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1 **ANALYSIS OF NON SCALAR CONTROL PROBLEMS FOR**
2 **PARABOLIC SYSTEMS BY THE BLOCK MOMENT METHOD**

3 FRANCK BOYER* AND MORGAN MORANCEY†

4 **Abstract.** This article deals with abstract linear time invariant controlled systems. In [*Annales*
5 *Henri Lebesgue*, **3** (2020), pp. 717–793], with A. Benabdallah, we introduced the block moment
6 method for scalar control operators. The principal aim of this method is to answer the question of
7 computing the minimal time needed to drive an initial condition (or a space of initial conditions) to
8 zero. The purpose of the present article is to push forward the analysis to deal with any admissible
9 control operator. The considered setting leads to applications to one dimensional parabolic-type
10 equations or coupled systems of such equations.

11 With such admissible control operator, the characterization of the minimal null control time
12 is obtained thanks to the resolution of an auxiliary vectorial block moment problem (*i.e.* set in
13 the control space) followed by a constrained optimization procedure of the cost of this resolution.
14 This leads to essentially sharp estimates on the resolution of the block moment problems which are
15 uniform with respect to the spectrum of the evolution operator in a certain class. This uniformity
16 allow the study of uniform controllability for various parameter dependent problems. We also deduce
17 estimates on the cost of controllability when the final time goes to the minimal null control time.

18 We provide applications on abstract controlled system to illustrate how the method works and
19 then deal with actual coupled systems of one dimensional parabolic partial differential equations.
20 Our strategy enables us to gather previous results obtained by different methods but to also tackle
21 controllability issues that seem out of reach by existing techniques.

22 **Key words.** Control theory, parabolic partial differential equations, minimal null control time,
23 block moment method

24 **AMS subject classifications.** 93B05, 93C20, 93C25, 30E05, 35K90, 35P10

25 **1. Introduction.**

26 **1.1. Problem under study and state of the art.**

27 In this paper we study the controllability properties of the following linear control
28 system

29 (1.1)
$$\begin{cases} y'(t) + \mathcal{A}y(t) = \mathcal{B}u(t), \\ y(0) = y_0. \end{cases}$$

30 The assumptions on the operator \mathcal{A} (see Section 1.3) will lead to applications to
31 linear parabolic-type equations or coupled systems of such equations mostly in the
32 one dimensional setting. In all this article the Hilbert space of control will be denoted
33 by U and the operator \mathcal{B} will be a general admissible operator.

34 The question we address is the characterization of the minimal null control time
35 (possibly zero or infinite) from y_0 that is: for a given initial condition y_0 , what
36 is the minimal time $T_0(y_0)$ such that, for any $T > T_0(y_0)$, there exists a control
37 $u \in L^2(0, T; U)$ such that the associated solution of (1.1) satisfies $y(T) = 0$.

38 For a presentation of null controllability of parabolic control problems as well as
39 the possible existence of a positive minimal null control time for such equations we
40 refer to [4] or [10, Section 1.1] and the references therein. Such a positive minimal null

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41 control time is due either to insufficient observation of eigenvectors, or to condensation
 42 of eigenvalues or to the geometry of generalized eigenspaces, or even to a combination
 43 of all those phenomena.

44 Under the considered assumptions on \mathcal{A} , the problem of characterizing the mini-
 45 mal null control time has been solved for scalar controls ($\dim U = 1$) in [10] where the
 46 *block moment method* has been introduced in that purpose. The aim of the present
 47 article is to push forward the analysis of [10] to extend it to any admissible control
 48 operator.

49 To present the general ideas, let us assume for simplicity that the operator \mathcal{A}^* has
 50 a sequence of positive eigenvalues Λ and that the associated eigenvectors ϕ_λ for $\lambda \in \Lambda$
 51 form a complete family of the state space (the precise functional setting is detailed in
 52 Section 1.3). Then, the solution of system (1.1) satisfies $y(T) = 0$ if and only if the
 53 control $u \in L^2(0, T; U)$ solves the following moment problem

$$54 \quad (1.2) \quad \int_0^T e^{-\lambda t} \langle u(T-t), \mathcal{B}^* \phi_\lambda \rangle_U dt = -e^{-\lambda T} \langle y_0, \phi_\lambda \rangle, \quad \forall \lambda \in \Lambda.$$

55 \star In the scalar case ($\dim U = 1$), provided that $\mathcal{B}^* \phi_\lambda \neq 0$, this moment problem
 56 reduces to

$$57 \quad (1.3) \quad \int_0^T e^{-\lambda t} u(T-t) dt = -e^{-\lambda T} \left\langle y_0, \frac{\phi_\lambda}{\mathcal{B}^* \phi_\lambda} \right\rangle, \quad \forall \lambda \in \Lambda.$$

58 This problem is usually solved by the construction of a biorthogonal family $(q_\lambda)_{\lambda \in \Lambda}$
 59 to the exponentials

$$60 \quad \{t \in (0, T) \mapsto e^{-\lambda t}; \lambda \in \Lambda\}$$

61 in $L^2(0, T; U)$, *i.e.*, a family $(q_\lambda)_{\lambda \in \Lambda}$ such that

$$62 \quad \int_0^T q_\lambda(t) e^{-\mu t} dt = \delta_{\lambda, \mu}, \quad \forall \lambda, \mu \in \Lambda.$$

63 From [35], the existence of such biorthogonal family is equivalent to the summability
 64 condition

$$65 \quad (1.4) \quad \sum_{\lambda \in \Lambda} \frac{1}{\lambda} < +\infty.$$

66 **REMARK 1.1.** *This condition (which will be assumed in the present article) is the*
 67 *main restriction to apply the moment method. Indeed, due to Weyl's law it imposes*
 68 *on many examples of partial differential equations of parabolic-type a restriction to the*
 69 *one dimensional setting. However, in some particular multi-dimensional geometries,*
 70 *the controllability problem can be transformed into a family of parameter dependent*
 71 *moment problems, each of them satisfying such assumption (see for instance [9, 3, 18]*
 72 *among others).*

73 With such a biorthogonal family, a formal solution of the moment problem (1.3)
 74 is given by

$$75 \quad u(T-t) = - \sum_{\lambda \in \Lambda} e^{-\lambda T} \left\langle y_0, \frac{\phi_\lambda}{\mathcal{B}^* \phi_\lambda} \right\rangle q_\lambda(t), \quad t \in (0, T).$$

76 Thus if, for any y_0 , the series defining u converges in $L^2(0, T; U)$ one obtains null
 77 controllability of system (1.1) in time T . To do so, it is crucial to prove upper bounds
 78 on $\|q_\lambda\|_{L^2(0, T)}$.

79 Suitable bounds on such biorthogonal families were provided in the pioneering
80 work of Fattorini and Russell [23] in the case where the eigenvalues of \mathcal{A}^* are well
81 separated *i.e.* satisfy the classical gap condition: $\inf \{|\lambda - \mu|; \lambda, \mu \in \Lambda, \lambda \neq \mu\} > 0$.
82 When the eigenvalues are allowed to condensate we refer to the work [5] for almost
83 sharp estimates implying the condensation index of the sequence Λ . A discussion on
84 other references providing estimates on biorthogonal families is detailed below. These
85 results have provided an optimal characterization of the minimal null control time
86 when the eigenvectors of \mathcal{A}^* form a Riesz basis of the state space (and thus do not
87 condensate).

88 However, as analyzed in [10], there are situations in which the eigenvectors also
89 condensate and for which providing estimates on biorthogonal families is not sufficient
90 to characterize the minimal null control time. In [10], it is assumed that the spec-
91 trum Λ can be decomposed as a union of well separated groups $(G_k)_{k \geq 1}$ of bounded
92 cardinality. Then, the control u is sought in the form

$$93 \quad u(T-t) = \sum_{k \geq 1} v_k(t),$$

94 where, for any $k \geq 1$, the function $v_k \in L^2(0, T; U)$ solves the block moment problem

$$95 \quad (1.5) \quad \begin{cases} \int_0^T e^{-\lambda t} v_k(t) dt = e^{-\lambda T} \left\langle y_0, \frac{\phi_\lambda}{\mathcal{B}^* \phi_\lambda} \right\rangle, & \forall \lambda \in G_k, \\ \int_0^T e^{-\lambda t} v_k(t) dt = 0, & \forall \lambda \notin G_k. \end{cases}$$

96 This enables to deal with the condensation of eigenvectors: the eigenvectors $(\phi_\lambda)_{\lambda \in \Lambda}$
97 are only assumed to form a complete family of the state space.

98 \star When the control is not scalar there are less available results in the literature.
99 Here again, these results rely on the existence of a biorthogonal family to the expo-
100 nentials with suitable bounds. For instance, in [7], null controllability in optimal time
101 is proved using a subtle decomposition of the moment problem into two families of
102 moment problems. In a more systematic way, one can take advantage of the biorthog-
103 onality in the time variable to seek for a solution u of the moment problem (1.2) in
104 the form

$$105 \quad u(T-t) = - \sum_{\lambda \in \Lambda} e^{-\lambda T} \langle y_0, \phi_\lambda \rangle \frac{\mathcal{B}^* \phi_\lambda}{\|\mathcal{B}^* \phi_\lambda\|_U^2}.$$

106 This strategy was introduced by Lagnese in [28] for a one dimensional wave equation
107 and used in the parabolic context for instance in [19, 2, 20, 3].

108 In the present article we deal with such general admissible control operators.
109 As the eigenvectors will only be assumed to form a complete family, for each initial
110 condition y_0 , we study its null control time for system (1.1) by solving block moment
111 problems of the following form

$$112 \quad (1.6) \quad \begin{cases} \int_0^T \langle V_k(t), e^{-\lambda t} \mathcal{B}^* \phi_\lambda \rangle_U dt = \langle y_0, e^{-\lambda T} \phi_\lambda \rangle, & \forall \lambda \in G_k, \\ \int_0^T \langle V_k(t), e^{-\lambda t} \mathcal{B}^* \phi_\lambda \rangle_U dt = 0, & \forall \lambda \notin G_k. \end{cases}$$

113 Let us recall that, for pedagogical purposes, we have restricted this first introductory
 114 subsection to the case of simple eigenvalues. The general form of block moment
 115 problems under study in this article is detailed in Section 1.4.

116 The strategy to solve such block moment problems and estimate its solution is
 117 presented on an example in Section 1.2 together with the structure of the article. Let
 118 us already notice that the geometry of the finite dimensional space $\text{Span}\{\mathcal{B}^*\phi_\lambda; \lambda \in$
 119 $G_k\}$ is crucial.

120 For instance, if this space is one dimensional, say generated by some $b \in U$, the
 121 strategy of Lagnese can be adapted if one seeks for V_k solution of the block moment
 122 problem (1.6) in the form

$$123 \quad V_k(t) = v_k(t)b,$$

124 where $v_k \in L^2(0, T; \mathbb{R})$ solves a scalar block moment problem of the same form as (1.5).

125 If, instead, the family $(\mathcal{B}^*\phi_\lambda)_{\lambda \in G_k}$ is composed of linearly independent vectors
 126 then it admits a biorthogonal family in U denoted by $(b_\lambda^*)_{\lambda \in G_k}$. Then, one can for
 127 instance seek for V_k solution of the block moment problem (1.6) in the form

$$128 \quad V_k(t) = v_k(t) \left(\sum_{\lambda \in G_k} b_\lambda^* \right).$$

129 where v_k solves a scalar block moment problem of the form (1.5).

130 In the general setting, taking into account the geometry of the observations of
 131 eigenvectors to solve block moment problems of the form (1.6) is a more intricate
 132 question that we solve in this article, still under the summability condition (1.4).

133 Finally, let us mention that we not only solve block moment problems of the
 134 form (1.6) but we also provide estimates on their solutions to ensure that the series
 135 defining the control converges. These estimates will provide an optimal characteriza-
 136 tion of the minimal null control time for each given problem.

137 We add also an extra care on these estimates so that they do not directly depend
 138 on the sequence Λ but are uniform for classes of such sequences. It is an important step
 139 to tackle uniform controllability for parameter dependent control problems. Estimates
 140 of this kind have already proved their efficiency in various contexts such as: numerical
 141 analysis of semi-discrete control problems [2], oscillating coefficients [32], analysis
 142 of degenerate control problems with respect to the degeneracy parameter [19, 20],
 143 analysis of higher dimensional controllability problems by reduction of families of one
 144 dimensional control problems [9, 1, 3, 18] or analysis of convergence of Robin-type
 145 controls to Dirichlet controls [12].

146 Another important feature of the estimates we obtain is to track the dependency
 147 with respect to the final time T when T goes to the minimal null control time. As pre-
 148 sented in Remark 1.7, this allows applications in higher dimensions (with a cylindrical
 149 geometry) or applications to nonlinear control problems.

150 Finally, let us recall some classical results providing estimates for biorthogonal
 151 families to a sequence of exponentials. Under the classical gap condition, uniform
 152 estimates for biorthogonal families were already obtained in [24] and sharp short-time
 153 estimates were obtained in [9]. In this setting, bounds with a detailed dependency
 154 with respect to parameters were given in [21]. In this work, the obtained bounds take
 155 into account the fact that the gap property between eigenvalues may be better in high
 156 frequencies.

157 Under a weak-gap condition of the form (1.21), that is when the eigenvalues
 158 can be gathered in blocks of bounded cardinality with a gap between blocks (which

159 is the setting of the present article), uniform estimates on biorthogonal sequences
 160 follow from the uniform estimates for the resolution of block moment problems proved
 161 in [10]. These estimates on biorthogonal family are improved with the dependency
 162 with respect to T in [26]. Let us mention that the estimates of [10] can also be
 163 supplemented with such dependency (see Theorem A.1) but only when the considered
 164 eigenvalues are assumed to be real (unlike the setting studied in [26]).

165 In the absence of any gap-type condition, estimates on biorthogonal families are
 166 proved in [5, 3].

167 1.2. Structure of the article and strategy of proof.

168 To highlight the ideas we develop in this article (without drowning them in tech-
 169 nicalities or notations), let us present our strategy of analysis of null controllability
 170 on an abstract simple example.

171 We consider $X = L^2(0, 1; \mathbb{R})^2$ and $\omega \subset (0, 1)$ a non empty open set. For a given
 172 $a > 0$ we define

$$173 \quad \Lambda = \left\{ \lambda_{k,1} := k^2, \lambda_{k,2} := k^2 + e^{-ak^2}; k \geq 1 \right\},$$

174 and take $(\varphi_k)_{k \geq 1}$ an Hilbert basis of X such that

$$175 \quad \inf_{k \geq 1} \|\varphi_k\|_{L^2(\omega)} > 0.$$

176 Let $\phi_{k,1} := \begin{pmatrix} \varphi_k \\ \varphi_k \end{pmatrix}$ and $\phi_{k,2} := \begin{pmatrix} 0 \\ \varphi_k \end{pmatrix}$. We define the operator \mathcal{A}^* in X by

$$177 \quad \mathcal{A}^* \phi_{k,1} = \lambda_{k,1} \phi_{k,1}, \quad \mathcal{A}^* \phi_{k,2} = \lambda_{k,2} \phi_{k,2},$$

178 with

$$179 \quad \mathcal{D}(\mathcal{A}^*) = \left\{ \sum_{k \geq 1} a_{k,1} \phi_{k,1} + a_{k,2} \phi_{k,2}; \sum_{k \geq 1} \lambda_{k,1}^2 a_{k,1}^2 + \lambda_{k,2}^2 a_{k,2}^2 < +\infty \right\}.$$

180 The control operator \mathcal{B} is defined by $U = L^2(0, 1; \mathbb{R})$ and

$$181 \quad \mathcal{B} : u \in U \mapsto \begin{pmatrix} 0 \\ \mathbf{1}_\omega u \end{pmatrix} \in X.$$

182 The condition $\inf_{k \geq 1} \|\varphi_k\|_{L^2(\omega)} > 0$ yields

$$183 \quad (1.7) \quad \mathcal{B}^* \phi_{k,1} = \mathcal{B}^* \phi_{k,2} \neq 0, \quad \forall k \geq 1.$$

184 This ensures approximate controllability of system (1.1).

185 We insist on the fact that the goal of this article is not to deal with this particular
 186 example but to develop a general methodology to analyze the null controllability of
 187 system (1.1).

188 • Let $y_0 \in X$. From Proposition 1.1 and the fact that $\{\phi_{k,1}, \phi_{k,2}; k \geq 1\}$ forms a
 189 complete family of X , system (1.1) is null controllable from y_0 at time T if and only
 190 if there exists $u \in L^2(0, T; U)$ such that for any $k \geq 1$ and any $j \in \{1, 2\}$,

$$191 \quad \int_0^T e^{-\lambda_{k,j}t} \langle u(T-t), \mathcal{B}^* \phi_{k,j} \rangle_U dt = -e^{-\lambda_{k,j}T} \langle y_0, \phi_{k,j} \rangle_X.$$

192 Following the idea developed in [10], we seek for a control u of the form

$$193 \quad (1.8) \quad u(t) = - \sum_{k \geq 1} v_k(T-t)$$

194 where, for each $k \geq 1$, v_k solves the block moment problem

$$195 \quad (1.9) \quad \begin{cases} \int_0^T e^{-\lambda_{k,j}t} \langle v_k(t), \mathcal{B}^* \phi_{k,j} \rangle_U dt = e^{-\lambda_{k,j}T} \langle y_0, \phi_{k,j} \rangle_X, & \forall j \in \{1, 2\}, \\ \int_0^T e^{-\lambda_{k',j}t} \langle v_k(t), \mathcal{B}^* \phi_{k',j} \rangle_U dt = 0, & \forall k' \neq k, \forall j \in \{1, 2\}. \end{cases}$$

196 • To solve (1.9), for a fixed k , we consider the following auxiliary block moment
197 problem in the space U

$$198 \quad (1.10) \quad \begin{cases} \int_0^T e^{-\lambda_{k,j}t} v_k(t) dt = \Omega_{k,j}, & \forall j \in \{1, 2\}, \\ \int_0^T e^{-\lambda_{k',j}t} v_k(t) dt = 0, & \forall k' \neq k, \forall j \in \{1, 2\}, \end{cases}$$

199 where $\Omega_{k,j} \in U$ have to be precised. If we impose that $\Omega_{k,1}$ and $\Omega_{k,2}$ satisfy the
200 constraints

$$201 \quad (1.11) \quad \langle \Omega_{k,j}, \mathcal{B}^* \phi_{k,j} \rangle_U = e^{-\lambda_{k,j}T} \langle y_0, \phi_{k,j} \rangle_X, \quad \forall j \in \{1, 2\},$$

202 we obtain that the solutions of (1.10) also solve (1.9). The existence of $\Omega_{k,1}$ and $\Omega_{k,2}$
203 satisfying the constraints (1.11) is ensured by (1.7), however there exist infinitely
204 many choices.

205 For any $\Omega_{k,1}, \Omega_{k,2} \in U$, applying the results of [10] component by component in
206 the finite dimensional subspace of U defined by $\text{Span}\{\Omega_{k,1}, \Omega_{k,2}\}$ leads to the existence
207 of $v_k \in L^2(0, T; U)$ satisfying (1.10). It also gives the following estimate

$$208 \quad (1.12) \quad \|v_k\|_{L^2(0, T; U)}^2 \leq C_{T, \varepsilon} e^{\varepsilon \lambda_{k,1}} F(\Omega_{k,1}, \Omega_{k,2}),$$

209 with

$$210 \quad F : (\Omega_{k,1}, \Omega_{k,2}) \in U^2 \mapsto \|\Omega_{k,1}\|_U^2 + \left\| \frac{\Omega_{k,2} - \Omega_{k,1}}{\lambda_{k,2} - \lambda_{k,1}} \right\|_U^2.$$

211 Using (1.12) and optimizing the function F under the constraints (1.11) we obtain
212 that there exists $v_k \in L^2(0, T; U)$ solution of the block moment problem (1.9) such
213 that

$$214 \quad (1.13) \quad \|v_k\|_{L^2(0, T; U)}^2 \leq C_{T, \varepsilon} e^{\varepsilon \lambda_{k,1}} \inf \{F(\Omega_{k,1}, \Omega_{k,2}) ; \Omega_{k,1}, \Omega_{k,2} \text{ satisfy (1.11)}\}.$$

215 The corresponding general statements of the resolution of block moment problems
216 are detailed in Section 1.4 (see Theorem 1.2) and proved in Section 2.

217 • Now that we can solve the block moment problems (1.9), a way to characterize the
218 minimal null control time is to estimate for which values of T the series (1.8) defining
219 the control u converges in $L^2(0, T; U)$.

220 To achieve this goal, we isolate in the estimate (1.13) the dependency with respect
221 to T . Notice that the function F does not depend on T but that the constraints (1.11)
222 does.

223 For any $k \geq 1$ and any $\Omega_{k,1}, \Omega_{k,2} \in U$ we set

$$224 \quad \tilde{\Omega}_{k,j} := e^{\lambda_{k,j}T} \Omega_{k,j}, \quad \forall j \in \{1, 2\}.$$

225 Then, there is equivalence between the constraints (1.11) and the new constraints

$$226 \quad (1.14) \quad \left\langle \tilde{\Omega}_{k,j}, \mathcal{B}^* \phi_{k,j} \right\rangle_U = \langle y_0, \phi_{k,j} \rangle_X, \quad \forall j \in \{1, 2\}.$$

227 Now these constraints are independent of the variable T . From the mean value theo-
228 rem we obtain

$$\begin{aligned} 229 \quad F(\Omega_{k,1}, \Omega_{k,2}) &= \left\| e^{-\lambda_{k,1}T} \tilde{\Omega}_{k,1} \right\|_U^2 + \left\| \frac{e^{-\lambda_{k,2}T} \tilde{\Omega}_{k,2} - e^{-\lambda_{k,1}T} \tilde{\Omega}_{k,1}}{\lambda_{k,2} - \lambda_{k,1}} \right\|_U^2 \\ 230 \quad &\leq e^{-2\lambda_{k,1}T} \left\| \tilde{\Omega}_{k,1} \right\|_U^2 + 2e^{-2\lambda_{k,2}T} \left\| \frac{\tilde{\Omega}_{k,2} - \tilde{\Omega}_{k,1}}{\lambda_{k,2} - \lambda_{k,1}} \right\|_U^2 \\ 231 \quad &\quad + 2 \left(\frac{e^{-\lambda_{k,2}T} - e^{-\lambda_{k,1}T}}{\lambda_{k,2} - \lambda_{k,1}} \right)^2 \left\| \tilde{\Omega}_{k,1} \right\|_U^2 \\ 232 \quad &\leq 2(1 + T^2)e^{-2\lambda_{k,1}T} F(\tilde{\Omega}_{k,1}, \tilde{\Omega}_{k,2}). \end{aligned}$$

234 The general statement of this estimate is given in Lemma 3.1.

235 Plugging this estimate into (1.12) and optimizing the function F under the con-
236 straints (1.14) yields

$$237 \quad (1.15) \quad \|v_k\|_{L^2(0,T;U)}^2 \leq C_{T,\varepsilon} e^{\varepsilon\lambda_{k,1}} e^{-2\lambda_{k,1}T} \mathcal{C}_k(y_0)$$

238 where

$$239 \quad (1.16) \quad \mathcal{C}_k(y_0) := \inf \left\{ \left\| \tilde{\Omega}_1 \right\|_U^2 + \left\| \frac{\tilde{\Omega}_2 - \tilde{\Omega}_1}{\lambda_{k,2} - \lambda_{k,1}} \right\|_U^2 ; \left\langle \tilde{\Omega}_j, \mathcal{B}^* \phi_{k,j} \right\rangle_U = \langle y_0, \phi_{k,j} \rangle_X, \right. \\ 240 \quad \left. \forall j \in \{1, 2\} \right\}.$$

243 Estimate (1.15) proves that for any time T such that

$$244 \quad T > \limsup_{k \rightarrow +\infty} \frac{\ln \mathcal{C}_k(y_0)}{2\lambda_{k,1}}$$

245 the series (1.8) defining the control u converges in $L^2(0, T; U)$. Thus, null controlla-
246 bility of (1.1) from y_0 holds for such T .

247 We also prove that the obtained estimate (1.15) is sufficiently sharp so that it
248 characterizes the minimal null control time from y_0 as

$$249 \quad (1.17) \quad T_0(y_0) = \limsup_{k \rightarrow +\infty} \frac{\ln \mathcal{C}_k(y_0)}{2\lambda_{k,1}}.$$

250 The corresponding general statements regarding the minimal null control time
251 together with bounds on the cost of controllability are detailed in Section 1.4 (see
252 Theorem 1.3) and proved in Section 3.

253 • At this stage we have characterized the minimal null control time as stated in (1.17).
 254 However to be able to estimate the actual value of $T_0(y_0)$ one should be able to
 255 estimate the quantity $C_k(y_0)$ as defined in (1.16). This formula is not very explicit
 256 and it does not get better in the general setting.

257 To do so, we remark that (1.16) is a finite dimensional optimization problem that
 258 we can explicitly solve in terms of eigenelements of \mathcal{A}^* and their observations through
 259 \mathcal{B}^* .

260 We obtain different results depending on the assumptions on the multiplicity
 261 of the eigenvalues of the considered blocks. The general statements of an explicit
 262 solution of the corresponding optimization problem are detailed in Section 1.5 (see
 263 Theorems 1.8 and 1.10) and proved in Section 4.

264 For the particular example we are considering here, the obtained formula reads

$$265 \quad C_k(y_0) = \frac{1}{\|\varphi_k\|_{L^2(\omega)}^2} \left\langle y_0, \begin{pmatrix} 0 \\ \varphi_k \end{pmatrix} \right\rangle_X^2 + \frac{e^{2ak^2}}{\|\varphi_k\|_{L^2(\omega)}^2} \left\langle y_0, \begin{pmatrix} \varphi_k \\ 0 \end{pmatrix} \right\rangle_X^2.$$

266 Then, the minimal null control time from X of this example is given by

$$267 \quad T_0(X) = a.$$

268 Notice, for instance, that this expression also gives that for a given y_0 if the set

$$269 \quad \left\{ k \in \mathbb{N}^* ; \left\langle y_0, \begin{pmatrix} \varphi_k \\ 0 \end{pmatrix} \right\rangle_X \neq 0 \right\}$$

270 is finite, then null controllability from y_0 holds in any positive time, *i.e.* $T_0(y_0) = 0$.

271 • Finally, we provide various examples of application of the results developed in this
 272 article. To highlight the ideas and phenomena we start with rather academic examples
 273 in Section 5. We then consider systems of coupled one dimensional linear parabolic
 274 equations with boundary or distributed controls in Section 6.

275 1.3. Framework, spectral assumptions and notations.

276 To state the main results of this article, we now detail the functional setting and
 277 assumptions we use.

278 1.3.1. Functional setting.

279 The functional setting for the study of system (1.1) is the same as in [10]. For
 280 the sake of completeness, let us detail it.

281 We consider X an Hilbert space, whose inner product and norm are denoted by
 282 $\langle \bullet, \bullet \rangle_X$ and $\|\bullet\|_X$ respectively. The space X is identified to its dual through the Riesz
 283 theorem. Let $(\mathcal{A}, D(\mathcal{A}))$ be an unbounded operator in X such that $-\mathcal{A}$ generates a
 284 C^0 -semigroup in X . Its adjoint in X is denoted by $(\mathcal{A}^*, D(\mathcal{A}^*))$. Up to a suitable
 285 translation, we can assume that 0 is in the resolvent set of \mathcal{A} .

286 We denote by X_1 (resp. X_1^*) the Hilbert space $D(\mathcal{A})$ (resp. $D(\mathcal{A}^*)$) equipped
 287 with the norm $\|x\|_1 := \|\mathcal{A}x\|_X$ (resp. $\|x\|_{1^*} := \|\mathcal{A}^*x\|_X$) and we define X_{-1} as the
 288 completion of X with respect to the norm

$$289 \quad \|y\|_{-1} := \sup_{z \in X_1^*} \frac{\langle y, z \rangle_X}{\|z\|_{1^*}}.$$

8

290 Notice that X_{-1} is isometrical to the topological dual of X_1^* using X as a pivot space
 291 (see for instance [37, Proposition 2.10.2]); the corresponding duality bracket will be
 292 denoted by $\langle \bullet, \bullet \rangle_{-1,1^*}$.

293 The control space U is an Hilbert space (that we will identify to its dual). Its inner
 294 product and norm are denoted by $\langle \bullet, \bullet \rangle_U$ and $\|\bullet\|_U$ respectively. Let $\mathcal{B} : U \rightarrow X_{-1}$ be
 295 a linear continuous control operator and denote by $\mathcal{B}^* : X_1^* \rightarrow U$ its adjoint in the
 296 duality described above.

297 Let $(X_\diamond^*, \|\cdot\|_{\diamond^*})$ be an Hilbert space such that $X_1^* \subset X_\diamond^* \subset X$ with dense and
 298 continuous embeddings. We assume that X_\diamond^* is stable by the semigroup generated by
 299 $-\mathcal{A}^*$. We also define $X_{-\diamond}$ as the subspace of X_{-1} defined by

$$300 \quad X_{-\diamond} := \left\{ y \in X_{-1} ; \|y\|_{-\diamond} := \sup_{z \in X_1^*} \frac{\langle y, z \rangle_{-1,1^*}}{\|z\|_{\diamond^*}} < +\infty \right\},$$

301 which is also isometrical to the dual of X_\diamond^* with X as a pivot space. The corresponding
 302 duality bracket will be denoted by $\langle \bullet, \bullet \rangle_{-\diamond,\diamond}$. Thus, we end up with the following five
 303 functional spaces

$$304 \quad X_1^* \subset X_\diamond^* \subset X \subset X_{-\diamond} \subset X_{-1}.$$

305 We say that the control operator \mathcal{B} is an admissible control operator for (1.1) with
 306 respect to the space $X_{-\diamond}$ if for any $T > 0$ there exists $C_T > 0$ such that

$$307 \quad (1.18) \quad \int_0^T \left\| \mathcal{B}^* e^{-(T-t)\mathcal{A}^*} z \right\|_U^2 dt \leq C_T \|z\|_{\diamond^*}^2, \quad \forall z \in X_1^*.$$

308 Notice that if (1.18) holds for some $T > 0$ it holds for any $T > 0$. The admissibility
 309 condition (1.18) implies that, by density, we can give a meaning to the map

$$310 \quad \left(t \mapsto \mathcal{B}^* e^{-(T-t)\mathcal{A}^*} z \right) \in L^2(0, T; U),$$

311 for any $z \in X_\diamond^*$. Then, we end up with the following well-posedness result (see [10,
 312 Proposition 1.2]).

