullet, \varphi_k \right\rangle_{L^2(0,1;\mathbb{R})} \varphi_k. \tag{5.2}$$

5.2. Spectral properties of Sturm-Liouville operators.

Let A be the Sturm-Liouville operator defined in (5.1). All the examples studied in this article are based on this operator. We recall here some spectral properties that will be used in our study.

From [2, Theorem 1.1], there exists $\varrho > 0$ such that

$$\varrho < \nu_{k+1} - \nu_k, \qquad \forall k \geq 1, \tag{5.3}$$

$$\inf_{k\geq 1} \frac{|\varphi_k'(x)|}{\sqrt{k}} > 0, \quad \forall x \in \{0,1\}, \tag{5.4}$$

and, for any non-empty open set $\omega \subset (0,1)$,

$$\inf_{k\geq 1} \|\varphi_k\|_\omega > 0. \tag{5.5}$$

Moreover, using [16, Theorem IV.1.3], we have

$$N_{(\nu_k)_k}(r) \leq C\sqrt{r}, \quad \forall r > 0. \tag{5.6}$$

To estimate various quantities, we will make an intensive use of the following lemma proved in [2, Lemma 2.3].

LEMMA 5.1. *Let A be the Sturm-Liouville operator defined in (5.1) and let $\lambda_0 > 0$. There exists $C > 0$ depending on γ, c and λ_0 such that, for any $\lambda \geq \lambda_0$, for any $F \in L^2(0,1;\mathbb{R})$, for any $x, y \in [0,1]$, for any u satisfying*

$$(A - \lambda)u = F \quad \text{ on } [x,y],$$

we have

$$|u(x)|^2 + \frac{\gamma(x)}{\lambda}|u'(x)|^2 \leq C\left(|u(y)|^2 + \frac{\gamma(y)}{\lambda}|u'(y)|^2 + \frac{1}{\lambda}\left|\int_x^y |F(s)|\mathrm{d}s\right|^2\right).$$

Applying Lemma 5.1 with $u = \varphi_k$, $F = 0$, $\lambda = \nu_k$ and integrating with respect to the variable $y \in (0,1)$ we obtain

$$|\varphi_k(x)|^2 + \frac{1}{\nu_k}|\varphi_k'(x)|^2 \leq C\left(1 + \frac{1}{\nu_k}\int_0^1 \gamma(y)|\varphi_k'(y)|^2\mathrm{d}y\right), \qquad \forall x \in (0,1),\ \forall k \geq 1.$$

Integrating by parts leads to

$$\int_0^1 \gamma(y)|\varphi_k'(y)|^2\mathrm{d}y = \int_0^1 (\nu_k - c(y))\varphi_k(y)^2\mathrm{d}y \leq \nu_k + \|c\|_\infty$$

which yields the existence of $C > 0$ such that

$$|\varphi_k(x)|^2 + \frac{1}{\nu_k}|\varphi_k'(x)|^2 \leq C, \qquad \forall x \in (0,1),\ \forall k \geq 1. \tag{5.7}$$

5.3. Perturbation of a 2x2 Jordan block.

Let $\omega \subset (0,1)$ be a non-empty open set and $U = L^2(\Omega)$. Let A be the Sturm-Liouville operator defined in (5.1) and $f(A)$ be the operator defined in (5.2) with $f : \mathrm{Sp}(A) \to \mathbb{R}$ satisfying

$$|f(\nu_k)| < \frac{\varrho}{2}, \qquad \forall k \geq 1.$$

We consider the operator $\mathcal{A}$ on $X = L^2(0,1;\mathbb{R})^2$ defined by

$$\mathcal{A} = \begin{pmatrix} A & I \\ 0 & A + f(A) \end{pmatrix}, \quad D(\mathcal{A}) = D(A) \times D(A), \tag{5.8}$$

and

$$\mathcal{B} : u \in U \mapsto \begin{pmatrix} 0 \\ \mathbf{1}_\omega u \end{pmatrix}. \tag{5.9}$$

Then,

$$\mathcal{B}^* : \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \in X \mapsto \mathbf{1}_\omega \varphi_2.$$

It is easy to see that $(-\mathcal{A}, D(\mathcal{A}))$ generates a $\mathcal{C}_0$-semigroup on X and that $\mathcal{B} : U \to X$ is bounded. Thus we consider for this example that $X^*_\diamond = X = X_{-\diamond}$.

PROPOSITION 5.2. *Let us consider the control system (1.1) with $\mathcal{A}$ and $\mathcal{B}$ given by (5.8)-(5.9). Then, null-controllability from $X_{-\diamond}$ holds in any time i.e. $T_0(X_{-\diamond}) = 0$.*

Proof. The spectrum of $(\mathcal{A}^*, D(\mathcal{A}))$ is given by

$$\Lambda = \{\nu_k \,;\, k \geq 1\} \cup \{\nu_k + f(\nu_k) \,;\, k \geq 1\}.$$

From (5.3) and (5.6) it comes that there exists $\bar{N} > 0$ such that $\Lambda \in \mathcal{L}\left(2, \frac{\varrho}{2}, \frac{1}{2}, \bar{N}\right)$.

An associated grouping is given by

$$\begin{cases} G_k := \{\lambda_{k,1} := \nu_k,\ \lambda_{k,2} := \nu_k + f(\nu_k)\}, & \text{if } f(\nu_k) \neq 0, \\ G_k := \{\lambda_{k,1} := \nu_k\}, & \text{if } f(\nu_k) = 0. \end{cases}$$

If $f(\nu_k) \neq 0$ the eigenvalues $\lambda_{k,1}$ and $\lambda_{k,2}$ are simple and we consider the associated eigenfunctions

$$\phi^0_{k,1} = \begin{pmatrix} -f(\nu_k) \\ 1 \end{pmatrix} \varphi_k, \qquad \phi^0_{k,2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varphi_k.$$

If $f(\nu_k) = 0$ the eigenvalue $\lambda_{k,1}$ is algebraically double and we consider the associated Jordan chain

$$\phi^0_{k,1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varphi_k, \qquad \phi^1_{k,1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \varphi_k.$$

From (5.5) it comes that (1.28) and (1.29) are satisfied. Thus, from Theorem 1.3, we obtain that for any $y_0 \in X_{-\diamond}$,

$$T_0(y_0) = \limsup_{k \to +\infty} \frac{\ln \mathcal{C}(G_k, y_0)}{2 \min G_k}.$$

Let us now conclude by estimating $\mathcal{C}(G_k, y_0)$.

- Consider first that $f(\nu_k) \neq 0$. Then, $\phi[\lambda_{k,1}, \lambda_{k,2}] = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \varphi_k$ and

$$b[\lambda_{k,1}, \lambda_{k,2}] = \mathcal{B}^* \phi[\lambda_{k,1}, \lambda_{k,2}] = \frac{\mathbf{1}_\omega \varphi_k - \mathbf{1}_\omega \varphi_k}{f(\nu_k)} = 0.$$

From Theorem 1.8 it comes that

$$\mathcal{C}(G_k, y_0) = \left\langle M^{-1} \xi, \xi \right\rangle$$

with

$$M = \mathrm{Gram}(b[\lambda_{k,1}], b[\lambda_{k,1}, \lambda_{k,2}]) + \mathrm{Gram}(0, b[\lambda_{k,2}]) = \begin{pmatrix} \|\varphi_k\|_\omega^2 & 0 \\ 0 & \|\varphi_k\|_\omega^2 \end{pmatrix}$$

and

$$\xi = \begin{pmatrix} \langle y_0, \phi[\lambda_{k,1}] \rangle_{-\diamond,\diamond} \\ \langle y_0, \phi[\lambda_{k,1}, \lambda_{k,2}] \rangle_{-\diamond,\diamond} \end{pmatrix} = \begin{pmatrix} \left\langle y_0, \begin{pmatrix} -f(\nu_k) \\ 1 \end{pmatrix} \varphi_k \right\rangle_{-\diamond,\diamond} \\ \left\langle y_0, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \varphi_k \right\rangle_{-\diamond,\diamond} \end{pmatrix}.$$

Thus,

$$\mathcal{C}(G_k, y_0) = \left\langle y_0, \begin{pmatrix} -f(\nu_k) \\ 1 \end{pmatrix} \frac{\varphi_k}{\|\varphi_k\|_\omega} \right\rangle_{-\diamond,\diamond}^2 + \left\langle y_0, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{\varphi_k}{\|\varphi_k\|_\omega} \right\rangle_{-\diamond,\diamond}^2.$$

- Consider now that $f(\nu_k) = 0$. Then, $b[\lambda_{k,1}, \lambda_{k,1}] = 0$. From Theorem 1.8 it comes that

$$\mathcal{C}(G_k, y_0) = \left\langle M^{-1} \xi, \xi \right\rangle$$

with

$$M_k = \mathrm{Gram}(b[\lambda_{k,1}], b[\lambda_{k,1}, \lambda_{k,1}]) + \mathrm{Gram}(0, b[\lambda_{k,1}]) = \begin{pmatrix} \|\varphi_k\|_\omega^2 & 0 \\ 0 & \|\varphi_k\|_\omega^2 \end{pmatrix}.$$

and

$$\xi = \begin{pmatrix} \langle y_0, \phi[\lambda_{k,1}] \rangle_{-\diamond,\diamond} \\ \langle y_0, \phi[\lambda_{k,1}, \lambda_{k,1}] \rangle_{-\diamond,\diamond} \end{pmatrix} = \begin{pmatrix} \left\langle y_0, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varphi_k \right\rangle_{-\diamond,\diamond} \\ \left\langle y_0, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \varphi_k \right\rangle_{-\diamond,\diamond} \end{pmatrix}.$$

As previously,

$$\mathcal{C}(G_k, y_0) = \left\langle y_0, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \frac{\varphi_k}{\|\varphi_k\|_\omega} \right\rangle_{-\diamond,\diamond}^2 + \left\langle y_0, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{\varphi_k}{\|\varphi_k\|_\omega} \right\rangle_{-\diamond,\diamond}^2.$$

Gathering both cases and using estimate (5.5) we obtain, for any $y_0 \in X_{-\diamond}$,

$$\mathcal{C}(G_k, y_0) \leq C \|y_0\|_{-\diamond}^2, \qquad \forall k \geq 1.$$

Thus,

$$T_0(y_0) = \limsup_{k \to +\infty} \frac{\ln \mathcal{C}(G_k, y_0)}{2 \min G_k} = 0.$$

□

5.4. Competition between different perturbations.

Let $\omega_1, \omega_2 \subset (0,1)$ be two open sets with $\omega_1 \neq \varnothing$ and $U = L^2(\Omega)^2$. Let $B_1, B_2 \in \mathbb{R}^3$. To simplify the computations, we assume that

$$B_i = \begin{pmatrix} 0 \\ B_{i,2} \\ B_{i,3} \end{pmatrix}.$$

Let $\alpha, \beta > 0$ with $\alpha \neq \beta$ and $f, g : \mathrm{Sp}(A) \to \mathbb{R}$ be defined by

$$f(\nu_k) = \frac{\varrho}{2} e^{-\alpha \nu_k}, \qquad g(\nu_k) = \frac{\varrho}{2} e^{-\beta \nu_k}.$$

As previously, we consider the associated operators $f(A)$ and $g(A)$ defined by the spectral theorem and we define the evolution operator $\mathcal{A}$ on $X = L^2(0,1;\mathbb{R})^3$ by

$$\mathcal{A} = \begin{pmatrix} A & I & 0 \\ 0 & A + f(A) & 0 \\ 0 & 0 & A + g(A) \end{pmatrix}, \quad D(\mathcal{A}) = D(A)^3, \tag{5.10}$$

and the control operator by

$$\mathcal{B} : \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in U \mapsto \mathbf{1}_{\omega_1} u_1 B_1 + \mathbf{1}_{\omega_2} u_2 B_2. \tag{5.11}$$

Then, the observation operator reads

$$\mathcal{B}^* : \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} \in X \mapsto \begin{pmatrix} \mathbf{1}_{\omega_1} \left(B_{1,2} \varphi_2 + B_{1,3} \varphi_3 \right) \\ \mathbf{1}_{\omega_2} \left(B_{2,2} \varphi_2 + B_{2,3} \varphi_3 \right) \end{pmatrix}.$$

PROPOSITION 5.3. *Let us consider the control system (1.1) with $\mathcal{A}$ and $\mathcal{B}$ given by (5.10)-(5.11).*

i. If $\omega_2 = \varnothing$, we assume that

$$B_{1,2} B_{1,3} \neq 0. \tag{5.12}$$

Then,

$$T_0(X_{-\diamond}) = \beta + \min\{\alpha, \beta\}.$$

ii. If $\omega_2 \neq \varnothing$, we assume that

$$\left(B_{1,2}^2 + B_{2,2}^2 \right) \left(B_{1,3}^2 + B_{2,3}^2 \right) \neq 0. \tag{5.13}$$

(a) If B_1 and B_2 are linearly independent, then,

$$T_0(X_{-\diamond}) = 0.$$

(b) If B_1 and B_2 are not linearly independent, then,

$$T_0(X_{-\diamond}) = \beta + \min\{\alpha, \beta\}.$$

Proof. It is easy to see that $(-\mathcal{A}, D(\mathcal{A}))$ generates a $\mathcal{C}_0$-semigroup on X and that $\mathcal{B} : U \to X$ is bounded. Thus we consider for this example that $X_\diamond^* = X = X_{-\diamond}$ and $Y_0 = X_{-\diamond}$.

The spectrum of $(\mathcal{A}^*, D(\mathcal{A}))$ is given by $\Lambda = \bigcup_{k\geq 1} G_k$ where

$$G_k := \{\lambda_{k,1} := \nu_k,\ \lambda_{k,2} := \nu_k + f(\nu_k),\ \lambda_{k,3} := \nu_k + g(\nu_k)\}.$$

From (5.3) and (5.6) it comes that there exists $\bar{N} > 0$ such that $\Lambda \in \mathcal{L}\left(3, \frac{\varrho}{2}, \frac{1}{2}, \bar{N}\right)$ and $(G_k)_{k\geq 1}$ is an associated grouping.

The eigenvalues are simple and the corresponding eigenvectors are given by

$$\phi^0_{k,1} = \begin{pmatrix} -f(\nu_k) \\ 1 \\ 0 \end{pmatrix} \varphi_k, \qquad \phi^0_{k,2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \varphi_k, \qquad \phi^0_{k,3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \varphi_k.$$

Thus, the assumption (1.29) hold. Moreover,

$$b_1 = b_2 = \begin{pmatrix} \mathbf{1}_{\omega_1} \varphi_k B_{1,2} \\ \mathbf{1}_{\omega_2} \varphi_k B_{2,2} \end{pmatrix}, \quad b_3 = \begin{pmatrix} \mathbf{1}_{\omega_1} \varphi_k B_{1,3} \\ \mathbf{1}_{\omega_2} \varphi_k B_{2,3} \end{pmatrix} \tag{5.14}$$

From (5.5) and (5.12) or (5.13) (depending on the assumption on ω_2) it comes that (1.28) is satisfied. Thus, from Theorem 1.3, it comes that for any $y_0 \in X_{-\diamond}$,

$$T_0(y_0) = \limsup_{k\to+\infty} \frac{\ln \mathcal{C}(G_k, y_0)}{2 \min G_k}.$$

Let us now estimate $\mathcal{C}(G_k, y_0)$. From Theorem 1.8 it comes that

$$\mathcal{C}(G_k, y_0) = \left\langle M^{-1}\xi, \xi \right\rangle$$

with

$$\begin{aligned} M =& \operatorname{Gram}\big(b[\lambda_{k,1}], b[\lambda_{k,1}, \lambda_{k,2}], b[\lambda_{k,1}, \lambda_{k,2}, \lambda_{k,3}]\big) \\ &+ \operatorname{Gram}\big(0, b[\lambda_{k,2}], b[\lambda_{k,2}, \lambda_{k,3}]\big) + \operatorname{Gram}\big(0, 0, b[\lambda_{k,3}]\big) \end{aligned}$$

and

$$\xi = \begin{pmatrix} \langle y_0, \phi[\lambda_{k,1}]\rangle_{-\diamond,\diamond} \\ \langle y_0, \phi[\lambda_{k,1}, \lambda_{k,2}]\rangle_{-\diamond,\diamond} \\ \langle y_0, \phi[\lambda_{k,1}, \lambda_{k,2}, \lambda_{k,3}]\rangle_{-\diamond,\diamond} \end{pmatrix}.$$

Explicit computations yield

$$\phi[\lambda_{k,1}] = \begin{pmatrix} -f(\nu_k) \\ 1 \\ 0 \end{pmatrix} \varphi_k, \qquad \phi[\lambda_{k,1}, \lambda_{k,2}] = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \varphi_k,$$

and

$$\phi[\lambda_{k,1}, \lambda_{k,2}, \lambda_{k,3}] = \frac{1}{g(\nu_k)\big(g(\nu_k) - f(\nu_k)\big)} \begin{pmatrix} f(\nu_k) - g(\nu_k) \\ -1 \\ 1 \end{pmatrix} \varphi_k.$$

i. Assume that $\omega_2 = \varnothing$.
After the change of variables

$$z = \operatorname{diag}\left(\frac{1}{B_{1,2}}, \frac{1}{B_{1,2}}, \frac{1}{B_{1,3}}\right) y,$$

the system under study reads

$$\begin{cases} \partial_t z + \begin{pmatrix} A & I & 0 \\ 0 & A+f(A) & 0 \\ 0 & 0 & A+g(A) \end{pmatrix} z = 1_{\omega_1} u_1(t,x) \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \\ z(t,0) = z(t,1) = 0. \end{cases}$$

This leads to

$$b[\lambda_{k,1}] = b[\lambda_{k,2}] = b[\lambda_{k,3}] = \mathbf{1}_{\omega_1} \varphi_k.$$

Thus, $M = \|\varphi_k\|_{\omega_1}^2 I_3$ and

$$\mathcal{C}(G_k, y_0) = \left\langle y_0, \begin{pmatrix} -f(\nu_k) \\ 1 \\ 0 \end{pmatrix} \frac{\varphi_k}{\|\varphi_k\|_{\omega_1}} \right\rangle_{-\diamond,\diamond}^2 + \left\langle y_0, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \frac{\varphi_k}{\|\varphi_k\|_{\omega_1}} \right\rangle_{-\diamond,\diamond}^2 + \left(\frac{1}{g(\nu_k)\big(g(\nu_k) - f(\nu_k)\big)} \right)^2 \left\langle y_0, \begin{pmatrix} f(\nu_k) - g(\nu_k) \\ -1 \\ 1 \end{pmatrix} \frac{\varphi_k}{\|\varphi_k\|_{\omega_1}} \right\rangle_{-\diamond,\diamond}^2.$$

From (5.5), we obtain for any $y_0 \in X_{-\diamond}$,

$$\mathcal{C}(G_k, y_0) \leq C \, \|y_0\|_{-\diamond}^2 \left(1 + \left(\frac{1}{g(\nu_k)\big(g(\nu_k) - f(\nu_k)\big)} \right)^2 \right).$$

This leads to

$$T_0(X_{-\diamond}) \leq \limsup_{k \to +\infty} \frac{-\ln \left| g(\nu_k)\big(g(\nu_k) - f(\nu_k)\big) \right|}{\nu_k}.$$

Conversely, with the particular choice

$$y_0 = \sum_{k \geq 1} \frac{1}{\nu_k} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \varphi_k,$$

we have

$$\mathcal{C}(G_k, y_0) = \frac{1}{\nu_k^2 \|\varphi_k\|_{\omega_1}^2} \left(\frac{1}{g(\nu_k)\big(g(\nu_k) - f(\nu_k)\big)} \right)^2.$$

Thus, from (5.5), we obtain

$$T_0(X_{-\diamond}) \geq T_0(y_0) = \limsup_{k \to +\infty} \frac{-\ln \left| g(\nu_k)\big(g(\nu_k) - f(\nu_k)\big) \right|}{\nu_k}$$

which gives

$$T_0(X_{-\diamond}) = \limsup_{k \to +\infty} \frac{-\ln \left| g(\nu_k)\big(g(\nu_k) - f(\nu_k)\big) \right|}{\nu_k}.$$

Then, the same computations as [10, Section 5.1.3] yield

$$T_0(X_{-\diamond}) = \beta + \min\{\alpha, \beta\}.$$

ii. We now consider the case $\omega_2 \neq \varnothing$.