313 **PROPOSITION 1.1.** *Assume that (1.18) holds. Then, for any $T > 0$, any $y_0 \in X_{-\diamond}$,
 314 and any $u \in L^2(0, T; U)$, there exists a unique $y \in C^0([0, T]; X_{-\diamond})$ solution to (1.1)
 315 in the sense that it satisfies for any $t \in [0, T]$ and any $z_t \in X_\diamond^*$,*

$$316 \quad \langle y(t), z_t \rangle_{-\diamond,\diamond} - \langle y_0, e^{-t\mathcal{A}^*} z_t \rangle_{-\diamond,\diamond} = \int_0^t \langle u(s), \mathcal{B}^* e^{-(t-s)\mathcal{A}^*} z_t \rangle_U ds.$$

317 Moreover there exists $C_T > 0$ such that

$$318 \quad \sup_{t \in [0, T]} \|y(t)\|_{-\diamond} \leq C_T (\|y_0\|_{-\diamond} + \|u\|_{L^2(0, T; U)}).$$

319 **REMARK 1.2.** *By analogy with the semigroup notation, when $u = 0$, we set for
 320 any $t \in [0, T]$, $e^{-t\mathcal{A}} y_0 := y(t)$. This extends the semigroup $e^{-\bullet\mathcal{A}}$ defined on X to $X_{-\diamond}$
 321 and implies that for any $z \in X_{-\diamond}$,*

$$322 \quad (1.19) \quad \langle e^{-T\mathcal{A}} z, \phi \rangle_{-\diamond,\diamond} = \langle z, e^{-T\mathcal{A}^*} \phi \rangle_{-\diamond,\diamond}, \quad \forall \phi \in X_\diamond^*.$$

323 With this notion of solution at hand, we finally define the minimal null control
 324 time from a subspace of initial conditions Y_0 .

325 DEFINITION 1.1. *Let Y_0 be a closed subspace of $X_{-\diamond}$ and let $T > 0$. The sys-*
 326 *tem (1.1) is said to be null controllable from Y_0 at time T if for any $y_0 \in Y_0$, there*
 327 *exists a control $u \in L^2(0, T; U)$ such that the associated solution of (1.1) satisfies*
 328 *$y(T) = 0$.*

329 *The minimal null control time $T_0(Y_0) \in [0, +\infty]$ is defined by*

- 330 • *for any $T > T_0(Y_0)$, system (1.1) is null controllable from Y_0 at time T ;*
- 331 • *for any $T < T_0(Y_0)$, system (1.1) is not null controllable from Y_0 at time T .*

332 To simplify the notations, for any $y_0 \in X_{-\diamond}$, we define $T_0(y_0) := T_0(\text{Span}\{y_0\})$. In
 333 the formulas given in this article, it can happen that $T_0(Y_0) < 0$. In this case, one
 334 should replace $T_0(Y_0)$ by 0.

335 1.3.2. Spectral assumptions.

336 In all this article we assume that the operators \mathcal{A} and \mathcal{B} satisfy the assumptions of
 337 Proposition 1.1. Moreover to solve the control problem we will need some additional
 338 spectral assumptions.

339 *★ Behavior of eigenvalues.*

340 We assume that the spectrum of \mathcal{A}^* , denoted by Λ , is only composed of (countably
 341 many) eigenvalues.

342 In what follows we assume that

$$343 \quad (1.20) \quad \Lambda \in (0, +\infty)^{\mathbb{N}}.$$

344 REMARK 1.3. *In [10], the assumption on Λ was slightly stronger. Namely, in that*
 345 *article it was assumed that $\Lambda \in (1, +\infty)^{\mathbb{N}}$. This stronger assumption was only used in*
 346 *the lower bound on the solution of scalar block moment problems (see estimate (A.4)).*
 347 *Thus the results of the article [10] that will be used in the present article remain valid*
 348 *under the assumption (1.20).*

349 *If necessary, one can replace the operator \mathcal{A} by $\mathcal{A} + \tau$ without modifying the*
 350 *controllability properties. Then, in the different estimates, the behavior with respect*
 351 *to τ can be carefully tracked if needed.*

352 Most of the results of this article (but not all) also holds when the eigenvalues in
 353 Λ are complex valued (yet with a dominant real part). To avoid confusion we stick
 354 with the assumption (1.20) and we only discuss in Section 7.1 which results hold in
 355 the complex setting and what are the necessary adjustments.

356 As in the case of a scalar control (see [10]) we assume that this spectrum satisfies
 357 a weak-gap condition. Namely, there exists $p \in \mathbb{N}^*$ and $\varrho > 0$ such that

$$358 \quad (1.21) \quad \#\left(\Lambda \cap [\mu, \mu + \varrho]\right) \leq p, \quad \forall \mu \in [0, +\infty).$$

359 This means that the eigenvalues are allowed to condensate by groups but the cardi-
 360 nality of these groups should be bounded. To precise this, let us recall the notion of
 361 groupings introduced in [10, Definition 1.6].

362 DEFINITION 1.2. *Let $p \in \mathbb{N}^*$ and $r, \varrho > 0$. A sequence of sets $(G_k)_{k \geq 1} \subset \mathcal{P}(\Lambda)$*
 363 *is said to be a grouping for Λ with parameters p, r, ϱ (which we denote by $(G_k)_k \in$*
 364 *$\mathcal{G}(\Lambda, p, r, \varrho)$) if it is a covering of Λ*

$$365 \quad \Lambda = \bigcup_{k \geq 1} G_k,$$

366 with the additional properties that, for every $k \geq 1$,

$$367 \quad \#G_k \leq p, \quad \sup(G_k) < \inf(G_{k+1}), \quad \text{dist}(G_k, G_{k+1}) \geq r$$

368 and

$$369 \quad \text{diam}(G_k) < \varrho.$$

370 As proved in [10, Proposition 7.1], the weak-gap condition (1.21) implies that

$$371 \quad \mathcal{G}\left(\Lambda, p, \frac{\varrho}{p}, \varrho\right) \neq \emptyset.$$

372

373 **REMARK 1.4.** For convenience, in the following we label the eigenvalues of a given
 374 group G in increasing order i.e. $G = \{\lambda_1, \dots, \lambda_g\}$ with $\lambda_k < \lambda_{k+1}$ but this is not
 375 mandatory.

376 Concerning the asymptotic behavior of the spectrum we will use the counting
 377 function associated to Λ defined by

$$378 \quad N_\Lambda : r > 0 \mapsto \#\{\lambda \in \Lambda ; \lambda \leq r\}.$$

379 When there is no ambiguity we drop the subscript Λ . We assume that there exists
 380 $\bar{N} > 0$ and $a \in (0, 1)$ such that

$$381 \quad (1.22) \quad N_\Lambda(r) \leq \bar{N}r^a, \quad \forall r > 0.$$

382 Notice that this condition is slightly stronger than the classical summability condi-
 383 tion (1.4) used for instance in [24, 5, 10] and many other works.

384 Notice also that (1.22), with $r = \min \Lambda$, implies the following lower bound on the
 385 bottom of the spectrum

$$386 \quad \min \Lambda \geq \bar{N}^{-a}.$$

387 Our goal is not only to study the controllability properties of our system but also to
 388 obtain estimates that are uniform in a way to be precised. To do so, we define the
 389 following class of sequences: let $p \in \mathbb{N}^*$, $\varrho, \bar{N} > 0$, $a \in (0, 1)$ and consider the class

$$390 \quad (1.23) \quad \mathcal{L}(p, \varrho, a, \bar{N}) := \{\Lambda \in (0, +\infty)^\mathbb{N} ; \Lambda \text{ satisfies (1.21) and (1.22)}\}.$$

391 In this work, we obtain sharper estimates when replacing (1.22) by the stronger
 392 assumption

$$393 \quad (1.24) \quad |N_\Lambda(r) - \bar{N}r^a| \leq \tilde{N}r^{a'}, \quad \forall r > 0,$$

394 with $\tilde{N} > 0$ and $a' \in [0, a)$. This motivates the definition of the class

$$395 \quad (1.25) \quad \mathcal{L}(p, \varrho, a, \bar{N}, a', \tilde{N}) := \{\Lambda \in (0, +\infty)^\mathbb{N} ; \Lambda \text{ satisfies (1.21) and (1.24)}\}.$$

396 Finally, we can also deal with the slightly larger class $\mathcal{L}(p, \varrho, \mathcal{N})$ used in [10]
 397 (see (A.1)) but this will not lead to explicit estimates with respect to the control
 398 time.

399 \star *Multiplicity of eigenvalues.*

400 In our study we allow both algebraic and geometric multiplicities for the eigenval-
 401 ues. We assume that these multiplicities are finite and that the algebraic multiplicity
 402 is globally bounded. More precisely, we assume that

$$403 \quad (1.26) \quad \gamma_\lambda := \dim \text{Ker}(\mathcal{A}^* - \lambda) < +\infty, \quad \forall \lambda \in \Lambda,$$

404 and that there exists $\eta \in \mathbb{N}^*$ such that

$$405 \quad (1.27) \quad \text{Ker}(\mathcal{A}^* - \lambda)^\eta = \text{Ker}(\mathcal{A}^* - \lambda)^{\eta+1}, \quad \forall \lambda \in \Lambda.$$

406 For any $\lambda \in \Lambda$ we denote by α_λ the smallest integer such that

$$407 \quad \text{Ker}(\mathcal{A}^* - \lambda)^{\alpha_\lambda} = \text{Ker}(\mathcal{A}^* - \lambda)^{\alpha_\lambda+1}$$

408 and set

$$409 \quad E_\lambda := \text{Ker}(\mathcal{A}^* - \lambda)^{\alpha_\lambda}.$$

410 For a given group $G = \{\lambda_1, \dots, \lambda_g\}$ we denote by $\alpha = (\alpha_1, \dots, \alpha_g)$ the multi-index of
 411 corresponding algebraic multiplicities.

412 \star *(Generalized) eigenvectors.*

413 To study null-controllability, we assume that the Fattorini-Hautus criterion is
 414 satisfied

$$415 \quad (1.28) \quad \text{Ker}(\mathcal{A}^* - \lambda) \cap \text{Ker} \mathcal{B}^* = \{0\}, \quad \forall \lambda \in \Lambda.$$

416 It is a necessary condition for approximate controllability. Note that, under additional
 417 assumptions on \mathcal{A} and \mathcal{B} it is also a sufficient condition for approximate controllability
 418 (see for instance [22, 33]). However, when studying null controllability of system (1.1)
 419 for initial conditions in a closed strict subspace Y_0 of $X_{-\diamond}$ the condition (1.28) can be
 420 too strong. This is discussed in Section 7.2.

421 We assume that the family of generalized eigenvectors of \mathcal{A}^*

$$422 \quad \Phi = \{\phi \in E_\lambda; \lambda \in \Lambda\} = \bigcup_{\lambda \in \Lambda} E_\lambda$$

423 is complete in X_{\diamond}^* i.e. for any $y \in X_{-\diamond}$,

$$424 \quad (1.29) \quad \left(\langle y, \phi \rangle_{-\diamond, \diamond} = 0, \quad \forall \phi \in \Phi \right) \implies y = 0.$$

425 In the following, to simplify the writing, we gather these assumptions and say that
 426 the operators \mathcal{A} and \mathcal{B} satisfy (H) if there exists $p \in \mathbb{N}^*$, $r, \varrho, \bar{N}, \tilde{N} > 0$, $a \in (0, 1)$,
 427 $a' \in [0, a)$ and $\mathcal{N} : (0, +\infty) \rightarrow \mathbb{R}$ such that

$$428 \quad (\text{H}) \quad \begin{cases} \mathcal{A} \text{ and } \mathcal{B} \text{ satisfy the assumptions of Proposition 1.1 ;} \\ \Lambda = \text{Sp}(\mathcal{A}^*) \text{ satisfies (1.20), (1.26), (1.27) and} \\ \Lambda \in \mathcal{L}(p, \varrho, \mathcal{N}) \cup \mathcal{L}(p, \varrho, a, \bar{N}) \cup \mathcal{L}(p, \varrho, a, \bar{N}, a', \tilde{N}) ; \\ \text{the associated (generalized) eigenvectors satisfy (1.28) and (1.29).} \end{cases}$$

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430

1.3.3. Notation.

We give here some notation that will be used throughout this article.

- For any integers $a, b \in \mathbb{N}$, we define the following subset of \mathbb{N} :

$$\llbracket a, b \rrbracket := [a, b] \cap \mathbb{N}.$$

431

- For any $s \in \mathbb{R}$ we denote by e_s the exponential function

432

$$e_s : (0, +\infty) \rightarrow \mathbb{R} \\ x \mapsto e^{-sx}.$$

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- We shall denote by $C_{\theta_1, \dots, \theta_l} > 0$ a constant possibly varying from one line to another but depending only on the parameters $\theta_1, \dots, \theta_l$.
- For any multi-index $\alpha \in \mathbb{N}^n$, we denote its length by $|\alpha| = \sum_{j=1}^n \alpha_j$ and its maximum by $|\alpha|_\infty = \max_{j \in \llbracket 1, n \rrbracket} \alpha_j$.
For $\alpha, \mu \in \mathbb{N}^n$, we say that $\mu \leq \alpha$ if and only if $\mu_j \leq \alpha_j$ for any $j \in \llbracket 1, n \rrbracket$.
- In all in this article the notation $f[\dots]$ stands for (generalized) divided differences of a set of values (x_j, f_j) . Let us recall that, for pairwise distinct $x_1, \dots, x_n \in \mathbb{R}$ and f_1, \dots, f_n in any vector space, the divided differences are defined by

442

$$f[x_j] = f_j, \quad f[x_1, \dots, x_j] = \frac{f[x_2, \dots, x_j] - f[x_1, \dots, x_{j-1}]}{x_j - x_1}.$$

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The two results that will be the most used in this article concerning divided differences are the Leibniz formula

445

$$(gf)[x_1, \dots, x_j] = \sum_{k=1}^j g[x_1, \dots, x_k] f[x_k, \dots, x_j],$$

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447

and the Lagrange theorem stating that, when $f_j = f(x_j)$ for a sufficiently regular function f , we have

448

$$f[x_1, \dots, x_j] = \frac{f^{(j-1)}(z)}{(j-1)!},$$

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with $z \in \text{Conv}\{x_1, \dots, x_j\}$. For more detailed statements and other useful properties as well as their generalizations when x_1, \dots, x_n are not assumed to be pairwise distinct we refer the reader to [10, Section 7.3]. This generalization is used in the present article whenever there are algebraically multiple eigenvalues.

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- For any closed subspace Y of $X_{-\diamond}$ we denote by P_Y the orthogonal projection in $X_{-\diamond}$ onto Y . We denote by $P_Y^* \in L(X_\diamond^*)$ its adjoint in the duality $X_{-\diamond}, X_\diamond^*$.

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1.4. Block moment problems and minimal time for null-controllability.

★ *Definition of block moment problems.*

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Using the notion of solution given in Proposition 1.1 and the assumption (1.29), null controllability from y_0 in time T reduces to the resolution of the following problem: find $u \in L^2(0, T; U)$ such that

462

$$(1.30) \quad \int_0^T \left\langle u(t), \mathcal{B}^* e^{-(T-t)\mathcal{A}^*} \phi \right\rangle_U dt = - \left\langle y_0, e^{-T\mathcal{A}^*} \phi \right\rangle_{-\diamond, \diamond}, \quad \forall \phi \in E_\lambda, \forall \lambda \in \Lambda.$$

463 Following the strategy initiated in [10] for scalar controls, we decompose this problem
 464 into block moments problem. Hence we look for a control of the form

$$465 \quad (1.31) \quad u = - \sum_{k \geq 1} v_k(T - \bullet)$$

466 where for every $k \in \mathbb{N}^*$, $v_k \in L^2(0, T; U)$ solves the moment problem in the group G_k
 467 *i.e.*

$$468 \quad (1.32a) \quad \int_0^T \langle v_k(t), \mathcal{B}^* e^{-t\mathcal{A}^*} \phi \rangle_U dt = \langle y_0, e^{-T\mathcal{A}^*} \phi \rangle_{-\diamond, \diamond}, \quad \forall \phi \in E_\lambda, \forall \lambda \in G_k,$$

$$469 \quad (1.32b) \quad \int_0^T \langle v_k(t), \mathcal{B}^* e^{-t\mathcal{A}^*} \phi \rangle_U dt = 0, \quad \forall \phi \in E_\lambda, \forall \lambda \in \Lambda \setminus G_k.$$

471 Let us rewrite the orthogonality condition between groups (1.32b) in a more conven-
 472 nient way. For any $\phi \in E_\lambda$, from [10, (1.22)], it comes that

$$473 \quad (1.33) \quad e^{-t\mathcal{A}^*} \phi = e^{-\lambda t} \sum_{r \geq 0} \frac{(-t)^r}{r!} (\mathcal{A}^* - \lambda)^r \phi = \sum_{r \geq 0} e_t \left[\lambda^{(r+1)} \right] (\mathcal{A}^* - \lambda)^r \phi,$$

474 where the sums are finite (and contains at most the first α_λ terms).

475 From (1.32) and (1.33), we study in this article the following block moment prob-
 476 lems for a given group G

$$477 \quad (1.34a) \quad \int_0^T \langle v(t), \mathcal{B}^* e^{-t\mathcal{A}^*} \phi \rangle_U dt = \langle e^{-T\mathcal{A}^*} y_0, \phi \rangle_{-\diamond, \diamond}, \quad \forall \phi \in E_\lambda, \forall \lambda \in G,$$

$$478 \quad (1.34b) \quad \int_0^T v(t) t^l e^{-\lambda t} dt = 0, \quad \forall \lambda \in \Lambda \setminus G, \forall l \in \llbracket 0, \eta - 1 \rrbracket.$$

480 where $e^{-T\mathcal{A}^*} y_0$ is defined in (1.19).

481 **REMARK 1.5.** *Thanks to (1.33), every solution of (1.34) solves (1.32). Yet, the*
 482 *orthogonality condition between groups (1.34b) is more restrictive than (1.32b): it is*
 483 *stated directly in U and each eigenvalue outside the group G is considered as if it has*
 484 *maximal algebraic multiplicity η . Those two choices allow a unification of the writing*
 485 *when the eigenvalues in different groups have different spectral behaviors and have no*
 486 *influence on the obtained results.*

487 **★ Resolution of block moment problems.**

488 In our setting, the block moment problem (1.34) is proved to be solvable for any
 489 $T > 0$. The resolution will follow from the scalar study done in [10] (see Theorem 1.2).

490 Due to (1.31), the main issue to prove null controllability of (1.1) is thus to sum
 491 these solutions to obtain a solution of (1.30). This is justified thanks to a precise
 492 estimate of the cost of the resolution of (1.34) for each group G that is the quantity

$$493 \quad \inf \left\{ \|v\|_{L^2(0, T; U)} ; v \text{ solution of (1.34)} \right\}.$$

494 To state this result, we introduce some additional notation.

495 To solve the moment problem (1.34) we lift it into a ‘vectorial block moment

496 *problem*' of the following form (see (2.1))

$$497 \quad \begin{cases} \int_0^T v(t) \frac{(-t)^l}{l!} e^{-\lambda t} dt = \Omega_\lambda^l, & \forall \lambda \in G, \forall l \in \llbracket 0, \alpha_\lambda - 1 \rrbracket, \\ \int_0^T v(t) t^l e^{-\lambda t} dt = 0, & \forall \lambda \in \Lambda \setminus G, \forall l \in \llbracket 0, \eta - 1 \rrbracket, \end{cases}$$

498 where Ω_λ^l belongs to U . Following (1.33), to recover a solution of (1.34), we need
 499 to impose some constraints on the right-hand side. Thus, for any $\lambda \in \Lambda$ and any
 500 $z \in X_{-\diamond}$, we set

$$501 \quad (1.35) \quad \mathcal{O}(\lambda, z) = \left\{ (\Omega^0, \dots, \Omega^{\alpha_\lambda - 1}) \in U^{\alpha_\lambda}; \sum_{l=0}^{\alpha_\lambda - 1} \langle \Omega^l, \mathcal{B}^*(\mathcal{A}^* - \lambda)^l \phi \rangle_U = \langle z, \phi \rangle_{-\diamond, \diamond}, \right. \\ \left. \forall \phi \in E_\lambda \right\}.$$

502
503

504 For a given group $G = \{\lambda_1, \dots, \lambda_g\}$ we set

$$505 \quad (1.36) \quad \mathcal{O}(G, z) = \mathcal{O}(\lambda_1, z) \times \dots \times \mathcal{O}(\lambda_g, z) \subset U^{|\alpha|}.$$

506 Recall that $\alpha = (\alpha_1, \dots, \alpha_g)$ is the multi-index of algebraic multiplicities. Consider
 507 any sequence of multi-indices $(\mu^l)_{l \in \llbracket 0, |\alpha| \rrbracket}$ such that

$$508 \quad (1.37) \quad \begin{cases} \mu^{l-1} \leq \mu^l, & \forall l \in \llbracket 1, |\alpha| \rrbracket, \\ |\mu^l| = l, & \forall l \in \llbracket 0, |\alpha| \rrbracket, \\ \mu^{|\alpha|} = \alpha. \end{cases}$$

509 To measure the cost associated to the group G let us define the following functional

$$510 \quad (1.38) \quad F : \Omega = (\Omega_1^0, \dots, \Omega_1^{\alpha_1 - 1}, \dots, \Omega_g^0, \dots, \Omega_g^{\alpha_g - 1}) \in U^{|\alpha|} \mapsto \sum_{l=1}^{|\alpha|} \left\| \Omega \left[\lambda^{\bullet} \right] \right\|_U^2.$$

511 The use of such functional to measure the cost comes from the analysis conducted
 512 in [10] (see Proposition 2.1).

513 The first main result of this article concerns the resolution of block moment
 514 problems of the form (1.34).

515 **THEOREM 1.2.** *Assume that the operators \mathcal{A} and \mathcal{B} satisfy the assumption (H)*
 516 *(see page 12) and let $(G_k)_{k \geq 1} \in \mathcal{G}(\Lambda, p, r, \varrho)$ be an associated grouping. Let $T \in$*
 517 *$(0, +\infty)$.*

518 *For any $G = \{\lambda_1, \dots, \lambda_g\} \in (G_k)_k$ and any $z \in X_{-\diamond}$, there exists $v \in L^2(0, T; U)$*
 519 *solution of*

$$520 \quad (1.39a) \quad \int_0^T \left\langle v(t), \mathcal{B}^* e^{-t\mathcal{A}^*} \phi \right\rangle_U dt = \langle z, \phi \rangle_{-\diamond, \diamond}, \quad \forall \phi \in E_{\lambda_j}, \forall j \in \llbracket 1, g \rrbracket,$$

$$521 \quad (1.39b) \quad \int_0^T v(t) t^l e^{-\lambda t} dt = 0, \quad \forall \lambda \in \Lambda \setminus G, \forall l \in \llbracket 0, \eta - 1 \rrbracket.$$

522

523 *Moreover, we have the following estimate*

$$524 \quad (1.40) \quad \|v\|_{L^2(0, T; U)}^2 \leq \mathcal{E}(\lambda_1) \mathcal{K}(T) \mathcal{C}(G, z),$$

525 where

$$526 \quad (1.41) \quad \mathcal{C}(G, z) := \inf \{F(\Omega) ; \Omega \in \mathcal{O}(G, z)\}$$

527 with F defined in (1.38), $\mathcal{O}(G, z)$ defined in (1.36) and the functions \mathcal{E} and \mathcal{K} satisfy
528 the bounds given in Theorem A.1.

529 Moreover, there exists $C_{p,\eta,\min \Lambda} > 0$ such that any $v \in L^2(0, T; U)$ solving (1.39a)
530 satisfies

$$531 \quad (1.42) \quad \|v\|_{L^2(0,T;U)}^2 \geq C_{p,\eta,\min \Lambda} \mathcal{C}(G, z).$$

532 Before giving the application of this resolution of block moment problems to null
533 controllability of problem (1.1), let us give some comments.

534 • As it was the case in [10], the considered setting allows for a wide variety
535 of applications. In (1.29) the generalized eigenvectors are only assumed to
536 form a complete family (and not a Riesz basis as in many previous works)
537 which is the minimal assumption to use a moment method-like strategy. The
538 weak gap condition (1.21) is also well adapted to study systems of coupled
539 one dimensional parabolic equations (see Section 6).

540 • The main restriction is the assumption (1.22) or (1.24) (or (A.1)). As detailed
541 in Section 1.1, this assumption is common to most of the results based on a
542 moment-like method.

543 Though restrictive, let us underline that the moment method is, to the best
544 of our knowledge, the most suitable method to capture very sensitive features
545 such as a minimal null control time for parabolic control problems without
546 constraints.

547 • The main novelty of this theorem is to ensure solvability of block moment
548 problems coming from control problems with control operators that are only
549 assumed to be admissible. In particular, the space U can be of infinite di-
550 mension.

551 • As proved by (1.42), for any fixed $T > 0$, up to the factor $\mathcal{E}(\lambda_1)$, the obtained
552 estimate (1.40) is optimal in the asymptotic $\min G \rightarrow +\infty$. This will be crucial
553 to completely characterize the minimal null control time in Theorem 1.3.
554 In the applications to control theory, this term $\mathcal{E}(\lambda_1)$ which accounts for the
555 orthogonality condition (1.39b), will always be negligible (see the bounds
556 given in Theorem A.1).

557 • The estimate (1.40) does not explicitly depend on the sequence of eigenval-
558 ues Λ but rather on some parameters such as the weak-gap parameters and
559 the asymptotic of the counting function. As presented in Section 1.1, the
560 uniformity of such bounds can be used to deal with parameter dependent
561 problems.

562 • Let us also underline that the obtained estimate (1.40) tracks the dependency
563 of the constants with respect to the controllability time T when

$$564 \quad \Lambda \in \mathcal{L}(p, \varrho, a, \bar{N}) \cup \mathcal{L}(p, \varrho, a, \bar{N}, a', \tilde{N}).$$

565 This will be crucial to estimate the cost of controllability in Proposition 1.5.
566 We refer to Remark 1.7 for possible applications of such estimates of the cost
567 of controllability.

568 • Though quite general and useful for the theoretical characterization of the
569 minimal null control time, the obtained estimate (1.40) is not very easy to

570 deal with on actual examples. With slightly stronger assumptions on the
571 eigenvalues of the group G we provide in Section 1.5 more explicit formulas.
572 • The formulation of the right-hand side of (1.39a) is not standard. Usually,
573 to set a moment problem, a specific basis of the generalized eigenspace is
574 exhibited. Here, in our study, we do not exhibit any particular generalized
575 eigenvector. This enables to choose different normalization condition on dif-
576 ferent examples and then simplify the computations on actual examples (see
577 Sections 5 and 6). Our study also leads to the resolution of block moment
578 problems with ‘standard’ right-hand sides. The obtained results are detailed
579 in Appendix C.

580 *★ Application to null controllability.*

581 The resolution of block moment problems stated in Theorem 1.2 allows to obtain
582 the following characterization of the minimal null control time from a given initial
583 condition.

584 **THEOREM 1.3.** *Assume that the operators \mathcal{A} and \mathcal{B} satisfy the assumption (H)*
585 *(see page 12) and let $(G_k)_{k \geq 1} \in \mathcal{G}(\Lambda, p, r, \varrho)$ be an associated grouping. Then, for any*
586 *$y_0 \in X_{-\diamond}$, the minimal null control time of (1.1) from y_0 is given by*

$$587 \quad (1.43) \quad T_0(y_0) = \limsup_{k \rightarrow +\infty} \frac{\ln \mathcal{C}(G_k, y_0)}{2 \min G_k}$$

588 where $\mathcal{C}(G_k, y_0)$ is defined in (1.41).

589 If one considers a space of initial conditions (instead of a single initial condition), the
590 characterization of the minimal null control time is given in the following corollary.

591 **COROLLARY 1.4.** *Let Y_0 be a closed subspace of $X_{-\diamond}$. Then, under the assump-*
592 *tions of Theorem 1.3, the minimal null control time from Y_0 is given by*

$$593 \quad T_0(Y_0) = \limsup_{k \rightarrow +\infty} \frac{\ln \mathcal{C}(G_k, Y_0)}{2 \min G_k}$$

594 with

$$595 \quad \mathcal{C}(G, Y_0) := \sup_{\substack{y_0 \in Y_0 \\ \|y_0\|_{-\diamond} = 1}} \mathcal{C}(G, y_0).$$

596 The remarks on the assumptions and their benefits and restrictions stated after
597 Theorem 1.2 remain valid.

598 When system (1.1) is null controllable, we obtain the following bound on the cost
599 of controllability.

600 **PROPOSITION 1.5.** *Assume that the operators \mathcal{A} and \mathcal{B} satisfy the assumption (H)*
601 *(see page 12) and let $(G_k)_{k \geq 1} \in \mathcal{G}(\Lambda, p, r, \varrho)$ be an associated grouping. Let Y_0 be a*
602 *closed subspace of $X_{-\diamond}$ and let $T > T_0(Y_0)$.*

603 *For any $y_0 \in Y_0$ with $\|y_0\|_{-\diamond} = 1$, there exists a control $u \in L^2(0, T; U)$ such that*
604 *the associated solution of (1.1) satisfies $y(T) = 0$ and*

$$605 \quad \|u\|_{L^2(0, T; U)}^2 \leq \mathcal{K}(T) \sum_{k \geq 1} (1 + T)^{2|\alpha_{G_k}|} \mathcal{E}(\min G_k) e^{-2(\min G_k)T} \mathcal{C}(G_k, Y_0),$$

606 where the functions \mathcal{E} and \mathcal{K} satisfy the bounds given in Theorem A.1.

607 Though quite general the above formula is not very explicit. More importantly,
 608 it is proved in [30, Theorem 1.1] that, with a suitable choice of \mathcal{A} and \mathcal{B} satisfying
 609 our assumptions, any blow-up of the cost of controllability can occur. We give below
 610 a setting (inspired from [30, Theorem 1.2]) in which this upper bound on the cost of
 611 controllability is simpler and can have some applications (see Remark 1.7).

612 **COROLLARY 1.6.** *Assume that the operators \mathcal{A} and \mathcal{B} satisfy the assumption (H)
 613 (see page 12) with*

$$614 \quad \Lambda \in \mathcal{L}(p, \varrho, a, \bar{N}, a', \tilde{N})$$

615 *as defined in (1.25). Let $\kappa > 0$. There exists $C > 0$ depending only on $\kappa, p, \varrho, \eta, a,$
 616 \bar{N}, a' and \tilde{N} such that for any $y_0 \in X_{-\diamond}$ satisfying*

$$617 \quad (1.44) \quad \mathcal{C}(G_k, y_0) \leq \kappa e^{2(\min G_k)T_0(y_0)} \|y_0\|_{-\diamond}^2, \quad \forall k \geq 1,$$

618 *for any $T > T_0(y_0)$ close enough to $T_0(y_0)$, there exists a control $u \in L^2(0, T; U)$ such
 619 that the associated solution of (1.1) satisfies $y(T) = 0$ and*

$$620 \quad \|u\|_{L^2(0, T; U)} \leq C \exp\left(\frac{C}{(T - T_0(y_0))^{\frac{a}{1-a}}}\right) \|y_0\|_{-\diamond}.$$

621 **REMARK 1.6.** *In the setting of Corollary 1.6, replacing the assumption (1.44) by*

$$622 \quad \mathcal{C}(G_k, y_0) \leq \kappa e^{C(\min G_k)^b} e^{2(\min G_k)T_0(y_0)} \|y_0\|_{-\diamond}^2, \quad \forall k \geq 1,$$

623 *with $b \in (0, 1)$ leads to the following estimate*

$$624 \quad \|u\|_{L^2(0, T; U)} \leq C \exp\left(\frac{C}{T^{\frac{a}{1-a}}} + \frac{C}{(T - T_0(y_0))^{\frac{\max(a, b)}{1 - \max(a, b)}}}\right) \|y_0\|_{-\diamond}.$$

625 **REMARK 1.7.** *Giving the best possible estimate on the cost of small time null
 626 controllability is a question that has drawn a lot of interest in the past years.*

627 *In classical cases, for instance for heat-like equations, null controllability holds
 628 in any positive time and the cost of controllability in small time behaves like $\exp\left(\frac{C}{T}\right)$
 629 (see for instance [36]). There are two mains applications of such estimate.*

630 • *Controllability in cylindrical domains.*

631 *It is proved in [9] that null controllability of parabolic problems in cylindrical
 632 geometries (with operators compatible with this geometry) with a boundary
 633 control located on the top of the cylinder can be proved thanks to null con-
 634 trollability of the associated problem in the transverse variable together with
 635 suitable estimates of the cost of controllability. Their proof relies on an adap-
 636 tation of the classical strategy of Lebeau and Robbiano [29] and thus uses
 637 an estimate of the cost of controllability in small time of the form $\exp\left(\frac{C}{T}\right)$.
 638 These ideas were already present in [11] and later generalized in an abstract
 639 setting in [1].*

640 • *Nonlinear control problems.*

641 *The source term method has been introduced in [31] to prove controllability
 642 of a nonlinear fluid-structure system (see also [8, Section 2] for a general
 643 presentation of this strategy). Roughly speaking it amounts to prove null con-
 644 trollability with a source term in suitable weighted spaces and then use a fixed
 645 point argument. The null controllability with a source term is here proved by
 646 an iterative process which strongly uses that the cost of controllability of the
 647 linearized system behaves like $\exp\left(\frac{C}{T}\right)$.*

648 Notice that from the upper bound given in Corollary 1.6, the cost of controllability in
649 small time can explode faster than $\exp\left(\frac{C}{T}\right)$. Yet, as studied in [34, Chapter 4], the
650 arguments of the two previous applications can be adapted with an explosion of the
651 cost of the form $\exp\left(\frac{C}{T^{1-a}}\right)$ with $a \in (0, 1)$.

652 However, these two applications uses a decomposition of the time interval $[0, T]$
653 into an infinite number of sub-intervals (which explains the use of the asymptotic of
654 the cost of controllability when the time goes to zero). Thus their extension in the
655 case of a minimal null control time is an open problem.

656 1.5. A more explicit formula.

657 Assume that the operators \mathcal{A} and \mathcal{B} satisfy the assumption (H) (see page 12).
658 Let $G = \{\lambda_1, \dots, \lambda_g\} \subset \Lambda$ be such that $\sharp G \leq p$ and $\text{diam } G \leq \varrho$. We have seen in
659 Theorem 1.3 that the key quantity to compute the minimal null control time from y_0
660 is

$$661 \quad \mathcal{C}(G, y_0) = \inf \{F(\Omega) ; \Omega \in \mathcal{O}(G, y_0)\}.$$

662 where the function F is defined in (1.38) and the constraints $\mathcal{O}(G, y_0)$ are defined
663 in (1.36).

664 Notice that, for any $z \in X_{-\diamond}$, the quantity $\mathcal{C}(G, z)$ can be expressed as a finite
665 dimensional constrained problem. Indeed, for a given group G we consider the finite
666 dimensional subspace

$$667 \quad (1.45) \quad U_G = \mathcal{B}^* \text{Span} \{\phi \in E_\lambda ; \lambda \in G\}$$

668 and P_{U_G} the orthogonal projection in U onto U_G . Then, for any $\Omega \in \mathcal{O}(G, z)$ it
669 comes that $P_{U_G} \Omega \in \mathcal{O}(G, z)$ and $F(P_{U_G} \Omega) \leq F(\Omega)$. Thus, the optimization problem
670 defining $\mathcal{C}(G, z)$ reduces to

$$671 \quad \mathcal{C}(G, z) = \inf \left\{ F(\Omega) ; \Omega \in \mathcal{O}(G, z) \cap U_G^{|\alpha|} \right\},$$

672 which is a finite dimensional optimization problem. From [10, Proposition 7.15], the
673 function F is coercive which implies that the infimum is attained:

$$674 \quad (1.46) \quad \mathcal{C}(G, z) = \min \left\{ F(\Omega) ; \Omega \in \mathcal{O}(G, z) \cap U_G^{|\alpha|} \right\}.$$

675 In this section, solving the optimization problem (1.46), we provide more explicit
676 formulas for this cost depending on stronger assumptions on the multiplicity of the
677 eigenvalues in the group G (and only in the group G).

678 **REMARK 1.8.** *All the results in this section only concern the group G . Then, the*
679 *assumption (H) is stronger than needed. For instance, one does not need the weak*
680 *gap condition (1.21) on the whole spectrum Λ but only that $\sharp G \leq p$ and $\text{diam } G \leq \varrho$.*
681 *However, to simplify the reading we stick with assumption (H).*

682 \star *A group G of geometrically simple eigenvalues.*

683 First, assume that the eigenvalues in G are all geometrically simple *i.e.* $\gamma_\lambda = 1$
684 for every $\lambda \in G$ where γ_λ is defined in (1.26).

685 For any $j \in \llbracket 1, g \rrbracket$ we denote by ϕ_j^0 an eigenvector of \mathcal{A}^* associated to the eigen-
686 value λ_j and by $(\phi_j^l)_{l \in \llbracket 0, \alpha_j - 1 \rrbracket}$ an associated Jordan chain *i.e.*

$$687 \quad (\mathcal{A}^* - \lambda_j) \phi_j^l = \phi_j^{l-1}, \quad \forall l \in \llbracket 1, \alpha_j - 1 \rrbracket.$$

688 To simplify the writing, we set

$$689 \quad b_j^l := \mathcal{B}^* \phi_j^l \in U, \quad \forall l \in \llbracket 0, \alpha_j - 1 \rrbracket, \forall j \in \llbracket 1, g \rrbracket.$$

690 Recall that the sequence of multi-index $(\mu^l)_{l \in \llbracket 0, |\alpha| \rrbracket}$ satisfy (1.37) and let

$$691 \quad (1.47) \quad M := \sum_{l=1}^{|\alpha|} \Gamma_\mu^l$$

692 with

$$693 \quad \Gamma_\mu^l := \text{Gram}_U \left(\underbrace{0, \dots, 0}_{l-1}, b \left[\lambda_{\bullet}^{(\mu^l - \mu^{l-1})} \right], \dots, b \left[\lambda_{\bullet}^{(\mu^{|\alpha|} - \mu^{l-1})} \right] \right)$$

694 where $\text{Gram}_U(\dots)$ denotes the Gram matrix of the arguments with respect to the
695 scalar product in U . To explicit the cost $\mathcal{C}(G, y_0)$, we will use the inverse of this
696 matrix. Its invertibility is guaranteed by the following proposition which is proved in
697 Section 4.2.

698 **PROPOSITION 1.7.** *Under condition (1.28), the matrix M defined in (1.47) is in-*
699 *vertible.*

700 The matrix M plays a crucial role in the computation of the cost $\mathcal{C}(G, y_0)$. Let us
701 give some comments. It is a sum of Gram matrices whose construction is summarized
702 in Figure 1 on an example with $G = \{\lambda_1, \lambda_2\}$ with $\alpha_1 = 3$ and $\alpha_2 = 2$. Each of these
703 matrices is of size $|\alpha|$ which is the number of eigenvalues (counted with their algebraic
704 multiplicities) that belong to the group G . Thus, on actual examples (see Section 6),
705 the size of these matrices is usually reasonably small.

706 Then, we obtain the following formula for the cost of a group of geometrically
707 simple eigenvalues.

708 **THEOREM 1.8.** *Assume that the operators \mathcal{A} and \mathcal{B} satisfy the assumption (H)*
709 *(see page 12). Let $G = \{\lambda_1, \dots, \lambda_g\} \subset \Lambda$ be such that $\sharp G \leq p$ and $\text{diam } G \leq \varrho$ and*
710 *assume that $\gamma_\lambda = 1$ for every $\lambda \in G$. Then, for any $y_0 \in X_{-\diamond}$, we have*

$$711 \quad \mathcal{C}(G, y_0) = \langle M^{-1} \xi, \xi \rangle, \quad \text{where } \xi = \begin{pmatrix} \left\langle y_0, \phi \left[\lambda_{\bullet}^{(\mu^1)} \right] \right\rangle_{-\diamond, \diamond} \\ \vdots \\ \left\langle y_0, \phi \left[\lambda_{\bullet}^{(\mu^{|\alpha|})} \right] \right\rangle_{-\diamond, \diamond} \end{pmatrix}$$

712 and M is defined in (1.47).

713 Moreover, if Y_0 is a closed subspace of $X_{-\diamond}$,

$$714 \quad (1.48) \quad \mathcal{C}(G, Y_0) = \rho \left(\text{Gram}_{X_{\diamond}^*}(\psi_1, \dots, \psi_{|\alpha|}) M^{-1} \right)$$

715 where $\psi_j := P_{Y_0}^* \phi \left[\lambda_{\bullet}^{(\mu^j)} \right]$ and, for any matrix M , the notation $\rho(M)$ denotes the
716 spectral radius of the matrix M .

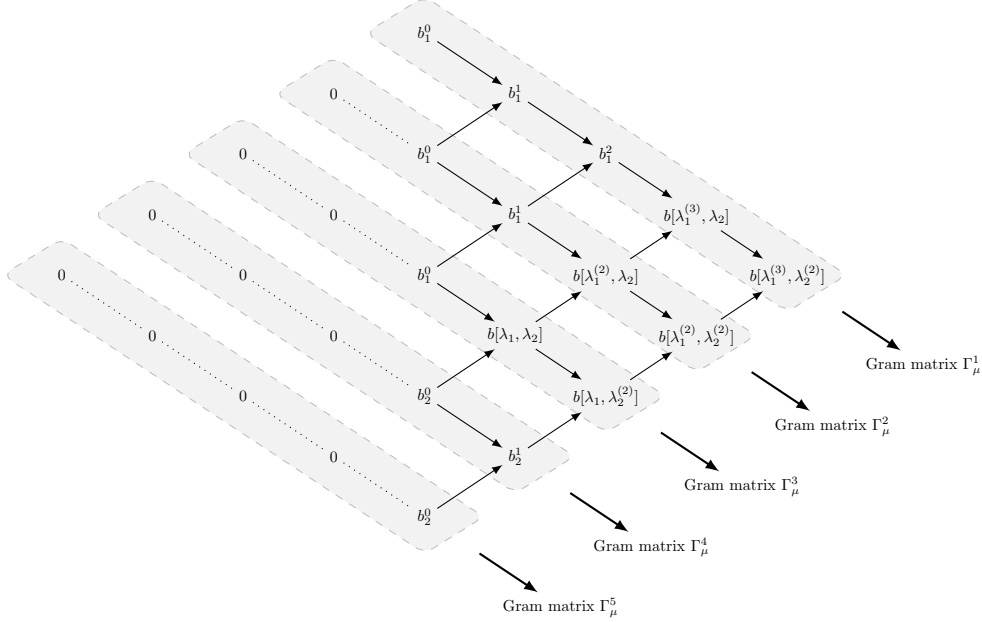


FIGURE 1. Construction of the Gram matrices Γ_μ^l in the case of a group $G = \{\lambda_1, \lambda_2\}$ with multiplicities $\alpha = (3, 2)$ and the sequence of multi-indices $\mu = ((0, 0), (1, 0), (2, 0), (3, 0), (3, 1), (3, 2))$

717 REMARK 1.9. Notice that we do not choose any particular eigenvector or Jordan
718 chain. To compute explicitly the cost $\mathcal{C}(G, y_0)$ on actual examples, we will often choose
719 them to satisfy

$$720 \quad \|b_j^0\|_U = 1, \quad \langle b_j^0, b_j^l \rangle_U = 0, \quad \forall l \in \llbracket 1, \alpha_j - 1 \rrbracket,$$

721 to simplify the Gram matrices. Obviously, as the quantity $\mathcal{C}(G, y_0)$ is independent of
722 this choice, we can choose any other specific Jordan chains or eigenvectors that are
723 more suitable to each problem.

724 REMARK 1.10. In the case where the eigenvalues of the considered group G are
725 also algebraically simple, then the expression of M given in (1.47) reduces to

$$726 \quad (1.49) \quad M = \sum_{l=1}^g \Gamma^l \quad \text{with} \quad \Gamma^l = \text{Gram}_U \left(\underbrace{0, \dots, 0}_{l-1}, b[\lambda_l], \dots, b[\lambda_l, \dots, \lambda_g] \right)$$

727 and the expression of ξ reduces to

$$728 \quad \xi = \begin{pmatrix} \langle y_0, \phi[\lambda_1] \rangle_{-\diamond, \diamond} \\ \vdots \\ \langle y_0, \phi[\lambda_1, \dots, \lambda_g] \rangle_{-\diamond, \diamond} \end{pmatrix}.$$

729 \star A group G of semi-simple eigenvalues.

730 We now assume that all the eigenvalues in G are semi-simple *i.e.* for any $\lambda \in G$
731 we have $\alpha_\lambda = 1$ where α_λ is defined in (1.27).

732 For any $j \in \llbracket 1, g \rrbracket$, we denote by $(\phi_{j,i})_{i \in \llbracket 1, \gamma_j \rrbracket}$ a basis of $\text{Ker}(\mathcal{A}^* - \lambda_j)$. To simplify
 733 the writing, we set

$$734 \quad b_{j,i} := \mathcal{B}^* \phi_{j,i}, \quad \forall j \in \llbracket 1, g \rrbracket, \forall i \in \llbracket 1, \gamma_j \rrbracket$$

735 and $\gamma_G := \gamma_1 + \dots + \gamma_g$.

736 For any $i \in \llbracket 1, g \rrbracket$, we set $\delta_1^i := 1$ and

$$737 \quad (1.50) \quad \delta_j^i := \prod_{k=1}^{j-1} (\lambda_i - \lambda_k), \quad \forall j \in \llbracket 2, g \rrbracket.$$

738 Notice that $\delta_j^i = 0$ as soon as $j > i$.

739 Let

$$740 \quad (1.51) \quad M = \sum_{l=1}^g \Gamma^l \text{ with } \Gamma^l = \text{Gram}_U(\delta_l^1 b_{1,1}, \dots, \delta_l^1 b_{1,\gamma_1}, \dots, \delta_l^g b_{g,1}, \dots, \delta_l^g b_{g,\gamma_g}).$$

741 Here again, to explicit the cost $\mathcal{C}(G, y_0)$ we will use the inverse of this matrix. Its
 742 invertibility is guaranteed by the following proposition which is proved in Section 4.3.

743 **PROPOSITION 1.9.** *Under condition (1.28), the matrix M defined in (1.51) is in-*
 744 *vertible.*

745 Notice that the square matrix Γ^l is of size γ_G and can be seen as a block matrix where
 746 the block (i, j) is

$$747 \quad \begin{pmatrix} \langle \delta_l^i b_{i,1}, \delta_l^j b_{j,1} \rangle_U & \cdots & \langle \delta_l^i b_{i,1}, \delta_l^j b_{j,\gamma_j} \rangle_U \\ \vdots & & \vdots \\ \langle \delta_l^i b_{i,\gamma_i}, \delta_l^j b_{j,1} \rangle_U & \cdots & \langle \delta_l^i b_{i,\gamma_i}, \delta_l^j b_{j,\gamma_j} \rangle_U \end{pmatrix}$$

748 Thus, the block (i, j) of Γ^l is identically 0 for $i, j \in \llbracket 1, l-1 \rrbracket$.

749 Then, we obtain the following formula for the cost of a group made of semi-simple
 750 eigenvalues.

751 **THEOREM 1.10.** *Assume that the operators \mathcal{A} and \mathcal{B} satisfy the assumption (H)*
 752 *(see page 12). Let $G = \{\lambda_1, \dots, \lambda_g\} \subset \Lambda$ be such that $\sharp G \leq p$ and $\text{diam } G \leq \varrho$ and*
 753 *assume that $\alpha_\lambda = 1$ for every $\lambda \in G$. Then, for any $y_0 \in X_{-\diamond}$, we have*

$$754 \quad \mathcal{C}(G, y_0) = \langle M^{-1} \xi, \xi \rangle$$

755 where

$$756 \quad \xi = \begin{pmatrix} \langle y_0, \phi_{1,1} \rangle_{-\diamond, \diamond} \\ \vdots \\ \langle y_0, \phi_{1,\gamma_1} \rangle_{-\diamond, \diamond} \\ \vdots \\ \langle y_0, \phi_{g,1} \rangle_{-\diamond, \diamond} \\ \vdots \\ \langle y_0, \phi_{g,\gamma_g} \rangle_{-\diamond, \diamond} \end{pmatrix}$$

757 and M is defined in (1.51).

758 Moreover, if Y_0 is a closed subspace of $X_{-\circ}$,

$$759 \quad (1.52) \quad \mathcal{C}(G, Y_0) = \rho(\text{Gram}_{X_\circ^*}(\psi_{1,1}, \dots, \psi_{1,\gamma_1}, \dots, \psi_{g,1}, \dots, \psi_{g,\gamma_g})M^{-1})$$

760 where $\psi_{j,i} := P_{Y_0}^* \phi_{j,i}$ and, for any matrix M , the notation $\rho(M)$ denotes the spectral
761 radius of the matrix M .

762 **REMARK 1.11.** When the eigenvalues of the group G are geometrically and alge-
763 braically simple, Theorem 1.10 gives a characterization of the cost of the block $\mathcal{C}(G, y_0)$
764 which is different from the one coming from Theorem 1.8 and detailed in Remark 1.10.
765 A direct proof of this equivalence (stated in Proposition D.3) using algebraic manipu-
766 lations is given in Appendix D.

767 \star Dealing simultaneously with geometric and algebraic multiplicity.

768 Combining Theorems 1.8 and 1.10, we can deal with operators \mathcal{A}^* which have
769 both groups of geometrically simple eigenvalues and groups of semi-simple eigenval-
770 ues. However, for technical reasons, in the case where both algebraic and geometric
771 multiplicities need to be taken into account into a group G we do obtain a general
772 formula for the cost of this group $\mathcal{C}(G, y_0)$. Nevertheless, if this situation occurs in
773 actual examples, computing this cost is a finite dimensional constrained optimization
774 problem which can be solved ‘by hand’. We present in Section 4.4 an example of such
775 resolution for a group G that does not satisfies the assumptions of Theorem 1.8 nor
776 of Theorem 1.10.

777 2. Resolution of block moment problems.

778 In this section we prove Theorem 1.2 that is we solve the block moment prob-
779 lem (1.39). To do so, we first consider a vectorial block moment problem (see (2.1)
780 below). We solve it in Section 2.1 with an estimate of the cost of this resolution.
781 Then, using the constraints (1.36), we prove that this implies Theorem 1.2. This is
782 detailed in Section 2.2.

783 2.1. An auxiliary vectorial block moment problem.

784 Let $\Lambda \in (0, +\infty)^{\mathbb{N}}$, $G = \{\lambda_1, \dots, \lambda_g\} \subset \Lambda$, $\eta \in \mathbb{N}^*$ and $\alpha = (\alpha_1, \dots, \alpha_g) \in \mathbb{N}^g$ with
785 $|\alpha|_\infty \leq \eta$. For any

$$786 \quad \Omega = (\Omega_1^0, \dots, \Omega_1^{\alpha_1-1}, \dots, \Omega_g^0, \dots, \Omega_g^{\alpha_g-1}) \in U^{|\alpha|},$$

787 we consider the following auxiliary vectorial block moment problem

$$788 \quad (2.1a) \quad \int_0^T v(t) \frac{(-t)^l}{l!} e^{-\lambda_j t} dt = \Omega_j^l, \quad \forall j \in \llbracket 1, g \rrbracket, \forall l \in \llbracket 0, \alpha_j - 1 \rrbracket,$$

$$789 \quad (2.1b) \quad \int_0^T v(t) t^l e^{-\lambda t} dt = 0, \quad \forall \lambda \in \Lambda \setminus G, \forall l \in \llbracket 0, \eta - 1 \rrbracket.$$

791 This block moment problem is said to be *vectorial* since the right-hand side Ω be-
792 longs to $U^{|\alpha|}$. Its resolution with (almost) sharp estimates is given in the following
793 proposition.

794 **PROPOSITION 2.1.** Let $p \in \mathbb{N}^*$, $r, \varrho, \bar{N}, \tilde{N} > 0$, $a \in (0, 1)$, $a' \in [0, a)$ and $\mathcal{N} : (0, +\infty) \rightarrow \mathbb{R}$. Assume that

$$796 \quad \Lambda \in \mathcal{L}(p, \varrho, \mathcal{N}) \cup \mathcal{L}(p, \varrho, a, \bar{N}) \cup \mathcal{L}(p, \varrho, a, \bar{N}, a', \tilde{N})$$

797 and let $(G_k)_{k \geq 1} \in \mathcal{G}(\Lambda, p, r, \varrho)$ be an associated grouping. Recall that these classes are
 798 defined in (A.1), (1.23) and (1.25).

799 Let $T \in (0, +\infty)$ and $\eta \in \mathbb{N}^*$. For any $G = \{\lambda_1, \dots, \lambda_g\} \in (G_k)_k$, for any
 800 multi-index $\alpha \in \mathbb{N}^g$ with $|\alpha|_\infty \leq \eta$ and any

$$801 \quad \Omega = (\Omega_1^0, \dots, \Omega_1^{\alpha_1-1}, \dots, \Omega_g^0, \dots, \Omega_g^{\alpha_g-1}) \in U^{|\alpha|},$$

802 there exists $v \in L^2(0, T; U)$ solution of (2.1) such that

$$803 \quad \|v\|_{L^2(0, T; U)}^2 \leq \mathcal{E}(\lambda_1) \mathcal{K}(T) F(\Omega),$$

804 where F is defined in (1.38) and the functions \mathcal{E} and \mathcal{K} satisfy the bounds given in
 805 Theorem A.1.

806 *Proof.* Let $(e_j)_{j \in \llbracket 1, d \rrbracket}$ be an orthonormal basis of the finite dimensional subspace
 807 of U

$$808 \quad \text{Span} \{ \Omega_j^l ; j \in \llbracket 1, g \rrbracket, l \in \llbracket 0, \alpha_j - 1 \rrbracket \}.$$

809 Then, we decompose Ω_j^l as

$$810 \quad \Omega_j^l = \sum_{i=1}^d a_{j,i}^l e_i.$$

811 From Theorem A.1, for any $i \in \llbracket 1, d \rrbracket$, there exists $v_i \in L^2(0, T; \mathbb{R})$ such that

$$812 \quad \begin{cases} \int_0^T v_i(t) \frac{(-t)^l}{l!} e^{-\lambda_j t} dt = a_{j,i}^l, & \forall j \in \llbracket 1, g \rrbracket, \forall l \in \llbracket 0, \alpha_j - 1 \rrbracket, \\ \int_0^T v_i(t) t^l e^{-\lambda t} dt = 0, & \forall \lambda \in \Lambda \setminus G, \forall l \in \llbracket 0, \eta - 1 \rrbracket, \end{cases}$$

813 and

$$814 \quad \|v_i\|_{L^2(0, T; \mathbb{R})}^2 \leq \mathcal{E}(\lambda_1) \mathcal{K}(T) \max_{\substack{\mu \in \mathbb{N}^g \\ \mu \leq \alpha}} \left| a_{\bullet, i} \left[\lambda_1^{(\mu_1)}, \dots, \lambda_g^{(\mu_g)} \right] \right|^2.$$

815 Setting

$$816 \quad v := \sum_{i=1}^d v_i e_i,$$

817 we get that v solves (2.1) and using [10, Proposition 7.15]

$$818 \quad \begin{aligned} \|v\|_{L^2(0, T; U)}^2 &= \sum_{i=1}^d \|v_i\|_{L^2(0, T; \mathbb{R})}^2 \\ 819 &\leq \mathcal{E}(\lambda_1) \mathcal{K}(T) \sum_{i=1}^d \max_{\substack{\mu \in \mathbb{N}^g \\ \mu \leq \alpha}} \left| a_{\bullet, i} \left[\lambda_1^{(\mu_1)}, \dots, \lambda_g^{(\mu_g)} \right] \right|^2 \\ 820 &\leq C_{p, \varrho, \eta} \mathcal{E}(\lambda_1) \mathcal{K}(T) \sum_{p=1}^{|\alpha|} \left(\sum_{i=1}^d \left| a_{\bullet, i} \left[\lambda_{\bullet}^{(\mu^p)} \right] \right|^2 \right) \\ 821 &= C_{p, \varrho, \eta} \mathcal{E}(\lambda_1) \mathcal{K}(T) \sum_{p=1}^{|\alpha|} \left\| \Omega \left[\lambda_{\bullet}^{(\mu^p)} \right] \right\|^2. \end{aligned}$$

823 Modifying the constants appearing in \mathcal{E} and \mathcal{K} (still satisfying the bounds given in
 824 Theorem A.1) ends the proof of Proposition 2.1. \square

825 **2.2. Solving the original moment problem.**

826 Through (1.33), when the right-hand side Ω of (2.1) satisfy the constraints (1.36),
 827 solving this vectorial block moment problem provides a solution of the original block
 828 moment problem (1.39). More precisely we have the following proposition

829 PROPOSITION 2.2. *Let $T > 0$ and $z \in X_{-\diamond}$. The following two statements are*
 830 *equivalent:*

- 831 *i. there exists $\Omega \in \mathcal{O}(G, z)$ such that the function $v \in L^2(0, T; U)$ solves (2.1);*
- 832 *ii. the function $v \in L^2(0, T; U)$ solves (1.39).*

833 *Proof.* Assume first that there exists $\Omega \in \mathcal{O}(G, z)$ and let $v \in L^2(0, T; U)$ be such
 834 that (2.1) holds.

835 Then, using (1.33), for any $j \in \llbracket 1, g \rrbracket$ and any $\phi \in E_{\lambda_j}$ we have

$$\begin{aligned}
 836 \quad \int_0^T \langle v(t), \mathcal{B}^* e^{-t\mathcal{A}^*} \phi \rangle_U dt &= \int_0^T \left\langle v(t), e^{-\lambda t} \sum_{l=0}^{\alpha_j-1} \frac{(-t)^l}{l!} (\mathcal{A}^* - \lambda_j)^l \phi \right\rangle_U dt \\
 837 \quad &= \sum_{l=0}^{\alpha_j-1} \left\langle \int_0^T v(t) \frac{(-t)^l}{l!} e^{-\lambda_j t} dt, (\mathcal{A}^* - \lambda_j)^l \phi \right\rangle_U \\
 838 \quad &= \sum_{l=0}^{\alpha_j-1} \langle \Omega_j^l, (\mathcal{A}^* - \lambda_j)^l \phi \rangle_U. \\
 839
 \end{aligned}$$

840 Since $(\Omega_j^0, \dots, \Omega_j^{\alpha_j-1}) \in \mathcal{O}(\lambda_j, z)$, this leads to

$$841 \quad \int_0^T \langle v(t), \mathcal{B}^* e^{-t\mathcal{A}^*} \phi \rangle_U dt = \langle z, \phi \rangle_{-\diamond, \diamond}, \quad \forall j \in \llbracket 1, g \rrbracket, \forall \phi \in E_{\lambda_j},$$

842 which proves that v solves (1.39).

843 Assume now that $v \in L^2(0, T; U)$ solves (1.39). Setting

$$844 \quad \Omega_j^l := \int_0^T v(t) \frac{(-t)^l}{l!} e^{-\lambda_j t} dt$$

845 we obtain that v solves (2.1). As in the previous step, the identity (1.33) implies that
 846 $\Omega \in \mathcal{O}(G, z)$. \square

847 Finally, to solve (1.39) we prove that there exists at least one Ω satisfying the
 848 constraints (1.36).

849 PROPOSITION 2.3. *Let $\lambda \in \Lambda$ and $z \in X_{-\diamond}$. Then, under assumption (1.28), we*
 850 *have*

$$851 \quad \mathcal{O}(\lambda, z) \neq \emptyset.$$

852 *Proof.* Let $T > 0$. From (1.33) the finite dimensional space E_λ is stable by the
 853 semigroup $e^{-\bullet\mathcal{A}^*}$. Using the approximate controllability assumption (1.28) we have
 854 that

$$855 \quad \phi \in E_\lambda \mapsto \left\| \mathcal{B}^* e^{-\bullet\mathcal{A}^*} \phi \right\|_{L^2(0, T, U)}$$

856 is a norm on E_λ . Then, the equivalence of norms in finite dimension implies that the
 857 following HUM-type functional

$$858 \quad J : \phi \in E_\lambda \mapsto \frac{1}{2} \left\| \mathcal{B}^* e^{-\bullet\mathcal{A}^*} \phi \right\|_{L^2(0, T, U)}^2 - \langle z, \phi \rangle_{-\diamond, \diamond}$$

859 is coercive. Let $\tilde{\phi} \in E_\lambda$ be such that

$$860 \quad J(\tilde{\phi}) = \inf_{\phi \in E_\lambda} J(\phi)$$

861 and $v := \mathcal{B}^* e^{-\bullet \mathcal{A}^*} \tilde{\phi}$. Then it comes that

$$862 \quad (2.2) \quad \int_0^T \langle v(t), \mathcal{B}^* e^{-t \mathcal{A}^*} \phi \rangle_U dt = \langle z, \phi \rangle_{-\diamond, \diamond}, \quad \forall \phi \in E_\lambda.$$

863 Finally, we set $\Omega := (\Omega^0, \dots, \Omega^{\alpha_\lambda - 1})$ with

$$864 \quad \Omega^l := \int_0^T v(t) \frac{(-t)^l}{l!} e^{-\lambda t} dt, \quad \forall l \in \llbracket 0, \alpha_\lambda - 1 \rrbracket.$$

865 Using (2.2) and following the computations of Proposition 2.2 we obtain that $\Omega \in$
866 $\mathcal{O}(\lambda, z)$. \square

867 We now have all the ingredients to prove Theorem 1.2.

868 *Proof (of Theorem 1.2).*

869 • From Proposition 2.3, we have $\mathcal{O}(G, z) \neq \emptyset$. Recall that, from (1.46), the opti-
870 mization problem defining $\mathcal{C}(G, z)$ can be reduced to a finite dimensional optimization
871 for which the infimum is attained. Thus, let $\Omega \in \mathcal{O}(G, z)$ be such that

$$872 \quad F(\Omega) = \mathcal{C}(G, z).$$

873 Let $v \in L^2(0, T; U)$ be the solution of (2.1) given by Proposition 2.1 with Ω as right-
874 hand side. As $\Omega \in \mathcal{O}(G, z)$, from Proposition 2.2 we deduce that v solves (1.39). The
875 upper bound (1.40) on $\|v\|_{L^2(0, T; U)}$ is given by Proposition 2.1.