(a) Assume that B_1 and B_2 are linearly independent. If necessary, we consider smaller control sets so that $\omega_1 \cap \omega_2 = \varnothing$. As we will prove that $T_0(X_{-\diamond}) = 0$, this is not a restrictive assumption.
To ease the reading we drop the index k in what follows. As previously, the vector ξ is not bounded. Let us consider the dilatation $D_\epsilon = \mathrm{diag}(1, 1, \epsilon)$ with

$$\epsilon = g(\nu)\big(g(\nu) - f(\nu)\big)$$

and $\widetilde{\xi} = D_\epsilon \xi$. Then, from Section D.1, it comes that

$$\mathcal{C}(G, y_0) = \left\langle \widetilde{M}^{-1}\widetilde{\xi}, \widetilde{\xi} \right\rangle$$

with

$$\begin{aligned}\widetilde{M} =& \mathrm{Gram}\big(b[\lambda_1], b[\lambda_1, \lambda_2], \epsilon b[\lambda_1, \lambda_2, \lambda_3]\big) \\ &+ \mathrm{Gram}\big(0, b[\lambda_2], \epsilon b[\lambda_2, \lambda_3]\big) + \mathrm{Gram}\big(0, 0, \epsilon b[\lambda_3]\big).\end{aligned}$$

As $\left\|\widetilde{\xi}\right\|$ is bounded, we simply give a lower bound on the smallest eigenvalue of $\widetilde{M}$. Using (5.14), it comes that

$$b[\lambda_1, \lambda_2] = 0, \quad b[\lambda_2, \lambda_3] = \frac{b_3 - b_1}{g(\nu) - f(\nu)}, \quad b[\lambda_1, \lambda_2, \lambda_3] = \frac{1}{\epsilon}(b_3 - b_1).$$

Thus,

$$\begin{aligned}\widetilde{M} =& \mathrm{Gram}\big(b_1, 0, b_3 - b_1\big) + \mathrm{Gram}\big(0, b_1, g(\nu)(b_3 - b_1)\big) \\ &+ \mathrm{Gram}\big(0, 0, \epsilon b_3\big).\end{aligned}$$

This gives that, for any $\tau \in \mathbb{R}^3$, we have

$$\begin{aligned}\left\langle \widetilde{M}\tau, \tau \right\rangle =& \|\tau_1 b_1 + \tau_3(b_3 - b_1)\|_U^2 + \|\tau_2 b_1 + g(\nu)\tau_3(b_3 - b_1)\|_U^2 \\ &+ \epsilon^2 \|\tau_3 b_3\|_U^2 .\end{aligned} \tag{5.15}$$

To obtain a lower bound on this quantity we use the following lemma.

Lemma 5.4. *There exists $C > 0$ (independent of k) such that for any θ_1, $\theta_3 \in \mathbb{R}$,*

$$\|\theta_1 b_1 + \theta_3 b_3\|_U^2 \geq C\left(\theta_1^2 + \theta_3^2\right).$$

Proof. As $\omega_1 \cap \omega_2 = \varnothing$,

$$\begin{aligned}\|\theta_1 b_1 + \theta_3 b_3\|_U^2 =& \left(B_{1,2}\theta_1 + B_{1,3}\theta_3\right)^2 \|\varphi_k\|_{\omega_1}^2 \\ &+ \left(B_{2,2}\theta_1 + B_{2,3}\theta_3\right)^2 \|\varphi_k\|_{\omega_2}^2.\end{aligned}$$

Using (5.5) it comes that

$$\begin{aligned}\|\theta_1 b_1 + \theta_3 b_3\|_U^2 &\geq C\left(\left(B_{1,2}\theta_1 + B_{1,3}\theta_3\right)^2 + \left(B_{2,2}\theta_1 + B_{2,3}\theta_3\right)^2\right) \\ &= \left\| \begin{pmatrix} B_{1,2} & B_{1,3} \\ B_{2,2} & B_{2,3} \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_3 \end{pmatrix} \right\|^2 .\end{aligned}$$

Since B_1 and B_2 are linearly independent, this ends the proof. ∎

Applying this lemma twice to (5.15) yield

$$\begin{aligned}\left\langle \widetilde{M}\tau,\tau\right\rangle &\geq C\left((\tau_1-\tau_3)^2+\tau_3^2+(\tau_2-g(\nu)\tau_3)^2+g(\nu)^2\tau_3^2+\epsilon^2\tau_3^2\right)\\ &\geq C\left((\tau_1-\tau_3)^2+\tau_3^2+(\tau_2-g(\nu)\tau_3)^2\right).\end{aligned}$$

Taking into account that $0<g(\nu)<\frac{1}{2}$ for ν large enough, the study of this quadratic form in $\mathbb{R}^3$ leads to

$$\left\langle \widetilde{M}\tau,\tau\right\rangle \geq C\left(\tau_1^2+\tau_2^2+\tau_3^2\right).$$

Thus the smallest eigenvalue of $\widetilde{M}$ is bounded from below. This leads to the boundedness of $\left\langle \widetilde{M}^{-1}\widetilde{\xi},\widetilde{\xi}\right\rangle$ which concludes the proof of case ii (a).

(b) Assume now that B_1 and B_2 are not linearly independent. Then, there exist x_1, $x_2\in\mathbb{R}$ such that

$$\begin{cases}x_1B_{1,2}+x_2B_{1,3}=0\\ x_1B_{2,2}+x_2B_{2,3}=0.\end{cases}$$

Up to a change of normalization of the eigenvectors (independent of k) we obtain

$$b_1=b_2=b_3=\begin{pmatrix}\mathbf{1}_{\omega_1}\varphi_k x_1B_{1,2}\\ \mathbf{1}_{\omega_2}\varphi_k x_1B_{2,2}\end{pmatrix}$$

and this amounts to case i. ∎

6. Analysis of controllability for systems of partial differential equations.

We now turn to the analysis of null controllability of actual partial differential equations. We consider cascade systems of two parabolic equations. First, we consider systems with different diffusion operators and constant zero order coupling term. Then, we consider the same diffusion operator with a space varying zero order coupling term.

6.1. Coupled heat equations with different diffusion coefficients.

In this example, we consider the Sturm Liouville operator A defined in (5.1) and we define in $X=L^2(0,1;\mathbb{R})^2$ the operator

$$\mathcal{A}=\begin{pmatrix}A & I\\ 0 & dA\end{pmatrix},\qquad D(\mathcal{A})=D(A)^2,$$

with $d>0$.

We study the following boundary control system

$$\begin{cases}\partial_t y+\mathcal{A}y=0, & t\in(0,T),\\ y(t,0)=B_0v_0(t), \quad y(t,1)=B_1v_1(t), & t\in(0,T),\end{cases}\tag{6.1}$$

with

$$B_0=\begin{pmatrix}1\\1\end{pmatrix}\quad\text{and}\quad B_1=\begin{pmatrix}1\\0\end{pmatrix}.\tag{6.2}$$

The control operator $\mathcal{B}$ is defined in a weak sense as in [37]. The expression of its adjoint is given by

$$\mathcal{B}^* : \begin{pmatrix} f \\ g \end{pmatrix} \in X_1^* \mapsto \begin{pmatrix} -B_0^* \begin{pmatrix} f'(0) \\ g'(0) \end{pmatrix} \\ B_1^* \begin{pmatrix} f'(1) \\ g'(1) \end{pmatrix} \end{pmatrix} = \begin{pmatrix} -(f'(0)+g'(0)) \\ f'(1) \end{pmatrix}.$$

Thus, setting $X_\diamond^* = H_0^1(0,1;\mathbb{R})^2$, we obtain that $\mathcal{B}$ is admissible with respect to $X_{-\diamond} = H^{-1}(0,1;\mathbb{R})^2$.

PROPOSITION 6.1. *For any $d > 0$, the minimal null control time of system (6.1) is given by $T_0(X_{-\diamond}) = 0$.*

REMARK 6.1. *The situation with a single control is quite different. Indeed, considering $B_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $B_1 = 0$, it is proved in [5] that, when A is the Dirichlet Laplace operator, approximate controllability holds if and only $\sqrt{d} \notin \mathbb{Q}$ and in this case that*

$$T_0(X_{-\diamond}) = \limsup_{\substack{\lambda \to \infty \\ \lambda \in \Lambda}} \frac{-\ln \operatorname{dist}\left(\lambda, \Lambda \backslash \{\lambda\}\right)}{\lambda}.$$

With this formula the authors prove that, for any $\tau \in [0, +\infty]$, there exists a diffusion ratio $d > 0$ such that the minimal null control time of system (6.1) satisfies $T_0(X_{-\diamond}) = \tau$.

REMARK 6.2. *The particular choice of B_0 and B_1 is done to simplify the computations. Notice that with this choice, it is not possible to steer to zero the second equation and then control the first equation. This would be the case with the simpler choice*

$$B_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad B_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Proof. The case $d = 1$ is very similar to the analysis conducted in Section 5.3 (with $f = 0$) and is not detailed here. We now assume that $d \neq 1$. Let

$$\Lambda_1 := \operatorname{Sp}(A) = \{\nu_k \, ; \, k \geq 1\}$$

and $\Lambda_2 := d\Lambda_1$. The spectrum of $\mathcal{A}^*$ is given by $\Lambda = \Lambda_1 \cup \Lambda_2$ which belongs to $\mathcal{L}\left(2, \varrho, \frac{1}{2}, \bar{N}\right)$ for some $\varrho, \bar{N} > 0$. For any $\lambda \in \Lambda$, there are three cases:

- $\lambda = \nu_k \in \Lambda_1 \backslash \Lambda_2$ is a simple eigenvalue. An eigenvector is given by

$$\phi_\lambda = \begin{pmatrix} 1 \\ \frac{1}{\nu_k(1-d)} \end{pmatrix} \varphi_k.$$

- $\lambda = d\nu_k \in \Lambda_2 \backslash \Lambda_1$ is a simple eigenvalue. An eigenvector is given by

$$\phi_\lambda = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varphi_k.$$

- $\lambda = \nu_k = d\nu_{k'} \in \Lambda_1 \cap \Lambda_2$ is a geometrically double eigenvalue. A basis of eigenvectors is given by

$$\phi_{\lambda,1} = \begin{pmatrix} 1 \\ \frac{1}{\nu_k(1-d)} \end{pmatrix} \varphi_k, \qquad \phi_{\lambda,2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varphi_{k'}.$$

Since $(\varphi_k)_{k\geq 1}$ is an Hilbert basis of X, this implies that (1.29) holds. Due to (5.4), it comes that (1.28) also holds. Then, from Theorem 1.3,

$$T_0(y_0) = \limsup_{k\to+\infty} \frac{\ln \mathcal{C}(G_k, y_0)}{2\min G_k}.$$

Using again (5.4), the blocks consisting of a single simple eigenvalue do not contribute to the minimal time.

⋆ Blocks of two simple eigenvalues. Assume that we have a block of the form

$$G := \{\lambda_1 := \nu_k,\ \lambda_2 := d\nu_{k'}\}.$$

As we will deal later on with semi-simple eigenvalues, we evaluate the contribution of the block G using Theorem 1.10. Then,

$$\mathcal{C}(G, y_0) = \left\langle M^{-1}\xi, \xi\right\rangle$$

with

$$M = \mathrm{Gram}\big(b[\lambda_1], b[\lambda_2]\big) + \mathrm{Gram}\big(0, (\lambda_2-\lambda_1)b[\lambda_2]\big)$$

and

$$\xi = \begin{pmatrix} \langle y_0, \phi_{\lambda_1}\rangle_{-\diamond,\diamond} \\ \langle y_0, \phi_{\lambda_2}\rangle_{-\diamond,\diamond}\end{pmatrix}.$$

For any $k \geq 1$, we define $\epsilon_k \in \mathbb{R}$ by $\varphi_k'(1) = \epsilon_k \varphi_k'(0)$. From Lemma 5.1, there exists $C > 0$ such that

$$\frac{1}{C} \leq |\epsilon_k| \leq C, \qquad \forall k \geq 1. \tag{6.3}$$

Then,

$$b[\lambda_1] = \mathcal{B}^*\phi_{\lambda_1} = \begin{pmatrix} -\varphi_k'(0) B_0^* \begin{pmatrix} 1 \\ \frac{1}{\nu_k(1-d)}\end{pmatrix} \\ \varphi_k'(1) B_1^* \begin{pmatrix} 1 \\ \frac{1}{\nu_k(1-d)}\end{pmatrix}\end{pmatrix} = -\varphi_k'(0) \begin{pmatrix} 1+\frac{1}{\nu_k(1-d)} \\ -\epsilon_k\end{pmatrix}$$

$$b[\lambda_2] = \mathcal{B}^*\phi_{\lambda_2} = \begin{pmatrix} -\varphi_{k'}'(0) B_0^* \begin{pmatrix} 0 \\ 1\end{pmatrix} \\ \varphi_{k'}'(1) B_1^* \begin{pmatrix} 0 \\ 1\end{pmatrix}\end{pmatrix} = -\varphi_{k'}'(0) \begin{pmatrix} 1 \\ 0\end{pmatrix}.$$

To ease the reading, we use the following change of normalization for the eigenvectors

$$\widetilde{\phi}_{\lambda_1} := \frac{\phi_{\lambda_1}}{-\varphi_k'(0)}, \qquad \widetilde{\phi}_{\lambda_2} := \frac{\phi_{\lambda_2}}{-\varphi_{k'}'(0)},$$

and we denote by $\widetilde{M}$ and $\tilde{\xi}$ the associated quantities. Notice that, due to (5.4), the quantity $\left\|\tilde{\xi}\right\|$ is bounded. Thus, to estimate $\mathcal{C}(G, y_0)$ we give a lower bound on the smallest eigenvalue of $\widetilde{M}$. We have

$$\begin{aligned}\widetilde{M} &= \mathrm{Gram}\big(\tilde{b}[\lambda_1], \tilde{b}[\lambda_2]\big) + \mathrm{Gram}\big(0, (\lambda_2-\lambda_1)\tilde{b}[\lambda_2]\big) \\ &= \underbrace{\begin{pmatrix} \epsilon_k^2 + \left(1+\frac{1}{\nu_k(1-d)}\right)^2 & 1+\frac{1}{\nu_k(1-d)} \\ 1+\frac{1}{\nu_k(1-d)} & 1\end{pmatrix}}_{=\Gamma^1} + \begin{pmatrix} 0 & 0 \\ 0 & (\lambda_2-\lambda_1)^2\end{pmatrix}.\end{aligned}$$

For any $\tau \in \mathbb{R}^2$, $\left\langle \widetilde{M}\tau, \tau \right\rangle \geq \left\langle \Gamma^1 \tau, \tau \right\rangle$. Then,

$$\min \mathrm{Sp}(\Gamma^1) \geq \frac{\det(\Gamma^1)}{\mathrm{tr}(\Gamma^1)} = \frac{\epsilon_k^2}{1+\epsilon_k^2+\left(1+\frac{1}{\nu_k(1-d)}\right)^2}$$

From (6.3), it comes that $\min \mathrm{Sp}(\Gamma^1)$ is bounded from below by a positive constant independent of G. Thus, the quantity $\mathcal{C}(G, y_0)$ is bounded.

$\star$ Blocks of a geometrically double eigenvalue.
Consider $G = \{\lambda\}$ with $\lambda = \nu_k = d\nu_{k'} \in \Lambda_1 \cap \Lambda_2$.
With the same notations as previously, Theorem 1.10 implies that

$$\mathcal{C}(G, y_0) = \left\langle \widetilde{M}^{-1}\tilde{\xi}, \tilde{\xi} \right\rangle$$

where

$$\tilde{\xi} = \begin{pmatrix} \left\langle y_0, \frac{\phi_{\lambda,1}}{-\varphi_k'(0)} \right\rangle_{-\diamond,\diamond} \\ \left\langle y_0, \frac{\phi_{\lambda,2}}{-\varphi_{k'}'(0)} \right\rangle_{-\diamond,\diamond} \end{pmatrix}$$

and

$$\begin{aligned} \widetilde{M} &= \mathrm{Gram}\left(\frac{\mathcal{B}^*\phi_{\lambda,1}}{-\varphi_k'(0)}, \frac{\mathcal{B}^*\phi_{\lambda,2}}{-\varphi_{k'}'(0)}\right) \\ &= \begin{pmatrix} \epsilon_k^2 + \left(1+\frac{1}{\nu_k(1-d)}\right)^2 & 1+\frac{1}{\nu_k(1-d)} \\ 1+\frac{1}{\nu_k(1-d)} & 1 \end{pmatrix} = \Gamma^1. \end{aligned}$$

Thus, the study of the previous item proves that $\min \mathrm{Sp}(\Gamma^1)$ is bounded from below by a positive constant independent of λ and that the quantity $\mathcal{C}(G, y_0)$ is bounded.