876 • We now turn to the lower bound (1.42). Let $v \in L^2(0, T; U)$ be any solution
877 of (1.39a). Let

$$878 \quad \Omega_j^l := \int_0^T v(t) \frac{(-t)^l}{l!} e^{-\lambda_j t} dt = \int_0^T v(t) e_t \left[\lambda_j^{(l+1)} \right] dt, \quad \forall j \in \llbracket 1, g \rrbracket, \forall l \in \llbracket 0, \alpha_j - 1 \rrbracket.$$

879 As in the proof of Proposition 2.2, the use of (1.33) implies that

$$880 \quad \Omega = (\Omega_1^0, \dots, \Omega_1^{\alpha_1 - 1}, \dots, \Omega_g^0, \dots, \Omega_g^{\alpha_g - 1}) \in \mathcal{O}(G, z).$$

881 Thus,

$$882 \quad (2.3) \quad \mathcal{C}(G, z) \leq F(\Omega) = \sum_{l=1}^{|\alpha|} \left\| \Omega \left[\lambda_{\bullet}^{(\mu^l)} \right] \right\|_U^2.$$

883 Notice that

$$884 \quad \Omega \left[\lambda_{\bullet}^{(\mu^l)} \right] = \int_0^T v(t) e_t \left[\lambda_{\bullet}^{(\mu^l)} \right] dt, \quad \forall l \in \llbracket 0, |\alpha| \rrbracket.$$

885 Using Lagrange theorem [10, Proposition 7.14] yields,

$$886 \quad \left| e_t \left[\lambda_{\bullet}^{(\mu^l)} \right] \right| \leq \frac{t^{l-1} e^{-\lambda_1 t}}{(l-1)!}.$$

887 Together with Cauchy-Schwarz inequality this implies

$$888 \quad \left\| \Omega \left[\lambda_{\bullet}^{(\mu^l)} \right] \right\|_U \leq \left(\int_0^{+\infty} \frac{t^{l-1} e^{-\lambda_1 t}}{(l-1)!} dt \right)^{\frac{1}{2}} \|v\|_{L^2(0, T; U)}.$$

889 Then, as $\lambda_1 \geq \min \Lambda$ and $|\alpha| \leq p\eta$, estimate (2.3) ends the proof of Theorem 1.2. \square

890 **3. Application to the determination of the minimal null control time.**

891 This section is dedicated to the consequences of Theorem 1.2 on the null control-
892 lability properties of system (1.1).

893 From Theorem 1.2, the resolution of block moment problems (1.34) associated
894 with null controllability of (1.1) will involve the quantity $\mathcal{C}(G, e^{-T\mathcal{A}}y_0)$. To formulate
895 the minimal null control time we isolate the dependency with respect to the variable
896 T leading to quantities involving $\mathcal{C}(G, y_0)$. The comparison between these two costs
897 is detailed in Section 3.1.

898 Then, this leads to the formulation of the minimal null control time stated in
899 Theorem 1.3. This is detailed in Section 3.2.

900 We then prove, in Section 3.3, the estimates on the cost of null controllability
901 stated in Proposition 1.5 and Corollary 1.6.

902 **3.1. Relating the different costs.**

903 The costs $\mathcal{C}(G, z)$ and $\mathcal{C}(G, e^{-T\mathcal{A}}z)$ satisfy the following estimates.

904 LEMMA 3.1. *Assume that the operators \mathcal{A} and \mathcal{B} satisfy the assumption (H) (see*
905 *page 12). There exists $C_{p,\varrho,\eta} > 0$ such that for any $G \subset \Lambda$ with $\sharp G \leq p$ and $\text{diam } G \leq$
906 ϱ , for any $T > 0$ and any $z \in X_{-\diamond}$,*

907 (3.1)
$$\mathcal{C}(G, e^{-T\mathcal{A}}z) \leq C_{p,\varrho,\eta}(1+T)^{2|\alpha|}e^{-2(\min G)T}\mathcal{C}(G, z)$$

908 and

909 (3.2)
$$\mathcal{C}(G, z) \leq C_{p,\varrho,\eta}(1+T)^{2|\alpha|}e^{2(\max G)T}\mathcal{C}(G, e^{-T\mathcal{A}}z).$$

910 *Proof.* Recall that from (1.19) we have

911
$$\left\langle y_0, e^{-T\mathcal{A}^*}\phi \right\rangle_{-\diamond,\diamond} = \left\langle e^{-T\mathcal{A}}y_0, \phi \right\rangle_{-\diamond,\diamond}, \quad \forall \phi \in X_{\diamond}^*.$$

912 We set $G = \{\lambda_1, \dots, \lambda_g\}$ with $\lambda_1 < \dots < \lambda_g$.

913 • We start with the proof of (3.1).

914 From (1.46), let $\tilde{\Omega} \in \mathcal{O}(G, z)$ be such that $F(\tilde{\Omega}) = \mathcal{C}(G, z)$. We define Ω by

915
$$\Omega_j^l := (e_T \tilde{\Omega}) \left[\lambda_j^{(l+1)} \right], \quad \forall j \in \llbracket 1, g \rrbracket, \forall l \in \llbracket 0, \alpha_j - 1 \rrbracket.$$

916 Let us prove that $\Omega \in \mathcal{O}(G, e^{-T\mathcal{A}}z)$. For any $j \in \llbracket 1, g \rrbracket$ and any $\phi \in E_{\lambda_j}$, using
917 Leibniz formula [10, Proposition 7.13] and $\tilde{\Omega} \in \mathcal{O}(G, z)$, we obtain

918
$$\begin{aligned} \sum_{l \geq 0} \langle \Omega_j^l, \mathcal{B}^*(\mathcal{A}^* - \lambda_j)^l \phi \rangle_U &= \sum_{l \geq 0} \sum_{r=0}^l e_T \left[\lambda_j^{(r+1)} \right] \left\langle \tilde{\Omega}_j^{l-r}, \mathcal{B}^*(\mathcal{A}^* - \lambda_j)^l \phi \right\rangle_U \\ 919 &= \sum_{r \geq 0} e_T \left[\lambda_j^{(r+1)} \right] \sum_{l \geq r} \left\langle \tilde{\Omega}_j^{l-r}, \mathcal{B}^*(\mathcal{A}^* - \lambda_j)^l \phi \right\rangle_U \\ 920 &= \sum_{r \geq 0} e_T \left[\lambda_j^{(r+1)} \right] \sum_{l \geq 0} \left\langle \tilde{\Omega}_j^l, \mathcal{B}^*(\mathcal{A}^* - \lambda_j)^{l+r} \phi \right\rangle_U \\ 921 &= \sum_{r \geq 0} e_T \left[\lambda_j^{(r+1)} \right] \langle z, ((\mathcal{A}^* - \lambda_j)^r \phi) \rangle_{-\diamond,\diamond} \\ 922 (3.3) &= \left\langle z, e^{-T\mathcal{A}^*} \phi \right\rangle_{-\diamond,\diamond} = \left\langle e^{-T\mathcal{A}}z, \phi \right\rangle_{-\diamond,\diamond}. \end{aligned}$$

924 This proves the claim.

925 Applying Leibniz formula [10, Proposition 7.13] and Lagrange theorem [10, Propo-
926 sition 7.14] we obtain,

$$927 \quad \left\| \Omega \left[\lambda_{\bullet}^{(\mu^l)} \right] \right\|_U = \left\| \sum_{q=1}^l e_T \left[\lambda_{\bullet}^{(\mu^l - \mu^{q-1})} \right] \tilde{\Omega} \left[\lambda_{\bullet}^{(\mu^q)} \right] \right\|$$

$$928 \quad \leq C_{p,\varrho,\eta} (1+T)^{|\alpha|} e^{-\lambda_1 T} \left(\sum_{q=1}^l \left\| \tilde{\Omega} \left[\lambda_{\bullet}^{(\mu^q)} \right] \right\|^2 \right)^{\frac{1}{2}}.$$

929

930 Thus,

$$931 \quad F(\Omega) \leq C_{p,\varrho,\eta} (1+T)^{2|\alpha|} e^{-2\lambda_1 T} F(\tilde{\Omega}) = C_{p,\varrho,\eta} (1+T)^{2|\alpha|} e^{-2\lambda_1 T} \mathcal{C}(G, z).$$

932 As $\Omega \in \mathcal{O}(G, e^{-T\mathcal{A}}z)$, this proves (3.1).

933 • The proof of (3.2) uses the same ingredients.

934 From (1.46), let $\Omega \in \mathcal{O}(G, e^{-T\mathcal{A}}z)$ be such that $F(\Omega) = \mathcal{C}(G, e^{-T\mathcal{A}}z)$. For any
935 $j \in \llbracket 1, g \rrbracket$ and any $l \in \llbracket 0, \alpha_j - 1 \rrbracket$, let

$$936 \quad \tilde{\Omega}_j^l := (e_{-T}\Omega) \left[\lambda_j^{(l+1)} \right].$$

937 As previously, applying Leibniz formula [10, Proposition 7.13] and Lagrange theo-
938 rem [10, Proposition 7.14], we obtain

$$939 \quad \left\| \tilde{\Omega} \left[\lambda_{\bullet}^{(\mu^l)} \right] \right\|_U \leq C_{p,\varrho,\eta} (1+T)^{2|\alpha|} e^{\lambda_g T} \left(\sum_{q=1}^l \left\| \Omega \left[\lambda_{\bullet}^{(\mu^q)} \right] \right\|^2 \right)^{\frac{1}{2}}.$$

940 The same computations as (3.3) give that $\Omega \in \mathcal{O}(G, z)$. Thus

$$941 \quad \mathcal{C}(G, z) \leq F(\tilde{\Omega}) \leq C_{p,\varrho,\eta} (1+T)^{2|\alpha|} e^{2\lambda_g T} F(\Omega) = C_{p,\varrho,\eta} (1+T)^{2|\alpha|} e^{2\lambda_g T} \mathcal{C}(G, e^{-T\mathcal{A}}z)$$

942 and (3.2) is proved. \square

943 3.2. The minimal null control time.

944 This section is dedicated to the proof of Theorem 1.3 and Corollary 1.4.

945 *Proof (of Theorem 1.3).*

946 • We start with the proof of null controllability in time $T > T_0(y_0)$. Let $k \in \mathbb{N}^*$ and
947 set $G_k := \{\lambda_1, \dots, \lambda_g\}$. Let $v_k \in L^2(0, T; U)$ be the solution of the block moment
948 problem (1.39) associated with $z = e^{-T\mathcal{A}}y_0$ given by Theorem 1.2 i.e.

$$949 \quad \int_0^T \left\langle v_k(t), \mathcal{B}^* e^{-t\mathcal{A}^*} \phi \right\rangle_U dt = \left\langle e^{-T\mathcal{A}}y_0, \phi \right\rangle_{-\diamond, \diamond}, \quad \forall \phi \in E_{\lambda_j}, \forall j \in \llbracket 1, g \rrbracket,$$

$$950 \quad \int_0^T v_k(t) t^l e^{-\lambda t} dt = 0, \quad \forall \lambda \in \Lambda \setminus G_k, \forall l \in \llbracket 0, \eta - 1 \rrbracket.$$

951

952 From (1.33), this implies that v_k solves (1.32). Thus, the only point left is to prove
953 that the series (1.31) defining the control u converges in $L^2(0, T; U)$.

954 From Theorem 1.2 we have that

$$955 \quad \|v_k\|_{L^2(0,T;U)}^2 \leq \mathcal{E}(\lambda_1) \mathcal{K}(T) \mathcal{C}(G_k, e^{-TA}y_0).$$

956 Using (3.1), up to a modification of the constants appearing in \mathcal{E} and \mathcal{K} (still depending
957 on the same parameters), we obtain

$$958 \quad (3.4) \quad \|v_k\|_{L^2(0,T;U)}^2 \leq (1+T)^{2|\alpha|} \mathcal{E}(\lambda_1) \mathcal{K}(T) e^{-2(\min G_k)T} \mathcal{C}(G_k, y_0).$$

959 Using the definition of $T_0(y_0)$ given in (1.43) and the bound on \mathcal{E} given in Theorem A.1,
960 it comes that the series

$$961 \quad \sum_{k \in \mathbb{N}^*} v_k(T - \bullet)$$

962 converges in $L^2(0, T; U)$ when $T > T_0(y_0)$ which proves null controllability of (1.1)
963 from y_0 in any time $T > T_0(y_0)$.

964 • We now end the proof of Theorem 1.3 by proving that null controllability does not
965 hold in time $T < T_0(y_0)$. The proof mainly relies on the optimality of the resolution
966 of the block moment problems given in Theorem 1.2 (see (1.42)).

967 Let $T > 0$. Assume that problem (1.1) is null controllable from y_0 in time T .
968 Thus there exists $u \in L^2(0, T; U)$ such that $y(T) = 0$ and

$$969 \quad \|u\|_{L^2(0,T;U)} \leq C_T \|y_0\|_{-\diamond}.$$

970 Let $v := -u(T - \bullet)$. Then, for any $k \in \mathbb{N}^*$, v satisfies (1.39a) with $z = e^{-TA}y_0$.
971 From (1.42), this implies

$$972 \quad (3.5) \quad C_T^2 \|y_0\|_{-\diamond}^2 \geq \|u\|_{L^2(0,T;U)}^2 = \|v\|_{L^2(0,T;U)}^2 \geq C_{p,\eta,\min \Lambda} \mathcal{C}(G_k, e^{-TA}y_0).$$

973 Applying (3.2) and using that $e^{2(\max G_k)T} \leq e^{2\varrho T} e^{2(\min G_k)T}$ we obtain

$$974 \quad \mathcal{C}(G_k, y_0) \leq C_{T,p,\varrho,\eta} e^{2(\min G_k)T} \mathcal{C}(G_k, e^{-TA}y_0).$$

975 Together with (3.5) this implies

$$976 \quad (3.6) \quad \mathcal{C}(G_k, y_0) \leq C_{T,p,\varrho,\eta,\min \Lambda} \|y_0\|_{-\diamond}^2 e^{2(\min G_k)T}.$$

977 Getting back to the definition of $T_0(y_0)$ given in (1.43), this implies that $T \geq T_0(y_0)$
978 and ends the proof of Theorem 1.3. \square

979 We end this subsection with the proof of Corollary 1.4.

980 *Proof (of Corollary 1.4).* By definition, we have $T_0(Y_0) = \sup_{y_0 \in Y_0} T_0(y_0)$. Using
981 the definition of $\mathcal{C}(G, Y_0)$ and Theorem 1.3, it directly comes that

$$982 \quad T_0(Y_0) \leq \limsup_{k \rightarrow +\infty} \frac{\ln \mathcal{C}(G_k, Y_0)}{2 \min G_k}.$$

983 We now focus on the converse inequality. Let

$$984 \quad T < \limsup_{k \rightarrow +\infty} \frac{\ln \mathcal{C}(G_k, Y_0)}{2 \min G_k}$$

985 and let us prove that $T \leq T_0(Y_0)$.

986 There exists $\varepsilon > 0$ and an increasing sequence of integers $(n_k)_{k \in \mathbb{N}^*}$ such that for
 987 any $k \in \mathbb{N}^*$, there exists $y_{0,k} \in Y_0$ with $\|y_{0,k}\|_{-\diamond} = 1$ satisfying

$$988 \quad (3.7) \quad T + \varepsilon < \frac{\ln \mathcal{C}(G_{n_k}, y_{0,k})}{2 \min G_{n_k}}.$$

989 By contradiction, assume that for any $y_0 \in Y_0$, we have $T > T_0(y_0)$. Thus,
 990 from (3.6), there exists $C_{T,p,\varrho,\eta,\min \Lambda} > 0$ such that for any $k \in \mathbb{N}^*$

$$991 \quad \frac{\ln \mathcal{C}(G_{n_k}, y_{0,k})}{2 \min G_{n_k}} \leq \frac{\ln C_{T,p,\varrho,\eta,\min \Lambda}}{2 \min G_{n_k}} + T.$$

992 Taking k sufficiently large, this is in contradiction with (3.7). \square

993 3.3. On the cost of controllability.

994 A careful inspection of the proof of null controllability in time $T > T_0(y_0)$ detailed
 995 in Section 3.2 allows to give a bound on the cost of controllability. Indeed, the proof
 996 of Proposition 1.5 follows directly from (1.31) and (3.4) and will not be more detailed.

997 Its consequences stated in Corollary 1.6 are proved below.

998 *Proof (of Corollary 1.6).* Let $y_0 \in X_{-\diamond}$ and $T > T_0(y_0)$. From Proposition 1.5 it
 999 comes that there exists $u \in L^2(0, T; U)$ such that $y(T) = 0$ and

$$1000 \quad \|u\|_{L^2(0,T;U)}^2 \leq \mathcal{K}(T) \sum_{k \geq 1} (1+T)^{2|\alpha_{G_k}|} \mathcal{E}(\min G_k) e^{-2(\min G_k)T} \mathcal{C}(G_k, y_0),$$

1001 The seeked estimate is only interesting when $T \rightarrow T_0(y_0)^+$ thus, as $|\alpha_{G_k}| \leq \eta p$, the
 1002 term $(1+T)^{2|\alpha_{G_k}|}$ is bounded. To simplify the writing we set $\lambda_{k,1} := \min G_k$. Due
 1003 to the assumption on Λ , it comes from Theorem A.1 that

$$1004 \quad \mathcal{E}(\lambda) \leq C \exp(C\lambda^a), \quad \forall \lambda \in \Lambda,$$

1005 and

$$1006 \quad \mathcal{K}(T) \leq C \exp\left(\frac{C}{T^{\frac{a}{1-a}}}\right), \quad \forall T > 0,$$

1007 where the constant C depends only on $p, r, \varrho, \eta, a, \bar{N}, a'$ and \tilde{N} . Thus, using the
 1008 assumption (1.44), we obtain

$$1009 \quad (3.8) \quad \|u\|_{L^2(0,T;U)}^2 \leq C \exp\left(\frac{C}{T^{\frac{a}{1-a}}}\right) \|y_0\|_{-\diamond}^2 \sum_{k \geq 1} e^{-\lambda_{k,1}(T-T_0(y_0))} e^{C\lambda_{k,1}^a - \lambda_{k,1}(T-T_0(y_0))}.$$

1010 The maximum of the function $x \mapsto Cx^a - x(T - T_0(y_0))$ is given by

$$1011 \quad (Ca)^{\frac{1}{1-a}} \left(\frac{1}{a} - 1\right) \frac{1}{(T - T_0(y_0))^{\frac{a}{1-a}}}.$$

1012 Thus, up to a modification of C (still depending on the same parameters),

$$1013 \quad (3.9) \quad e^{C\lambda_{k,1}^a - \lambda_{k,1}(T-T_0(y_0))} \leq \exp\left(\frac{C}{(T - T_0(y_0))^{\frac{a}{1-a}}}\right), \quad \forall k \geq 1.$$

1014 From [16, Propositions A.6.20 and A.6.21], it comes that

$$1015 \quad (3.10) \quad \sum_{k \geq 1} e^{-\lambda_{k,1}(T-T_0(y_0))} \leq \frac{C_{a,\bar{N}}}{T-T_0(y_0)}.$$

1016 Plugging (3.10) and (3.9) into (3.8) ends the proof of Corollary 1.6. \square

1017 4. Computation of the cost of a block.

1018 In this section we prove more explicit formulas to estimate the cost $\mathcal{C}(G, y_0)$
 1019 of the resolution of a block moment problem depending on the assumptions on the
 1020 eigenvalues in the group G . More precisely, we prove here Theorems 1.8 and 1.10.
 1021 For pedagogical purpose, we start in Section 4.1 with Theorem 1.8 for algebraically
 1022 (and geometrically) simple eigenvalues *i.e.* when $\alpha_\lambda = \gamma_\lambda = 1$ for any $\lambda \in G$. Then,
 1023 in Section 4.2, we prove the general statement of Theorem 1.8 that is when all the
 1024 eigenvalues in the group are geometrically simple *i.e.* $\gamma_\lambda = 1$ for any $\lambda \in G$.

1025 The formula for the cost $\mathcal{C}(G, y_0)$ when all the eigenvalues in the group G are
 1026 semi-simple (*i.e.* $\alpha_\lambda = 1$ for any $\lambda \in G$) stated in Theorem 1.10 is then proved in
 1027 Section 4.3. The extension to spaces of initial conditions (1.48) and (1.52) does not
 1028 depend on the matrix M and follows directly from Lemma B.1. Thus, their proofs
 1029 are not detailed here.

1030 When both algebraic and geometric multiplicities appear in the same group we
 1031 do not get a general formula but describe the procedure on an example in Section 4.4.

1032 Recall that from (1.46), computing $\mathcal{C}(G, y_0)$ is a finite dimensional optimization
 1033 problem given by

$$1034 \quad \mathcal{C}(G, y_0) = \min \left\{ F(\Omega) ; \Omega \in \mathcal{O}(G, y_0) \cap U_G^{|\alpha|} \right\}$$

1035 where the function F is defined in (1.38), the constraints associated with $\mathcal{O}(G, y_0)$ are
 1036 defined in (1.36) and U_G is defined in (1.45).

1037 4.1. The case of simple eigenvalues.

1038 In all this section, we consider the simpler case where $\alpha_\lambda = \gamma_\lambda = 1$ for every
 1039 $\lambda \in G$. Thus, in the rest of this section, we drop the superscript 0 associated to
 1040 eigenvectors.

1041 We start with the proof of the invertibility of the matrix M stated in Proposi-
 1042 tion 1.7.

1043 *Proof.* Recall that, as $\alpha_\lambda = \gamma_\lambda = 1$, the positive semi-definite matrix M is defined
 1044 in (1.49). Let $\tau \in \mathbb{R}^g$ be such that $\langle M\tau, \tau \rangle = 0$. Then, for each $l \in \llbracket 1, g \rrbracket$, we have

$$1045 \quad \langle \Gamma^l \tau, \tau \rangle = 0.$$

1046 We prove that $\tau = 0$. By contradiction let

$$1047 \quad l = \max\{j \in \llbracket 1, g \rrbracket ; \tau_j \neq 0\}.$$

1048 Then from (1.49) this leads to $\langle \Gamma^l \tau, \tau \rangle = \|b[\lambda_l]\|_U^2 \tau_l^2$. Using (1.28) implies $\tau_l = 0$.
 1049 This is in contradiction with the definition of l which proves the invertibility of M . \square

1050 We now prove Theorem 1.8.

1051 *Proof.* First of all, notice that the function F to minimize reduces to

$$1052 \quad F(\Omega) = \sum_{j=1}^g \|\Omega[\lambda_1, \dots, \lambda_j]\|^2$$

1053 and, as $\gamma_\lambda = \alpha_\lambda = 1$, the constraints defining the set $\mathcal{O}(\lambda_j, y_0)$ reduce to

$$1054 \quad \langle \Omega_j, b_j \rangle_U = \langle y_0, \phi_j \rangle_{-\diamond, \diamond}.$$

1055 Thus, the minimization problem reduces to

$$1056 \quad (4.1) \quad \mathcal{C}(G, y_0) = \min \left\{ F(\Omega) ; \Omega = (\Omega_1, \dots, \Omega_g) \in U_G^g \text{ such that} \right.$$

$$1058 \quad \left. \langle \Omega_j, b_j \rangle_U = \langle y_0, \phi_j \rangle_{-\diamond, \diamond}, \forall j \in \llbracket 1, g \rrbracket \right\}.$$

1060 For the sake of generality, let us consider for this proof any $\omega_1, \dots, \omega_g \in \mathbb{R}$ and the
1061 more general constraints

$$1062 \quad (4.2) \quad \langle \Omega_j, b_j \rangle_U = \omega_j, \quad \forall j \in \llbracket 1, g \rrbracket.$$

1063 Using the formalism of divided differences, this is equivalent to the family of con-
1064 straints

$$1065 \quad (4.3) \quad \langle \Omega, b \rangle_U [\lambda_1, \dots, \lambda_j] = \omega[\lambda_1, \dots, \lambda_j], \quad \forall j \in \llbracket 1, g \rrbracket.$$

1066 Denote by $(m_j)_{j \in \llbracket 1, g \rrbracket}$ the Lagrange multipliers associated with the minimization
1067 problem

$$1068 \quad \min \{ F(\Omega) ; \Omega = (\Omega_1, \dots, \Omega_g) \in U_G^g \text{ such that (4.3) holds} \}.$$

1069 Then, we obtain that the minimum satisfies

$$1070 \quad (4.4) \quad \sum_{j=1}^g \langle \Omega[\lambda_1, \dots, \lambda_j], H[\lambda_1, \dots, \lambda_j] \rangle_U = \sum_{j=1}^g m_j \langle H, b \rangle_U [\lambda_1, \dots, \lambda_j],$$

1071 for any $H_1, \dots, H_g \in U_G$.

1072 Then, for a given $q \in \llbracket 1, g \rrbracket$, using Leibniz formula [10, Proposition 7.7], the
1073 constraints (4.3) can be rewritten as

$$1074 \quad (4.5) \quad \omega[\lambda_1, \dots, \lambda_q] = \langle \Omega, b \rangle_U [\lambda_1, \dots, \lambda_q] = \sum_{j=1}^q \langle \Omega[\lambda_1, \dots, \lambda_j], b[\lambda_j, \dots, \lambda_q] \rangle_U$$

1075 To relate (4.5) and (4.4), we set $H_1, \dots, H_g \in U_G$ such that

$$1076 \quad H[\lambda_1, \dots, \lambda_j] = \begin{cases} b[\lambda_j, \dots, \lambda_q], & \text{for } j \leq q, \\ 0, & \text{for } j > q. \end{cases}$$

1077 This can be done defining $H_1 = b[\lambda_1, \dots, \lambda_q]$ and, from the interpolation formula [10,
1078 Proposition 7.6], defining H_j by the formula

$$1079 \quad H_j = \sum_{i=1}^j \left(\prod_{k=1}^{j-1} (\lambda_i - \lambda_k) \right) H[\lambda_1, \dots, \lambda_i], \quad \forall j \in \llbracket 2, g \rrbracket.$$

1080 Then, from (4.5) we obtain

$$1081 \quad \omega[\lambda_1, \dots, \lambda_q] = \sum_{j=1}^g \langle \Omega[\lambda_1, \dots, \lambda_j], H[\lambda_1, \dots, \lambda_j] \rangle_U.$$

1082 Now relation (4.4) leads to

$$1083 \quad \omega[\lambda_1, \dots, \lambda_q] = \sum_{j=1}^g m_j \langle H, b \rangle_U [\lambda_1, \dots, \lambda_j].$$

1084 The application of Leibniz formula [10, Proposition 7.7] yields

$$\begin{aligned} 1085 \quad \omega[\lambda_1, \dots, \lambda_q] &= \sum_{j=1}^g m_j \left(\sum_{l=1}^j \langle H[\lambda_1, \dots, \lambda_l], b[\lambda_l, \dots, \lambda_j] \rangle_U \right) \\ 1086 \quad &= \sum_{j=1}^g m_j \left(\sum_{l=1}^{\min(j,q)} \langle b[\lambda_l, \dots, \lambda_q], b[\lambda_l, \dots, \lambda_j] \rangle_U \right) \\ 1087 \quad &= \sum_{l=1}^g \sum_{j=1}^g m_j \Gamma_{q,j}^l = (Mm)_q \\ 1088 \end{aligned}$$

1089 where Γ^l and M are defined in (1.49).

1090 Let

$$1091 \quad \xi := \begin{pmatrix} \omega[\lambda_1] \\ \vdots \\ \omega[\lambda_1, \dots, \lambda_g] \end{pmatrix}.$$

1092 As $m = M^{-1}\xi$, getting back to (4.4) with $H = \Omega$ together with the constraints (4.3),
1093 we obtain

$$1094 \quad F(\Omega) = \sum_{j=1}^g m_j \langle \Omega, b \rangle_U [\lambda_1, \dots, \lambda_j] = \sum_{j=1}^g (M^{-1}\xi)_j \xi_j = \langle M^{-1}\xi, \xi \rangle.$$

1095 With the specific choice, $\omega_j = \langle y_0, \phi_j \rangle_{-\diamond, \diamond}$, this ends the proof of Theorem 1.8 with
1096 the extra assumption that $\alpha_\lambda = 1$ for all $\lambda \in G$. \square

1097 **REMARK 4.1.** *As mentioned in Remark 1.9, estimate (4.1) implies that the cost*
1098 *of the block G (i.e. the quantity $\langle M^{-1}\xi, \xi \rangle$) can be estimated using any eigenvectors:*
1099 *there is no normalization condition.*

1100 **REMARK 4.2.** *Rewriting the constraints in the form (4.3) is not mandatory but,*
1101 *as the function to minimize involves divided differences, it leads to more exploitable*
1102 *formulas and will ease the writing when dealing with algebraic multiplicity of eigen-*
1103 *values. Dealing directly with (4.2) would lead to the expression (D.9) for the cost of*
1104 *the block G as it will appear in the proof of Theorem 1.10.*

1105 **4.2. The case of geometrically simple eigenvalues.**

1106 The proof of Proposition 1.7 and Theorem 1.8 under the sole assumption $\gamma_\lambda = 1$
1107 for any $\lambda \in G$ follows closely the proof done in Section 4.1. The main difference is
1108 the use of generalized divided differences (see [10, Section 7.3]) instead of classical
1109 divided differences as detailed below.