Gathering both cases, this implies that

$$T_0(y_0) = \limsup_{k \to +\infty} \frac{\ln \mathcal{C}(G_k, y_0)}{2 \min G_k} = 0.$$

∎

6.2. A system with two different potentials.

Let us consider the following control system

(6.4)
$$\begin{cases} \partial_t y + \begin{pmatrix} -\partial_{xx} + c_1(x) & 1 \\ 0 & -\partial_{xx} + c_2(x) \end{pmatrix} y = \begin{pmatrix} 0 \\ \mathbf{1}_\omega u(t,x) \end{pmatrix}, \quad (t,x) \in (0,T) \times (0,1), \\ y(t,0) = y(t,1) = 0, \quad t \in (0,T), \\ y(0,x) = y_0(x), \end{cases}$$

where $c_1, c_2 \in L^2(0,1;\mathbb{R})$.

With the technics developed in this article, one can prove the following controllability result.

PROPOSITION 6.2. *For any non-negative potentials c_1, c_2, system (6.4) is null controllable in any time $T > 0$ from $L^2(0,1;\mathbb{R})^2$.*

The proof follows closely the computations done for the same system with a boundary control in [10, Section 5.2.1]. The only difference is that the contributions of terms of the form $\|\mathcal{B}^*\bullet\|_U = \|\bullet\|_\omega$ are estimated using Lemma 5.1. As the result stated in Proposition 6.2 is already known (it is for instance an application of [25] with a proof based on Carleman estimates), we do not detail the proof here to lighten this article.

6.3. A system with a space varying zero order coupling term.

Let us consider A the Sturm-Liouville operator defined in (5.1). In this section we study the null controllability of the following control system

$$\begin{cases} \partial_t y + \begin{pmatrix} A & q(x) \\ 0 & A \end{pmatrix} y = \begin{pmatrix} 0 \\ \mathbf{1}_\omega u(t,x) \end{pmatrix}, \quad (t,x) \in (0,T)\times(0,1), \\ y(t,0) = y(t,1) = 0, \quad t \in (0,T), \\ y(0,x) = y_0(x), \end{cases} \tag{6.5}$$

where the coupling function q belongs to $L^\infty(0,1;\mathbb{R})$ and $\omega \subset (0,1)$ is a non empty open set.

To fit in the formalism of system (1.1) the evolution operator $\mathcal{A}$ is defined by

$$\mathcal{A} = \begin{pmatrix} A & q \\ 0 & A \end{pmatrix}, \qquad D(\mathcal{A}) = D(A)^2$$

and the control operator $\mathcal{B}$ is defined by

$$\mathcal{B} : u \in U = L^2((0,1);\mathbb{R}) \mapsto \begin{pmatrix} 0 \\ \mathbf{1}_\omega u \end{pmatrix}.$$

As the control operator is bounded we consider in this example $X_{-\diamond} = X = X_\diamond^* = L^2(0,1;\mathbb{R})^2$. Recall that the eigenvalues ν_k and the associated eigenvectors φ_k of the operator A satisfy (5.3), (5.5) and (5.6).

6.3.1. Already known controllability results for this system.

• *Approximate controllability.*

For any $k \in \mathbb{N}^*$, let us define $\widetilde{\varphi}_k$ as the unique solution of the Cauchy problem

$$\begin{cases} (A-\nu_k)\widetilde{\varphi}_k = 0, \\ \widetilde{\varphi}_k(0) = 1, \qquad \widetilde{\varphi}_k'(0) = 0. \end{cases} \tag{6.6}$$

For any $F \in L^2(0,1;\mathbb{R})$, we also define the quantities

$$\mathcal{M}_{k,1}(F,\omega) = \sup\left\{ \left| \int_{\mathfrak{C}} F\varphi_k \right| ; \mathfrak{C} \text{ connected component of } \overline{(0,1)\backslash\omega} \right\} \tag{6.7a}$$

$$\mathcal{M}_{k,2}(F,\omega) = \sup\left\{ \left| \int_{\mathfrak{C}} F\widetilde{\varphi}_k \right| ; \mathfrak{C} \text{ connected component of } \overline{(0,1)\backslash\omega} \text{ such that } \mathfrak{C} \cap \{0,1\} = \varnothing \right\} \tag{6.7b}$$

and

$$\mathcal{M}_k(F,\omega) = \max\left\{\mathcal{M}_{k,1}(F,\omega),\ \mathcal{M}_{k,2}(F,\omega)\right\}. \tag{6.8}$$

It is proved in [17, Theorem 3.2] that, if $\mathrm{Supp}(q)\cap\omega=\varnothing$, approximate controllability of (6.5) holds if and only if

$$\mathcal{M}_k(q\varphi_k,\omega)\neq 0,\qquad \forall k\geq 1. \tag{6.9}$$

Notice also that applying [17, Theorem 2.2] we obtain that

- If $\mathrm{Supp}(q)\cap\omega\neq\varnothing$, approximate controllability of (6.5) holds without any other condition.
- If $\mathrm{Supp}(q)\cap\omega=\varnothing$, approximate controllability of (6.5) holds if and only if

$$\mathcal{M}_k\left((I_k(q)-q)\varphi_k,\omega\right)\neq 0,\qquad \forall k\geq 1, \tag{6.10}$$

where

$$I_k(q)=\int_0^1 q(x)\varphi_k^2(x)\mathrm{d}x. \tag{6.11}$$

Rewriting the condition this way is more coherent with the expression of the minimal null control time that is proved in what follows (see Section 6.3.3).

• *Null controllability under a sign assumption.*

If there exists $\omega_0\subset\omega$ such that q has a strict sign inside ω_0 then it follows from [25] that null controllability holds in any arbitrary time. The proof is based on Carleman estimates.

• *Null controllability with disjoint control and coupling domains.*

System (6.5) was then studied in the case where $A=-\partial_{xx}$ and $\omega=(a,b)$ is an interval such that $\mathrm{Supp}(q)\cap\omega=\varnothing$.

First, it was proved in [6] that if $\mathrm{Supp}(q)\subset(b,1)$ then, approximate controllability holds if and only if

$$I_k(q)\neq 0,\qquad \forall k\geq 1.$$

This condition is equivalent to (6.9). In this case the authors proved that

$$T_0(X_{-\diamond})=\limsup_{k\to+\infty}\frac{-\ln|I_k(q)|}{\nu_k}.$$

Later on, it was proved in [7] that if $\mathrm{Supp}(q)\subset\big((0,a)\cup(b,1)\big)$, then approximate controllability holds if and only if

$$|I_{1,k}(q)|+|I_{2,k}(q)|\neq 0,\qquad \forall k\geq 1,$$

where

$$I_{1,k}(q)=\int_0^a q(x)\varphi_k^2(x)\mathrm{d}x,\qquad I_{2,k}(q)=\int_b^1 q(x)\varphi_k^2(x)\mathrm{d}x. \tag{6.12}$$

In this case the authors proved that

$$T_0(X_{-\diamond})=\limsup_{k\to+\infty}\frac{-\ln\max\left\{|I_k(q)|,\ |I_{1,k}(q)|,\ |I_{2,k}(q)|\right\}}{\nu_k}.$$

Moreover, the authors proved that for any $\tau_0\in[0,+\infty]$ there exists a coupling function q such that $T_0(X_{-\diamond})=\tau_0$. Let us underline that these results are the first results exhibiting a positive minimal null control time for a system of coupled parabolic

equations with a distributed control. Due to the assumption $\mathrm{Supp}(q) \cap \omega = \varnothing$, the strategy based on Carleman estimates is inefficient. The proofs of [6, 7] are based on the moments method.

In what follows we give a minimal time characterization that unifies all these results (see Section 6.3.4). It also extends these results to a general Sturm-Liouville operator and enables us to deal with new geometric configurations for ω and the support of q (see Proposition 6.8).

6.3.2. A first formula for the minimal time.

To compute the minimal null control time let us detail the spectral analysis of the operator $\mathcal{A}^*$.

For any $k \in \mathbb{N}^*$, we define ψ_k as the unique solution of

$$\begin{cases} (A - \nu_k)\psi_k = \big(I_k(q) - q\big)\varphi_k, \\ \psi_k(0) = \psi_k(1) = 0, \\ \langle \varphi_k, \psi_k \rangle_\omega = 0. \end{cases} \tag{6.13}$$

This is possible since

$$\int_0^1 \big(I_k(q) - q(x)\big)\varphi_k(x)\varphi_k(x)\mathrm{d}x = 0.$$

We have $\Lambda = (\nu_k)_{k\in\mathbb{N}^*}$ and we distinguish the following cases.

$\star$ If $I_k(q) \neq 0$ then ν_k is algebraically double and geometrically simple. A Jordan chain is given by

$$\phi_k^0 = \begin{pmatrix} 0 \\ \varphi_k \end{pmatrix}, \qquad \phi_k^1 = \frac{1}{I_k(q)} \begin{pmatrix} \varphi_k \\ \psi_k \end{pmatrix} \tag{6.14}$$

where ψ_k is given by (6.13).

$\star$ If $I_k(q) = 0$ then ν_k is geometrically double and a basis of eigenvectors is given by

$$\phi_{k,1}^0 = \begin{pmatrix} 0 \\ \varphi_k \end{pmatrix}, \qquad \phi_{k,2}^0 = \begin{pmatrix} \varphi_k \\ \psi_k \end{pmatrix} \tag{6.15}$$

where ψ_k is given by (6.13).

Remark 6.3. *In the definition of ψ_k, the choice of normalization $\langle \varphi_k, \psi_k \rangle_\omega = 0$ is done to simplify the computations. It ensures orthogonality between observations of (generalized) eigenvectors.*

Applying Theorem 1.3 we obtain the following characterization of the minimal null control time.

Proposition 6.3. *Let $\omega \subset (0,1)$ be a non empty open set and let $q \in L^\infty(0,1;\mathbb{R})$. Assume that either $\mathrm{Supp}(q) \cap \omega \neq \varnothing$ or that (6.10) holds. Then, the minimal null control time for system (6.5) is given by*

$$T_0(X_{-\diamond}) = \limsup_{k\to+\infty} \frac{-\ln\big(I_k(q)^2\|\varphi_k\|_\omega^2 + \|\psi_k\|_\omega^2\big)}{2\nu_k}$$

This formula is valid in a general geometric configuration for ω and the support of q and unifies the different results obtained in the literature for the study of null controllability of system (6.5). For example, even if it is not straightforward, we prove in Section 6.3.4 that it allows to recover the null controllability of (6.5) in any time proved in [25] when q has a strict sign on $\omega_0 \subset \omega$.

We also give in Section 6.3.3 another formula which is more convenient to deal with but necessitates the geometric assumption $\mathrm{Supp}(q) \cap \omega = \varnothing$.

Proof. Since $\Lambda = (\nu_k)_{k\in\mathbb{N}^*}$, it belongs to $\mathcal{L}\left(1, \varrho, \frac{1}{2}, \bar{N}\right)$ for some $\bar{N} > 0$ and ϱ defined in (5.3). Thus, a suitable grouping is given by $(\{\nu_k\})_{k\geq 1}$.

Due to (6.14) and (6.15) we obtain that (1.29) holds. As mentioned previously, the approximate controllability assumption (1.28) follows from (6.10) and [17, Theorem 2.2].

Then, from Theorem 1.3, for any $y_0 \in X_{-\diamond}$, we have

$$T_0(y_0) = \limsup_{k\to+\infty} \frac{\ln \mathcal{C}(\{\nu_k\}, y_0)}{2\nu_k}.$$

To compute $\mathcal{C}(\{\nu_k\}, y_0)$ we distinguish two cases.

$\star$ Assume that $I_k(q) \neq 0$. Then, from Theorem 1.8, it comes that

$$\mathcal{C}(\{\nu_k\}, y_0) = \left\langle M^{-1}\xi, \xi \right\rangle$$

where

$$M = \mathrm{Gram}\big(\mathcal{B}^*\phi_k^0, \mathcal{B}^*\phi_k^1\big) + \mathrm{Gram}\big(0, \mathcal{B}^*\phi_k^0\big) = \begin{pmatrix} \|\varphi_k\|_\omega^2 & 0 \\ 0 & \|\varphi_k\|_\omega^2 + \frac{1}{I_k(q)^2}\|\psi_k\|_\omega^2 \end{pmatrix}$$

and

$$\xi = \begin{pmatrix} \langle y_0, \phi_k^0\rangle_{-\diamond,\diamond} \\ \langle y_0, \phi_k^1\rangle_{-\diamond,\diamond} \end{pmatrix}.$$

Thus,

$$\mathcal{C}(\{\nu_k\}, y_0) = \frac{1}{\|\varphi_k\|_\omega^2}\left\langle y_0, \begin{pmatrix} 0 \\ \varphi_k \end{pmatrix}\right\rangle_{-\diamond,\diamond}^2 + \frac{1}{I_k(q)^2\|\varphi_k\|_\omega^2 + \|\psi_k\|_\omega^2}\left\langle y_0, \begin{pmatrix} \varphi_k \\ \psi_k \end{pmatrix}\right\rangle_{-\diamond,\diamond}^2.$$

$\star$ Assume that $I_k(q) = 0$. Then, from Theorem 1.10, it comes that

$$\mathcal{C}(\{\nu_k\}, y_0) = \left\langle M^{-1}\xi, \xi \right\rangle$$

where

$$M = \mathrm{Gram}\big(\mathcal{B}^*\phi_{k,1}^0, \mathcal{B}^*\phi_{k,2}^0\big) = \begin{pmatrix} \|\varphi_k\|_\omega^2 & 0 \\ 0 & \|\psi_k\|_\omega^2 \end{pmatrix}$$

and

$$\xi = \begin{pmatrix} \langle y_0, \phi_{k,1}\rangle_{-\diamond,\diamond} \\ \langle y_0, \phi_{k,2}\rangle_{-\diamond,\diamond} \end{pmatrix}.$$

Thus,

$$\mathcal{C}(\{\nu_k\}, y_0) = \frac{1}{\|\varphi_k\|_\omega^2}\left\langle y_0, \begin{pmatrix} 0 \\ \varphi_k \end{pmatrix}\right\rangle_{-\diamond,\diamond}^2 + \frac{1}{\|\psi_k\|_\omega^2}\left\langle y_0, \begin{pmatrix} \varphi_k \\ \psi_k \end{pmatrix}\right\rangle_{-\diamond,\diamond}^2.$$

Finally, in both cases, the cost of the group $\{\nu_k\}$ is given by

(6.16)
$$\mathcal{C}(\{\nu_k\}, y_0) = \frac{1}{\|\varphi_k\|_\omega^2} \left\langle y_0, \begin{pmatrix} 0 \\ \varphi_k \end{pmatrix} \right\rangle_{-\diamond,\diamond}^2 + \frac{1}{I_k(q)^2\|\varphi_k\|_\omega^2 + \|\psi_k\|_\omega^2} \left\langle y_0, \begin{pmatrix} \varphi_k \\ \psi_k \end{pmatrix} \right\rangle_{-\diamond,\diamond}^2.$$

We now evaluate the different contributions of the terms in the previous right-hand side.

Recall that $\|\varphi_k\|_{(0,1)} = 1$ and that, from (5.5),

$$\|\varphi_k\|_\omega \geq C > 0, \qquad \forall k \geq 1.$$

We now prove that $\|\psi_k\|_{\diamond^*} = \|\psi_k\|_{(0,1)}$ is bounded. Let

$$\widetilde{\psi}_k := \psi_k - \frac{\psi_k'(0)}{\varphi_k'(0)}\varphi_k.$$

Then, the function $\widetilde{\psi}_k$ satisfies

$$\begin{cases} (A - \nu_k)\widetilde{\psi}_k = \big(I_k(q) - q\big)\varphi_k, \\ \widetilde{\psi}_k(0) = \widetilde{\psi}_k(1) = 0, \\ \widetilde{\psi}_k'(0) = 0. \end{cases}$$

From Lemma 5.1 it comes that

$$\left|\widetilde{\psi}_k(x)\right|^2 + \frac{\gamma(x)}{\nu_k}\left|\widetilde{\psi}_k'(x)\right|^2 \leq \frac{C}{\nu_k}, \quad \forall x \in (0,1),\ \forall k \geq 1.$$

which yields

$$\left\|\widetilde{\psi}_k\right\|_{(0,1)} \leq C, \qquad \forall k \geq 1.$$

Notice that, by definition of $\widetilde{\psi}_k$, we have $\big(\psi_k - \widetilde{\psi}_k\big) \in \mathbb{R}\varphi_k$. Then, multiplying by φ_k, integrating over ω and recalling that $\langle \psi_k, \varphi_k \rangle_\omega = 0$, we obtain that

$$\psi_k = \widetilde{\psi}_k - \frac{\left\langle \widetilde{\psi}_k, \varphi_k \right\rangle_\omega}{\|\varphi_k\|_\omega^2}\varphi_k.$$

This implies that

$$\|\psi_k\|_{(0,1)} \leq \left\|\widetilde{\psi}_k\right\|_{(0,1)} \left(1 + \frac{1}{\|\varphi_k\|_\omega}\right) \leq C, \qquad \forall k \geq 1.$$

Now, getting back to (6.16), we obtain that

$$\mathcal{C}(\{\nu_k\}, y_0) \leq C\,\|y_0\|_{-\diamond}^2 \left(1 + \frac{1}{I_k(q)^2\|\varphi_k\|_\omega^2 + \|\psi_k\|_\omega^2}\right), \quad \forall k \geq 1,\ \forall y_0 \in X_{-\diamond},$$

which proves that

$$T_0(X_{-\diamond}) \leq \limsup_{k \to +\infty} \frac{-\ln\left(I_k(q)^2\|\varphi_k\|_\omega^2 + \|\psi_k\|_\omega^2\right)}{2\nu_k}.$$

To prove the converse inequality let us choose

$$y_0 = \sum_{k\geq 1} \frac{1}{\nu_k} \begin{pmatrix} \varphi_k \\ 0 \end{pmatrix}.$$

From (6.16) we obtain that for this particular choice of y_0,

$$\mathcal{C}(\{\nu_k\}, y_0) = \frac{1}{\nu_k^2} \frac{1}{I_k(q)^2 \|\varphi_k\|_\omega^2 + \|\psi_k\|_\omega^2}, \quad \forall k \geq 1.$$

Thus,

$$T_0(X_{-\diamond}) \geq T_0(y_0) = \limsup_{k\to+\infty} \frac{-\ln\left(I_k(q)^2\|\varphi_k\|_\omega^2 + \|\psi_k\|_\omega^2\right)}{2\nu_k}.$$

This ends the proof of Proposition 6.3. ∎

6.3.3. A second formula for the minimal time with disjoint control and coupling domains.

The main result of this section is the following characterization of the minimal null control time.