1110 *Proof (of Proposition 1.7).* Due to (1.37), for any $l \in \llbracket 1, |\alpha| \rrbracket$ the multi-index
 1111 $\mu^l - \mu^{l-1}$ is composed of only one 1 and $g - 1$ zeros. Thus,

$$1112 \quad b \left[\lambda_{\bullet}^{(\mu^l - \mu^{l-1})} \right] = b_j^0$$

1113 for a certain $j \in \llbracket 1, g \rrbracket$. From (1.28) it comes that

$$1114 \quad b \left[\lambda_{\bullet}^{(\mu^l - \mu^{l-1})} \right] \neq 0, \quad \forall l \in \llbracket 1, |\alpha| \rrbracket.$$

1115 The rest of the proof follows as in Section 4.1. \square

1116 *Proof (of Theorem 1.8).* As $\gamma_\lambda = 1$, the constraints defining the set $\mathcal{O}(\lambda_j, y_0)$
 1117 reduce to

$$1118 \quad \sum_{r=0}^l \langle \Omega_j^r, b_j^{l-r} \rangle_U = \sum_{r=0}^l \langle \Omega_j^r, \mathcal{B}^*(\mathcal{A}^* - \lambda_j)^r \phi_j^l \rangle_U$$

$$1119 \quad = \langle y_0, \phi_j^l \rangle_{-\diamond, \diamond}, \quad \forall l \in \llbracket 0, \alpha_j - 1 \rrbracket.$$

1121 Using Leibniz formula [10, Proposition 7.13] this is equivalent to

$$1122 \quad \langle \Omega, b \rangle_U \left[\lambda_j^{(l+1)} \right] = \langle y_0, \phi_j^l \rangle_{-\diamond, \diamond}, \quad \forall l \in \llbracket 0, \alpha_j - 1 \rrbracket.$$

1123 Thus,

$$1124 \quad (4.6) \quad \mathcal{C}(G, y_0) = \min \left\{ F(\Omega); \Omega = (\Omega_1^0, \dots, \Omega_1^{\alpha_1-1}, \dots, \Omega_g^0, \dots, \Omega_g^{\alpha_g-1}) \in U_G^{|\alpha|} \right.$$

$$1126 \quad \left. \text{such that } \langle \Omega, b \rangle_U \left[\lambda_j^{(l+1)} \right] = \langle y_0, \phi_j^l \rangle_{-\diamond, \diamond}, \quad \forall j \in \llbracket 1, g \rrbracket, \forall l \in \llbracket 0, \alpha_j - 1 \rrbracket \right\}.$$

1128 For the sake of generality, let us consider for this proof any

$$1129 \quad (\omega_1^0, \dots, \omega_1^{\alpha_1-1}, \dots, \omega_g^0, \dots, \omega_g^{\alpha_g-1}) \in \mathbb{R}^{|\alpha|}$$

1130 and the more general constraints

$$1131 \quad \langle \Omega, b \rangle_U \left[\lambda_j^{(l+1)} \right] = \omega_j^l, \quad \forall j \in \llbracket 1, g \rrbracket, \forall l \in \llbracket 0, \alpha_j - 1 \rrbracket.$$

1132 From (1.37), this is equivalent to the family of constraints

$$1133 \quad \langle \Omega, b \rangle_U \left[\lambda_{\bullet}^{(\mu^p)} \right] = \omega \left[\lambda_{\bullet}^{(\mu^p)} \right], \quad \forall p \in \llbracket 1, |\alpha| \rrbracket,$$

1134 and we proceed as in Section 4.1. The only difference is the use of generalized divided
 1135 differences. For instance, the equation (4.4) defining the Lagrange multipliers now
 1136 reads

$$1137 \quad \sum_{l=1}^{|\alpha|} \langle \Omega[\lambda^{(\mu^l)}], H[\lambda^{(\mu^l)}] \rangle = \sum_{l=1}^{|\alpha|} m_l \langle H, b \rangle[\lambda^{(\mu^l)}], \quad \forall H = (H_j^l) \in U_G^{|\alpha|},$$

1138 and the Leibniz formula [10, Proposition 7.7] is replaced by its generalization [10,
 1139 Proposition 7.13]. The rest of the proof remains unchanged. \square

1140 **REMARK 4.3.** As mentioned in Remark 1.9, estimate (4.6) implies that the cost
 1141 of the block G (i.e. the quantity $\langle M^{-1}\xi, \xi \rangle$) can be estimated using any eigenvectors
 1142 and any associated Jordan chains.

1143 **4.3. The case of semi-simple eigenvalues.**

1144 We start with the proof of Proposition 1.9.

1145 *Proof (of Proposition 1.9).* Recall that the positive semi-definite matrix M is de-
 1146 fined in (1.51). Let $\tau \in \mathbb{R}^{\gamma_G}$ be such that $\langle M\tau, \tau \rangle = 0$. Then, for any $l \in \llbracket 1, g \rrbracket$,
 1147 $\langle \Gamma^l \tau, \tau \rangle = 0$. We prove that $\tau = 0$. By contradiction let

1148
$$\tilde{l} = \max\{j \in \llbracket 1, \gamma_G \rrbracket ; \tau_j \neq 0\}$$

1149 and $l \in \llbracket 1, g \rrbracket$ be such that

1150
$$\gamma_1 + \dots + \gamma_{l-1} < \tilde{l} \leq \gamma_1 + \dots + \gamma_l$$

1151 with the convention that $l = 1$ when $\tilde{l} \leq \gamma_1$. We denote by $\sigma \in \mathbb{R}^{\gamma_l}$ the l^{th} block of τ
 1152 *i.e.*

1153
$$\sigma = \begin{pmatrix} \tau_{\gamma_1 + \dots + \gamma_{l-1} + 1} \\ \vdots \\ \tau_{\gamma_1 + \dots + \gamma_l} \end{pmatrix}.$$

1154 From (1.50) we have $\delta_i^i = 0$ when $i < l$. Thus all the blocks (i, j) of Γ^l are equal to 0
 1155 when $i, j \in \llbracket 1, l-1 \rrbracket$. This leads to

1156
$$\langle \Gamma^l \tau, \tau \rangle = \left\langle (\delta_l^l)^2 \text{Gram}_U(b_{l,1}, \dots, b_{l,\gamma_l}) \sigma, \sigma \right\rangle.$$

1157 As the eigenvalues $\lambda_1, \dots, \lambda_g$ are distinct it comes that $\delta_l^l \neq 0$ (see (1.50)) which
 1158 implies

1159
$$\langle \text{Gram}_U(b_{l,1}, \dots, b_{l,\gamma_l}) \sigma, \sigma \rangle = 0.$$

1160 From (1.28), we have that $b_{l,1}, \dots, b_{l,\gamma_l}$ are linearly independent. This proves the
 1161 invertibility of $\text{Gram}_U(b_{l,1}, \dots, b_{l,\gamma_l})$ and gives $\sigma = 0$. This is in contradiction with
 1162 the definition of \tilde{l} which proves the invertibility of M . \square

1163 We now turn to the proof of Theorem 1.10

1164 *Proof (of Theorem 1.10).* First of all, notice that the function F to minimize
 1165 reduces to

1166
$$F(\Omega) = \sum_{j=1}^g \|\Omega[\lambda_1, \dots, \lambda_j]\|^2$$

1167 and, as $\alpha_\lambda = 1$, the constraints defining the set $\mathcal{O}(\lambda_j, y_0)$ reduce to

1168
$$\langle \Omega_j, \mathcal{B}^* \phi \rangle_U = \langle y_0, \phi \rangle_{-\diamond, \diamond}, \quad \forall \phi \in \text{Ker}(\mathcal{A}^* - \lambda_j).$$

1169 To simplify the writing, let us consider the maps

1170
$$B_j := \begin{pmatrix} \langle \mathcal{B}^* \phi_{j,1}, \bullet \rangle_U \\ \vdots \\ \langle \mathcal{B}^* \phi_{j,\gamma_j}, \bullet \rangle_U \end{pmatrix} \in \mathcal{L}(U, \mathbb{R}^{\gamma_j}).$$

1171 Then the constraints defining $\mathcal{O}(\lambda_j, y_0)$ can be rewritten as the equality

1172 (4.7)
$$B_j \Omega_j = \begin{pmatrix} \langle y_0, \phi_{j,1} \rangle_{-\diamond, \diamond} \\ \vdots \\ \langle y_0, \phi_{j,\gamma_j} \rangle_{-\diamond, \diamond} \end{pmatrix}.$$

1173 Thus,

1174

1175 (4.8) $\mathcal{C}(G, y_0) = \min \left\{ F(\Omega) ; \Omega = (\Omega_1, \dots, \Omega_g) \in U_G^g \right.$

1176
1177

such that (4.7) holds for any $j \in \llbracket 1, g \rrbracket \}$.

1178 For the sake of generality, let us consider for this proof any

1179

$$(\omega_{1,1}, \dots, \omega_{1,\gamma_1}, \dots, \omega_{g,1}, \dots, \omega_{g,\gamma_g}) \in \mathbb{R}^{\gamma_G}$$

1180 and the more general constraints

1181

$$B_j \Omega_j = \omega_j, \quad \forall j \in \llbracket 1, g \rrbracket,$$

1182 where ω_j denotes $\begin{pmatrix} \omega_{j,1} \\ \vdots \\ \omega_{j,\gamma_j} \end{pmatrix} \in \mathbb{R}^{\gamma_j}$.

1183 As the ω_j 's have different sizes we avoid in this proof the use of divided differences
1184 to rewrite the constraints. This is why we end up with the formula (1.51) rather than
1185 an adaptation of (1.49) (see also the discussion in Remark 4.2).

1186 Denoting by $m_j \in \mathbb{R}^{\gamma_j}$ the Lagrange multipliers we obtain that the minimum
1187 satisfies

1188 (4.9) $\sum_{j=1}^g \langle \Omega[\lambda_1, \dots, \lambda_j], H[\lambda_1, \dots, \lambda_j] \rangle_U = \sum_{j=1}^g \langle m_j, B_j H_j \rangle, \quad \forall H_1, \dots, H_g \in U_G.$

1189 Recall that in (1.50) we defined

1190

$$\delta_j^i = \prod_{l \in \llbracket 1, j-1 \rrbracket} (\lambda_i - \lambda_l), \quad \forall j \in \llbracket 2, g \rrbracket.$$

1191 Then, from the interpolation formula [10, Proposition 7.6], we obtain that

1192 (4.10)
$$\Omega_i = \sum_{l=1}^i \delta_l^i \Omega[\lambda_1, \dots, \lambda_l].$$

1193 For any $H \in U_G$ and $i \in \llbracket 1, g \rrbracket$, let us design $H_1^{(i)}, \dots, H_g^{(i)} \in U_G$ such that

1194 (4.11)

$$H^{(i)}[\lambda_1, \dots, \lambda_l] = \delta_l^i H, \quad \forall l \in \llbracket 1, i \rrbracket.$$

1195 We set $H_1^{(i)} = H$ and, using the interpolation formula [10, Proposition 7.6], we define
1196 recursively

1197
$$H_j^{(i)} = \sum_{l=1}^j \delta_l^j H^{(i)}[\lambda_1, \dots, \lambda_l] = \left(\sum_{l=1}^j \delta_l^i \delta_l^j \right) H = \theta_j^{(i)} H$$

1198 with

1199 (4.12)
$$\theta_j^{(i)} := \sum_{l=1}^g \delta_l^i \delta_l^j = \sum_{l=1}^{\min(i,j)} \delta_l^i \delta_l^j.$$

1200 This ensures (4.11). Taking into account (4.10), plugging $H_j^{(i)}$ in (4.9) leads to

$$1201 \quad (4.13) \quad \Omega_i = \sum_{j=1}^g \theta_j^{(i)} B_j^* m_j.$$

1202 Together with (4.7), using (4.12), we obtain that

$$\begin{aligned} 1203 \quad \omega_i &= \sum_{j=1}^g \theta_j^{(i)} B_i B_j^* m_j \\ 1204 \quad &= \sum_{l=1}^g \sum_{j=1}^g (\delta_l^i B_i) (\delta_l^j B_j)^* m_j \\ 1205 \quad &= (Mm)_i \end{aligned}$$

1207 where M is defined in (1.51) and $(Mm)_i \in \mathbb{R}^{\gamma_i}$ denotes the i^{th} block of $Mm \in \mathbb{R}^{\gamma_G}$.

1208 Finally, if we set

$$1209 \quad \xi := \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_g \end{pmatrix} \in \mathbb{R}^{\gamma_G}.$$

1210 we have proved that the Lagrange multipliers are given by $m = M^{-1}\xi$. Applying (4.9)
1211 with $H_j = \Omega_j$ and using the constraints (4.7) leads to

$$1212 \quad F(\Omega) = \sum_{j=1}^g \|\Omega[\lambda_1, \dots, \lambda_j]\|^2 = \sum_{j=1}^g \langle (M^{-1}\xi)_j, \xi_j \rangle = \langle M^{-1}\xi, \xi \rangle,$$

1213 which proves the claim. □

1214 **REMARK 4.4.** From (4.13) and $m = M^{-1}\xi$ it comes that

1215

$$1216 \quad \mathcal{C}(G, y_0) = \min \left\{ F(\Omega) ; \Omega = (\Omega_1, \dots, \Omega_g) \in U_G^g \right. \\ 1217 \quad \left. \text{such that (4.7) holds for any } j \in \llbracket 1, g \rrbracket \right\}$$

1218
1219 is attained for

$$1220 \quad \Omega_i = \sum_{j=1}^g \left(\sum_{l=1}^g \delta_l^i \delta_l^j \right) B_j^* (M^{-1}\xi)_j.$$

1221 **REMARK 4.5.** As mentioned in Remark 1.9, estimate (4.8) implies that the cost
1222 of the block G (i.e. the quantity $\langle M^{-1}\xi, \xi \rangle$) can be estimated using any basis of
1223 eigenvectors.

1224 **4.4. Dealing simultaneously with algebraic and geometric multiplici-** 1225 **ties.**

1226 The proof of Theorem 1.8 strongly relies on the use of divided differences to rewrite
1227 the constraints whereas the proof of Theorem 1.10 is based on the vectorial writing
1228 of the constraints through the operators $B_j \in \mathcal{L}(U; \mathbb{R}^{\gamma_j})$. As the target spaces of
1229 these operators do not have the same dimension, one cannot directly compute divided

1230 differences. Thus, the setting we developed to compute the cost of a given block does
 1231 not lead to a general formula when both kind of multiplicities need to be taken into
 1232 account in the same group. However, for actual problems, the computation of this
 1233 cost is a finite dimensional constrained optimization problem which can be explicitly
 1234 solved.

1235 Let us give an example of such a group that does not fit into Theorem 1.8 nor
 1236 into Theorem 1.10 but for which we manage to compute the cost ‘by hand’.

1237 We consider a group $G = \{\lambda_1, \lambda_2\}$ such that $\gamma_{\lambda_1} = \alpha_{\lambda_1} = 2$ and $\gamma_{\lambda_2} = \alpha_{\lambda_2} = 1$.
 1238 Let $(\phi_{1,1}^0, \phi_{1,2}^0)$ be a basis of $\text{Ker}(\mathcal{A}^* - \lambda_1)$ and $\phi_{2,1}^0$ be an eigenvector of \mathcal{A}^* associated
 1239 to the eigenvalue λ_2 . Assume that the generalized eigenvector $\phi_{1,1}^1$ is such that

$$1240 \quad (\mathcal{A}^* - \lambda_1) \phi_{1,1}^1 = \phi_{1,1}^0,$$

1241 and that $(\phi_{1,1}^0, \phi_{1,1}^1, \phi_{1,2}^0)$ forms a basis of $\text{Ker}(\mathcal{A}^* - \lambda_1)^2$.

1242 For this group, in the same spirit as in Theorems 1.8 and 1.10, we obtain the
 1243 following result.

1244 PROPOSITION 4.1. *For any $y_0 \in X_{-\diamond}$, we have*

$$1245 \quad \mathcal{C}(G, y_0) = \langle M^{-1}\xi, \xi \rangle \quad \text{where} \quad \xi = \begin{pmatrix} \langle y_0, \phi_{1,1}^0 \rangle_{-\diamond, \diamond} \\ \langle y_0, \phi_{1,2}^0 \rangle_{-\diamond, \diamond} \\ \langle y_0, \phi_{1,1}^1 \rangle_{-\diamond, \diamond} \\ \langle y_0, \phi_{2,1}^0 \rangle_{-\diamond, \diamond} \end{pmatrix}$$

1246 and M is the invertible matrix defined by

$$1247 \quad M = \text{Gram}_U(b_{1,1}^0, b_{1,2}^0, b_{1,1}^1, b_{2,1}^0) \\
 1248 \quad \quad \quad + \text{Gram}_U(0, 0, b_{1,1}^0, \delta b_{2,1}^0) \\
 1249 \quad \quad \quad + \text{Gram}_U(0, 0, 0, \delta^2 b_{2,1}^0)$$

1251 with $\delta = \lambda_2 - \lambda_1$.

1252 *Proof.* Let

$$1253 \quad (\omega_{1,1}^0, \omega_{1,2}^0, \omega_{1,1}^1, \omega_{2,1}^0)^t \in \mathbb{R}^4.$$

1254 As in the proofs of Theorems 1.8 and 1.10, the goal is to compute the minimum of
 1255 the function

$$1256 \quad F : (\Omega_1^0, \Omega_1^1, \Omega_2^0) \in U_G^3 \mapsto \|\Omega_1^0\|^2 + \|\Omega_1^1\|^2 + \|\Omega[\lambda_1, \lambda_1, \lambda_2]\|^2,$$

1257 under the 4 constraints

$$1258 \quad \langle \Omega_j^0, b_{j,i}^0 \rangle_U = \omega_{j,i}^0, \quad \forall i \in \llbracket 1, \gamma_j \rrbracket, \forall j \in \llbracket 1, 2 \rrbracket, \\
 1259 \quad \langle \Omega_1^0, b_{1,1}^1 \rangle_U + \langle \Omega_1^1, b_{1,1}^0 \rangle_U = \omega_{1,1}^1.$$

1261 Then, the Lagrange multipliers $m_{1,1}^0, m_{1,2}^0, m_{1,1}^1$ and $m_{2,1}^0$ satisfy the equations

$$1262 \quad (4.14) \quad \langle \Omega_1^0, H_1^0 \rangle_U + \langle \Omega_1^1, H_1^1 \rangle_U + \langle \Omega[\lambda_1, \lambda_1, \lambda_2], H[\lambda_1, \lambda_1, \lambda_2] \rangle_U = m_{1,1}^0 \langle H_1^0, b_{1,1}^0 \rangle_U \\
 1263 \quad \quad \quad + m_{1,2}^0 \langle H_1^0, b_{1,2}^0 \rangle_U + m_{1,1}^1 \left(\langle H_1^0, b_{1,1}^1 \rangle_U + \langle H_1^1, b_{1,1}^0 \rangle_U \right) + m_{2,1}^0 \langle H_2^0, b_{2,1}^0 \rangle_U,$$

1266 for every $H_1^0, H_1^1, H_2^0 \in U_G$. Considering successively

$$1267 \quad H_1^0 = b_{1,1}^0, \quad H_1^1 = 0, \quad H_2^0 = b_{1,1}^0,$$

1268

$$1269 \quad H_1^0 = b_{1,2}^0, \quad H_1^1 = 0, \quad H_2^0 = b_{1,2}^0,$$

1270

$$1271 \quad H_1^0 = b_{1,1}^1, \quad H_1^1 = b_{1,1}^0, \quad H_2^0 = b_{1,1}^1 + (\lambda_2 - \lambda_1)b_{1,1}^0,$$

1272 and

$$1273 \quad H_1^0 = b_{2,1}^0, \quad H_1^1 = (\lambda_2 - \lambda_1)b_{2,1}^0, \quad H_2^0 = (1 + (\lambda_2 - \lambda_1)^2 + (\lambda_2 - \lambda_1)^4) b_{2,1}^0,$$

1274 and plugging it into (4.14), we obtain that $\begin{pmatrix} \omega_{1,1}^0 \\ \omega_{1,2}^0 \\ \omega_{1,1}^1 \\ \omega_{2,1}^0 \end{pmatrix} = M \begin{pmatrix} m_{1,1}^0 \\ m_{1,2}^0 \\ m_{1,1}^1 \\ m_{2,1}^0 \end{pmatrix}$ which ends the

1275 proof. \square

1276 5. Application to the study of null controllability of academic exam- 1277 ples.

1278 In this section we provide examples to illustrate how to use the formulas obtained
1279 in Theorems 1.3, 1.8 and 1.10 in order to compute the minimal null control time.

1280 We start with academic examples for which computations are simpler. Then, in
1281 Section 6, we study coupled systems of actual partial differential equations of parabolic
1282 type.

1283 5.1. Setting and notations.

1284 Let A be the unbounded Sturm-Liouville operator defined in $L^2(0, 1; \mathbb{R})$ by

$$1285 \quad (5.1) \quad D(A) = H^2(0, 1; \mathbb{R}) \cap H_0^1(0, 1; \mathbb{R}), \quad A\bullet = -\partial_x(\gamma\partial_x\bullet) + c\bullet,$$

1286 with $c \in L^\infty(0, 1; \mathbb{R})$ satisfying $c \geq 0$ and $\gamma \in C^1([0, 1]; \mathbb{R})$ satisfying $\inf_{[0,1]} \gamma > 0$.

1287 The operator A admits an increasing sequence of eigenvalues denoted by $(\nu_k)_{k \in \mathbb{N}^*}$.
1288 The associated normalized eigenvectors $(\varphi_k)_{k \in \mathbb{N}^*}$ form an Hilbert basis of $L^2(0, 1; \mathbb{R})$.

1289 **REMARK 5.1.** *The assumption $c \geq 0$ ensures that for any $k \geq 1$, the eigenvalues*
1290 *satisfies $\nu_k > 0$. From Remark 1.3, the controllability results proved in the present*
1291 *article still hold when the function c is bounded from below.*

1292 To lighten the notations, for any $I \subset (0, 1)$ we set $\|\bullet\|_I = \|\bullet\|_{L^2(I)}$.

1293 Let $f : \text{Sp}(A) \rightarrow \mathbb{R}$ be a bounded function. Associated to this function we consider
1294 the operator $f(A)$ defined on $D(A)$ by the spectral theorem by

$$1295 \quad (5.2) \quad f(A) = \sum_{k \geq 1} f(\nu_k) \langle \bullet, \varphi_k \rangle_{L^2(0,1;\mathbb{R})} \varphi_k.$$

1296 **5.2. Spectral properties of Sturm-Liouville operators.**

1297 Let A be the Sturm-Liouville operator defined in (5.1). All the examples studied
 1298 in this article are based on this operator. We recall here some spectral properties that
 1299 will be used in our study.

1300 From [2, Theorem 1.1], there exists $\varrho > 0$ such that

1301 (5.3)
$$\varrho < \nu_{k+1} - \nu_k, \quad \forall k \geq 1,$$

1302

1303 (5.4)
$$\inf_{k \geq 1} \frac{|\varphi'_k(x)|}{\sqrt{k}} > 0, \quad \forall x \in \{0, 1\},$$

1304 and, for any non-empty open set $\omega \subset (0, 1)$,

1305 (5.5)
$$\inf_{k \geq 1} \|\varphi_k\|_\omega > 0.$$

1306 Moreover, using [16, Theorem IV.1.3], we have

1307 (5.6)
$$N_{(\nu_k)_k}(r) \leq C\sqrt{r}, \quad \forall r > 0.$$

1308 To estimate various quantities, we will make an intensive use of the following
 1309 lemma proved in [2, Lemma 2.3].

1310 **LEMMA 5.1.** *Let A be the Sturm-Liouville operator defined in (5.1) and let $\lambda_0 > 0$.
 1311 There exists $C > 0$ depending on γ, c and λ_0 such that, for any $\lambda \geq \lambda_0$, for any
 1312 $F \in L^2(0, 1; \mathbb{R})$, for any $x, y \in [0, 1]$, for any u satisfying*

1313
$$(A - \lambda)u = F \quad \text{on } [x, y],$$

1314 *we have*

1315
$$|u(x)|^2 + \frac{\gamma(x)}{\lambda} |u'(x)|^2 \leq C \left(|u(y)|^2 + \frac{\gamma(y)}{\lambda} |u'(y)|^2 + \frac{1}{\lambda} \left| \int_x^y |F(s)| ds \right|^2 \right).$$

1316 Applying Lemma 5.1 with $u = \varphi_k, F = 0, \lambda = \nu_k$ and integrating with respect to
 1317 the variable $y \in (0, 1)$ we obtain

1318
$$|\varphi_k(x)|^2 + \frac{1}{\nu_k} |\varphi'_k(x)|^2 \leq C \left(1 + \frac{1}{\nu_k} \int_0^1 \gamma(y) |\varphi'_k(y)|^2 dy \right), \quad \forall x \in (0, 1), \forall k \geq 1.$$

1319 Integrating by parts leads to

1320
$$\int_0^1 \gamma(y) |\varphi'_k(y)|^2 dy = \int_0^1 (\nu_k - c(y)) \varphi_k(y)^2 dy \leq \nu_k + \|c\|_\infty$$

1321 which yields the existence of $C > 0$ such that

1322 (5.7)
$$|\varphi_k(x)|^2 + \frac{1}{\nu_k} |\varphi'_k(x)|^2 \leq C, \quad \forall x \in (0, 1), \forall k \geq 1.$$

1323 **5.3. Perturbation of a 2x2 Jordan block.**

1324 Let $\omega \subset (0, 1)$ be a non-empty open set and $U = L^2(\Omega)$. Let A be the Sturm-
 1325 Liouville operator defined in (5.1) and $f(A)$ be the operator defined in (5.2) with
 1326 $f : \text{Sp}(A) \rightarrow \mathbb{R}$ satisfying

$$1327 \quad |f(\nu_k)| < \frac{\varrho}{2}, \quad \forall k \geq 1.$$

1328 We consider the operator \mathcal{A} on $X = L^2(0, 1; \mathbb{R})^2$ defined by

$$1329 \quad (5.8) \quad \mathcal{A} = \begin{pmatrix} A & I \\ 0 & A + f(A) \end{pmatrix}, \quad D(\mathcal{A}) = D(A) \times D(A),$$

1330 and

$$1331 \quad (5.9) \quad \mathcal{B} : u \in U \mapsto \begin{pmatrix} 0 \\ \mathbf{1}_\omega u \end{pmatrix}.$$

1332 Then,

$$1333 \quad \mathcal{B}^* : \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \in X \mapsto \mathbf{1}_\omega \varphi_2.$$

1334 It is easy to see that $(-\mathcal{A}, D(\mathcal{A}))$ generates a \mathcal{C}_0 -semigroup on X and that $\mathcal{B} : U \rightarrow X$
 1335 is bounded. Thus we consider for this example that $X_\diamond^* = X = X_{-\diamond}$.

1336 **PROPOSITION 5.2.** *Let us consider the control system (1.1) with \mathcal{A} and \mathcal{B} given*
 1337 *by (5.8)-(5.9). Then, null-controllability from $X_{-\diamond}$ holds in any time i.e. $T_0(X_{-\diamond}) =$*
 1338 *0.*

1339 *Proof.* The spectrum of $(\mathcal{A}^*, D(\mathcal{A}))$ is given by

$$1340 \quad \Lambda = \{\nu_k ; k \geq 1\} \cup \{\nu_k + f(\nu_k) ; k \geq 1\}.$$

1341 From (5.3) and (5.6) it comes that there exists $\bar{N} > 0$ such that $\Lambda \in \mathcal{L}(2, \frac{\varrho}{2}, \frac{1}{2}, \bar{N})$.

1342 An associated grouping is given by

$$1343 \quad \begin{cases} G_k := \{\lambda_{k,1} := \nu_k, \lambda_{k,2} := \nu_k + f(\nu_k)\}, & \text{if } f(\nu_k) \neq 0, \\ G_k := \{\lambda_{k,1} := \nu_k\}, & \text{if } f(\nu_k) = 0. \end{cases}$$

1344 If $f(\nu_k) \neq 0$ the eigenvalues $\lambda_{k,1}$ and $\lambda_{k,2}$ are simple and we consider the associated
 1345 eigenfunctions

$$1346 \quad \phi_{k,1}^0 = \begin{pmatrix} -f(\nu_k) \\ 1 \end{pmatrix} \varphi_k, \quad \phi_{k,2}^0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varphi_k.$$

1347 If $f(\nu_k) = 0$ the eigenvalue $\lambda_{k,1}$ is algebraically double and we consider the associated
 1348 Jordan chain

$$1349 \quad \phi_{k,1}^0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varphi_k, \quad \phi_{k,1}^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \varphi_k.$$

1350 From (5.5) it comes that (1.28) and (1.29) are satisfied. Thus, from Theorem 1.3, we
 1351 obtain that for any $y_0 \in X_{-\diamond}$,

$$1352 \quad T_0(y_0) = \limsup_{k \rightarrow +\infty} \frac{\ln \mathcal{C}(G_k, y_0)}{2 \min G_k}.$$

1353 Let us now conclude by estimating $\mathcal{C}(G_k, y_0)$.

1354 • Consider first that $f(\nu_k) \neq 0$. Then, $\phi[\lambda_{k,1}, \lambda_{k,2}] = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \varphi_k$ and

1355
$$b[\lambda_{k,1}, \lambda_{k,2}] = \mathcal{B}^* \phi[\lambda_{k,1}, \lambda_{k,2}] = \frac{\mathbf{1}_\omega \varphi_k - \mathbf{1}_\omega \varphi_k}{f(\nu_k)} = 0.$$

1356 From Theorem 1.8 it comes that

1357
$$\mathcal{C}(G_k, y_0) = \langle M^{-1} \xi, \xi \rangle$$

1358 with

1359
$$M = \text{Gram}(b[\lambda_{k,1}], b[\lambda_{k,1}, \lambda_{k,2}]) + \text{Gram}(0, b[\lambda_{k,2}]) = \begin{pmatrix} \|\varphi_k\|_\omega^2 & 0 \\ 0 & \|\varphi_k\|_\omega^2 \end{pmatrix}$$

1360 and

1361
$$\xi = \begin{pmatrix} \langle y_0, \phi[\lambda_{k,1}] \rangle_{-\diamond, \diamond} \\ \langle y_0, \phi[\lambda_{k,1}, \lambda_{k,2}] \rangle_{-\diamond, \diamond} \end{pmatrix} = \begin{pmatrix} \left\langle y_0, \begin{pmatrix} -f(\nu_k) \\ 1 \end{pmatrix} \varphi_k \right\rangle_{-\diamond, \diamond} \\ \left\langle y_0, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \varphi_k \right\rangle_{-\diamond, \diamond} \end{pmatrix}.$$

1362 Thus,

1363
$$\mathcal{C}(G_k, y_0) = \left\langle y_0, \begin{pmatrix} -f(\nu_k) \\ 1 \end{pmatrix} \frac{\varphi_k}{\|\varphi_k\|_\omega} \right\rangle_{-\diamond, \diamond}^2 + \left\langle y_0, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{\varphi_k}{\|\varphi_k\|_\omega} \right\rangle_{-\diamond, \diamond}^2.$$

1364 • Consider now that $f(\nu_k) = 0$. Then, $b[\lambda_{k,1}, \lambda_{k,1}] = 0$. From Theorem 1.8 it comes that

1365
$$\mathcal{C}(G_k, y_0) = \langle M^{-1} \xi, \xi \rangle$$

1366 with

1367
$$M_k = \text{Gram}(b[\lambda_{k,1}], b[\lambda_{k,1}, \lambda_{k,1}]) + \text{Gram}(0, b[\lambda_{k,1}]) = \begin{pmatrix} \|\varphi_k\|_\omega^2 & 0 \\ 0 & \|\varphi_k\|_\omega^2 \end{pmatrix}.$$

1368 and

1369
$$\xi = \begin{pmatrix} \langle y_0, \phi[\lambda_{k,1}] \rangle_{-\diamond, \diamond} \\ \langle y_0, \phi[\lambda_{k,1}, \lambda_{k,1}] \rangle_{-\diamond, \diamond} \end{pmatrix} = \begin{pmatrix} \left\langle y_0, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varphi_k \right\rangle_{-\diamond, \diamond} \\ \left\langle y_0, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \varphi_k \right\rangle_{-\diamond, \diamond} \end{pmatrix}.$$

1370 As previously,

1371
$$\mathcal{C}(G_k, y_0) = \left\langle y_0, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \frac{\varphi_k}{\|\varphi_k\|_\omega} \right\rangle_{-\diamond, \diamond}^2 + \left\langle y_0, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{\varphi_k}{\|\varphi_k\|_\omega} \right\rangle_{-\diamond, \diamond}^2.$$

1372 Gathering both cases and using estimate (5.5) we obtain, for any $y_0 \in X_{-\diamond}$,

1373
$$\mathcal{C}(G_k, y_0) \leq C \|y_0\|_{-\diamond}^2, \quad \forall k \geq 1.$$

1374 Thus,

1375
$$T_0(y_0) = \limsup_{k \rightarrow +\infty} \frac{\ln \mathcal{C}(G_k, y_0)}{2 \min G_k} = 0.$$

1376 \square

1379 **5.4. Competition between different perturbations.**

1380 Let $\omega_1, \omega_2 \subset (0, 1)$ be two open sets with $\omega_1 \neq \emptyset$ and $U = L^2(\Omega)^2$. Let $B_1, B_2 \in$
 1381 \mathbb{R}^3 . To simplify the computations, we assume that

$$1382 \quad B_i = \begin{pmatrix} 0 \\ B_{i,2} \\ B_{i,3} \end{pmatrix}.$$

1383 Let $\alpha, \beta > 0$ with $\alpha \neq \beta$ and $f, g : \text{Sp}(A) \rightarrow \mathbb{R}$ be defined by

$$1384 \quad f(\nu_k) = \frac{\alpha}{2} e^{-\alpha\nu_k}, \quad g(\nu_k) = \frac{\beta}{2} e^{-\beta\nu_k}.$$

1385 As previously, we consider the associated operators $f(A)$ and $g(A)$ defined by the
 1386 spectral theorem and we define the evolution operator \mathcal{A} on $X = L^2(0, 1; \mathbb{R})^3$ by

$$1387 \quad (5.10) \quad \mathcal{A} = \begin{pmatrix} A & I & 0 \\ 0 & A + f(A) & 0 \\ 0 & 0 & A + g(A) \end{pmatrix}, \quad D(\mathcal{A}) = D(A)^3,$$

1388 and the control operator by

$$1389 \quad (5.11) \quad \mathcal{B} : \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in U \mapsto \mathbf{1}_{\omega_1} u_1 B_1 + \mathbf{1}_{\omega_2} u_2 B_2.$$

1390 Then, the observation operator reads

$$1391 \quad \mathcal{B}^* : \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} \in X \mapsto \begin{pmatrix} \mathbf{1}_{\omega_1} (B_{1,2}\varphi_2 + B_{1,3}\varphi_3) \\ \mathbf{1}_{\omega_2} (B_{2,2}\varphi_2 + B_{2,3}\varphi_3) \end{pmatrix}.$$

1392

1393 **PROPOSITION 5.3.** *Let us consider the control system (1.1) with \mathcal{A} and \mathcal{B} given*
 1394 *by (5.10)-(5.11).*

1395 *i. If $\omega_2 = \emptyset$, we assume that*

$$1396 \quad (5.12) \quad B_{1,2}B_{1,3} \neq 0.$$

1397 *Then,*

$$1398 \quad T_0(X_{-\diamond}) = \beta + \min\{\alpha, \beta\}.$$

1399 *ii. If $\omega_2 \neq \emptyset$, we assume that*

$$1400 \quad (5.13) \quad (B_{1,2}^2 + B_{2,2}^2) (B_{1,3}^2 + B_{2,3}^2) \neq 0.$$

1401 *(a) If B_1 and B_2 are linearly independent, then,*

$$1402 \quad T_0(X_{-\diamond}) = 0.$$

1403 *(b) If B_1 and B_2 are not linearly independent, then,*

$$1404 \quad T_0(X_{-\diamond}) = \beta + \min\{\alpha, \beta\}.$$

1405 *Proof.* It is easy to see that $(-\mathcal{A}, D(\mathcal{A}))$ generates a \mathcal{C}_0 -semigroup on X and that
 1406 $\mathcal{B} : U \rightarrow X$ is bounded. Thus we consider for this example that $X_\diamond^* = X = X_{-\diamond}$ and
 1407 $Y_0 = X_{-\diamond}$.

1408 The spectrum of $(\mathcal{A}^*, D(\mathcal{A}))$ is given by $\Lambda = \bigcup_{k \geq 1} G_k$ where

$$1409 \quad G_k := \{\lambda_{k,1} := \nu_k, \lambda_{k,2} := \nu_k + f(\nu_k), \lambda_{k,3} := \nu_k + g(\nu_k)\}.$$

1410 From (5.3) and (5.6) it comes that there exists $\bar{N} > 0$ such that $\Lambda \in \mathcal{L}(3, \frac{g}{2}, \frac{1}{2}, \bar{N})$
 1411 and $(G_k)_{k \geq 1}$ is an associated grouping.

1412 The eigenvalues are simple and the corresponding eigenvectors are given by

$$1413 \quad \phi_{k,1}^0 = \begin{pmatrix} -f(\nu_k) \\ 1 \\ 0 \end{pmatrix} \varphi_k, \quad \phi_{k,2}^0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \varphi_k, \quad \phi_{k,3}^0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \varphi_k.$$

1414 Thus, the assumption (1.29) hold. Moreover,

$$1415 \quad (5.14) \quad b_1 = b_2 = \begin{pmatrix} \mathbf{1}_{\omega_1} \varphi_k B_{1,2} \\ \mathbf{1}_{\omega_2} \varphi_k B_{2,2} \end{pmatrix}, \quad b_3 = \begin{pmatrix} \mathbf{1}_{\omega_1} \varphi_k B_{1,3} \\ \mathbf{1}_{\omega_2} \varphi_k B_{2,3} \end{pmatrix}$$

1416 From (5.5) and (5.12) or (5.13) (depending on the assumption on ω_2) it comes
 1417 that (1.28) is satisfied. Thus, from Theorem 1.3, it comes that for any $y_0 \in X_{-\diamond}$,

$$1418 \quad T_0(y_0) = \limsup_{k \rightarrow +\infty} \frac{\ln \mathcal{C}(G_k, y_0)}{2 \min G_k}.$$

1419 Let us now estimate $\mathcal{C}(G_k, y_0)$. From Theorem 1.8 it comes that

$$1420 \quad \mathcal{C}(G_k, y_0) = \langle M^{-1} \xi, \xi \rangle$$

1421 with

$$1422 \quad M = \text{Gram}(b[\lambda_{k,1}], b[\lambda_{k,1}, \lambda_{k,2}], b[\lambda_{k,1}, \lambda_{k,2}, \lambda_{k,3}]) \\ 1423 \quad + \text{Gram}(0, b[\lambda_{k,2}], b[\lambda_{k,2}, \lambda_{k,3}]) + \text{Gram}(0, 0, b[\lambda_{k,3}])$$

1425 and

$$1426 \quad \xi = \begin{pmatrix} \langle y_0, \phi[\lambda_{k,1}] \rangle_{-\diamond, \diamond} \\ \langle y_0, \phi[\lambda_{k,1}, \lambda_{k,2}] \rangle_{-\diamond, \diamond} \\ \langle y_0, \phi[\lambda_{k,1}, \lambda_{k,2}, \lambda_{k,3}] \rangle_{-\diamond, \diamond} \end{pmatrix}.$$

1427 Explicit computations yield

$$1428 \quad \phi[\lambda_{k,1}] = \begin{pmatrix} -f(\nu_k) \\ 1 \\ 0 \end{pmatrix} \varphi_k, \quad \phi[\lambda_{k,1}, \lambda_{k,2}] = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \varphi_k,$$

1429 and

$$1430 \quad \phi[\lambda_{k,1}, \lambda_{k,2}, \lambda_{k,3}] = \frac{1}{g(\nu_k)(g(\nu_k) - f(\nu_k))} \begin{pmatrix} f(\nu_k) - g(\nu_k) \\ -1 \\ 1 \end{pmatrix} \varphi_k.$$

1431 i. Assume that $\omega_2 = \emptyset$.

1432 After the change of variables

$$1433 \quad z = \text{diag} \left(\frac{1}{B_{1,2}}, \frac{1}{B_{1,2}}, \frac{1}{B_{1,3}} \right) y,$$

1434

the system under study reads

$$\begin{cases} \partial_t z + \begin{pmatrix} A & I & 0 \\ 0 & A + f(A) & 0 \\ 0 & 0 & A + g(A) \end{pmatrix} z = \mathbf{1}_{\omega_1} u_1(t, x) \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \\ z(t, 0) = z(t, 1) = 0. \end{cases}$$

1436

This leads to

1437

$$b[\lambda_{k,1}] = b[\lambda_{k,2}] = b[\lambda_{k,3}] = \mathbf{1}_{\omega_1} \varphi_k.$$

1438

Thus, $M = \|\varphi_k\|_{\omega_1}^2 I_3$ and

1439

$$\begin{aligned} \mathcal{C}(G_k, y_0) &= \left\langle y_0, \begin{pmatrix} -f(\nu_k) \\ 1 \\ 0 \end{pmatrix} \frac{\varphi_k}{\|\varphi_k\|_{\omega_1}} \right\rangle_{-\diamond, \diamond}^2 + \left\langle y_0, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \frac{\varphi_k}{\|\varphi_k\|_{\omega_1}} \right\rangle_{-\diamond, \diamond}^2 \\ &+ \left(\frac{1}{g(\nu_k)(g(\nu_k) - f(\nu_k))} \right)^2 \left\langle y_0, \begin{pmatrix} f(\nu_k) - g(\nu_k) \\ -1 \\ 1 \end{pmatrix} \frac{\varphi_k}{\|\varphi_k\|_{\omega_1}} \right\rangle_{-\diamond, \diamond}^2. \end{aligned}$$

1441

1442

1443

From (5.5), we obtain for any $y_0 \in X_{-\diamond}$,

$$\mathcal{C}(G_k, y_0) \leq C \|y_0\|_{-\diamond}^2 \left(1 + \left(\frac{1}{g(\nu_k)(g(\nu_k) - f(\nu_k))} \right)^2 \right).$$

1445

This leads to

$$T_0(X_{-\diamond}) \leq \limsup_{k \rightarrow +\infty} \frac{-\ln |g(\nu_k)(g(\nu_k) - f(\nu_k))|}{\nu_k}.$$

1447

Conversely, with the particular choice

$$y_0 = \sum_{k \geq 1} \frac{1}{\nu_k} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \varphi_k,$$

1449

we have

$$\mathcal{C}(G_k, y_0) = \frac{1}{\nu_k^2 \|\varphi_k\|_{\omega_1}^2} \left(\frac{1}{g(\nu_k)(g(\nu_k) - f(\nu_k))} \right)^2.$$

1451

Thus, from (5.5), we obtain

$$T_0(X_{-\diamond}) \geq T_0(y_0) = \limsup_{k \rightarrow +\infty} \frac{-\ln |g(\nu_k)(g(\nu_k) - f(\nu_k))|}{\nu_k}$$

1453

which gives

$$T_0(X_{-\diamond}) = \limsup_{k \rightarrow +\infty} \frac{-\ln |g(\nu_k)(g(\nu_k) - f(\nu_k))|}{\nu_k}.$$

1455

Then, the same computations as [10, Section 5.1.3] yield

$$T_0(X_{-\diamond}) = \beta + \min\{\alpha, \beta\}.$$

1457 ii. We now consider the case $\omega_2 \neq \emptyset$.

1458 (a) Assume that B_1 and B_2 are linearly independent. If necessary, we con-
 1459 sider smaller control sets so that $\omega_1 \cap \omega_2 = \emptyset$. As we will prove that
 1460 $T_0(X_{-\diamond}) = 0$, this is not a restrictive assumption.

1461 To ease the reading we drop the index k in what follows. As pre-
 1462 viously, the vector ξ is not bounded. Let us consider the dilatation
 1463 $D_\epsilon = \text{diag}(1, 1, \epsilon)$ with

$$1464 \quad \epsilon = g(\nu)(g(\nu) - f(\nu))$$

1465 and $\tilde{\xi} = D_\epsilon \xi$. Then, from Section D.1, it comes that

$$1466 \quad \mathcal{C}(G, y_0) = \left\langle \widetilde{M}^{-1} \tilde{\xi}, \tilde{\xi} \right\rangle$$

1467 with

$$1468 \quad \begin{aligned} \widetilde{M} = & \text{Gram}(b[\lambda_1], b[\lambda_1, \lambda_2], \epsilon b[\lambda_1, \lambda_2, \lambda_3]) \\ & + \text{Gram}(0, b[\lambda_2], \epsilon b[\lambda_2, \lambda_3]) + \text{Gram}(0, 0, \epsilon b[\lambda_3]). \end{aligned}$$

1471 As $\|\tilde{\xi}\|$ is bounded, we simply give a lower bound on the smallest eigen-
 1472 value of \widetilde{M} . Using (5.14), it comes that

$$1473 \quad b[\lambda_1, \lambda_2] = 0, \quad b[\lambda_2, \lambda_3] = \frac{b_3 - b_1}{g(\nu) - f(\nu)}, \quad b[\lambda_1, \lambda_2, \lambda_3] = \frac{1}{\epsilon}(b_3 - b_1).$$

1474 Thus,

$$1475 \quad \begin{aligned} \widetilde{M} = & \text{Gram}(b_1, 0, b_3 - b_1) + \text{Gram}(0, b_1, g(\nu)(b_3 - b_1)) \\ & + \text{Gram}(0, 0, \epsilon b_3). \end{aligned}$$

1478 This gives that, for any $\tau \in \mathbb{R}^3$, we have

$$1480 \quad (5.15) \quad \left\langle \widetilde{M} \tau, \tau \right\rangle = \|\tau_1 b_1 + \tau_3 (b_3 - b_1)\|_U^2 + \|\tau_2 b_1 + g(\nu) \tau_3 (b_3 - b_1)\|_U^2 \\ 1481 \quad \quad \quad + \epsilon^2 \|\tau_3 b_3\|_U^2.$$

1483 To obtain a lower bound on this quantity we use the following lemma.

1484 LEMMA 5.4. *There exists $C > 0$ (independent of k) such that for any*
 1485 $\theta_1, \theta_3 \in \mathbb{R}$,

$$1486 \quad \|\theta_1 b_1 + \theta_3 b_3\|_U^2 \geq C (\theta_1^2 + \theta_3^2).$$

1487 *Proof.* As $\omega_1 \cap \omega_2 = \emptyset$,

$$1488 \quad \begin{aligned} \|\theta_1 b_1 + \theta_3 b_3\|_U^2 = & (B_{1,2} \theta_1 + B_{1,3} \theta_3)^2 \|\varphi_k\|_{\omega_1}^2 \\ & + (B_{2,2} \theta_1 + B_{2,3} \theta_3)^2 \|\varphi_k\|_{\omega_2}^2. \end{aligned}$$

1491 Using (5.5) it comes that

$$1492 \quad \begin{aligned} \|\theta_1 b_1 + \theta_3 b_3\|_U^2 \geq & C \left((B_{1,2} \theta_1 + B_{1,3} \theta_3)^2 + (B_{2,2} \theta_1 + B_{2,3} \theta_3)^2 \right) \\ & = \left\| \begin{pmatrix} B_{1,2} & B_{1,3} \\ B_{2,2} & B_{2,3} \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_3 \end{pmatrix} \right\|^2. \end{aligned}$$

1493 Since B_1 and B_2 are linearly independent, this ends the proof. \square

1496 Applying this lemma twice to (5.15) yield

$$\begin{aligned}
 1497 \quad \langle \widetilde{M}\tau, \tau \rangle &\geq C((\tau_1 - \tau_3)^2 + \tau_3^2 + (\tau_2 - g(\nu)\tau_3)^2 + g(\nu)^2\tau_3^2 + \epsilon^2\tau_3^2) \\
 1498 \quad &\geq C((\tau_1 - \tau_3)^2 + \tau_3^2 + (\tau_2 - g(\nu)\tau_3)^2).
 \end{aligned}$$

1500 Taking into account that $0 < g(\nu) < \frac{1}{2}$ for ν large enough, the study of
 1501 this quadratic form in \mathbb{R}^3 leads to

$$1502 \quad \langle \widetilde{M}\tau, \tau \rangle \geq C(\tau_1^2 + \tau_2^2 + \tau_3^2).$$

1503 Thus the smallest eigenvalue of \widetilde{M} is bounded from below. This leads to
 1504 the boundedness of $\langle \widetilde{M}^{-1}\widetilde{\xi}, \widetilde{\xi} \rangle$ which concludes the proof of case ii (a).

1505 (b) Assume now that B_1 and B_2 are not linearly independent. Then, there
 1506 exist $x_1, x_2 \in \mathbb{R}$ such that

$$1507 \quad \begin{cases} x_1 B_{1,2} + x_2 B_{1,3} = 0 \\ x_1 B_{2,2} + x_2 B_{2,3} = 0. \end{cases}$$

1508 Up to a change of normalization of the eigenvectors (independent of k)
 1509 we obtain

$$1510 \quad b_1 = b_2 = b_3 = \begin{pmatrix} \mathbf{1}_{\omega_1} \varphi_k x_1 B_{1,2} \\ \mathbf{1}_{\omega_2} \varphi_k x_1 B_{2,2} \end{pmatrix}$$

1511 and this amounts to case i. □

1512 **6. Analysis of controllability for systems of partial differential equa-** 1513 **tions.**

1514 We now turn to the analysis of null controllability of actual partial differential
 1515 equations. We consider cascade systems of two parabolic equations. First, we consider
 1516 systems with different diffusion operators and constant zero order coupling term.
 1517 Then, we consider the same diffusion operator with a space varying zero order coupling
 1518 term.

1519 **6.1. Coupled heat equations with different diffusion coefficients.**

1520 In this example, we consider the Sturm Liouville operator A defined in (5.1) and
 1521 we define in $X = L^2(0, 1; \mathbb{R})^2$ the operator

$$1522 \quad \mathcal{A} = \begin{pmatrix} A & I \\ 0 & dA \end{pmatrix}, \quad D(\mathcal{A}) = D(A)^2,$$

1523 with $d > 0$.

1524 We study the following boundary control system

$$1525 \quad (6.1) \quad \begin{cases} \partial_t y + \mathcal{A}y = 0, & t \in (0, T), \\ y(t, 0) = B_0 v_0(t), \quad y(t, 1) = B_1 v_1(t), & t \in (0, T), \end{cases}$$

1526 with

$$1527 \quad (6.2) \quad B_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad B_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

1528 The control operator \mathcal{B} is defined in a weak sense as in [37]. The expression of its
 1529 adjoint is given by

$$1530 \quad \mathcal{B}^* : \begin{pmatrix} f \\ g \end{pmatrix} \in X_1^* \mapsto \begin{pmatrix} -B_0^* \begin{pmatrix} f'(0) \\ g'(0) \end{pmatrix} \\ B_1^* \begin{pmatrix} f'(1) \\ g'(1) \end{pmatrix} \end{pmatrix} = \begin{pmatrix} -(f'(0) + g'(0)) \\ f'(1) \end{pmatrix}.$$

1531 Thus, setting $X_\diamond^* = H_0^1(0, 1; \mathbb{R})^2$, we obtain that \mathcal{B} is admissible with respect to
 1532 $X_{-\diamond} = H^{-1}(0, 1; \mathbb{R})^2$.

1533 PROPOSITION 6.1. *For any $d > 0$, the minimal null control time of system (6.1)
 1534 is given by $T_0(X_{-\diamond}) = 0$.*

1535 REMARK 6.1. *The situation with a single control is quite different. Indeed, con-
 1536 sidering $B_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $B_1 = 0$, it is proved in [5] that, when A is the Dirichlet
 1537 Laplace operator, approximate controllability holds if and only if $\sqrt{d} \notin \mathbb{Q}$ and in this
 1538 case that*

$$1539 \quad T_0(X_{-\diamond}) = \limsup_{\substack{\lambda \rightarrow \infty \\ \lambda \in \Lambda}} \frac{-\ln \text{dist}(\lambda, \Lambda \setminus \{\lambda\})}{\lambda}.$$

1540 *With this formula the authors prove that, for any $\tau \in [0, +\infty]$, there exists a diffusion
 1541 ratio $d > 0$ such that the minimal null control time of system (6.1) satisfies $T_0(X_{-\diamond}) =$
 1542 τ .*

1543 REMARK 6.2. *The particular choice of B_0 and B_1 is done to simplify the com-
 1544 putations. Notice that with this choice, it is not possible to steer to zero the second
 1545 equation and then control the first equation. This would be the case with the simpler
 1546 choice*

$$1547 \quad B_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad B_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

1548 *Proof.* The case $d = 1$ is very similar to the analysis conducted in Section 5.3
 1549 (with $f = 0$) and is not detailed here. We now assume that $d \neq 1$. Let

$$1550 \quad \Lambda_1 := \text{Sp}(A) = \{\nu_k ; k \geq 1\}$$

1551 and $\Lambda_2 := d\Lambda_1$. The spectrum of \mathcal{A}^* is given by $\Lambda = \Lambda_1 \cup \Lambda_2$ which belongs to
 1552 $\mathcal{L}(2, \varrho, \frac{1}{2}, \bar{N})$ for some $\varrho, \bar{N} > 0$. For any $\lambda \in \Lambda$, there are three cases:

1553 • $\lambda = \nu_k \in \Lambda_1 \setminus \Lambda_2$ is a simple eigenvalue. An eigenvector is given by

$$1554 \quad \phi_\lambda = \begin{pmatrix} 1 \\ \frac{1}{\nu_k(1-d)} \end{pmatrix} \varphi_k.$$

1555 • $\lambda = d\nu_k \in \Lambda_2 \setminus \Lambda_1$ is a simple eigenvalue. An eigenvector is given by

$$1556 \quad \phi_\lambda = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varphi_k.$$

1557 • $\lambda = \nu_k = d\nu_{k'} \in \Lambda_1 \cap \Lambda_2$ is a geometrically double eigenvalue. A basis of
 1558 eigenvectors is given by

$$1559 \quad \phi_{\lambda,1} = \begin{pmatrix} 1 \\ \frac{1}{\nu_k(1-d)} \end{pmatrix} \varphi_k, \quad \phi_{\lambda,2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varphi_{k'}.$$

1560 Since $(\varphi_k)_{k \geq 1}$ is an Hilbert basis of X , this implies that (1.29) holds. Due to (5.4), it
 1561 comes that (1.28) also holds. Then, from Theorem 1.3,

$$1562 \quad T_0(y_0) = \limsup_{k \rightarrow +\infty} \frac{\ln \mathcal{C}(G_k, y_0)}{2 \min G_k}.$$

1563 Using again (5.4), the blocks consisting of a single simple eigenvalue do not contribute
 1564 to the minimal time.

1565 \star Blocks of two simple eigenvalues. Assume that we have a block of the form

$$1566 \quad G := \{\lambda_1 := \nu_k, \lambda_2 := d\nu_k\}.$$

1567 As we will deal later on with semi-simple eigenvalues, we evaluate the contribution of
 1568 the block G using Theorem 1.10. Then,

$$1569 \quad \mathcal{C}(G, y_0) = \langle M^{-1}\xi, \xi \rangle$$

1570 with

$$1571 \quad M = \text{Gram}(b[\lambda_1], b[\lambda_2]) + \text{Gram}(0, (\lambda_2 - \lambda_1)b[\lambda_2])$$

1572 and

$$1573 \quad \xi = \begin{pmatrix} \langle y_0, \phi_{\lambda_1} \rangle_{-\diamond, \diamond} \\ \langle y_0, \phi_{\lambda_2} \rangle_{-\diamond, \diamond} \end{pmatrix}.$$

1574 For any $k \geq 1$, we define $\epsilon_k \in \mathbb{R}$ by $\varphi'_k(1) = \epsilon_k \varphi'_k(0)$. From Lemma 5.1, there exists
 1575 $C > 0$ such that

$$1576 \quad (6.3) \quad \frac{1}{C} \leq |\epsilon_k| \leq C, \quad \forall k \geq 1.$$

1577 Then,

$$1578 \quad b[\lambda_1] = \mathcal{B}^* \phi_{\lambda_1} = \begin{pmatrix} -\varphi'_k(0)B_0^* \begin{pmatrix} 1 \\ \frac{1}{\nu_k(1-d)} \end{pmatrix} \\ \varphi'_k(1)B_1^* \begin{pmatrix} 1 \\ \frac{1}{\nu_k(1-d)} \end{pmatrix} \end{pmatrix} = -\varphi'_k(0) \begin{pmatrix} 1 + \frac{1}{\nu_k(1-d)} \\ -\epsilon_k \end{pmatrix}$$

1579

$$1580 \quad b[\lambda_2] = \mathcal{B}^* \phi_{\lambda_2} = \begin{pmatrix} -\varphi'_{k'}(0)B_0^* \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \varphi'_{k'}(1)B_1^* \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} = -\varphi'_{k'}(0) \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

1581 To ease the reading, we use the following change of normalization for the eigenvectors

$$1582 \quad \tilde{\phi}_{\lambda_1} := \frac{\phi_{\lambda_1}}{-\varphi'_k(0)}, \quad \tilde{\phi}_{\lambda_2} := \frac{\phi_{\lambda_2}}{-\varphi'_{k'}(0)},$$

1583 and we denote by \tilde{M} and $\tilde{\xi}$ the associated quantities. Notice that, due to (5.4), the
 1584 quantity $\|\tilde{\xi}\|$ is bounded. Thus, to estimate $\mathcal{C}(G, y_0)$ we give a lower bound on the
 1585 smallest eigenvalue of \tilde{M} . We have

$$1586 \quad \tilde{M} = \text{Gram}(\tilde{b}[\lambda_1], \tilde{b}[\lambda_2]) + \text{Gram}(0, (\lambda_2 - \lambda_1)\tilde{b}[\lambda_2])$$

$$1587 \quad = \underbrace{\begin{pmatrix} \epsilon_k^2 + \left(1 + \frac{1}{\nu_k(1-d)}\right)^2 & 1 + \frac{1}{\nu_k(1-d)} \\ 1 + \frac{1}{\nu_k(1-d)} & 1 \end{pmatrix}}_{=\Gamma^1} + \begin{pmatrix} 0 & 0 \\ 0 & (\lambda_2 - \lambda_1)^2 \end{pmatrix}.$$

1588

1589 For any $\tau \in \mathbb{R}^2$, $\langle \widetilde{M}\tau, \tau \rangle \geq \langle \Gamma^1 \tau, \tau \rangle$. Then,

$$1590 \quad \min \text{Sp}(\Gamma^1) \geq \frac{\det(\Gamma^1)}{\text{tr}(\Gamma^1)} = \frac{\epsilon_k^2}{1 + \epsilon_k^2 + \left(1 + \frac{1}{\nu_k(1-d)}\right)^2}$$

1591 From (6.3), it comes that $\min \text{Sp}(\Gamma^1)$ is bounded from below by a positive constant
1592 independent of G . Thus, the quantity $\mathcal{C}(G, y_0)$ is bounded.

1593 \star Blocks of a geometrically double eigenvalue.

1594 Consider $G = \{\lambda\}$ with $\lambda = \nu_k = d\nu_{k'} \in \Lambda_1 \cap \Lambda_2$.

1595 With the same notations as previously, Theorem 1.10 implies that

$$1596 \quad \mathcal{C}(G, y_0) = \langle \widetilde{M}^{-1} \tilde{\xi}, \tilde{\xi} \rangle$$

1597 where

$$1598 \quad \tilde{\xi} = \begin{pmatrix} \langle y_0, \frac{\phi_{\lambda,1}}{-\varphi'_k(0)} \rangle_{-\diamond, \diamond} \\ \langle y_0, \frac{\phi_{\lambda,2}}{-\varphi'_{k'}(0)} \rangle_{-\diamond, \diamond} \end{pmatrix}$$

1599 and

$$1600 \quad \widetilde{M} = \text{Gram} \left(\frac{\mathcal{B}^* \phi_{\lambda,1}}{-\varphi'_k(0)}, \frac{\mathcal{B}^* \phi_{\lambda,2}}{-\varphi'_{k'}(0)} \right)$$

$$1601 \quad = \begin{pmatrix} \epsilon_k^2 + \left(1 + \frac{1}{\nu_k(1-d)}\right)^2 & 1 + \frac{1}{\nu_k(1-d)} \\ 1 + \frac{1}{\nu_k(1-d)} & 1 \end{pmatrix} = \Gamma^1.$$

1603 Thus, the study of the previous item proves that $\min \text{Sp}(\Gamma^1)$ is bounded from below
1604 by a positive constant independent of λ and that the quantity $\mathcal{C}(G, y_0)$ is bounded.