PROPOSITION 6.4. *Let $\omega \subset (0,1)$ be a non empty open set with a finite number of connected components. Let $q \in L^\infty(0,1;\mathbb{R})$ be such that $\mathrm{Supp}(q) \cap \omega = \varnothing$. Assume that (6.10) holds. Then, the minimal null control time for system (6.5) is given by*

$$T_0(X_{-\diamond}) = \limsup_{k\to+\infty} \frac{-\ln \mathcal{M}_k\big((I_k(q)-q)\varphi_k,\, \omega\big)}{\nu_k}.$$

The main advantage of this formulation with respect to the one proved in Proposition 6.3 is that it does not involve ψ_k. As we prove in Section 6.3.4, this formula allows to recover the values of $T_0(X_{-\diamond})$ proved in the literature for various geometric configurations. It also allows to prove new results for this system (see Proposition 6.8).

REMARK 6.4. *Notice that the assumption $\mathrm{Supp}(q) \cap \omega = \varnothing$ is necessary for this formulation. Indeed, if $q = 1$ and ω is an interval then, from [25], null controllability holds in any time $T > 0$ but $I_k(q) - q = 0$ for any $k \geq 1$. However, the assumption $\mathrm{Supp}(q) \cap \omega = \varnothing$ is not restrictive for our study as it is the setting in which a minimal null control time can occur.*

Using Proposition 6.3, the proof of Proposition 6.4 consists in comparing the asymptotic behaviors of $\mathcal{M}_k\big((I_k(q)-q)\varphi_k, \omega\big)$ and

$$I_k(q)^2\|\varphi_k\|_\omega^2 + \|\psi_k\|_\omega^2.$$

To do so we will use the following technical lemma. To improve the readability we postpone its proof to Appendix E.

LEMMA 6.5. *Let $\omega \subset (0,1)$ be a non empty open set with a finite number of connected components.*

i. There exists $K \in \mathbb{N}^*$ and $C > 0$ such that for any $k \geq K$, any $F \in L^2(0,1;\mathbb{R})$ and any u satisfying the differential equation

$$(A - \nu_k)u = F,$$

we have

$$\mathcal{M}_k(F,\omega) \leq C\Big(\sqrt{\nu_k}\|u\|_\omega + \sqrt{\nu_k}\,(|u(0)| + |u(1)|) + \|F\|_\omega\Big).$$

ii. There exists $K \in \mathbb{N}^*$ and $C > 0$ such that for any $k \geq K$ and any $F \in L^2(0,1;\mathbb{R})$, there exists u satisfying

$$\begin{cases} (A - \nu_k)u = F, \\ u(0) = u(1) = 0, \end{cases}$$

such that

$$\left(\sqrt{\nu_k}\|u\|_\omega - \|F\|_\omega\right) \leq C\mathcal{M}_k(F,\omega).$$

We now turn to the proof of Proposition 6.4.

Proof (of Proposition 6.4). Recall that, from Proposition 6.3,

$$T_0(X_{-\diamond}) = \limsup_{k\to+\infty} \frac{-\ln\left(I_k(q)^2\|\varphi_k\|_\omega^2 + \|\psi_k\|_\omega^2\right)}{2\nu_k}.$$

As $\mathrm{Supp}(q) \cap \omega = \varnothing$, applying point i. of Lemma 6.5 to ψ_k yields, for $k \geq K$,

$$\begin{aligned} \mathcal{M}_k\left((I_k(q) - q)\varphi_k, \omega\right)^2 &\leq C\left(\nu_k\|\psi_k\|_\omega^2 + \|(I_k(q) - q)\varphi_k\|_\omega^2\right) \\ &\leq C\nu_k\left(\|\psi_k\|_\omega^2 + I_k(q)^2\|\varphi_k\|_\omega^2\right). \end{aligned}$$

Thus,

$$T_0(X_{-\diamond}) \leq \limsup_{k\to+\infty} \frac{-\ln \mathcal{M}_k\big((I_k(q) - q)\varphi_k,\, \omega\big)}{\nu_k}.$$

We now prove the converse inequality. Let u be the function given by the point ii. of Lemma 6.5 with $F = \big(I_k(q) - q\big)\varphi_k$. Notice that there exists $\alpha \in \mathbb{R}$ such that $u = \psi_k + \alpha\varphi_k$. Then, as $\langle\varphi_k, \psi_k\rangle_\omega = 0$, we have $\|\psi_k\|_\omega \leq \|u\|_\omega$. Thus, using the estimate given by point ii. of Lemma 6.5 and the assumption $\mathrm{Supp}(q) \cap \omega = \varnothing$, we obtain that, for any $k \geq K$,

$$C\nu_k\|\psi_k\|_\omega^2 \leq \mathcal{M}_k(F,\omega)^2 + \|F\|_\omega^2 \leq \mathcal{M}_k(F,\omega)^2 + I_k(q)^2\|\varphi_k\|_\omega^2.$$

This yields

$$\|\psi_k\|_\omega^2 + I_k(q)^2\|\varphi_k\|_\omega^2 \leq C\left(\mathcal{M}_k(F,\omega)^2 + I_k(q)^2\|\varphi_k\|_\omega^2\right). \tag{6.17}$$

We denote by $\mathfrak{C}_1, \ldots, \mathfrak{C}_N$ the connected components of $\overline{(0,1)\backslash\omega}$. As $\mathrm{Supp}(q) \cap \omega = \varnothing$, notice that

$$\begin{aligned} \sum_{j=1}^N \int_{\mathfrak{C}_j} F(x)\varphi_k(x)\mathrm{d}x &= I_k(q)\sum_{j=1}^N \int_{\mathfrak{C}_j} \varphi_k^2(x)\mathrm{d}x - \sum_{j=1}^N \int_{\mathfrak{C}_j} q(x)\varphi_k^2(x)\mathrm{d}x \\ &= I_k(q)\left(1 - \|\varphi_k\|_\omega^2\right) - I_k(q) \\ &= -I_k(q)\|\varphi_k\|_\omega^2. \end{aligned}$$

Thus, from (5.5) we deduce that

$$|I_k(q)| \leq C\mathcal{M}_k(F,\omega).$$

Plugging it into (6.17) we obtain

$$\|\psi_k\|_\omega^2 + I_k(q)^2\|\varphi_k\|_\omega^2 \leq C\mathcal{M}_k(F,\omega)^2.$$

This implies that

$$T_0(X_{-\diamond}) \geq \limsup_{k\to+\infty} \frac{-\ln \mathcal{M}_k\big((I_k(q)-q)\varphi_k,\, \omega\big)}{\nu_k}$$

and ends the proof of Proposition 6.4. ∎

6.3.4. Application of the minimal null control time formulas.

Unification of previously known results.

Let us prove that the obtained results unifies previous characterizations given in the literature and stated in Section 6.3.1. Recall that φ_k satisfies (5.5).

• Let us consider the setting studied in [6] *i.e.* $\omega = (a,b)$ and $\mathrm{Supp}(q) \subset (b,1)$.

In this case, $\overline{(0,1)\backslash\omega}$ has at most two connected components both touching the boundary of $(0,1)$. Thus, setting

$$F = (I_k(q)-q)\varphi_k$$

we obtain

$$\mathcal{M}_k\big(F,\omega\big) = \max\left\{\left|\int_0^a F(x)\varphi_k(x)\mathrm{d}x\right|, \left|\int_b^1 F(x)\varphi_k(x)\mathrm{d}x\right|\right\}.$$

Using the assumption $\mathrm{Supp}(q) \subset (b,1)$ we get

$$\left|\int_0^a F(x)\varphi_k(x)\mathrm{d}x\right| = |I_k(q)| \int_0^a \varphi_k^2(x)\mathrm{d}x,$$

and

$$\left|\int_b^1 F(x)\varphi_k(x)\mathrm{d}x\right| = \left|I_k(q)\int_b^1 \varphi_k^2(x)\mathrm{d}x - I_k(q)\right| = |I_k(q)| \int_0^b \varphi_k^2(x)\mathrm{d}x.$$

Thus,

$$\mathcal{M}_k\big(F,\omega\big) = |I_k(q)| \int_0^b \varphi_k^2(x)\mathrm{d}x.$$

Recall that from (5.5)

$$\inf_{k\geq 1} \int_0^b \varphi_k^2(x)\mathrm{d}x > 0.$$

This implies that approximate controllability holds if and only if

$$I_k(q) \neq 0, \qquad \forall k \geq 1,$$

and in this case that

$$T_0(X_{-\diamond}) = \limsup_{k\to+\infty} \frac{-\ln |I_k(q)|}{\nu_k}.$$

Thus we recover the result proved in [6] and extend it to a general Sturm-Liouville operator.

• Let us now consider the setting studied in [7] *i.e.* $\omega = (a,b)$ and $\mathrm{Supp}(q) \cap \omega = \varnothing$. Again, setting

$$F = (I_k(q)-q)\varphi_k$$

we obtain

$$\mathcal{M}_k\big(F,\omega\big) = \max\left\{\left|\int_0^a F(x)\varphi_k(x)\mathrm{d}x\right|, \left|\int_b^1 F(x)\varphi_k(x)\mathrm{d}x\right|\right\}.$$

Using the notations introduced in (6.12) we have

$$\int_0^a F(x)\varphi_k(x)\mathrm{d}x = I_k(q)\int_0^a \varphi_k^2(x)\mathrm{d}x - I_{1,k}(q) \tag{6.18}$$

and

$$\int_b^1 F(x)\varphi_k(x)\mathrm{d}x = I_k(q)\int_b^1 \varphi_k^2(x)\mathrm{d}x - I_{2,k}(q). \tag{6.19}$$

Thus,

$$\mathcal{M}_k\big(F,\omega\big) \leq 2\max\left\{|I_k(q)|, |I_{1,k}(q)|, |I_{2,k}(q)|\right\}.$$

Conversely, using (6.18) and (6.19) we have

$$\begin{aligned}
&\int_0^a F(x)\varphi_k(x)\mathrm{d}x + \int_b^1 F(x)\varphi_k(x)\mathrm{d}x\\
&= I_k(q)\left(\int_0^a \varphi_k^2(x)\mathrm{d}x + \int_b^1 \varphi_k^2(x)\mathrm{d}x\right) - (I_{1,k}(q) + I_{2,k}(q))\\
&= -I_k(q)\int_a^b \varphi_k(x)^2\mathrm{d}x
\end{aligned}$$

where we have used that $I_k(q) = I_{1,k}(q) + I_{2,k}(q)$. Thus, from (5.5) we get

$$|I_k(q)| \leq C\mathcal{M}_k\big(F,\omega\big).$$

Using (6.18) or (6.19) and the previous inequality we obtain

$$|I_{j,k}(q)| \leq C\mathcal{M}_k\big(F,\omega\big), \qquad \forall j \in \{1,2\}.$$

Thus,

$$\max\left\{|I_k(q)|, |I_{1,k}(q)|, |I_{2,k}(q)|\right\} \leq C\mathcal{M}_k\big(F,\omega\big).$$

This implies that approximate controllability holds if and only if

$$\max\left\{|I_k(q)|, |I_{1,k}(q)|, |I_{2,k}(q)|\right\} \neq 0, \qquad \forall k \geq 1$$

and in this case

$$T_0(X_{-\diamond}) = \limsup_{k\to+\infty} \frac{-\ln\max\left\{|I_k(q)|, |I_{1,k}(q)|, |I_{2,k}(q)|\right\}}{\nu_k}.$$

Thus we recover the result proved in [7] and extend it to a general Sturm-Liouville operator.

- Let us finally consider the setting studied in [25].

PROPOSITION 6.6. *Assume that there exists an open set $\omega_0 \subset \omega$ such that $q(x) \geq q_0 > 0$ for almost every $x \in \omega_0$. Then, system (6.5) is null controllable in any time $T > 0$.*

Even though this result is already known from [25], we provide here a proof without Carleman estimates. The same result holds if there exists an open set $\omega_0 \subset \omega$ such that $q(x) \leq q_0 < 0$ for almost every $x \in \omega_0$.

Proof. Here we consider the minimal time characterization given by Proposition 6.3 and prove that

$$I_k(q)^2\|\varphi_k\|_\omega^2 + \|\psi_k\|_\omega^2 \geq r_k \quad \text{with} \quad \limsup_{k\to+\infty} \frac{-\ln r_k}{2\nu_k} = 0.$$

As we seek for a lower bound, due to our assumptions, we can restrict ω to an interval (a, b) such that $q(x) \geq q_0 > 0$ for almost every $x \in \omega$. We now cut ω into the following pieces

- $\omega_1 = (a, a + \ell)$ with ℓ sufficiently small (to be determined later on);
- $\omega_2 = (b - \ell, b)$ with ℓ sufficiently small (to be determined later on);
- $\widetilde{\omega} = \omega_1 \cup \omega_2$;
- $\mathfrak{C}_0 = \left[\frac{a+b}{2} - \frac{b-a}{6}, \frac{a+b}{2} + \frac{b-a}{6}\right]$;
- $\mathfrak{C} = [a + \ell, b - \ell]$.

This procedure is summarized in Figure 2

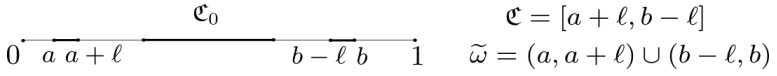

FIGURE 2. *Cutting of* $\omega = (a, b)$

Notice that for any $k \geq 1$,

$$I_k(q)^2\|\varphi_k\|_\omega^2 + \|\psi_k\|_\omega^2 \geq I_k(q)^2\|\varphi_k\|_{\widetilde{\omega}}^2 + \|\psi_k\|_{\widetilde{\omega}}^2.$$

From (5.5), there exists $\alpha_1 > 0$ depending on γ, c and $\mathfrak{C}_0$ such that

$$\int_{\mathfrak{C}_0} \varphi_k^2(x)\mathrm{d}x \geq \alpha_1, \qquad \forall k \geq 1. \tag{6.20}$$

Following closely the proof of item i of Lemma 6.5 with a careful tracking of the dependency with respect to l we prove the following lemma.

LEMMA 6.7. *There exists $\alpha_2 > 0$ depending on γ and c such that for any $\ell < \frac{b-a}{3}$, there exists $K \in \mathbb{N}^*$ such that for any $k \geq K$, any $F \in L^2(0, 1; \mathbb{R})$ and any u satisfying the differential equation*

$$(A - \nu_k)u = F,$$

we have

$$\sqrt{\ell}\left|\int_{\mathfrak{C}} F(x)\varphi_k(x)\mathrm{d}x\right| \leq \alpha_2\sqrt{\nu_k}\|u\|_{\widetilde{\omega}} + \alpha_2\ell\|F\|_{\widetilde{\omega}}.$$

To improve the reading, the proof of Lemma 6.7 is postponed at the end of the current proof (see page 62).

Let $\alpha_2 > 0$ be the constant given by Lemma 6.7 and assume in all what follows that $\ell > 0$ is fixed such that

$$\sqrt{\ell} < \min\left\{\sqrt{\frac{b-a}{3}}, \frac{q_0\alpha_1}{2\alpha_2\|q\|_{L^\infty(0,1)}}\right\}. \tag{6.21}$$

Let $K \in \mathbb{N}^*$ be the index given by Lemma 6.7. In the rest of the proof, we assume that $k \geq K$.

$\star$ Let us first consider the case $I_k(q) = 0$. Applying Lemma 6.7, with $u = \psi_k$ and $F = -q\varphi_k$ we obtain

$$\sqrt{\ell}\left|\int_{\mathfrak{C}} q(x)\varphi_k^2(x)\mathrm{d}x\right| \leq \alpha_2\sqrt{\nu_k}\|\psi_k\|_{\widetilde{\omega}} + \alpha_2\ell\|q\varphi_k\|_{\widetilde{\omega}}.$$

As $\ell < \frac{b-a}{3}$ we have $\mathfrak{C}_0 \subset \mathfrak{C}$ and thus

$$\left|\int_{\mathfrak{C}} q(x)\varphi_k^2(x)\mathrm{d}x\right| \geq q_0\int_{\mathfrak{C}_0}\varphi_k^2(x)\mathrm{d}x \geq q_0\alpha_1.$$

Notice also that, since $\|\varphi_k\|_{(0,1)} = 1$, we have

$$\|q\varphi_k\|_{\widetilde{\omega}} \leq \|q\|_{L^\infty(0,1)}.$$

Gathering these estimates and using (6.21) we obtain

$$\begin{aligned}\alpha_2\sqrt{\nu_k}\|\psi_k\|_{\widetilde{\omega}} &\geq \sqrt{\ell}\left|\int_{\mathfrak{C}} q(x)\varphi_k^2(x)\mathrm{d}x\right| - \alpha_2\ell\|q\varphi_k\|_{\widetilde{\omega}}\\ &\geq \sqrt{\ell}\left(q_0\alpha_1 - \alpha_2\sqrt{\ell}\|q\|_{L^\infty(0,1)}\right) \geq \sqrt{\ell}\,\frac{q_0\alpha_1}{2}\end{aligned}$$

which gives the desired estimate in the case $I_k(q) = 0$.

$\star$ Let us now consider the case $I_k(q) \neq 0$. Let u be the solution of the Cauchy problem

$$\begin{cases}(A-\nu_k)u = -q\varphi_k,\\ u(0) = \psi_k(0),\\ u'(0) = \psi_k'(0).\end{cases}$$

The same analysis as in the previous step yields

$$\alpha_2\sqrt{\nu_k}\|u\|_{\widetilde{\omega}} \geq \sqrt{\ell}\,\frac{q_0\alpha_1}{2}. \tag{6.22}$$

Notice that

$$\begin{cases}(A-\nu_k)(\psi_k - u) = I_k(q)\varphi_k,\\ (\psi_k-u)(0) = 0,\\ (\psi_k-u)'(0) = 0.\end{cases}$$

From Lemma 5.1, there exists $C > 0$ depending only on γ and c such that

$$\|\psi_k - u\|_{L^\infty(0,1)} \leq \frac{C}{\sqrt{\nu_k}}|I_k(q)|.$$

Then, from (5.5), we deduce that

$$\begin{aligned}\|u\|_{\widetilde{\omega}}^2 &\leq 2\big(\|\psi_k\|_{\widetilde{\omega}}^2 + \|\psi_k - u\|_{L^\infty(0,1)}^2\big)\\ &\leq 2\left(\|\psi_k\|_{\widetilde{\omega}}^2 + \frac{C}{\nu_k}|I_k(q)|^2\right)\\ &\leq C\left(\|\psi_k\|_{\widetilde{\omega}}^2 + |I_k(q)|^2\|\varphi_k\|_{\widetilde{\omega}}^2\right).\end{aligned}$$

Together with (6.22) this gives the desired estimate in the case $I_k(q) \neq 0$ and ends the proof of Proposition 6.6. ∎

To complete the proof of Proposition 6.6 let us prove Lemma 6.7.