1605 Gathering both cases, this implies that

$$1606 \quad T_0(y_0) = \limsup_{k \rightarrow +\infty} \frac{\ln \mathcal{C}(G_k, y_0)}{2 \min G_k} = 0. \quad \square$$

1607 **6.2. A system with two different potentials.**

1608 Let us consider the following control system

$$1609 \quad (6.4) \quad \begin{cases} \partial_t y + \begin{pmatrix} -\partial_{xx} + c_1(x) & 1 \\ 0 & -\partial_{xx} + c_2(x) \end{pmatrix} y = \begin{pmatrix} 0 \\ \mathbf{1}_\omega u(t, x) \end{pmatrix}, & (t, x) \in (0, T) \times (0, 1), \\ y(t, 0) = y(t, 1) = 0, & t \in (0, T), \\ y(0, x) = y_0(x), \end{cases}$$

1610 where $c_1, c_2 \in L^2(0, 1; \mathbb{R})$.

1611 With the technics developed in this article, one can prove the following control-
1612 lability result.

1613 **PROPOSITION 6.2.** *For any non-negative potentials c_1, c_2 , system (6.4) is null*
1614 *controllable in any time $T > 0$ from $L^2(0, 1; \mathbb{R})^2$.*

1615 The proof follows closely the computations done for the same system with a boundary
 1616 control in [10, Section 5.2.1]. The only difference is that the contributions of terms
 1617 of the form $\|\mathcal{B}^* \bullet\|_U = \|\bullet\|_\omega$ are estimated using Lemma 5.1. As the result stated in
 1618 Proposition 6.2 is already known (it is for instance an application of [25] with a proof
 1619 based on Carleman estimates), we do not detail the proof here to lighten this article.

1620 6.3. A system with a space varying zero order coupling term.

1621 Let us consider A the Sturm-Liouville operator defined in (5.1). In this section
 1622 we study the null controllability of the following control system

$$1623 \quad (6.5) \quad \begin{cases} \partial_t y + \begin{pmatrix} A & q(x) \\ 0 & A \end{pmatrix} y = \begin{pmatrix} 0 \\ \mathbf{1}_\omega u(t, x) \end{pmatrix}, & (t, x) \in (0, T) \times (0, 1), \\ y(t, 0) = y(t, 1) = 0, & t \in (0, T), \\ y(0, x) = y_0(x), \end{cases}$$

1624 where the coupling function q belongs to $L^\infty(0, 1; \mathbb{R})$ and $\omega \subset (0, 1)$ is a non empty
 1625 open set.

1626 To fit in the formalism of system (1.1) the evolution operator \mathcal{A} is defined by

$$1627 \quad \mathcal{A} = \begin{pmatrix} A & q \\ 0 & A \end{pmatrix}, \quad D(\mathcal{A}) = D(A)^2$$

1628 and the control operator \mathcal{B} is defined by

$$1629 \quad \mathcal{B} : u \in U = L^2((0, 1); \mathbb{R}) \mapsto \begin{pmatrix} 0 \\ \mathbf{1}_\omega u \end{pmatrix}.$$

1630 As the control operator is bounded we consider in this example $X_{-\diamond} = X = X_\diamond^* =$
 1631 $L^2(0, 1; \mathbb{R})^2$. Recall that the eigenvalues ν_k and the associated eigenvectors φ_k of the
 1632 operator A satisfy (5.3), (5.5) and (5.6).

1633 6.3.1. Already known controllability results for this system.

- 1634 • *Approximate controllability.*

1635 For any $k \in \mathbb{N}^*$, let us define $\tilde{\varphi}_k$ as the unique solution of the Cauchy problem

$$1636 \quad (6.6) \quad \begin{cases} (A - \nu_k)\tilde{\varphi}_k = 0, \\ \tilde{\varphi}_k(0) = 1, \quad \tilde{\varphi}'_k(0) = 0. \end{cases}$$

1637 For any $F \in L^2(0, 1; \mathbb{R})$, we also define the quantities

$$1638 \quad (6.7a) \quad \mathcal{M}_{k,1}(F, \omega) = \sup \left\{ \left| \int_{\mathfrak{C}} F \varphi_k \right| ; \mathfrak{C} \text{ connected component of } \overline{(0, 1) \setminus \omega} \right\}$$

$$1639 \quad (6.7b) \quad \mathcal{M}_{k,2}(F, \omega) = \sup \left\{ \left| \int_{\mathfrak{C}} F \tilde{\varphi}_k \right| ; \mathfrak{C} \text{ connected component of } \overline{(0, 1) \setminus \omega} \right.$$

$$1640 \quad \left. \text{such that } \mathfrak{C} \cap \{0, 1\} = \emptyset \right\}$$

1641 and

$$1643 \quad (6.8) \quad \mathcal{M}_k(F, \omega) = \max \{ \mathcal{M}_{k,1}(F, \omega), \mathcal{M}_{k,2}(F, \omega) \}.$$

1644 It is proved in [17, Theorem 3.2] that, if $\text{Supp}(q) \cap \omega = \emptyset$, approximate controllability
 1645 of (6.5) holds if and only if

$$1646 \quad (6.9) \quad \mathcal{M}_k(q\varphi_k, \omega) \neq 0, \quad \forall k \geq 1.$$

1647 Notice also that applying [17, Theorem 2.2] we obtain that

- 1648 • If $\text{Supp}(q) \cap \omega \neq \emptyset$, approximate controllability of (6.5) holds without any
- 1649 other condition.
- 1650 • If $\text{Supp}(q) \cap \omega = \emptyset$, approximate controllability of (6.5) holds if and only if

$$1651 \quad (6.10) \quad \mathcal{M}_k((I_k(q) - q)\varphi_k, \omega) \neq 0, \quad \forall k \geq 1,$$

1652 where

$$1653 \quad (6.11) \quad I_k(q) = \int_0^1 q(x)\varphi_k^2(x)dx.$$

1654 Rewriting the condition this way is more coherent with the expression of the
 1655 minimal null control time that is proved in what follows (see Section 6.3.3).

- 1656 • *Null controllability under a sign assumption.*

1657 If there exists $\omega_0 \subset \omega$ such that q has a strict sign inside ω_0 then it follows
 1658 from [25] that null controllability holds in any arbitrary time. The proof is based on
 1659 Carleman estimates.

- 1660 • *Null controllability with disjoint control and coupling domains.*

1661 System (6.5) was then studied in the case where $A = -\partial_{xx}$ and $\omega = (a, b)$ is an
 1662 interval such that $\text{Supp}(q) \cap \omega = \emptyset$.

1663 First, it was proved in [6] that if $\text{Supp}(q) \subset (b, 1)$ then, approximate controllability
 1664 holds if and only if

$$1665 \quad I_k(q) \neq 0, \quad \forall k \geq 1.$$

1666 This condition is equivalent to (6.9). In this case the authors proved that

$$1667 \quad T_0(X_{-\diamond}) = \limsup_{k \rightarrow +\infty} \frac{-\ln |I_k(q)|}{\nu_k}.$$

1668 Later on, it was proved in [7] that if $\text{Supp}(q) \subset ((0, a) \cup (b, 1))$, then approximate
 1669 controllability holds if and only if

$$1670 \quad |I_{1,k}(q)| + |I_{2,k}(q)| \neq 0, \quad \forall k \geq 1,$$

1671 where

$$1672 \quad (6.12) \quad I_{1,k}(q) = \int_0^a q(x)\varphi_k^2(x)dx, \quad I_{2,k}(q) = \int_b^1 q(x)\varphi_k^2(x)dx.$$

1673 In this case the authors proved that

$$1674 \quad T_0(X_{-\diamond}) = \limsup_{k \rightarrow +\infty} \frac{-\ln \max\{|I_k(q)|, |I_{1,k}(q)|, |I_{2,k}(q)|\}}{\nu_k}.$$

1675 Moreover, the authors proved that for any $\tau_0 \in [0, +\infty]$ there exists a coupling function
 1676 q such that $T_0(X_{-\diamond}) = \tau_0$. Let us underline that these results are the first results
 1677 exhibiting a positive minimal null control time for a system of coupled parabolic

1678 equations with a distributed control. Due to the assumption $\text{Supp}(q) \cap \omega = \emptyset$, the
 1679 strategy based on Carleman estimates is inefficient. The proofs of [6, 7] are based on
 1680 the moments method.

1681 In what follows we give a minimal time characterization that unifies all these
 1682 results (see Section 6.3.4). It also extends these results to a general Sturm-Liouville
 1683 operator and enables us to deal with new geometric configurations for ω and the
 1684 support of q (see Proposition 6.8).

1685 6.3.2. A first formula for the minimal time.

1686 To compute the minimal null control time let us detail the spectral analysis of
 1687 the operator \mathcal{A}^* .

1688 For any $k \in \mathbb{N}^*$, we define ψ_k as the unique solution of

$$1689 \quad (6.13) \quad \begin{cases} (A - \nu_k)\psi_k = (I_k(q) - q)\varphi_k, \\ \psi_k(0) = \psi_k(1) = 0, \\ \langle \varphi_k, \psi_k \rangle_\omega = 0. \end{cases}$$

1690 This is possible since

$$1691 \quad \int_0^1 (I_k(q) - q(x))\varphi_k(x)\varphi_k(x)dx = 0.$$

1692 We have $\Lambda = (\nu_k)_{k \in \mathbb{N}^*}$ and we distinguish the following cases.

1693 \star If $I_k(q) \neq 0$ then ν_k is algebraically double and geometrically simple. A Jordan
 1694 chain is given by

$$1695 \quad (6.14) \quad \phi_k^0 = \begin{pmatrix} 0 \\ \varphi_k \end{pmatrix}, \quad \phi_k^1 = \frac{1}{I_k(q)} \begin{pmatrix} \varphi_k \\ \psi_k \end{pmatrix}$$

1696 where ψ_k is given by (6.13).

1697 \star If $I_k(q) = 0$ then ν_k is geometrically double and a basis of eigenvectors is given
 1698 by

$$1699 \quad (6.15) \quad \phi_{k,1}^0 = \begin{pmatrix} 0 \\ \varphi_k \end{pmatrix}, \quad \phi_{k,2}^0 = \begin{pmatrix} \varphi_k \\ \psi_k \end{pmatrix}$$

1700 where ψ_k is given by (6.13).

1701 **REMARK 6.3.** *In the definition of ψ_k , the choice of normalization $\langle \varphi_k, \psi_k \rangle_\omega = 0$*
 1702 *is done to simplify the computations. It ensures orthogonality between observations of*
 1703 *(generalized) eigenvectors.*

1704 Applying Theorem 1.3 we obtain the following characterization of the minimal
 1705 null control time.

1706 **PROPOSITION 6.3.** *Let $\omega \subset (0, 1)$ be a non empty open set and let $q \in L^\infty(0, 1; \mathbb{R})$.*
 1707 *Assume that either $\text{Supp}(q) \cap \omega \neq \emptyset$ or that (6.10) holds. Then, the minimal null*
 1708 *control time for system (6.5) is given by*

$$1709 \quad T_0(X_{-\diamond}) = \limsup_{k \rightarrow +\infty} \frac{-\ln(I_k(q)^2 \|\varphi_k\|_\omega^2 + \|\psi_k\|_\omega^2)}{2\nu_k}$$

1710 This formula is valid in a general geometric configuration for ω and the support
 1711 of q and unifies the different results obtained in the literature for the study of null
 1712 controllability of system (6.5). For example, even if it is not straightforward, we prove
 1713 in Section 6.3.4 that it allows to recover the null controllability of (6.5) in any time
 1714 proved in [25] when q has a strict sign on $\omega_0 \subset \omega$.

1715 We also give in Section 6.3.3 another formula which is more convenient to deal
 1716 with but necessitates the geometric assumption $\text{Supp}(q) \cap \omega = \emptyset$.

1717 *Proof.* Since $\Lambda = (\nu_k)_{k \in \mathbb{N}^*}$, it belongs to $\mathcal{L}(1, \varrho, \frac{1}{2}, \bar{N})$ for some $\bar{N} > 0$ and ϱ
 1718 defined in (5.3). Thus, a suitable grouping is given by $(\{\nu_k\})_{k \geq 1}$.

1719 Due to (6.14) and (6.15) we obtain that (1.29) holds. As mentioned previously, the
 1720 approximate controllability assumption (1.28) follows from (6.10) and [17, Theorem
 1721 2.2].

1722 Then, from Theorem 1.3, for any $y_0 \in X_{-\diamond}$, we have

$$1723 \quad T_0(y_0) = \limsup_{k \rightarrow +\infty} \frac{\ln \mathcal{C}(\{\nu_k\}, y_0)}{2\nu_k}.$$

1724 To compute $\mathcal{C}(\{\nu_k\}, y_0)$ we distinguish two cases.

1725 \star Assume that $I_k(q) \neq 0$. Then, from Theorem 1.8, it comes that

$$1726 \quad \mathcal{C}(\{\nu_k\}, y_0) = \langle M^{-1}\xi, \xi \rangle$$

1727 where

$$1728 \quad M = \text{Gram}(\mathcal{B}^* \phi_k^0, \mathcal{B}^* \phi_k^1) + \text{Gram}(0, \mathcal{B}^* \phi_k^0) = \begin{pmatrix} \|\varphi_k\|_\omega^2 & 0 \\ 0 & \|\varphi_k\|_\omega^2 + \frac{1}{I_k(q)^2} \|\psi_k\|_\omega^2 \end{pmatrix}$$

1729 and

$$1730 \quad \xi = \begin{pmatrix} \langle y_0, \phi_k^0 \rangle_{-\diamond, \diamond} \\ \langle y_0, \phi_k^1 \rangle_{-\diamond, \diamond} \end{pmatrix}.$$

1731 Thus,

$$1732 \quad \mathcal{C}(\{\nu_k\}, y_0) = \frac{1}{\|\varphi_k\|_\omega^2} \left\langle y_0, \begin{pmatrix} 0 \\ \varphi_k \end{pmatrix} \right\rangle_{-\diamond, \diamond}^2 + \frac{1}{I_k(q)^2 \|\varphi_k\|_\omega^2 + \|\psi_k\|_\omega^2} \left\langle y_0, \begin{pmatrix} \varphi_k \\ \psi_k \end{pmatrix} \right\rangle_{-\diamond, \diamond}^2.$$

1733 \star Assume that $I_k(q) = 0$. Then, from Theorem 1.10, it comes that

$$1734 \quad \mathcal{C}(\{\nu_k\}, y_0) = \langle M^{-1}\xi, \xi \rangle$$

1735 where

$$1736 \quad M = \text{Gram}(\mathcal{B}^* \phi_{k,1}^0, \mathcal{B}^* \phi_{k,2}^0) = \begin{pmatrix} \|\varphi_k\|_\omega^2 & 0 \\ 0 & \|\psi_k\|_\omega^2 \end{pmatrix}$$

1737 and

$$1738 \quad \xi = \begin{pmatrix} \langle y_0, \phi_{k,1} \rangle_{-\diamond, \diamond} \\ \langle y_0, \phi_{k,2} \rangle_{-\diamond, \diamond} \end{pmatrix}.$$

1739 Thus,

$$1740 \quad \mathcal{C}(\{\nu_k\}, y_0) = \frac{1}{\|\varphi_k\|_\omega^2} \left\langle y_0, \begin{pmatrix} 0 \\ \varphi_k \end{pmatrix} \right\rangle_{-\diamond, \diamond}^2 + \frac{1}{\|\psi_k\|_\omega^2} \left\langle y_0, \begin{pmatrix} \varphi_k \\ \psi_k \end{pmatrix} \right\rangle_{-\diamond, \diamond}^2.$$

1741 Finally, in both cases, the cost of the group $\{\nu_k\}$ is given by
 1742 (6.16)

$$1742 \quad \mathcal{C}(\{\nu_k\}, y_0) = \frac{1}{\|\varphi_k\|_\omega^2} \left\langle y_0, \begin{pmatrix} 0 \\ \varphi_k \end{pmatrix} \right\rangle_{-\diamond, \diamond}^2 + \frac{1}{I_k(q)^2 \|\varphi_k\|_\omega^2 + \|\psi_k\|_\omega^2} \left\langle y_0, \begin{pmatrix} \varphi_k \\ \psi_k \end{pmatrix} \right\rangle_{-\diamond, \diamond}^2.$$

1743 We now evaluate the different contributions of the terms in the previous right-hand
 1744 side.

1745 Recall that $\|\varphi_k\|_{(0,1)} = 1$ and that, from (5.5),

$$1746 \quad \|\varphi_k\|_\omega \geq C > 0, \quad \forall k \geq 1.$$

1747 We now prove that $\|\psi_k\|_{\diamond^*} = \|\psi_k\|_{(0,1)}$ is bounded. Let

$$1748 \quad \tilde{\psi}_k := \psi_k - \frac{\psi'_k(0)}{\varphi'_k(0)} \varphi_k.$$

1749 Then, the function $\tilde{\psi}_k$ satisfies

$$1750 \quad \begin{cases} (A - \nu_k) \tilde{\psi}_k = (I_k(q) - q) \varphi_k, \\ \tilde{\psi}_k(0) = \tilde{\psi}_k(1) = 0, \\ \tilde{\psi}'_k(0) = 0. \end{cases}$$

1751 From Lemma 5.1 it comes that

$$1752 \quad \left| \tilde{\psi}_k(x) \right|^2 + \frac{\gamma(x)}{\nu_k} \left| \tilde{\psi}'_k(x) \right|^2 \leq \frac{C}{\nu_k}, \quad \forall x \in (0, 1), \forall k \geq 1.$$

1753 which yields

$$1754 \quad \left\| \tilde{\psi}_k \right\|_{(0,1)} \leq C, \quad \forall k \geq 1.$$

1755 Notice that, by definition of $\tilde{\psi}_k$, we have $(\psi_k - \tilde{\psi}_k) \in \mathbb{R}\varphi_k$. Then, multiplying by φ_k ,
 1756 integrating over ω and recalling that $\langle \psi_k, \varphi_k \rangle_\omega = 0$, we obtain that

$$1757 \quad \psi_k = \tilde{\psi}_k - \frac{\langle \tilde{\psi}_k, \varphi_k \rangle_\omega}{\|\varphi_k\|_\omega^2} \varphi_k.$$

1758 This implies that

$$1759 \quad \|\psi_k\|_{(0,1)} \leq \left\| \tilde{\psi}_k \right\|_{(0,1)} \left(1 + \frac{1}{\|\varphi_k\|_\omega} \right) \leq C, \quad \forall k \geq 1.$$

1760 Now, getting back to (6.16), we obtain that

$$1761 \quad \mathcal{C}(\{\nu_k\}, y_0) \leq C \|y_0\|_{-\diamond}^2 \left(1 + \frac{1}{I_k(q)^2 \|\varphi_k\|_\omega^2 + \|\psi_k\|_\omega^2} \right), \quad \forall k \geq 1, \forall y_0 \in X_{-\diamond},$$

1762 which proves that

$$1763 \quad T_0(X_{-\diamond}) \leq \limsup_{k \rightarrow +\infty} \frac{-\ln(I_k(q)^2 \|\varphi_k\|_\omega^2 + \|\psi_k\|_\omega^2)}{2\nu_k}.$$

1764 To prove the converse inequality let us choose

$$1765 \quad y_0 = \sum_{k \geq 1} \frac{1}{\nu_k} \begin{pmatrix} \varphi_k \\ 0 \end{pmatrix}.$$

1766 From (6.16) we obtain that for this particular choice of y_0 ,

$$1767 \quad \mathcal{C}(\{\nu_k\}, y_0) = \frac{1}{\nu_k^2} \frac{1}{I_k(q)^2 \|\varphi_k\|_\omega^2 + \|\psi_k\|_\omega^2}, \quad \forall k \geq 1.$$

1768 Thus,

$$1769 \quad T_0(X_{-\diamond}) \geq T_0(y_0) = \limsup_{k \rightarrow +\infty} \frac{-\ln(I_k(q)^2 \|\varphi_k\|_\omega^2 + \|\psi_k\|_\omega^2)}{2\nu_k}.$$

1770 This ends the proof of Proposition 6.3. \square

1771 6.3.3. A second formula for the minimal time with disjoint control and 1772 coupling domains.

1773 The main result of this section is the following characterization of the minimal
1774 null control time.

1775 PROPOSITION 6.4. *Let $\omega \subset (0, 1)$ be a non empty open set with a finite number of
1776 connected components. Let $q \in L^\infty(0, 1; \mathbb{R})$ be such that $\text{Supp}(q) \cap \omega = \emptyset$. Assume
1777 that (6.10) holds. Then, the minimal null control time for system (6.5) is given by*

$$1778 \quad T_0(X_{-\diamond}) = \limsup_{k \rightarrow +\infty} \frac{-\ln \mathcal{M}_k((I_k(q) - q)\varphi_k, \omega)}{\nu_k}.$$

1779 The main advantage of this formulation with respect to the one proved in Proposi-
1780 tion 6.3 is that it does not involve ψ_k . As we prove in Section 6.3.4, this formula
1781 allows to recover the values of $T_0(X_{-\diamond})$ proved in the literature for various geometric
1782 configurations. It also allows to prove new results for this system (see Proposition 6.8).

1783 REMARK 6.4. *Notice that the assumption $\text{Supp}(q) \cap \omega = \emptyset$ is necessary for this
1784 formulation. Indeed, if $q = 1$ and ω is an interval then, from [25], null controllability
1785 holds in any time $T > 0$ but $I_k(q) - q = 0$ for any $k \geq 1$. However, the assumption
1786 $\text{Supp}(q) \cap \omega = \emptyset$ is not restrictive for our study as it is the setting in which a minimal
1787 null control time can occur.*

1788 Using Proposition 6.3, the proof of Proposition 6.4 consists in comparing the
1789 asymptotic behaviors of $\mathcal{M}_k((I_k(q) - q)\varphi_k, \omega)$ and

$$1790 \quad I_k(q)^2 \|\varphi_k\|_\omega^2 + \|\psi_k\|_\omega^2.$$

1791 To do so we will use the following technical lemma. To improve the readability we
1792 postpone its proof to Appendix E.

1793 LEMMA 6.5. *Let $\omega \subset (0, 1)$ be a non empty open set with a finite number of
1794 connected components.*

1795 *i. There exists $K \in \mathbb{N}^*$ and $C > 0$ such that for any $k \geq K$, any $F \in L^2(0, 1; \mathbb{R})$
1796 and any u satisfying the differential equation*

$$1797 \quad (A - \nu_k)u = F,$$

1798 *we have*

$$1799 \quad \mathcal{M}_k(F, \omega) \leq C \left(\sqrt{\nu_k} \|u\|_\omega + \sqrt{\nu_k} (|u(0)| + |u(1)|) + \|F\|_\omega \right).$$

1800 *ii. There exists $K \in \mathbb{N}^*$ and $C > 0$ such that for any $k \geq K$ and any $F \in$
 1801 $L^2(0, 1; \mathbb{R})$, there exists u satisfying*

$$1802 \quad \begin{cases} (A - \nu_k)u = F, \\ u(0) = u(1) = 0, \end{cases}$$

1803 *such that*

$$1804 \quad \left(\sqrt{\nu_k} \|u\|_\omega - \|F\|_\omega \right) \leq C \mathcal{M}_k(F, \omega).$$

1805 We now turn to the proof of Proposition 6.4.

1806 *Proof (of Proposition 6.4).* Recall that, from Proposition 6.3,

$$1807 \quad T_0(X_{-\diamond}) = \limsup_{k \rightarrow +\infty} \frac{-\ln(I_k(q)^2 \|\varphi_k\|_\omega^2 + \|\psi_k\|_\omega^2)}{2\nu_k}.$$

1808 As $\text{Supp}(q) \cap \omega = \emptyset$, applying point i. of Lemma 6.5 to ψ_k yields, for $k \geq K$,

$$1809 \quad \begin{aligned} \mathcal{M}_k((I_k(q) - q)\varphi_k, \omega)^2 &\leq C(\nu_k \|\psi_k\|_\omega^2 + \|(I_k(q) - q)\varphi_k\|_\omega^2) \\ 1810 &\leq C\nu_k(\|\psi_k\|_\omega^2 + I_k(q)^2 \|\varphi_k\|_\omega^2). \end{aligned}$$

1812 Thus,

$$1813 \quad T_0(X_{-\diamond}) \leq \limsup_{k \rightarrow +\infty} \frac{-\ln \mathcal{M}_k((I_k(q) - q)\varphi_k, \omega)}{\nu_k}.$$

1814 We now prove the converse inequality. Let u be the function given by the point ii.
 1815 of Lemma 6.5 with $F = (I_k(q) - q)\varphi_k$. Notice that there exists $\alpha \in \mathbb{R}$ such that
 1816 $u = \psi_k + \alpha\varphi_k$. Then, as $\langle \varphi_k, \psi_k \rangle_\omega = 0$, we have $\|\psi_k\|_\omega \leq \|u\|_\omega$. Thus, using the
 1817 estimate given by point ii. of Lemma 6.5 and the assumption $\text{Supp}(q) \cap \omega = \emptyset$, we
 1818 obtain that, for any $k \geq K$,

$$1819 \quad C\nu_k \|\psi_k\|_\omega^2 \leq \mathcal{M}_k(F, \omega)^2 + \|F\|_\omega^2 \leq \mathcal{M}_k(F, \omega)^2 + I_k(q)^2 \|\varphi_k\|_\omega^2.$$

1820 This yields

$$1821 \quad (6.17) \quad \|\psi_k\|_\omega^2 + I_k(q)^2 \|\varphi_k\|_\omega^2 \leq C(\mathcal{M}_k(F, \omega)^2 + I_k(q)^2 \|\varphi_k\|_\omega^2).$$

1822 We denote by $\mathfrak{C}_1, \dots, \mathfrak{C}_N$ the connected components of $\overline{(0, 1) \setminus \omega}$. As $\text{Supp}(q) \cap \omega = \emptyset$,
 1823 notice that

$$1824 \quad \begin{aligned} \sum_{j=1}^N \int_{\mathfrak{C}_j} F(x)\varphi_k(x)dx &= I_k(q) \sum_{j=1}^N \int_{\mathfrak{C}_j} \varphi_k^2(x)dx - \sum_{j=1}^N \int_{\mathfrak{C}_j} q(x)\varphi_k^2(x)dx \\ 1825 &= I_k(q)(1 - \|\varphi_k\|_\omega^2) - I_k(q) \\ 1826 &= -I_k(q)\|\varphi_k\|_\omega^2. \end{aligned}$$

1828 Thus, from (5.5) we deduce that

$$1829 \quad |I_k(q)| \leq C \mathcal{M}_k(F, \omega).$$

1830 Plugging it into (6.17) we obtain

$$1831 \quad \|\psi_k\|_\omega^2 + I_k(q)^2 \|\varphi_k\|_\omega^2 \leq C \mathcal{M}_k(F, \omega)^2.$$

1832 This implies that

$$1833 \quad T_0(X_{-\diamond}) \geq \limsup_{k \rightarrow +\infty} \frac{-\ln \mathcal{M}_k((I_k(q) - q)\varphi_k, \omega)}{\nu_k}$$

1834 and ends the proof of Proposition 6.4. □

1835 **6.3.4. Application of the minimal null control time formulas.**

1836 *Unification of previously known results.*

1837 Let us prove that the obtained results unifies previous characterizations given in
1838 the literature and stated in Section 6.3.1. Recall that φ_k satisfies (5.5).

1839 • Let us consider the setting studied in [6] *i.e.* $\omega = (a, b)$ and $\text{Supp}(q) \subset (b, 1)$.

1840 In this case, $(0, 1) \setminus \omega$ has at most two connected components both touching the
1841 boundary of $(0, 1)$. Thus, setting

$$1842 \quad F = (I_k(q) - q)\varphi_k$$

1843 we obtain

$$1844 \quad \mathcal{M}_k(F, \omega) = \max \left\{ \left| \int_0^a F(x)\varphi_k(x)dx \right|, \left| \int_b^1 F(x)\varphi_k(x)dx \right| \right\}.$$

1845 Using the assumption $\text{Supp}(q) \subset (b, 1)$ we get

$$1846 \quad \left| \int_0^a F(x)\varphi_k(x)dx \right| = |I_k(q)| \int_0^a \varphi_k^2(x)dx,$$

1847 and

$$1848 \quad \left| \int_b^1 F(x)\varphi_k(x)dx \right| = \left| I_k(q) \int_b^1 \varphi_k^2(x)dx - I_k(q) \right| = |I_k(q)| \int_0^b \varphi_k^2(x)dx.$$

1850 Thus,

$$1851 \quad \mathcal{M}_k(F, \omega) = |I_k(q)| \int_0^b \varphi_k^2(x)dx.$$

1852 Recall that from (5.5)

$$1853 \quad \inf_{k \geq 1} \int_0^b \varphi_k^2(x)dx > 0.$$

1854 This implies that approximate controllability holds if and only if

$$1855 \quad I_k(q) \neq 0, \quad \forall k \geq 1,$$

1856 and in this case that

$$1857 \quad T_0(X_{-\diamond}) = \limsup_{k \rightarrow +\infty} \frac{-\ln |I_k(q)|}{\nu_k}.$$

1858 Thus we recover the result proved in [6] and extend it to a general Sturm-Liouville
1859 operator.

1860 • Let us now consider the setting studied in [7] *i.e.* $\omega = (a, b)$ and $\text{Supp}(q) \cap \omega = \emptyset$.