Proof (of Lemma 6.7). From (5.7), there exists $C > 0$ depending on γ and c such that

$$\|\varphi_k\|_{L^\infty(0,1)} + \frac{1}{\sqrt{\nu_k}}\|\varphi_k'\|_{L^\infty(0,1)} \leq C, \qquad \forall k \geq 1. \tag{6.23}$$

Integrating by parts, we obtain

$$\begin{aligned}\int_{\mathfrak{C}} F(x)\varphi_k(x)\mathrm{d}x &= \int_{a+\ell}^{b-\ell} (A-\nu_k)u(x)\varphi_k(x)\mathrm{d}x \\ &= -\big(\gamma u'\varphi_k\big)(b-\ell) + \big(\gamma u'\varphi_k\big)(a+\ell) \\ &\quad + \big(u\gamma\varphi_k'\big)(b-\ell) - \big(u\gamma\varphi_k'\big)(a+\ell).\end{aligned}$$

Using (6.23) we obtain

$$\begin{aligned}\frac{1}{\sqrt{\nu_k}}\left|\int_{\mathfrak{C}} F(x)\varphi_k(x)\mathrm{d}x\right| &\leq C\|\sqrt{\gamma}\|_{L^\infty}\left(|u(a+\ell)| + \frac{\sqrt{\gamma(a+\ell)}}{\sqrt{\nu_k}}|u'(a+\ell)|\right) \\ &\quad + C\|\sqrt{\gamma}\|_{L^\infty}\left(|u(b-\ell)| + \frac{\sqrt{\gamma(b-\ell)}}{\sqrt{\nu_k}}|u'(b-\ell)|\right).\end{aligned}$$

Let $\lambda_0 > 0$ by given Corollary E.1 (for an interval of length ℓ) and let $K \in \mathbb{N}^*$ be such that

$$k \geq K \quad \Longrightarrow \quad \nu_k \geq \lambda_0.$$

Assume that $k \geq K$. As $a + \ell \in \overline{\omega_1}$ the application of Corollary E.1 yields

$$|u(a+\ell)| + \frac{\sqrt{\gamma(a+\ell)}}{\sqrt{\nu_k}}|u'(a+\ell)| \leq \frac{C}{\sqrt{\ell}}\|u\|_{\omega_1} + \frac{C\sqrt{\ell}}{\sqrt{\nu_k}}\|F\|_{\omega_1}.$$

As $b - \ell \in \overline{\omega_2}$ the application of Corollary E.1 yields

$$|u(b-\ell)| + \frac{\sqrt{\gamma(b-\ell)}}{\sqrt{\nu_k}}|u'(b-\ell)| \leq \frac{C}{\sqrt{\ell}}\|u\|_{\omega_2} + \frac{C\sqrt{\ell}}{\sqrt{\nu_k}}\|F\|_{\omega_2}$$

which proves Lemma 6.7. ∎

Dealing with new geometric configurations.

As proved in the previous paragraph, the obtained characterization of the minimal null control time unifies the different known results for system (6.5). It also enables to study new geometric configurations for example when ω is not an interval and $\mathrm{Supp}(q) \cap \omega = \varnothing$. We provide below an example inspired by [17].

PROPOSITION 6.8. *Let $\gamma = 1$ and $c = 0$ (i.e. A is the Dirichlet Laplace operator) and let*

$$q : x \in (0,1) \mapsto \left(x - \frac{1}{2}\right)\mathbf{1}_{\left(\frac{1}{4},\frac{3}{4}\right)}(x).$$

i. If $\omega \subset \left(\frac{3}{4}, 1\right)$, then approximate controllability for system (6.5) does not hold.

ii. If $\omega = \left(0, \frac{1}{4}\right) \cup \left(\frac{3}{4}, 1\right)$, then system (6.5) is null controllable from $L^2(0,1;\mathbb{R})$ in any time $T > 0$.

Proof. In this case, we have for any $k \geq 1$,

$$\nu_k = k^2\pi^2, \qquad \varphi_k = \sqrt{2}\sin(k\pi\bullet), \qquad \widetilde{\varphi}_k = \sqrt{2}\cos(k\pi\bullet).$$

The proof of item i can be found in [17, Section 3.3.1] and relies on explicit computations: due to symmetry it comes that $I_k(q) = 0$ for any $k \geq 1$. This implies that

$$\int_0^{\inf(\omega)} q(x)\varphi_k(x)\varphi_k(x)\mathrm{d}x = I_k(q) = 0.$$

Let $\mathfrak{C}$ be any other connected component of $\overline{(0,1)\backslash\omega}$. Then $\mathfrak{C} \subset \left(\frac{3}{4}, 1\right)$. This means that $q = 0$ on $\mathfrak{C}$ which gives

$$\int_{\mathfrak{C}} q(x)\varphi_k(x)\varphi_k(x)\mathrm{d}x = \int_{\mathfrak{C}} q(x)\varphi_k(x)\widetilde{\varphi}_k(x)\mathrm{d}x = 0.$$

Thus,

$$\mathcal{M}_k(q\varphi_k, \omega) = 0, \qquad \forall k \geq 1.$$

We now turn to item ii. In this case $\overline{(0,1)\backslash\omega}$ has only one connected component which is $\left[\frac{1}{4}, \frac{3}{4}\right]$ but the key point is that it does not touch the boundary of $(0,1)$. Approximate controllability in this case was also studied in [17, Section 3.3.1]. Again for symmetry reasons we have

$$\int_{\frac{1}{4}}^{\frac{3}{4}} q(x)\varphi_k(x)\varphi_k(x)\mathrm{d}x = 0, \qquad \forall k \geq 1,$$

but

$$\int_{\frac{1}{4}}^{\frac{3}{4}} q(x)\varphi_k(x)\widetilde{\varphi}_k(x)\mathrm{d}x = \begin{cases} -\dfrac{(-1)^{\frac{k-1}{2}}}{2\pi^2k^2}, & \text{if } k \text{ is odd,} \\ -\dfrac{(-1)^{\frac{k}{2}}}{4\pi k}, & \text{if } k \text{ is even.} \end{cases}$$

This implies that for any $k \geq 1$,

$$\mathcal{M}_k\big((I_k(q) - q)\varphi_k, \omega\big) = \begin{cases} \dfrac{1}{2\pi^2k^2}, & \text{if } k \text{ is odd,} \\ \dfrac{1}{4\pi k}, & \text{if } k \text{ is even.} \end{cases}$$

Thus, from Proposition 6.4

$$T_0(X_{-\diamond}) = \limsup_{k\to+\infty} \frac{-\ln \mathcal{M}_k\big((I_k(q) - q)\varphi_k,\, \omega\big)}{\nu_k} = 0.$$

∎

7. Some extensions.

7.1. Dealing with complex valued eigenvalues.

In this section we allow the eigenvalues in Λ to be complex valued with a dominant real part (see the precise assumptions below). The resolution of block moment problems, the estimate of the cost of a block and thus the minimal null control time are obtained with really few adjustments that we detail in this section. However, as it is

already true for scalar control operators, the dependency of the cost of controllability with respect to the final time T is lost. Indeed it follows from Proposition A.2 which deeply uses real analysis. Thus in the case of complex valued sequences there is no need to consider the adaptation of the classes $\mathcal{L}(p,\varrho,a,\bar{N})$ or $\mathcal{L}(p,\varrho,a,\bar{N},a',\widetilde{N})$ which are mainly introduced in this article to obtain sharper bounds on the cost of controllability. We restrict ourselves to sequences of eigenvalues Λ in the class $\mathcal{L}_{\mathbb{C}}(\delta,p,\varrho,\mathcal{N})$ which for $\delta,\varrho>0$, $p\in\mathbb{N}^*$ and $\mathcal{N}:\mathbb{R}^+\to\mathbb{R}$ is the class of sequences $\Lambda\in\mathbb{C}^{\mathbb{N}}$ satisfying

- Parabolicity condition:
$$\Re\lambda\geq\delta|\lambda|,\quad\forall\lambda\in\Lambda.$$
- Asymptotic behavior: for any $\varepsilon>0$, we have
$$\sum_{\substack{\lambda\in\Lambda\\|\lambda|>\mathcal{N}(\varepsilon)}}\frac{1}{|\lambda|}\leq\varepsilon.$$
- Weak gap condition with parameters $\varrho>0$ and $p\in\mathbb{N}^*$:
$$\#\Lambda\cap([\mu,\mu+\varrho]+i\mathbb{R})\leq p,\quad\forall\mu>0.$$

In that case, a grouping $(G_k)_k$ should satisfy

$$\Lambda=\bigcup_{k\geq1}G_k,\quad\#G_k\leq p,\quad\operatorname{diam}(G_k)<\varrho,\inf(\Re G_{k+1})-\sup(\Re G_k)>r.$$

In this setting replacing the auxiliary moment problem (2.1) by

$$\begin{cases}\displaystyle\int_0^T v(t)\frac{(-t)^l}{l!}e^{-\overline{\lambda_j}t}\mathrm{d}t=\Omega_j^l,\quad\forall j\in[\![1,g]\!],\,\forall l\in[\![0,\alpha_j-1]\!],\\ \displaystyle\int_0^T v(t)t^le^{-\overline{\lambda}t}\mathrm{d}t=0,\quad\forall\lambda\in\Lambda\backslash G,\,\forall l\in[\![0,\eta-1]\!],\end{cases}$$

we obtain that the resolution of block moment problems stated in Theorem 1.2 still holds. The minimal null control time given in Theorem 1.3 is replaced by

$$T_0(y_0)=\limsup_{k\to+\infty}\frac{\ln\mathcal{C}(G_k,y_0)}{2\min\Re G_k}$$

with $\mathcal{C}(G,z)$ still defined in (1.41) and the function F is now defined by

$$F:\Omega=\left(\Omega_1^0,\dots,\Omega_1^{\alpha_1-1},\dots,\Omega_g^0,\dots,\Omega_g^{\alpha_g-1}\right)\in U^{|\alpha|}\mapsto\sum_{l=1}^{|\alpha|}\left\|\Omega\left[\overline{\lambda}_\bullet^{(\mu^l)}\right]\right\|_U^2.$$

In the course of the proof, one should replace in the estimates λ by $\Re\lambda$ and the use of Lagrange theorem [10, Proposition 7.14] by an inequality proved by Jensen in [27] (and recalled in [10, Proposition 6.1]).

Similarly, replacing λ by $\overline{\lambda}$ the more explicit formulas for $\mathcal{C}(G_k,y_0)$ provided by Theorems 1.8 and 1.10 also holds.

7.2. Weakening the Fattorini-Hautus test.

When studying null controllability from a closed strict subspace Y_0 of $X_{-\diamond}$, the Fattorini-Hautus test (1.28) can be a too strong requirement. Let us underline that this condition (1.28) is only used in this article for two purposes:

1. to prove that the set $\mathcal{O}(\lambda, z)$ defining the constraints is not empty (see Proposition 2.3);
2. to prove that the matrices M defined either by (1.47) or (1.51) are invertible (see Propositions 1.7 and 1.9).

Let Y_0 be a closed subspace of $X_{-\diamond}$. Following for instance [15, Proposition 1.17], it can be proved that if every assumption of (H) except (1.28) hold, then a necessary condition for approximate controllability from Y_0 can be expressed as follows: for any $\lambda \in \Lambda$, for any $\phi \in E_\lambda$ and for any $l \in [\![1, \alpha_\lambda]\!]$,

$$\Big(\mathcal{B}^*(\mathcal{A}^* - \lambda)^{\alpha_\lambda - r}\phi = 0,\ \forall r \in [\![1, l]\!]\Big) \implies \Big(P^*_{Y_0}(\mathcal{A}^* - \lambda)^{\alpha_\lambda - r}\phi = 0,\ \forall r \in [\![1, l]\!]\Big). \tag{7.1}$$

Notice that when $\dim U = 1$, this condition is equivalent to the condition formulated in [10, Section 6.2].

Let us prove that the results of this article hold (for null controllability from Y_0) under assumption (7.1) instead of (1.28), at least in the case when every eigenvalue of $\mathcal{A}^*$ is either geometrically simple or semi-simple. Let $\lambda \in \Lambda$ be such that

$$\operatorname{Ker}(\mathcal{A}^* - \lambda) \cap \operatorname{Ker} \mathcal{B}^* \neq \{0\}.$$

• If $E_\lambda \subset \operatorname{Ker} \mathcal{B}^*$ then from (7.1) it follows that $E_\lambda \subset \operatorname{Ker} P^*_{Y_0}$. Then, the equations associated with λ in the moment problem (1.30) are automatically satisfied. We can then simply forget about the eigenvalue λ in the analysis, which amounts to replace Λ by $\Lambda \backslash \{\lambda\}$ in our study.

• Otherwise we distinguish the two cases $\alpha_\lambda = 1$ and $\gamma_\lambda = 1$.

i. Assume that $\alpha_\lambda = 1$ and let I be a (non-empty) set of indices such that the family $(\mathcal{B}^*\phi_{\lambda,i})_{i\in I}$ is a basis of $\mathcal{B}^* E_\lambda$. Then, the moment equations associated to λ in the moment problem (1.30) can be reduced to

$$\int_0^T e^{-\lambda(T-t)} \langle u(t), \mathcal{B}^*\phi_{\lambda,i}\rangle_U \,\mathrm{d}t = -\left\langle y_0, e^{-T\mathcal{A}^*}\phi_{\lambda,i}\right\rangle_{-\diamond,\diamond}, \quad \forall i \in I.$$

Indeed, from (7.1), it comes that the moment equations associated with $\phi_{\lambda,j}$ for $j \in [\![1, \gamma_\lambda]\!] \backslash I$ are automatically satisfied and can be forgotten. This leads to a decrease of the geometric multiplicity replacing γ_λ by $|I|$.

ii. Assume that $\gamma_\lambda = 1$ and let us consider a basis of E_λ formed by a Jordan chain $\phi^0_\lambda, \dots, \phi^{\alpha_\lambda - 1}_\lambda$.
Let $j \in [\![1, \alpha_\lambda]\!]$ be the first index such that $\mathcal{B}^*\phi^j_\lambda \neq 0$. From (7.1), it comes that

$$P^*_{Y_0}\phi^0_\lambda = \cdots = P^*_{Y_0}\phi^{j-1}_\lambda = 0$$

and thus the moment equations associated to λ in the moment problem (1.30) can be reduced to

$$\int_0^T \left\langle u(t), e^{-(T-t)\mathcal{A}^*}\phi\right\rangle_U \mathrm{d}t = -\left\langle y_0, e^{-T\mathcal{A}^*}\phi\right\rangle_{-\diamond,\diamond}, \quad \forall \phi \in (\mathcal{A}^* - \lambda)^j E_\lambda.$$

This leads to a decrease of the algebraic multiplicity replacing α_λ by $\alpha_\lambda - j$ and shifting the associated Jordan chain.

This proves the claim.

Finally let us notice that, if on a given example there exists $\lambda \in \Lambda$ with $\alpha_\lambda \geq 2$, $\gamma_\lambda \geq 2$,

$$\operatorname{Ker}(\mathcal{A}^* - \lambda) \cap \operatorname{Ker} \mathcal{B}^* \neq \{0\},$$

then under assumption (7.1) one can iterate a finite number of steps of the form *i.* or *ii.* to reduce the moment equations associated to λ in the moment problem (1.30) to a solvable moment problem.

Appendix A. Some refinements in the case of scalar controls.

In [10], the block moment method was introduced to solve null controllability problems with scalar controls ($U = \mathbb{R}$). With respect to block moment problems, the main result of this paper is [10, Theorem 4.1]. In this work there were no assumptions on the counting function. The spectrum Λ was only assumed to satisfy $\sum_{\lambda\in\Lambda} \frac{1}{\lambda} < +\infty$. More precisely, for $p \in \mathbb{N}^*$, $\varrho > 0$ and $\mathcal{N} : (0, +\infty) \to \mathbb{R}$ it is defined in [10, Definition 2.1] the following classes

$$\mathcal{L}(p, \varrho, \mathcal{N}) = \left\{ \Lambda \in (0, +\infty)^{\mathbb{N}} \, ; \, \Lambda \text{ satisfies (1.21) and } \sum_{\substack{\lambda\in\Lambda \\ \lambda\geq\mathcal{N}(\varepsilon)}} \frac{1}{\lambda} < \varepsilon \right\}. \tag{A.1}$$

Using the slightly more restrictive condition (1.22) (or (1.24)) we can adapt the resolution of scalar block moment problems to obtain better estimates on the cost of this resolution. In particular, this allows to explicit the dependency of the various estimates with respect to the variable T (see Remark 1.7 for possible applications of such estimates). Namely, we obtain the following result.

THEOREM A.1. *Let $p \in \mathbb{N}^*$, $r, \varrho, \bar{N}, \widetilde{N} > 0$, $a \in (0, 1)$, $a' \in [0, a)$ and $\mathcal{N} : (0, +\infty) \to \mathbb{R}$. Assume that*

$$\Lambda \in \mathcal{L}(p, \varrho, \mathcal{N}) \cup \mathcal{L}(p, \varrho, a, \bar{N}) \cup \mathcal{L}(p, \varrho, a, \bar{N}, a', \widetilde{N})$$

and let $(G_k)_{k\geq 1} \in \mathcal{G}(\Lambda, p, r, \varrho)$ be an associated grouping. Recall that these classes are defined in (A.1), (1.23) and (1.25). Let $\eta \in \mathbb{N}^*$ and $T \in (0, +\infty)$.

For any $G = \{\lambda_1, \dots, \lambda_g\} \in (G_k)_k$, for any multi-index $\alpha \in \mathbb{N}^g$ with $|\alpha|_\infty \leq \eta$ and any $\omega \in \mathbb{R}^{|\alpha|}$ there exists $v_G \in L^2(0, T; \mathbb{R})$ satisfying

$$\int_0^T v_G(t) \frac{(-t)^l}{l!} e^{-\lambda_j t} \mathrm{d}t = \omega_j^l, \quad \forall j \in [\![1, g]\!], \, \forall l \in [\![0, \alpha_j - 1]\!], \tag{A.2a}$$

$$\int_0^T v_G(t) \frac{(-t)^l}{l!} e^{-\lambda t} \mathrm{d}t = 0, \quad \forall \lambda \in \Lambda \backslash G, \, \forall l \in [\![0, \eta - 1]\!], \tag{A.2b}$$

and the bound

$$\|v_G\|_{L^2(0,T;\mathbb{R})} \leq \mathcal{E}(\lambda_1) \, \mathcal{K}(T) \max_{\substack{\mu\in\mathbb{N}^g \\ \mu\leq\alpha}} \left| \omega \left[\lambda_\bullet^{(\mu)} \right] \right| \tag{A.3}$$

where the functions $\mathcal{E}$ and $\mathcal{K}$ are such that

i. for any $T \in (0, +\infty)$, for any $\varepsilon > 0$, there exists $C > 0$ depending only on ε, T, p, r, ϱ, η and $\mathcal{N}$ such that for any $\Lambda \in \mathcal{L}(p, \varrho, \mathcal{N})$ we have

$$\mathcal{E}(\lambda) \leq C \exp(\varepsilon\lambda), \quad \forall \lambda \in \Lambda;$$

ii. there exists $C > 0$ depending only on p, r, ϱ, η, a and $\bar{N}$ such that for any $\Lambda \in \mathcal{L}(p, \varrho, a, \bar{N})$ we have

$$\mathcal{E}(\lambda) \leq C \exp\left(C\lambda^a \left(1 + \log(\lambda)\right)\right), \quad \forall \lambda \in \Lambda,$$

$$\mathcal{K}(T) \leq C \exp\left(\frac{C}{T^{\frac{a}{1-a}}}\right), \quad \forall T \in (0, +\infty);$$

iii. there exists $C > 0$ depending only on p, r, ϱ, η, a, $\bar{N}$, a' and $\widetilde{N}$ such that for any $\Lambda \in \mathcal{L}(p, \varrho, a, \bar{N}, a', \widetilde{N})$ we have

$$\mathcal{E}(\lambda) \leq C \exp\left(C\lambda^a\right), \quad \forall \lambda \in \Lambda,$$

$$\mathcal{K}(T) \leq C \exp\left(\frac{C}{T^{\frac{a}{1-a}}}\right), \quad \forall T \in (0, +\infty).$$

Moreover, up to the factor $\mathcal{E}(\lambda_1)$, the estimate (A.3) is sharp: there exists a constant $C_{p,\eta,\min\Lambda} > 0$ such that any solution v_G of (A.2a) satisfy

$$\|v_G\|_{L^2(0,T;\mathbb{R})} \geq C_{p,\eta,\min\Lambda} \max_{\substack{\mu \in \mathbb{N}^g \\ \mu \leq \alpha}} \left|\omega\left[\lambda_\bullet^{(\mu)}\right]\right|. \tag{A.4}$$

The statement i. is exactly the one given in [10, Theorem 4.1]. Let us detail the necessary adjustments to obtain Theorem A.1 ii. and Theorem A.1 iii..

• Dependency with respect to the variable T

One key point of the strategy developed in [10] is to solve the scalar block moment problem in infinite time horizon (see [10, Proposition 4.5]) and then use a uniform bound on the inverse of the restriction map

$$R_{\Lambda,T} : f \in A(\Lambda, +\infty) \mapsto f_{|(0,T)} \in A(\Lambda, T),$$

where for any $T \in (0, +\infty]$,

$$A(\Lambda, T) := \overline{\mathrm{Span}\left\{t \mapsto e^{-\lambda t}\ ;\ \lambda \in \Lambda\right\}}^{L^2(0,T;\mathbb{C})}.$$

This uniform bound (see [10, Proposition 2.9]) was proved by contradiction and thus its dependency with respect to T was not explicit.

Actually, one can use instead the following result (which at that time was not known to the authors of [10]).

PROPOSITION A.2. *Let $a \in (0,1)$, $\varrho, \bar{N} > 0$ and $\Lambda \in (0, +\infty)^{\mathbb{N}}$ be a family whose counting function satisfies (1.22). There exists a constant $C_{a,\bar{N}} > 0$ depending only on a and $\bar{N}$ such that, for any $T > 0$,*

$$\|f\|_{L^2(0,+\infty)} \leq C_{a,\bar{N}} \exp\left(\frac{C_{a,\bar{N}}}{T^{\frac{a}{1-a}}}\right) \|f\|_{L^2(0,T)}, \quad \forall f \in A(\Lambda, \infty).$$

This general result is a consequence of the study of Remez-type inequalities in Müntz spaces that can be found for instance in [13, 14]. A detailed proof is proposed in [16, Theorem IV.1.18].

Then, following [10, Section 4], but using Proposition A.2 instead of [10, Proposition 2.9] leads to the bounds on $\mathcal{K}(T)$ given in cases ii. and iii..

• Bounds on $\mathcal{E}(\lambda)$.

In the estimate (A.3), the term $\mathcal{E}(\lambda)$ comes from the orthogonality condition given by the equations (A.2b).

In the proof of [10, Theorem 4.1], the bound of case i. is given by [10, Proposition 2.8]. Following exactly the proof given in [10, Section 2.1.6] but using [16, Proposition IV.1.14 and Corollary IV.1.16] (see also [16, Theorem IV.1.10] for an explicit formula for the function $\mathcal{E}$) instead of [10, Proposition 2.8] directly yield the bounds on $\mathcal{E}(\lambda)$ given in Theorem A.1 cases ii. and iii..

Appendix B. An auxiliary optimization argument.

LEMMA B.1. *Let Y be a closed subspace of $X_{-\diamond}$. Let $g \in \mathbb{N}^*$ and $\psi_1, \dots, \psi_g \in P_Y^* X_\diamond^*$. For any $y \in Y$, let*

$$\xi_y = \begin{pmatrix} \langle y, \psi_1 \rangle_{-\diamond,\diamond} \\ \vdots \\ \langle y, \psi_g \rangle_{-\diamond,\diamond} \end{pmatrix}.$$

Then, for any positive semi-definite symmetric square matrix $M \in \mathcal{M}_g(\mathbb{R})$, we have

$$\sup_{\substack{y \in Y \\ \|y\|_{-\diamond}=1}} \langle M\xi_y, \xi_y \rangle = \rho(\mathsf{G}_\psi M) \tag{B.1}$$

with $\mathsf{G}_\psi = \mathrm{Gram}_{X_\diamond^*}(\psi_1, \dots, \psi_g)$.

In the course of the proof we will use that there exists an isometric bijection $I : X_{-\diamond} \mapsto X_\diamond^*$ such that

$$\langle y, \varphi \rangle_{-\diamond,\diamond} = (Iy, \varphi)_{\diamond^*}, \quad \forall y \in X_{-\diamond}, \forall \varphi \in X_\diamond^*.$$

Note that it satisfies

$$(Iy, \varphi)_{\diamond^*} = \left(y, I^{-1}\varphi\right)_{-\diamond}, \quad \forall y \in X_{-\diamond}, \forall \varphi \in X_\diamond^*.$$

Proof. Let S be the value of the supremum in the left-hand side of (B.1). By assumption on the $(\psi_i)_i$, we first observe that the supremum can be taken on the whole space $X_{-\diamond}$ instead of Y without changing its value. Then, for any $1 \le i \le g$, we have

$$\langle y, \psi_i \rangle_{-\diamond,\diamond} = \left(y, I^{-1}\psi_i\right)_{-\diamond},$$

and therefore the value of S does not change if we take the supremum over the set

$$\widetilde{\Psi} = \mathrm{Span}(\widetilde{\psi}_1, \dots, \widetilde{\psi}_g) \subset X_{-\diamond},$$

with

$$\widetilde{\psi}_i = I^{-1}\psi_i. \tag{B.2}$$

We write any element $y \in \widetilde{\Psi}$ as follows $y = \sum_{i=1}^g x_i \widetilde{\psi}_i$, with $x = (x_j)_{j \in [\![1,g]\!]} \in \mathbb{R}^g$ so that we can compute

$$\left(y, \widetilde{\psi}_i\right)_{-\diamond} = \sum_{j=1}^g x_j \left(\widetilde{\psi}_i, \widetilde{\psi}_j\right)_{-\diamond} = (\mathsf{G}_{\widetilde{\psi}} x)_i, \quad \forall i \in [\![1, g]\!],$$

$$(y,y)_{-\diamond} = \sum_{i=1}^{g}\sum_{j=1}^{g} x_i x_j \left(\widetilde{\psi_i},\widetilde{\psi_j}\right)_{-\diamond} = \left\langle \mathsf{G}_{\widetilde{\psi}} x, x\right\rangle,$$

where $\mathsf{G}_{\widetilde{\psi}}$ is the Gram matrix in $X_{-\diamond}$ of the family $\{\widetilde{\psi}_1,...,\widetilde{\psi}_g\}$. Using that I is an isometry from $X_{-\diamond}$ onto $X_{\diamond}^*$ it actually appears that

$$\mathsf{G}_{\widetilde{\psi}} = \mathsf{G}_{\psi}.$$

Finally, we have proved that

$$\xi_y = \mathsf{G}_\psi x, \quad \text{and} \quad \|y\|_{-\diamond}^2 = \langle \mathsf{G}_\psi x, x\rangle .$$

The supremum we are looking for thus reads

$$S = \sup_{\substack{x\in\mathbb{R}^g \\ \langle \mathsf{G}_\psi x, x\rangle = 1}} \langle M\mathsf{G}_\psi x, \mathsf{G}_\psi x\rangle.$$

- By compactness, we know that this supremum is actually achieved at some point $x_0 \in \mathbb{R}^g$, that is

$$\langle M\mathsf{G}_\psi x_0, \mathsf{G}_\psi x_0\rangle = S, \quad \text{and} \quad \langle \mathsf{G}_\psi x_0, x_0\rangle = 1.$$

The Lagrange multiplier theorem gives that there exists $\lambda \in \mathbb{R}$ such that

$$\langle M\mathsf{G}_\psi x_0, \mathsf{G}_\psi h\rangle = \lambda \langle \mathsf{G}_\psi x_0, h\rangle, \quad \forall h \in \mathbb{R}^g. \tag{B.3}$$

We get

$$\mathsf{G}_\psi M \mathsf{G}_\psi x_0 = \lambda \mathsf{G}_\psi x_0,$$

and since $\mathsf{G}_\psi x_0 \neq 0$ (we recall that $\langle \mathsf{G}_\psi x_0, x_0\rangle = 1$), we deduce that λ is an eigenvalue of $\mathsf{G}_\psi M$ and therefore

$$\lambda \leq \rho(\mathsf{G}_\psi M).$$

Moreover, taking $h = x_0$ in (B.3), we get

$$\langle M\mathsf{G}_\psi x_0, \mathsf{G}_\psi x_0\rangle = \lambda \langle \mathsf{G}_\psi x_0, x_0\rangle = \lambda,$$

and thus $\lambda = S$. We have thus proved that

$$S \leq \rho(\mathsf{G}_\psi M).$$

- If $\rho(\mathsf{G}_\psi M) = 0$, the claim is proved. If not, we set

$$\lambda = \rho(\mathsf{G}_\psi M) = \rho(M\mathsf{G}_\psi) = \rho\left(\mathsf{G}_\psi^{\frac{1}{2}} M \mathsf{G}_\psi^{\frac{1}{2}}\right),$$

which is positive and which is an eigenvalue of the three matrices above. In particular, there exists $x_0 \in \mathbb{R}^g \setminus \{0\}$ such that

$$M\mathsf{G}_\psi x_0 = \lambda x_0.$$

Taking the inner product with $\mathsf{G}_\psi x_0$ we obtain

$$\langle M\mathsf{G}_\psi x_0, \mathsf{G}_\psi x_0\rangle = \lambda \langle x_0, \mathsf{G}_\psi x_0\rangle,$$

and since $\langle x_0, \mathsf{G}_\psi x_0\rangle = \left\|\mathsf{G}_\psi^{\frac{1}{2}} x_0\right\|$ cannot be equal to zero, we deduce that

$$\lambda \leq S,$$

and the proof is complete. ∎

Appendix C. Solving general block moment problems.

As this paper is oriented towards control theory we do not deal with the most general block moment problems. Indeed, in Theorem 1.2, the considered block moment problems have a specific right-hand side which is a linear form. This formalism is chosen in order to avoid exhibiting a particular basis of the generalized eigenspaces. The price to pay is this restriction on the considered right-hand sides. However the proofs detailed in Sections 2 and 4 directly lead to the following more general results.

The study with a group composed of geometrically simple eigenvalues (see Sections 4.1 and 4.2) leads to the following theorem.

THEOREM C.1. *Let $p \in \mathbb{N}^*$, $r, \varrho, \bar{N}, \widetilde{N} > 0$, $a \in (0,1)$, $a' \in [0,a)$ and $\mathcal{N} : (0,+\infty) \to \mathbb{R}$. Assume that*

$$\Lambda \in \mathcal{L}(p,\varrho,\mathcal{N}) \cup \mathcal{L}(p,\varrho,a,\bar{N}) \cup \mathcal{L}(p,\varrho,a,\bar{N},a',\widetilde{N})$$

and let $(G_k)_{k\geq 1} \in \mathcal{G}(\Lambda,p,r,\varrho)$ be an associated grouping. Recall that these classes are defined in (A.1), (1.23) and (1.25). Let $\eta \in \mathbb{N}^*$ and $T \in (0,+\infty)$.

For any $G = \{\lambda_1,\dots,\lambda_g\} \in (G_k)_k$, for any multi-index $\alpha \in \mathbb{N}^g$ with $|\alpha|_\infty \leq \eta$, any $\omega \in \mathbb{R}^{|\alpha|}$ and any $b \in U^{|\alpha|}$ with

$$b_j^0 \neq 0, \qquad \forall j \in [\![1,g]\!],$$

there exists $v \in L^2(0,T;U)$ satisfying

$$\int_0^T \left\langle v(t), (e_t b)\left[\lambda_j^{(l+1)}\right]\right\rangle_U \mathrm{d}t = \omega_j^l, \quad \forall j \in [\![1,g]\!],\ \forall l \in [\![0,\alpha_j-1]\!], \tag{C.1a}$$

$$\int_0^T v(t) t^l e^{-\lambda t}\,\mathrm{d}t = 0, \quad \forall \lambda \in \Lambda \setminus G, \forall l \in [\![0,\eta-1]\!]. \tag{C.1b}$$

Moreover, we have the following estimate

$$\|v\|^2_{L^2(0,T;U)} \leq \mathcal{E}(\lambda_1)\,\mathcal{K}(T)\,\left\langle M^{-1}\xi,\xi\right\rangle, \qquad \text{where } \xi := \begin{pmatrix} \omega\left[\lambda_\bullet^{(\mu^1)}\right] \\ \vdots \\ \omega\left[\lambda_\bullet^{\left(\mu^{|\alpha|}\right)}\right] \end{pmatrix},$$

the sequence $(\mu^p)_{p\in[\![0,|\alpha_k|]\!]}$ is defined in (1.37), the associated matrix M is defined in (1.47) and the functions $\mathcal{E}$ and $\mathcal{K}$ satisfy the bounds given in Theorem A.1.

Moreover, any $v \in L^2(0,T;U)$ such that (C.1a) holds satisfy

$$\|v\|^2_{L^2(0,T;U)} \geq C_{p,\eta,\min\Lambda}\left\langle M^{-1}\xi,\xi\right\rangle$$

for some $C_{p,\eta,\min\Lambda} > 0$.

REMARK C.1. *As detailed in Remark 1.10, when the eigenvalues in G are also algebraically simple, i.e. $\alpha_\lambda = \gamma_\lambda = 1$ for any $\lambda \in G$, the expression of ξ reduces to*

$$\xi := \begin{pmatrix} \omega[\lambda_1] \\ \vdots \\ \omega[\lambda_1,\dots,\lambda_g] \end{pmatrix},$$

and the expression of M reduces to the one given in (1.49).

The study with a group composed of semi-simple eigenvalues (see Section 4.3) leads to the following theorem.

THEOREM C.2. *Let $p \in \mathbb{N}^*$, $r, \varrho, \bar{N}, \widetilde{N} > 0$, $a \in (0,1)$, $a' \in [0,a)$ and $\mathcal{N} : (0,+\infty) \to \mathbb{R}$. Assume that*

$$\Lambda \in \mathcal{L}(p,\varrho,\mathcal{N}) \cup \mathcal{L}(p,\varrho,a,\bar{N}) \cup \mathcal{L}(p,\varrho,a,\bar{N},a',\widetilde{N})$$

and let $(G_k)_{k\geq 1} \in \mathcal{G}(\Lambda,p,r,\varrho)$ be an associated grouping. Recall that these classes are defined in (A.1), (1.23) and (1.25). Let $\gamma_1,\dots,\gamma_g \in \mathbb{N}^*$ and $\gamma_G = \gamma_1 + \cdots + \gamma_g$. Let $\eta \in \mathbb{N}^*$ and $T \in (0,+\infty)$.

For any $G = \{\lambda_1,\dots,\lambda_g\} \in (G_k)_k$, for any $\left(\omega_{j,i}\right)_{j\in[\![1,g]\!],i\in[\![1,\gamma_j]\!]} \in U^{\gamma_G}$ and any $\left(b_{j,i}\right)_{j\in[\![1,g]\!],i\in[\![1,\gamma_j]\!]} \in U^{\gamma_G}$ such that $b_{j,1},\dots,b_{j,\gamma_j}$ are linearly independent for every $j \in [\![1,g]\!]$, there exists $v \in L^2(0,T;U)$ satisfying

$$\int_0^T \left\langle v(t), e^{-\lambda_j t} b_{j,i} \right\rangle_U \mathrm{d}t = \omega_{j,i}, \quad \forall j \in [\![1,g]\!],\ \forall i \in [\![1,\gamma_j]\!], \tag{C.2a}$$

$$\int_0^T v(t) t^l e^{-\lambda t} \mathrm{d}t = 0, \quad \forall \lambda \in \Lambda \backslash G,\ \forall l \in [\![0,\eta-1]\!]. \tag{C.2b}$$

Moreover, we have the following estimate

$$\|v\|^2_{L^2(0,T;U)} \leq \mathcal{E}(\lambda_1)\,\mathcal{K}(T)\,\left\langle M^{-1}\xi,\xi\right\rangle,$$

where $\xi \in \mathbb{R}^{\gamma_G}$ is defined by blocks with

$$\xi_j := \begin{pmatrix} \omega_{j,1} \\ \vdots \\ \omega_{j,g} \end{pmatrix},$$

the matrix M is defined in (1.51) and the functions $\mathcal{E}$ and $\mathcal{K}$ satisfy the bounds given in Theorem A.1.

Moreover, any $v \in L^2(0,T;U)$ such that (C.2a) holds satisfy

$$\|v\|^2_{L^2(0,T;U)} \geq C_{p,\min\Lambda} \left\langle M^{-1}\xi,\xi\right\rangle$$

for some $C_{p,\min\Lambda} > 0$.

Appendix D. Post-processing formulas.

The minimal null control time given in Theorem 1.3, together with the computation of the contribution of each group given in Theorems 1.8 and 1.10, allow to answer the question of minimal null control time for a wide variety of one dimensional parabolic control problems. However, for a given problem, the precise estimate of the quantity of interest $\left\langle M^{-1}\xi,\xi\right\rangle$ can remain a tricky question.

There is no normalization condition on the eigenvectors and no uniqueness of the considered Jordan chains. Thus, it happens that there are choices for which the quantity of interest $\left\langle M^{-1}\xi,\xi\right\rangle$ is easier to compute (see for instance Remark 1.9). We gather here some results that are use in Sections 5 and 6 to estimate such quantities.

We will make an intensive use of the following reformulation. Let $n \in \mathbb{N}^*$ and let $T, M \in \mathrm{GL}_n(\mathbb{R})$. For any $\xi \in \mathbb{R}^n$, let $\tilde{\xi} := T\xi$. Then,

$$\left\langle M^{-1}\xi,\xi\right\rangle = \left\langle M^{-1}T^{-1}\tilde{\xi}, T^{-1}\tilde{\xi}\right\rangle = \left\langle \widetilde{M}^{-1}\tilde{\xi},\tilde{\xi}\right\rangle \tag{D.1}$$

where

$$\widetilde{M} := TM^tT. \tag{D.2}$$

As the matrix M is a sum of Gram matrices we will also use the following lemma.

LEMMA D.1. *Let X be an Hilbert space. Let $n \in \mathbb{N}^*$ and $e = (e_1, \dots, e_n) \in X^n$. Let $T \in \mathcal{M}_n(\mathbb{R})$. Then,*

$$T\mathrm{Gram}_X(e_1, \dots, e_n)^tT = \mathrm{Gram}_X\big((Te)_1, \dots, (Te)_n\big)$$

where, for any $i \in [\![1, n]\!]$, $(Te)_i$ is defined by

$$(Te)_i := \sum_{j=1}^{n} T_{i,j}e_j.$$

Proof. For any $\omega \in \mathbb{R}^n$, it comes that

$$\langle T\mathrm{Gram}_X(e_1, \dots, e_n)^tT\omega, \omega\rangle = \langle \mathrm{Gram}_X(e_1, \dots, e_n)\left({}^tT\omega\right), \left({}^tT\omega\right)\rangle \tag{D.3}$$

$$= \left\| \sum_{i=1}^{n} ({}^tT\omega)_i e_i \right\|^2 \tag{D.4}$$

$$= \left\| \sum_{i=1}^{n}\sum_{j=1}^{n} T_{j,i}\omega_j e_i \right\|^2 \tag{D.5}$$

$$= \left\| \sum_{j=1}^{n} \omega_j (Te)_j \right\|^2 \tag{D.6}$$

$$= \langle \mathrm{Gram}_X\big((Te)_1, \dots, (Te)_n\big)\omega, \omega\rangle. \tag{D.7}$$

∎

Depending on the phenomenon at stake on actual examples, with a suitable choice of $\tilde{\xi}$ (i.e. of T), the quantity $\langle \widetilde{M}^{-1}\tilde{\xi}, \tilde{\xi}\rangle$ can be easier to estimate than $\langle M^{-1}\xi, \xi\rangle$.

D.1. Dilatations.

Notice that

$$\left\langle \widetilde{M}^{-1}\tilde{\xi}, \tilde{\xi} \right\rangle \leq \|\widetilde{M}^{-1}\| \, \|\tilde{\xi}\|^2.$$

When the minimal null control time can be estimated with rough estimates (this can only characterize the minimal time when $T_0 = 0$), it can simplify the computations to have a bounded $\|\tilde{\xi}\|$. To do so, it is convenient to consider dilatations of ξ.

Let X be an Hilbert space. Let $n \in \mathbb{N}^*$ and $e_1, \dots, e_n \in X$. Let $\xi \in \mathbb{R}^n$ and $\beta \in \mathbb{R}^n$ with non-zero entries. Let

$$T = D_\beta := \mathrm{diag}(\beta) \in \mathrm{GL}_n(\mathbb{R}), \qquad \tilde{\xi} = T\xi.$$

Then, from Lemma D.1, it comes that

$$T\mathrm{Gram}_X(e_1, \dots, e_n)^tT = \mathrm{Gram}_X\big(\beta_1 e_1, \dots, \beta_n e_n\big).$$

D.2. Invariance by scale change.

In our assumptions there is no normalization condition on the eigenvectors (see Remark 1.9). This allows to have simpler expressions for these eigenvectors. Actually, the computation of $\left\langle M^{-1}\xi,\xi\right\rangle$ can be done with a different scale change on every generalized eigenvector as detailed in the following proposition.

PROPOSITION D.2. *Let M and ξ be as defined in Theorem 1.8. Let $\beta\in\mathbb{R}^{|\alpha|}$ be such that $\beta_j^0\neq 0$ for all $j\in[\![1,g]\!]$. Set*

$$\tilde{\xi}=\begin{pmatrix}\left\langle y_0,(\beta\phi)\left[\lambda^{(\mu^1)}\right]\right\rangle_{-\diamond,\diamond}\\ \vdots\\ \left\langle y_0,(\beta\phi)\left[\lambda^{(\mu^{|\alpha|})}\right]\right\rangle_{-\diamond,\diamond}\end{pmatrix}.$$

Then,

$$\left\langle M^{-1}\xi,\xi\right\rangle=\left\langle \widetilde{M}^{-1}\tilde{\xi},\tilde{\xi}\right\rangle$$

where

$$\widetilde{M}:=\sum_{l=1}^{|\alpha|}\mathrm{Gram}_U\left(\underbrace{0,\dots,0}_{l-1},(\beta b)\left[\lambda^{(\mu^l-\mu^{l-1})}\right],\dots,(\beta b)\left[\lambda^{(\mu^{|\alpha_k|}-\mu^{l-1})}\right]\right). \tag{D.8}$$

Proof. From Leibniz formula [10, Proposition 7.13], it comes that for any $p\in[\![1,|\alpha|]\!]$,

$$(\beta\phi)\left[\lambda^{(\mu^p)}\right]=\sum_{q=1}^{|\mu^p|}\beta\left[\lambda^{(\mu^p-\mu^{q-1})}\right]\ \phi\left[\lambda^{(\mu^q)}\right].$$

Thus, $\tilde{\xi}=T\xi$ where T is the following lower triangular matrix

$$T=\left(\mathbf{1}_{q\leq p}\beta\left[\lambda^{(\mu^p-\mu^{q-1})}\right]\right)_{p,q\in[\![1,|\alpha|]\!]}.$$

The diagonal entries of this lower triangular matrix are β_j^0 and thus $T\in\mathrm{GL}_{|\alpha|}(\mathbb{C})$. From (D.2),

$$\widetilde{M}:=\sum_{l=1}^{|\alpha|}T\mathrm{Gram}_U\left(\underbrace{0,\dots,0}_{l-1},b\left[\lambda^{(\mu^l-\mu^{l-1})}\right],\dots,b\left[\lambda^{(\mu^{|\alpha|}-\mu^{l-1})}\right]\right){}^tT.$$

Let $l\in[\![1,|\alpha|]\!]$ and

$$e_1=\cdots=e_{l-1}=0,$$
$$e_p=b\left[\lambda^{(\mu^p-\mu^{l-1})}\right],\quad\forall p\in[\![l,|\alpha|]\!].$$

Then, for any $p\in[\![1,|\alpha|]\!]$,

$$(Te)_p=\sum_{q=1}^{|\alpha|}\mathbf{1}_{q\leq p}\beta\left[\lambda^{(\mu^p-\mu^{q-1})}\right]e_q.$$

Thus, $(Te)_1 = \cdots = (Te)_{l-1} = 0$ and, for any $p \in [\![l, |\alpha|]\!]$,

$$(Te)_p = \sum_{q=1}^{|\alpha|} \mathbf{1}_{q\le p} \beta \left[\lambda^{(\mu^p - \mu^{q-1})}\right] e_q = \sum_{q=l}^{p} \beta \left[\lambda^{(\mu^p - \mu^{q-1})}\right] b \left[\lambda^{(\mu^q - \mu^{l-1})}\right].$$

Let

$$\kappa^j := \mu^{l-1+j} - \mu^{l-1}, \qquad \forall j \in [\![1, p-l+1]\!].$$

Then, using again Leibniz formula [10, Proposition 7.13], we obtain

$$\begin{aligned}
(\beta b)\left[\lambda^{(\mu^p - \mu^{l-1})}\right] &= (\beta b)\left[\lambda^{(\kappa^{p-l+1})}\right] \\
&= \sum_{r=1}^{p-l+1} \beta \left[\lambda^{(\kappa^{p-l+1} - \kappa^{r-1})}\right] b \left[\lambda^{(\kappa^r)}\right] \\
&= \sum_{r=1}^{p-l+1} \beta \left[\lambda^{(\mu^p - \mu^{l-1+r-1})}\right] b \left[\lambda^{(\mu^{l-1+r} - \mu^{l-1})}\right] \\
&= \sum_{q=l}^{p} \beta \left[\lambda^{(\mu^p - \mu^{q-1})}\right] b \left[\lambda^{(\mu^q - \mu^{l-1})}\right] \\
&= (Te)_p.
\end{aligned}$$

Finally, applying (D.1) and Lemma D.1 ends the proof of Proposition D.2. ∎

REMARK D.1. *As there is no normalization condition on the eigenvectors a similar statement automatically holds with M and ξ defined in Theorem 1.10.*

D.3. An equivalent formula for simple eigenvalues.

In this section, we consider the case of a group of simple eigenvalues *i.e.* $\alpha_\lambda = \gamma_\lambda = 1$ for every $\lambda \in G$. In that case, the cost of the group G can be computed either using the formula of Theorem 1.8 for geometrically simple eigenvalues or the formula of Theorem 1.10 for semi-simple eigenvalues. Even though these theorems imply that those two formulas coincide (as they are both the cost of the group) we give a direct proof of this statement.

PROPOSITION D.3. *Let M and ξ be the matrix and the vector given in Theorem 1.8 i.e.*

$$M := \sum_{l=1}^{g} \mathrm{Gram}_U \left(\underbrace{0, \dots, 0}_{l-1}, b[\lambda_l], \dots, b[\lambda_l, \dots, \lambda_g] \right)$$

and

$$\xi = \begin{pmatrix} \langle y_0, \phi[\lambda_1] \rangle_{-\diamond,\diamond} \\ \vdots \\ \langle y_0, \phi[\lambda_1, \dots, \lambda_g] \rangle_{-\diamond,\diamond} \end{pmatrix}.$$

Let $\widetilde{M}$ and $\tilde{\xi}$ be the matrix and the vector given in Theorem 1.10 i.e.

$$\widetilde{M} := \sum_{l=1}^{g} \mathrm{Gram}_U \left(\delta_l^1 b[\lambda_1], \dots, \delta_l^g b[\lambda_g] \right) \quad \text{and} \quad \tilde{\xi} := \begin{pmatrix} \langle y_0, \phi[\lambda_1] \rangle_{-\diamond,\diamond} \\ \vdots \\ \langle y_0, \phi[\lambda_g] \rangle_{-\diamond,\diamond} \end{pmatrix}. \tag{D.9}$$

Then,

$$\left\langle M^{-1}\xi,\xi\right\rangle = \left\langle \widetilde{M}^{-1}\tilde{\xi},\tilde{\xi}\right\rangle$$

Proof. The usual interpolation formula [10, Proposition 7.6] gives

$$\phi[\lambda_i] = \sum_{j=1}^{i}\left(\prod_{k=1}^{j-1}\lambda_i-\lambda_k\right)\phi[\lambda_1,\dots,\lambda_j]. \tag{D.10}$$

Recall that the notation δ_j^i has been introduced in (1.50). With these notations, $\tilde{\xi} = T\xi$ where T is the following lower triangular matrix

$$T = \left(\delta_j^i\right)_{i,j\in[\![1,g]\!]} \in \mathrm{GL}_g(\mathbb{C}).$$

From (D.2), we define

$$\widehat{M} := \sum_{l=1}^{g} T\mathrm{Gram}_U\left(\underbrace{0,\dots,0}_{l-1},b\left[\lambda_l\right],\dots,b\left[\lambda_l,\dots,\lambda_g\right]\right){}^tT.$$

Let $l\in[\![1,g]\!]$ and

$$\begin{aligned} e_1 = \cdots = e_{l-1} &= 0,\\ e_j = b[\lambda_l,\dots,\lambda_j],&\quad \forall j\in[\![l,g]\!]. \end{aligned}$$

Then, $(Te)_1 = \cdots = (Te)_{l-1} = 0$ and for $i\in[\![l,g]\!]$, using again the interpolation property [10, Proposition 7.6], we obtain

$$\begin{aligned} (Te)_i &= \sum_{j=l}^{g}\delta_j^i b[\lambda_l,\dots,\lambda_j]\\ &= \sum_{j=l}^{i}\delta_j^i b[\lambda_l,\dots,\lambda_j]\\ &= \delta_l^i\sum_{j=l}^{i}\left(\prod_{k=l}^{j-1}\lambda_i-\lambda_k\right)b[\lambda_l,\dots,\lambda_j]\\ &= \delta_l^i b[\lambda_i]. \end{aligned}$$

Recalling that $\delta_l^1 = \cdots = \delta_l^{l-1} = 0$, we thus obtain

$$(Te)_i = \delta_l^i b[\lambda_i],\qquad \forall i[\![1,g]\!].$$

Finally, from Lemma D.1, we deduce that $\widehat{M} = \widetilde{M}$ which ends the proof of Proposition D.3. ∎

Appendix E. Technical estimates for Sturm-Liouville operators.

In this appendix, we prove Lemma 6.5.

To do so, we start with the following corollary of Lemma 5.1.

COROLLARY E.1. *Let A be the Sturm-Liouville operator defined in (5.1). There exists $C > 0$ such that for any $0 < a < b < 1$ there exists $\lambda_0 > 0$ (depending on $b-a$) such that for any $\lambda \geq \lambda_0$, for any $F \in L^2(0,1;\mathbb{R})$, for any u satisfying*

$$(A - \lambda)u = F \qquad \text{on } [a,b],$$

and for any $x \in [a,b]$, we have

$$|u(x)|^2 + \frac{\gamma(x)}{\lambda}|u'(x)|^2 \leq \frac{C}{b-a}\left(\|u\|^2_{(a,b)} + \frac{(b-a)^2}{\lambda}\|F\|^2_{(a,b)}\right).$$

Proof. We start with the proof in the case $F = 0$.

Let $\chi_0 \in C^\infty(\mathbb{R};\mathbb{R})$ be a cut-off function such that $0 \leq \chi_0 \leq 1$ and

- $\chi_0(x) = 1$ for every $x \in [1/4, 3/4]$,
- $\chi_0(x) = 0$ for every $x \notin (0,1)$.

We then set

$$\chi(x) = \chi_0\left(\frac{x-a}{b-a}\right),$$

in such a way that, if we set $\alpha = a + \frac{b-a}{4}$ and $\beta = b - \frac{b-a}{4}$, we have

- $\chi(x) = 1$ for every $x \in [\alpha, \beta]$,
- $\chi(x) = 0$ for every $x \notin (a,b)$.

Let $C_1 > 0$ be the constant given by Lemma 5.1 with $\lambda_0 = 1$. In the rest of this proof we set

$$\lambda_0 = \max\left\{1\,,\, 2C_1^2\,\|\chi'\|^2_{L^\infty}\right\} \tag{E.1}$$

and assume that $\lambda \geq \lambda_0$. Note that λ_0 only depends on $b-a$ since $\|\chi'\|_{L^\infty} = \|\chi_0'\|_{L^\infty}(b-a)^{-1}$.

Let $x \in [a,b]$. We apply Lemma 5.1 and integrate in the variable $y \in (\alpha,\beta)$ to obtain

$$\frac{b-a}{2}\left(|u(x)|^2 + \frac{\gamma(x)}{\lambda}|u'(x)|^2\right) \leq C_1\left(\|u\|^2_{(a,b)} + \frac{1}{\lambda}\int_\alpha^\beta \gamma(y)|u'(y)|^2\mathrm{d}y\right). \tag{E.2}$$

Then integrating by parts, using $(A-\lambda)u = 0$ and Cauchy-Schwarz inequality yield

$$\begin{aligned}
\frac{1}{\lambda}\int_\alpha^\beta \gamma(y)|u'(y)|^2\mathrm{d}y &\leq \frac{1}{\lambda}\int_a^b \chi(y)\gamma(y)|u'(y)|^2\mathrm{d}y \\
&= -\frac{1}{\lambda}\int_a^b \chi'(y)(\gamma u')(y)u(y)\mathrm{d}y + \frac{1}{\lambda}\int_a^b \chi(y)\left(\lambda - c(y)\right)|u(y)|^2\mathrm{d}y \\
&\leq \left(\frac{\|\chi'\|_{L^\infty}}{\lambda}\|\sqrt{\gamma}u'\|_{(a,b)}\right)\left(\|\sqrt{\gamma}\|_{L^\infty}\|u\|_{(a,b)}\right) \\
&\quad + \int_a^b \chi(y)\left|1 - \frac{c(y)}{\lambda}\right||u(y)|^2\mathrm{d}y \\
&\leq \left(1 + \|c\|_{L^\infty} + \frac{\|\gamma\|_{L^\infty}}{2}\right)\|u\|^2_{(a,b)} + \frac{\|\chi'\|^2_{L^\infty}}{2\lambda^2}\|\sqrt{\gamma}u'\|^2_{(a,b)}.
\end{aligned}$$

Plugging it into estimate (E.4) we obtain

$$\frac{b-a}{2}\left(|u(x)|^2 + \frac{\gamma(x)}{\lambda}|u'(x)|^2\right) \leq C\|u\|^2_{(a,b)} + \frac{C_1\,\|\chi'\|^2_{L^\infty}}{2\lambda^2}\|\sqrt{\gamma}u'\|^2_{(a,b)}. \tag{E.3}$$

Applying again Lemma 5.1 gives that, for any $y \in (a,b)$,

$$\frac{\gamma(y)}{\lambda}|u'(y)|^2 \leq C_1\left(|u(x)|^2 + \frac{\gamma(x)}{\lambda}|u'(x)|^2\right).$$

Integrating in the variable $y \in (a,b)$ and using the definition of λ_0 given in (E.1) we obtain

$$\begin{aligned}\frac{C_1\|\chi'\|_{L^\infty}^2}{2\lambda^2}\|\sqrt{\gamma}u'\|_{(a,b)}^2 &\leq \frac{C_1^2\|\chi'\|_{L^\infty}^2}{2\lambda}(b-a)\left(|u(x)|^2 + \frac{\gamma(x)}{\lambda}|u'(x)|^2\right)\\ &\leq \frac{b-a}{4}\left(|u(x)|^2 + \frac{\gamma(x)}{\lambda}|u'(x)|^2\right).\end{aligned}$$

Plugging it into (E.3) ends the proof of Corollary E.1 in the case $F = 0$.

Assume now that $F \neq 0$. Let $x \in [a,b]$ and define $\tilde{u}$ as the solution of the Cauchy problem

$$\begin{cases}(A-\lambda)\tilde{u} = 0,\\ \tilde{u}(x) = u(x),\\ \tilde{u}'(x) = u'(x).\end{cases}$$

From the case $F = 0$ we deduce that

$$|u(x)|^2 + \frac{\gamma(x)}{\lambda}|u'(x)|^2 = |\tilde{u}(x)|^2 + \frac{\gamma(x)}{\lambda}|\tilde{u}'(x)|^2 \leq \frac{C}{b-a}\|\tilde{u}\|_{(a,b)}^2. \tag{E.4}$$

Notice that

$$\begin{cases}(A-\lambda)(u-\tilde{u}) = F,\\ (u-\tilde{u})(x) = (u-\tilde{u})'(x) = 0.\end{cases}$$

Thus, from Lemma 5.1 we obtain that for any $y \in [a,b]$

$$|(u-\tilde{u})(y)|^2 \leq \frac{C}{\lambda}\left|\int_x^y |F(s)|\mathrm{d}s\right|^2 \leq C\frac{b-a}{\lambda}\|F\|_{(a,b)}^2$$

which implies

$$\|u-\tilde{u}\|_{(a,b)}^2 \leq C\frac{(b-a)^2}{\lambda}\|F\|_{(a,b)}^2.$$

Thus, we deduce that

$$\|\tilde{u}\|_{(a,b)}^2 \leq C\left(\|u\|_{(a,b)}^2 + \frac{(b-a)^2}{\lambda}\|F\|_{(a,b)}^2\right).$$

Together with (E.4) this ends the proof of Corollary E.1. ∎

REMARK E.1. *Notice that, when $\gamma \in C^2([0,1];\mathbb{R})$, for λ large enough, for any u satisfying $(A-\lambda)u = 0$, there exist α and β in (a,b) such that*

$$u(\alpha) = u(\beta) = 0, \qquad \beta - \alpha \geq \frac{b-a}{2}.$$

This is a consequence of the Sturm comparison theorem (see for instance [16, Corollary A.5.16]). With this choice one does not need to introduce the cut-off function χ in the first step of the proof of Corollary E.1 which simplifies the proof.

We now have all the ingredients to prove Lemma 6.5.

Proof (of Lemma 6.5). We denote by $\omega_1, \dots, \omega_N$ the connected components of ω labeled such that

$$\sup \omega_j \leq \inf \omega_{j+1}, \qquad \forall j \in [\![1, N-1]\!].$$

For any $j \in [\![1, N]\!]$, let $\lambda_0(\omega_j) > 0$ be the constant given by Corollary E.1 applied to the interval $\overline{\omega_j}$. Let

$$\lambda_0 = \max_{j \in [\![1,N]\!]} \lambda_0(\omega_j)$$

and, let $K > 0$ be such that

$$k \geq K \quad \Longrightarrow \quad \nu_k \geq \lambda_0.$$

We start with the proof of item i.

Let $\mathfrak{C} = [a, b]$ be a connected component of $\overline{(0,1)\backslash\omega}$. Integrating by parts we obtain

$$\int_{\mathfrak{C}} F(x)\varphi_k(x)\mathrm{d}x = -(\gamma u'\varphi_k)(b) + (\gamma u'\varphi_k)(a) + (u\gamma\varphi_k)'(b) - (u\gamma\varphi_k')(a).$$

Recall that from (5.7),

$$|\varphi_k(x)| + \frac{1}{\sqrt{\nu_k}}|\varphi_k'(x)| \leq C, \qquad \forall x \in (0,1),\ \forall k \geq 1.$$

Similarly, applying Lemma 5.1 with $y = 0$ we obtain

$$|\widetilde{\varphi}_k(x)| + \frac{1}{\sqrt{\nu_k}}|\widetilde{\varphi}_k'(x)| \leq C, \qquad \forall x \in (0,1),\ \forall k \geq 1. \tag{E.5}$$

Thus,

$$\frac{1}{\sqrt{\nu_k}}\left|\int_{\mathfrak{C}} F(x)\varphi_k(x)\mathrm{d}x\right| \leq C\left(|u(a)| + \frac{\sqrt{\gamma(a)}}{\sqrt{\nu_k}}|u'(a)|\right) + C\left(|u(b)| + \frac{\sqrt{\gamma(b)}}{\sqrt{\nu_k}}|u'(b)|\right).$$

- If $\mathfrak{C} \cap \{0,1\} = \varnothing$, then there exists $j \in [\![2, N]\!]$ such that $a \in \overline{\omega_{j-1}}$ and $b \in \overline{\omega_j}$. Applying twice Corollary E.1 we obtain

$$|u(a)| + \frac{\sqrt{\gamma(a)}}{\sqrt{\nu_k}}|u'(a)| \leq C\left(\sqrt{\nu_k}\|u\|_{\omega_{j-1}} + \|F\|_{\omega_{j-1}}\right),$$

and

$$|u(b)| + \frac{\sqrt{\gamma(b)}}{\sqrt{\nu_k}}|u'(b)| \leq C\left(\sqrt{\nu_k}\|u\|_{\omega_j} + \|F\|_{\omega_j}\right)$$

where C now also depends on ω. This implies

$$\left|\int_{\mathfrak{C}} F(x)\varphi_k(x)\mathrm{d}x\right| \leq C\left(\sqrt{\nu_k}\|u\|_{\omega} + \|F\|_{\omega}\right)$$

The same computations hold for $\left|\int_{\mathfrak{C}} F(x)\widetilde{\varphi}_k(x)\mathrm{d}x\right|$.

- Now, if $a = 0$, taking into account the boundary condition $\varphi_k(a) = 0$, the same computations yields

$$\frac{1}{\sqrt{\nu_k}}\left|\int_{\mathfrak{C}} F(x)\varphi_k(x)\mathrm{d}x\right| \leq C|u(0)| + C\left(|u(b)| + \frac{\sqrt{\gamma(b)}}{\sqrt{\nu_k}}|u'(b)|\right).$$

As $b \in \overline{\omega}$, applying Corollary E.1 we obtain

$$|u(b)| + \frac{\sqrt{\gamma(b)}}{\sqrt{\nu_k}}|u'(b)| \leq C\left(\sqrt{\nu_k}\|u\|_\omega + \|F\|_\omega\right)$$

which implies

$$\left|\int_{\mathfrak{C}} F(x)\varphi_k(x)\mathrm{d}x\right| \leq C\big(\sqrt{\nu_k}\|u\|_\omega + \sqrt{\nu_k}|u(0)| + \|F\|_\omega\big).$$

- Similarly, if $b = 1$, we prove that

$$\left|\int_{\mathfrak{C}} F(x)\varphi_k(x)\mathrm{d}x\right| \leq C\big(\sqrt{\nu_k}\|u\|_\omega + \sqrt{\nu_k}|u(1)| + \|F\|_\omega\big).$$

Gathering these results proves item i.

We now turn to the proof of item ii.

- We start designing u a solution of

$$\begin{cases}(A-\nu_k)u = F,\\ u(0) = u(1) = 0,\end{cases}$$

such that

$$|u(x)| + \frac{\sqrt{\gamma(x)}}{\sqrt{\nu_k}}|u'(x)| \leq \frac{C}{\sqrt{\nu_k}}\left(\mathcal{M}_k(F,\omega) + \|F\|_{\omega_1}\right), \qquad \forall x \in \overline{\omega_1}. \tag{E.6}$$

Let $\overline{u}$ be any solution of

$$\begin{cases}(A-\nu_k)\overline{u} = F,\\ \overline{u}(0) = \overline{u}(1) = 0.\end{cases}$$

If $0 \notin \overline{\omega_1}$ we set $b = \inf\omega_1$ whereas if $0 \in \overline{\omega_1}$ we set $b \in \omega_1$. Notice that in both cases

$$\int_0^b F(x)\varphi_k(x)\mathrm{d}x = \overline{u}(b)\gamma(b)\varphi_k'(b) - \gamma(b)\overline{u}'(b)\varphi_k(b).$$

Applying Lemma 5.1 with $y = b$, integrating with respect to the variable $x \in (0,1)$ and using $\|\varphi_k\|_{(0,1)} = 1$ we obtain that there exists $C > 0$ such that

$$|\varphi_k(b)| + \frac{\sqrt{\gamma(b)}}{\sqrt{\nu_k}}|\varphi_k'(b)| \geq C.$$

– If $|\varphi_k(b)| \geq \frac{C}{2}$, we set $u = \overline{u} - \frac{\overline{u}(b)}{\varphi_k(b)}\varphi_k$.

Thus, we have $u(b) = 0$ which implies

$$\sqrt{\gamma(b)}u'(b) = \frac{-1}{\sqrt{\gamma(b)\varphi_k(b)}} \int_0^b F(x)\varphi_k(x)\mathrm{d}x.$$

Thus,

$$|u(b)| + \frac{\sqrt{\gamma(b)}}{\sqrt{\nu_k}}|u'(b)| \leq \frac{C}{\sqrt{\nu_k}} \left| \int_0^b F(x)\varphi_k(x)\mathrm{d}x \right|. \tag{E.7}$$

– Otherwise, we have $\frac{\sqrt{\gamma(b)}}{\sqrt{\nu_k}}|\varphi_k'(b)| \geq \frac{C}{2}$. Setting $u = \overline{u} - \frac{\overline{u}'(b)}{\varphi_k'(b)}\varphi_k$, the same computations also imply (E.7).

We now prove that (E.7) implies (E.6).
As $b \in \overline{\omega_1}$, applying Lemma 5.1 and (E.7) we obtain for any $x \in \overline{\omega_1}$,

$$\begin{aligned} |u(x)| + \frac{\sqrt{\gamma(x)}}{\sqrt{\nu_k}}|u'(x)| &\leq C \left(|u(b)| + \frac{\sqrt{\gamma(b)}}{\sqrt{\nu_k}}|u'(b)| + \frac{1}{\sqrt{\nu_k}}\|F\|_{\omega_1} \right) \\ &\leq \frac{C}{\sqrt{\nu_k}} \left(\left| \int_0^b F(x)\varphi_k(x)\mathrm{d}x \right| + \|F\|_{\omega_1} \right). \end{aligned}$$

– Assume first that $0 \notin \overline{\omega_1}$ and recall that $b = \inf \omega_1$. Then, by definition of $\mathcal{M}_k(F, \omega)$ (see (6.8)), we have

$$\left| \int_0^b F(x)\varphi_k(x)\mathrm{d}x \right| \leq \mathcal{M}_k(F, \omega).$$

Thus, for any $x \in \overline{\omega_1}$,

$$|u(x)| + \frac{\sqrt{\gamma(x)}}{\sqrt{\nu_k}}|u'(x)| \leq \frac{C}{\sqrt{\nu_k}} \big(\mathcal{M}_k(F, \omega) + \|F\|_{\omega_1} \big).$$

– Otherwise, $0 \in \overline{\omega_1}$ and we have set $b \in \omega_1$. Then, since $(0, b) \subset \omega_1$ and $\|\varphi_k\|_{(0,1)} = 1$, we have

$$\left| \int_0^b F(x)\varphi_k(x)\mathrm{d}x \right| \leq \|F\|_{\omega_1}.$$

Thus, for any $x \in \overline{\omega_1}$,

$$|u(x)| + \frac{\sqrt{\gamma(x)}}{\sqrt{\nu_k}}|u'(x)| \leq \frac{C}{\sqrt{\nu_k}} \|F\|_{\omega_1}.$$

Gathering these two cases proves (E.6).

- We prove by induction that the function u designed at the previous step satisfies

$$|u(x)| + \frac{\sqrt{\gamma(x)}}{\sqrt{\nu_k}}|u'(x)| \leq \frac{C}{\sqrt{\nu_k}} \big(\mathcal{M}_k(F, \omega) + \|F\|_{\omega} \big), \qquad \forall x \in \overline{\omega_j}. \tag{E.8}$$

The case $j = 1$ was proved in the previous step. Let $j \in [\![2, N]\!]$ be such that

$$|u(x)| + \frac{\sqrt{\gamma(x)}}{\sqrt{\nu_k}}|u'(x)| \leq \frac{C}{\sqrt{\nu_k}}\big(\mathcal{M}_k(F,\omega) + \|F\|_\omega\big), \qquad \forall x \in \overline{\omega_{j-1}}.$$

Let $a_j = \sup \omega_{j-1}$ and $b_j = \inf \omega_j$. Integrating by parts we obtain

$$\begin{aligned}\frac{1}{\sqrt{\nu_k}}\int_{a_j}^{b_j} F(x)\varphi_k(x)\mathrm{d}x = u(b_j)\frac{\gamma(b_j)\varphi_k'(b_j)}{\sqrt{\nu_k}} - \frac{\gamma(b_j)u'(b_j)}{\sqrt{\nu_k}}\varphi_k(b_j)\\ - u(a_j)\frac{\gamma(a_j)\varphi_k'(a_j)}{\sqrt{\nu_k}} - \frac{\gamma(a_j)u'(a_j)}{\sqrt{\nu_k}}\varphi_k(a_j).\end{aligned}$$

The same computations hold replacing φ_k by $\widetilde{\varphi}_k$. This can be rewritten in matrix form as

(E.9)
$$\begin{pmatrix} \frac{\gamma(b_j)\varphi_k'(b_j)}{\sqrt{\nu_k}} & -\varphi_k(b_j) \\ \frac{\gamma(b_j)\widetilde{\varphi}_k'(b_j)}{\sqrt{\nu_k}} & -\widetilde{\varphi}_k(b_j)\end{pmatrix}\begin{pmatrix} u(b_j) \\ \frac{\gamma(b_j)u'(b_j)}{\sqrt{\nu_k}}\end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{\nu_k}}\int_{a_j}^{b_j} F(x)\varphi_k(x)\mathrm{d}x \\ \frac{1}{\sqrt{\nu_k}}\int_{a_j}^{b_j} F(x)\widetilde{\varphi}_k(x)\mathrm{d}x\end{pmatrix} + R_j$$

with

$$\|R_j\| \leq C\left(|u(a_j)| + \frac{\sqrt{\gamma(a_j)}}{\sqrt{\nu_k}}|u'(a_j)|\right).$$

Notice that the determinant of the matrix appearing in (E.9) is a wronskian which is constant. Thus,

$$\det\begin{pmatrix} \frac{\gamma(b_j)\varphi_k'(b_j)}{\sqrt{\nu_k}} & -\varphi_k(b_j) \\ \frac{\gamma(b_j)\widetilde{\varphi}_k'(b_j)}{\sqrt{\nu_k}} & -\widetilde{\varphi}_k(b_j)\end{pmatrix} = -\frac{\gamma(0)\varphi_k'(0)}{\sqrt{\nu_k}}$$

which is bounded from below. From (5.7) and (E.5), all the coefficients of this matrix are also bounded. Thus, we obtain

$$|u(b_j)| + \frac{\sqrt{\gamma(b_j)}}{\sqrt{\nu_k}}|u'(b_j)| \leq \frac{C}{\sqrt{\nu_k}}\mathcal{M}_k(F,\omega) + C\|R_j\|.$$

As $a_j \in \overline{\omega_{j-1}}$ the induction hypothesis imply

$$\|R_j\| \leq \frac{C}{\sqrt{\nu_k}}\big(\mathcal{M}_k(F,\omega) + \|F\|_\omega\big)$$

and thus

$$|u(b_j)| + \frac{\sqrt{\gamma(b_j)}}{\sqrt{\nu_k}}|u'(b_j)| \leq \frac{C}{\sqrt{\nu_k}}\left(\mathcal{M}_k(F,\omega) + \|F\|_\omega\right).$$

As $b_j \in \overline{\omega_j}$, applying Lemma 5.1 we obtain for any $x \in \overline{\omega_j}$

$$\begin{aligned}|u(x)| + \frac{\sqrt{\gamma(x)}}{\sqrt{\nu_k}}|u'(x)| &\leq C\left(|u(b_j)| + \frac{\sqrt{\gamma(b_j)}}{\sqrt{\nu_k}}|u'(b_j)| + \frac{1}{\sqrt{\nu_k}}\|F\|_{\omega_j}\right)\\ &\leq \frac{C}{\sqrt{\nu_k}}\big(\mathcal{M}_k(F,\omega) + \|F\|_\omega\big)\end{aligned}$$

This proves (E.8).

- Conclusion. From (E.8) we obtain

$$|u(x)| \leq \frac{C}{\sqrt{\nu_k}}\big(\mathcal{M}_k(F,\omega) + \|F\|_\omega\big), \qquad \forall x \in \omega_j,\ \forall j \in [\![1,N]\!].$$

This leads to

$$\|u\|_{\omega_j} \leq \frac{C}{\sqrt{\nu_k}}\big(\mathcal{M}_k(F,\omega) + \|F\|_\omega\big), \qquad \forall j \in [\![1,N]\!]$$

and ends the proof of item ii. ∎

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