1861 Again, setting

$$1862 \quad F = (I_k(q) - q)\varphi_k$$

1863 we obtain

$$1864 \quad \mathcal{M}_k(F, \omega) = \max \left\{ \left| \int_0^a F(x) \varphi_k(x) dx \right|, \left| \int_b^1 F(x) \varphi_k(x) dx \right| \right\}.$$

1865 Using the notations introduced in (6.12) we have

$$1866 \quad (6.18) \quad \int_0^a F(x) \varphi_k(x) dx = I_k(q) \int_0^a \varphi_k^2(x) dx - I_{1,k}(q)$$

1867 and

$$1868 \quad (6.19) \quad \int_b^1 F(x) \varphi_k(x) dx = I_k(q) \int_b^1 \varphi_k^2(x) dx - I_{2,k}(q).$$

1869 Thus,

$$1870 \quad \mathcal{M}_k(F, \omega) \leq 2 \max \{ |I_k(q)|, |I_{1,k}(q)|, |I_{2,k}(q)| \}.$$

1871 Conversely, using (6.18) and (6.19) we have

$$\begin{aligned} 1872 \quad & \int_0^a F(x) \varphi_k(x) dx + \int_b^1 F(x) \varphi_k(x) dx \\ 1873 \quad &= I_k(q) \left(\int_0^a \varphi_k^2(x) dx + \int_b^1 \varphi_k^2(x) dx \right) - (I_{1,k}(q) + I_{2,k}(q)) \\ 1874 \quad &= -I_k(q) \int_a^b \varphi_k(x)^2 dx \\ 1875 \end{aligned}$$

1876 where we have used that $I_k(q) = I_{1,k}(q) + I_{2,k}(q)$. Thus, from (5.5) we get

$$1877 \quad |I_k(q)| \leq C \mathcal{M}_k(F, \omega).$$

1878 Using (6.18) or (6.19) and the previous inequality we obtain

$$1879 \quad |I_{j,k}(q)| \leq C \mathcal{M}_k(F, \omega), \quad \forall j \in \{1, 2\}.$$

1880 Thus,

$$1881 \quad \max \{ |I_k(q)|, |I_{1,k}(q)|, |I_{2,k}(q)| \} \leq C \mathcal{M}_k(F, \omega).$$

1882 This implies that approximate controllability holds if and only if

$$1883 \quad \max \{ |I_k(q)|, |I_{1,k}(q)|, |I_{2,k}(q)| \} \neq 0, \quad \forall k \geq 1$$

1884 and in this case

$$1885 \quad T_0(X_{-\infty}) = \limsup_{k \rightarrow +\infty} \frac{-\ln \max \{ |I_k(q)|, |I_{1,k}(q)|, |I_{2,k}(q)| \}}{\nu_k}.$$

1886 Thus we recover the result proved in [7] and extend it to a general Sturm-Liouville
1887 operator.

1888 • Let us finally consider the setting studied in [25].

1889 PROPOSITION 6.6. *Assume that there exists an open set $\omega_0 \subset \omega$ such that $q(x) \geq$
1890 $q_0 > 0$ for almost every $x \in \omega_0$. Then, system (6.5) is null controllable in any time
1891 $T > 0$.*

1892 Even though this result is already known from [25], we provide here a proof without
 1893 Carleman estimates. The same result holds if there exists an open set $\omega_0 \subset \omega$ such
 1894 that $q(x) \leq q_0 < 0$ for almost every $x \in \omega_0$.

1895 *Proof.* Here we consider the minimal time characterization given by Proposi-
 1896 tion 6.3 and prove that

$$1897 \quad I_k(q)^2 \|\varphi_k\|_\omega^2 + \|\psi_k\|_\omega^2 \geq r_k \quad \text{with} \quad \limsup_{k \rightarrow +\infty} \frac{-\ln r_k}{2\nu_k} = 0.$$

1898 As we seek for a lower bound, due to our assumptions, we can restrict ω to an interval
 1899 (a, b) such that $q(x) \geq q_0 > 0$ for almost every $x \in \omega$. We now cut ω into the following
 1900 pieces

- 1901 • $\omega_1 = (a, a + \ell)$ with ℓ sufficiently small (to be determined later on);
- 1902 • $\omega_2 = (b - \ell, b)$ with ℓ sufficiently small (to be determined later on);
- 1903 • $\tilde{\omega} = \omega_1 \cup \omega_2$;
- 1904 • $\mathfrak{C}_0 = [\frac{a+b}{2} - \frac{b-a}{6}, \frac{a+b}{2} + \frac{b-a}{6}]$;
- 1905 • $\mathfrak{C} = [a + \ell, b - \ell]$.

1906 This procedure is summarized in Figure 2

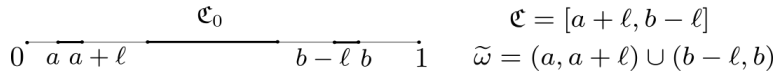


FIGURE 2. Cutting of $\omega = (a, b)$

1907 Notice that for any $k \geq 1$,

$$1908 \quad I_k(q)^2 \|\varphi_k\|_\omega^2 + \|\psi_k\|_\omega^2 \geq I_k(q)^2 \|\varphi_k\|_{\tilde{\omega}}^2 + \|\psi_k\|_{\tilde{\omega}}^2.$$

1909 From (5.5), there exists $\alpha_1 > 0$ depending on γ, c and \mathfrak{C}_0 such that

$$1910 \quad (6.20) \quad \int_{\mathfrak{C}_0} \varphi_k^2(x) dx \geq \alpha_1, \quad \forall k \geq 1.$$

1911 Following closely the proof of item i of Lemma 6.5 with a careful tracking of the
 1912 dependency with respect to l we prove the following lemma.

1913 LEMMA 6.7. *There exists $\alpha_2 > 0$ depending on γ and c such that for any $\ell < \frac{b-a}{3}$,
 1914 there exists $K \in \mathbb{N}^*$ such that for any $k \geq K$, any $F \in L^2(0, 1; \mathbb{R})$ and any u satisfying
 1915 the differential equation*

$$1916 \quad (A - \nu_k)u = F,$$

1917 we have

$$1918 \quad \sqrt{\ell} \left| \int_{\mathfrak{C}} F(x) \varphi_k(x) dx \right| \leq \alpha_2 \sqrt{\nu_k} \|u\|_{\tilde{\omega}} + \alpha_2 \ell \|F\|_{\tilde{\omega}}.$$

1919 To improve the reading, the proof of Lemma 6.7 is postponed at the end of the current
 1920 proof (see page 62).

1921 Let $\alpha_2 > 0$ be the constant given by Lemma 6.7 and assume in all what follows
 1922 that $\ell > 0$ is fixed such that

$$1923 \quad (6.21) \quad \sqrt{\ell} < \min \left\{ \sqrt{\frac{b-a}{3}}, \frac{q_0 \alpha_1}{2\alpha_2 \|q\|_{L^\infty(0,1)}} \right\}.$$

1924 Let $K \in \mathbb{N}^*$ be the index given by Lemma 6.7. In the rest of the proof, we assume
 1925 that $k \geq K$.

1926 \star Let us first consider the case $I_k(q) = 0$. Applying Lemma 6.7, with $u = \psi_k$ and
 1927 $F = -q\varphi_k$ we obtain

$$1928 \quad \sqrt{\ell} \left| \int_{\mathfrak{C}} q(x) \varphi_k^2(x) dx \right| \leq \alpha_2 \sqrt{\nu_k} \|\psi_k\|_{\tilde{\omega}} + \alpha_2 \ell \|q\varphi_k\|_{\tilde{\omega}}.$$

1929 As $\ell < \frac{b-a}{3}$ we have $\mathfrak{C}_0 \subset \mathfrak{C}$ and thus

$$1930 \quad \left| \int_{\mathfrak{C}} q(x) \varphi_k^2(x) dx \right| \geq q_0 \int_{\mathfrak{C}_0} \varphi_k^2(x) dx \geq q_0 \alpha_1.$$

1931 Notice also that, since $\|\varphi_k\|_{(0,1)} = 1$, we have

$$1932 \quad \|q\varphi_k\|_{\tilde{\omega}} \leq \|q\|_{L^\infty(0,1)}.$$

1933 Gathering these estimates and using (6.21) we obtain

$$1934 \quad \begin{aligned} \alpha_2 \sqrt{\nu_k} \|\psi_k\|_{\tilde{\omega}} &\geq \sqrt{\ell} \left| \int_{\mathfrak{C}} q(x) \varphi_k^2(x) dx \right| - \alpha_2 \ell \|q\varphi_k\|_{\tilde{\omega}} \\ 1935 \quad &\geq \sqrt{\ell} \left(q_0 \alpha_1 - \alpha_2 \sqrt{\ell} \|q\|_{L^\infty(0,1)} \right) \geq \sqrt{\ell} \frac{q_0 \alpha_1}{2} \end{aligned}$$

1937 which gives the desired estimate in the case $I_k(q) = 0$.

1938 \star Let us now consider the case $I_k(q) \neq 0$. Let u be the solution of the Cauchy
 1939 problem

$$1940 \quad \begin{cases} (A - \nu_k)u = -q\varphi_k, \\ u(0) = \psi_k(0), \\ u'(0) = \psi_k'(0). \end{cases}$$

1941 The same analysis as in the previous step yields

$$1942 \quad (6.22) \quad \alpha_2 \sqrt{\nu_k} \|u\|_{\tilde{\omega}} \geq \sqrt{\ell} \frac{q_0 \alpha_1}{2}.$$

1943 Notice that

$$1944 \quad \begin{cases} (A - \nu_k)(\psi_k - u) = I_k(q)\varphi_k, \\ (\psi_k - u)(0) = 0, \\ (\psi_k - u)'(0) = 0. \end{cases}$$

1945 From Lemma 5.1, there exists $C > 0$ depending only on γ and c such that

$$1946 \quad \|\psi_k - u\|_{L^\infty(0,1)} \leq \frac{C}{\sqrt{\nu_k}} |I_k(q)|.$$

1947 Then, from (5.5), we deduce that

$$1948 \quad \begin{aligned} \|u\|_{\tilde{\omega}}^2 &\leq 2(\|\psi_k\|_{\tilde{\omega}}^2 + \|\psi_k - u\|_{L^\infty(0,1)}^2) \\ 1949 \quad &\leq 2 \left(\|\psi_k\|_{\tilde{\omega}}^2 + \frac{C}{\nu_k} |I_k(q)|^2 \right) \\ 1950 \quad &\leq C (\|\psi_k\|_{\tilde{\omega}}^2 + |I_k(q)|^2 \|\varphi_k\|_{\tilde{\omega}}^2). \end{aligned}$$

1952 Together with (6.22) this gives the desired estimate in the case $I_k(q) \neq 0$ and ends
 1953 the proof of Proposition 6.6. \square

1954 To complete the proof of Proposition 6.6 let us prove Lemma 6.7.

1955 *Proof (of Lemma 6.7).* From (5.7), there exists $C > 0$ depending on γ and c such
 1956 that

$$1957 \quad (6.23) \quad \|\varphi_k\|_{L^\infty(0,1)} + \frac{1}{\sqrt{\nu_k}} \|\varphi'_k\|_{L^\infty(0,1)} \leq C, \quad \forall k \geq 1.$$

1958 Integrating by parts, we obtain

$$1959 \quad \int_{\mathfrak{e}} F(x) \varphi_k(x) dx = \int_{a+\ell}^{b-\ell} (A - \nu_k) u(x) \varphi_k(x) dx$$

$$1960 \quad = -(\gamma u' \varphi_k)(b-\ell) + (\gamma u' \varphi_k)(a+\ell)$$

$$1961 \quad + (u \gamma \varphi'_k)(b-\ell) - (u \gamma \varphi'_k)(a+\ell).$$

1963 Using (6.23) we obtain

$$1964 \quad \frac{1}{\sqrt{\nu_k}} \left| \int_{\mathfrak{e}} F(x) \varphi_k(x) dx \right| \leq C \|\sqrt{\gamma}\|_{L^\infty} \left(|u(a+\ell)| + \frac{\sqrt{\gamma(a+\ell)}}{\sqrt{\nu_k}} |u'(a+\ell)| \right)$$

$$1965 \quad + C \|\sqrt{\gamma}\|_{L^\infty} \left(|u(b-\ell)| + \frac{\sqrt{\gamma(b-\ell)}}{\sqrt{\nu_k}} |u'(b-\ell)| \right).$$

1967 Let $\lambda_0 > 0$ by given Corollary E.1 (for an interval of length ℓ) and let $K \in \mathbb{N}^*$ be such
 1968 that

$$1969 \quad k \geq K \implies \nu_k \geq \lambda_0.$$

1970 Assume that $k \geq K$. As $a + \ell \in \overline{\omega_1}$ the application of Corollary E.1 yields

$$1971 \quad |u(a+\ell)| + \frac{\sqrt{\gamma(a+\ell)}}{\sqrt{\nu_k}} |u'(a+\ell)| \leq \frac{C}{\sqrt{\ell}} \|u\|_{\omega_1} + \frac{C\sqrt{\ell}}{\sqrt{\nu_k}} \|F\|_{\omega_1}.$$

1972 As $b - \ell \in \overline{\omega_2}$ the application of Corollary E.1 yields

$$1973 \quad |u(b-\ell)| + \frac{\sqrt{\gamma(b-\ell)}}{\sqrt{\nu_k}} |u'(b-\ell)| \leq \frac{C}{\sqrt{\ell}} \|u\|_{\omega_2} + \frac{C\sqrt{\ell}}{\sqrt{\nu_k}} \|F\|_{\omega_2}$$

1974 which proves Lemma 6.7. □

1975 *Dealing with new geometric configurations.*

1976 As proved in the previous paragraph, the obtained characterization of the minimal
 1977 null control time unifies the different known results for system (6.5). It also enables
 1978 to study new geometric configurations for example when ω is not an interval and
 1979 $\text{Supp}(q) \cap \omega = \emptyset$. We provide below an example inspired by [17].

1980 PROPOSITION 6.8. Let $\gamma = 1$ and $c = 0$ (i.e. A is the Dirichlet Laplace operator)
 1981 and let

$$1982 \quad q : x \in (0, 1) \mapsto \left(x - \frac{1}{2}\right) \mathbf{1}_{\left(\frac{1}{4}, \frac{3}{4}\right)}(x).$$

- 1983 *i.* If $\omega \subset \left(\frac{3}{4}, 1\right)$, then approximate controllability for system (6.5) does not hold.
 1984 *ii.* If $\omega = \left(0, \frac{1}{4}\right) \cup \left(\frac{3}{4}, 1\right)$, then system (6.5) is null controllable from $L^2(0, 1; \mathbb{R})$
 1985 in any time $T > 0$.

1986 *Proof.* In this case, we have for any $k \geq 1$,

1987
$$\nu_k = k^2\pi^2, \quad \varphi_k = \sqrt{2}\sin(k\pi\bullet), \quad \tilde{\varphi}_k = \sqrt{2}\cos(k\pi\bullet).$$

1988 The proof of item i can be found in [17, Section 3.3.1] and relies on explicit com-
 1989 putations: due to symmetry it comes that $I_k(q) = 0$ for any $k \geq 1$. This implies
 1990 that

1991
$$\int_0^{\inf(\omega)} q(x)\varphi_k(x)\varphi_k(x)dx = I_k(q) = 0.$$

1992 Let \mathfrak{C} be any other connected component of $\overline{(0,1)\setminus\omega}$. Then $\mathfrak{C} \subset (\frac{3}{4}, 1)$. This means
 1993 that $q = 0$ on \mathfrak{C} which gives

1994
$$\int_{\mathfrak{C}} q(x)\varphi_k(x)\varphi_k(x)dx = \int_{\mathfrak{C}} q(x)\varphi_k(x)\tilde{\varphi}_k(x)dx = 0.$$

1995 Thus,

1996
$$\mathcal{M}_k(q\varphi_k, \omega) = 0, \quad \forall k \geq 1.$$

1997 We now turn to item ii. In this case $\overline{(0,1)\setminus\omega}$ has only one connected component
 1998 which is $[\frac{1}{4}, \frac{3}{4}]$ but the key point is that it does not touch the boundary of $(0, 1)$.
 1999 Approximate controllability in this case was also studied in [17, Section 3.3.1]. Again
 2000 for symmetry reasons we have

2001
$$\int_{\frac{1}{4}}^{\frac{3}{4}} q(x)\varphi_k(x)\varphi_k(x)dx = 0, \quad \forall k \geq 1,$$

2002 but

2003
$$\int_{\frac{1}{4}}^{\frac{3}{4}} q(x)\varphi_k(x)\tilde{\varphi}_k(x)dx = \begin{cases} -\frac{(-1)^{\frac{k-1}{2}}}{2\pi^2k^2}, & \text{if } k \text{ is odd,} \\ -\frac{(-1)^{\frac{k}{2}}}{4\pi k}, & \text{if } k \text{ is even.} \end{cases}$$

2004 This implies that for any $k \geq 1$,

2005
$$\mathcal{M}_k((I_k(q) - q)\varphi_k, \omega) = \begin{cases} \frac{1}{2\pi^2k^2}, & \text{if } k \text{ is odd,} \\ \frac{1}{4\pi k}, & \text{if } k \text{ is even.} \end{cases}$$

2006 Thus, from Proposition 6.4

2007
$$T_0(X_{-\diamond}) = \limsup_{k \rightarrow +\infty} \frac{-\ln \mathcal{M}_k((I_k(q) - q)\varphi_k, \omega)}{\nu_k} = 0. \quad \square$$

2008 7. Some extensions.

2009 7.1. Dealing with complex valued eigenvalues.

2010 In this section we allow the eigenvalues in Λ to be complex valued with a dominant
 2011 real part (see the precise assumptions below). The resolution of block moment prob-
 2012 lems, the estimate of the cost of a block and thus the minimal null control time are
 2013 obtained with really few adjustments that we detail in this section. However, as it is

2014 already true for scalar control operators, the dependency of the cost of controllability
 2015 with respect to the final time T is lost. Indeed it follows from Proposition A.2 which
 2016 deeply uses real analysis. Thus in the case of complex valued sequences there is no
 2017 need to consider the adaptation of the classes $\mathcal{L}(p, \varrho, a, \bar{N})$ or $\mathcal{L}(p, \varrho, a, \bar{N}, a', \tilde{N})$ which
 2018 are mainly introduced in this article to obtain sharper bounds on the cost of control-
 2019 lability. We restrict ourselves to sequences of eigenvalues Λ in the class $\mathcal{L}_{\mathbb{C}}(\delta, p, \varrho, \mathcal{N})$
 2020 which for $\delta, \varrho > 0, p \in \mathbb{N}^*$ and $\mathcal{N} : \mathbb{R}^+ \rightarrow \mathbb{R}$ is the class of sequences $\Lambda \in \mathbb{C}^{\mathbb{N}}$ satisfying

- Parabolicity condition:

$$\Re \lambda \geq \delta |\lambda|, \quad \forall \lambda \in \Lambda.$$

- Asymptotic behavior: for any $\varepsilon > 0$, we have

$$\sum_{\substack{\lambda \in \Lambda \\ |\lambda| > \mathcal{N}(\varepsilon)}} \frac{1}{|\lambda|} \leq \varepsilon.$$

- Weak gap condition with parameters $\varrho > 0$ and $p \in \mathbb{N}^*$:

$$\#\Lambda \cap ([\mu, \mu + \varrho] + i\mathbb{R}) \leq p, \quad \forall \mu > 0.$$

In that case, a grouping $(G_k)_k$ should satisfy

$$\Lambda = \bigcup_{k \geq 1} G_k, \quad \#G_k \leq p, \quad \text{diam}(G_k) < \varrho, \quad \inf(\Re G_{k+1}) - \sup(\Re G_k) > r.$$

2021 In this setting replacing the auxiliary moment problem (2.1) by

$$2022 \quad \begin{cases} \int_0^T v(t) \frac{(-t)^l}{l!} e^{-\bar{\lambda}_j t} dt = \Omega_j^l, & \forall j \in \llbracket 1, g \rrbracket, \forall l \in \llbracket 0, \alpha_j - 1 \rrbracket, \\ \int_0^T v(t) t^l e^{-\bar{\lambda} t} dt = 0, & \forall \lambda \in \Lambda \setminus G, \forall l \in \llbracket 0, \eta - 1 \rrbracket, \end{cases}$$

2023 we obtain that the resolution of block moment problems stated in Theorem 1.2 still
 2024 holds. The minimal null control time given in Theorem 1.3 is replaced by

$$2025 \quad T_0(y_0) = \limsup_{k \rightarrow +\infty} \frac{\ln \mathcal{C}(G_k, y_0)}{2 \min \Re G_k}$$

2026 with $\mathcal{C}(G, z)$ still defined in (1.41) and the function F is now defined by

$$2027 \quad F : \Omega = (\Omega_1^0, \dots, \Omega_1^{\alpha_1 - 1}, \dots, \Omega_g^0, \dots, \Omega_g^{\alpha_g - 1}) \in U^{|\alpha|} \mapsto \sum_{l=1}^{|\alpha|} \left\| \Omega \left[\bar{\lambda}^{\bullet} \right] \right\|_U^2.$$

2028 In the course of the proof, one should replace in the estimates λ by $\Re \lambda$ and the use
 2029 of Lagrange theorem [10, Proposition 7.14] by an inequality proved by Jensen in [27]
 2030 (and recalled in [10, Proposition 6.1]).

2031 Similarly, replacing λ by $\bar{\lambda}$ the more explicit formulas for $\mathcal{C}(G_k, y_0)$ provided by
 2032 Theorems 1.8 and 1.10 also holds.

2033 7.2. Weakening the Fattorini-Hautus test.

2034 When studying null controllability from a closed strict subspace Y_0 of $X_{-\infty}$, the
 2035 Fattorini-Hautus test (1.28) can be a too strong requirement. Let us underline that
 2036 this condition (1.28) is only used in this article for two purposes:

- 2037 1. to prove that the set $\mathcal{O}(\lambda, z)$ defining the constraints is not empty (see Propo-
 2038 sition 2.3);
 2039 2. to prove that the matrices M defined either by (1.47) or (1.51) are invertible
 2040 (see Propositions 1.7 and 1.9).

2041 Let Y_0 be a closed subspace of $X_{-\diamond}$. Following for instance [15, Proposition 1.17],
 2042 it can be proved that if every assumption of (H) except (1.28) hold, then a necessary
 2043 condition for approximate controllability from Y_0 can be expressed as follows: for any
 2044 $\lambda \in \Lambda$, for any $\phi \in E_\lambda$ and for any $l \in \llbracket 1, \alpha_\lambda \rrbracket$,

$$2045 \quad (7.1) \quad \left(\mathcal{B}^*(\mathcal{A}^* - \lambda)^{\alpha_\lambda - r} \phi = 0, \forall r \in \llbracket 1, l \rrbracket \right) \implies \left(P_{Y_0}^*(\mathcal{A}^* - \lambda)^{\alpha_\lambda - r} \phi = 0, \forall r \in \llbracket 1, l \rrbracket \right).$$

2046 Notice that when $\dim U = 1$, this condition is equivalent to the condition formulated
 2047 in [10, Section 6.2].

2048 Let us prove that the results of this article hold (for null controllability from Y_0)
 2049 under assumption (7.1) instead of (1.28), at least in the case when every eigenvalue
 2050 of \mathcal{A}^* is either geometrically simple or semi-simple. Let $\lambda \in \Lambda$ be such that

$$2051 \quad \text{Ker}(\mathcal{A}^* - \lambda) \cap \text{Ker } \mathcal{B}^* \neq \{0\}.$$

2052 • If $E_\lambda \subset \text{Ker } \mathcal{B}^*$ then from (7.1) it follows that $E_\lambda \subset \text{Ker } P_{Y_0}^*$. Then, the equations
 2053 associated with λ in the moment problem (1.30) are automatically satisfied. We can
 2054 then simply forget about the eigenvalue λ in the analysis, which amounts to replace
 2055 Λ by $\Lambda \setminus \{\lambda\}$ in our study.

2056 • Otherwise we distinguish the two cases $\alpha_\lambda = 1$ and $\gamma_\lambda = 1$.
 2057 i. Assume that $\alpha_\lambda = 1$ and let I be a (non-empty) set of indices such that the
 2058 family $(\mathcal{B}^* \phi_{\lambda, i})_{i \in I}$ is a basis of $\mathcal{B}^* E_\lambda$. Then, the moment equations associated
 2059 to λ in the moment problem (1.30) can be reduced to

$$2060 \quad \int_0^T e^{-\lambda(T-t)} \langle u(t), \mathcal{B}^* \phi_{\lambda, i} \rangle_U dt = - \left\langle y_0, e^{-T\mathcal{A}^*} \phi_{\lambda, i} \right\rangle_{-\diamond, \diamond}, \quad \forall i \in I.$$

2061 Indeed, from (7.1), it comes that the moment equations associated with $\phi_{\lambda, j}$
 2062 for $j \in \llbracket 1, \gamma_\lambda \rrbracket \setminus I$ are automatically satisfied and can be forgotten. This leads
 2063 to a decrease of the geometric multiplicity replacing γ_λ by $|I|$.

2064 ii. Assume that $\gamma_\lambda = 1$ and let us consider a basis of E_λ formed by a Jordan
 2065 chain $\phi_\lambda^0, \dots, \phi_\lambda^{\alpha_\lambda - 1}$.

2066 Let $j \in \llbracket 1, \alpha_\lambda \rrbracket$ be the first index such that $\mathcal{B}^* \phi_\lambda^j \neq 0$. From (7.1), it comes
 2067 that

$$2068 \quad P_{Y_0}^* \phi_\lambda^0 = \dots = P_{Y_0}^* \phi_\lambda^{j-1} = 0$$

2069 and thus the moment equations associated to λ in the moment problem (1.30)
 2070 can be reduced to

$$2071 \quad \int_0^T \left\langle u(t), e^{-(T-t)\mathcal{A}^*} \phi \right\rangle_U dt = - \left\langle y_0, e^{-T\mathcal{A}^*} \phi \right\rangle_{-\diamond, \diamond}, \quad \forall \phi \in (\mathcal{A}^* - \lambda)^j E_\lambda.$$

2072 This leads to a decrease of the algebraic multiplicity replacing α_λ by $\alpha_\lambda - j$
 2073 and shifting the associated Jordan chain.

2074 This proves the claim.

2075 Finally let us notice that, if on a given example there exists $\lambda \in \Lambda$ with $\alpha_\lambda \geq 2$,
 2076 $\gamma_\lambda \geq 2$,

$$2077 \quad \text{Ker}(\mathcal{A}^* - \lambda) \cap \text{Ker} \mathcal{B}^* \neq \{0\},$$

2078 then under assumption (7.1) one can iterate a finite number of steps of the form *i.* or
 2079 *ii.* to reduce the moment equations associated to λ in the moment problem (1.30) to
 2080 a solvable moment problem.

2081 **Appendix A. Some refinements in the case of scalar controls.**

2082 In [10], the block moment method was introduced to solve null controllability
 2083 problems with scalar controls ($U = \mathbb{R}$). With respect to block moment problems, the
 2084 main result of this paper is [10, Theorem 4.1]. In this work there were no assumptions
 2085 on the counting function. The spectrum Λ was only assumed to satisfy $\sum_{\lambda \in \Lambda} \frac{1}{\lambda} <$
 2086 $+\infty$. More precisely, for $p \in \mathbb{N}^*$, $\varrho > 0$ and $\mathcal{N} : (0, +\infty) \rightarrow \mathbb{R}$ it is defined in [10,
 2087 Definition 2.1] the following classes

$$2088 \quad (\text{A.1}) \quad \mathcal{L}(p, \varrho, \mathcal{N}) = \left\{ \Lambda \in (0, +\infty)^{\mathbb{N}}; \Lambda \text{ satisfies (1.21) and } \sum_{\substack{\lambda \in \Lambda \\ \lambda \geq \mathcal{N}(\varepsilon)}} \frac{1}{\lambda} < \varepsilon \right\}.$$

2089 Using the slightly more restrictive condition (1.22) (or (1.24)) we can adapt the res-
 2090 olution of scalar block moment problems to obtain better estimates on the cost of
 2091 this resolution. In particular, this allows to explicit the dependency of the various
 2092 estimates with respect to the variable T (see Remark 1.7 for possible applications of
 2093 such estimates). Namely, we obtain the following result.

2094 **THEOREM A.1.** *Let $p \in \mathbb{N}^*$, $r, \varrho, \bar{N}, \tilde{N} > 0$, $a \in (0, 1)$, $a' \in [0, a)$ and $\mathcal{N} :$
 2095 $(0, +\infty) \rightarrow \mathbb{R}$. Assume that*

$$2096 \quad \Lambda \in \mathcal{L}(p, \varrho, \mathcal{N}) \cup \mathcal{L}(p, \varrho, a, \bar{N}) \cup \mathcal{L}(p, \varrho, a, \bar{N}, a', \tilde{N})$$

2097 and let $(G_k)_{k \geq 1} \in \mathcal{G}(\Lambda, p, r, \varrho)$ be an associated grouping. Recall that these classes are
 2098 defined in (A.1), (1.23) and (1.25). Let $\eta \in \mathbb{N}^*$ and $T \in (0, +\infty)$.

2099 For any $G = \{\lambda_1, \dots, \lambda_g\} \in (G_k)_k$, for any multi-index $\alpha \in \mathbb{N}^g$ with $|\alpha|_\infty \leq \eta$
 2100 and any $\omega \in \mathbb{R}^{|\alpha|}$ there exists $v_G \in L^2(0, T; \mathbb{R})$ satisfying

$$2101 \quad (\text{A.2a}) \quad \int_0^T v_G(t) \frac{(-t)^l}{l!} e^{-\lambda_j t} dt = \omega_j^l, \quad \forall j \in \llbracket 1, g \rrbracket, \forall l \in \llbracket 0, \alpha_j - 1 \rrbracket,$$

$$2102 \quad (\text{A.2b}) \quad \int_0^T v_G(t) \frac{(-t)^l}{l!} e^{-\lambda t} dt = 0, \quad \forall \lambda \in \Lambda \setminus G, \forall l \in \llbracket 0, \eta - 1 \rrbracket,$$

2103 and the bound

$$2104 \quad (\text{A.3}) \quad \|v_G\|_{L^2(0, T; \mathbb{R})} \leq \mathcal{E}(\lambda_1) \mathcal{K}(T) \max_{\substack{\mu \in \mathbb{N}^g \\ \mu \leq \alpha}} \left| \omega \left[\lambda_{\bullet}^{(\mu)} \right] \right|$$

2105 where the functions \mathcal{E} and \mathcal{K} are such that

2106 *i.* for any $T \in (0, +\infty)$, for any $\varepsilon > 0$, there exists $C > 0$ depending only on ε ,
 2107 T , p , r , ϱ , η and \mathcal{N} such that for any $\Lambda \in \mathcal{L}(p, \varrho, \mathcal{N})$ we have

$$2108 \quad \mathcal{E}(\lambda) \leq C \exp(\varepsilon \lambda), \quad \forall \lambda \in \Lambda;$$

2110 *ii. there exists $C > 0$ depending only on p, r, ϱ, η, a and \bar{N} such that for any*
 2111 *$\Lambda \in \mathcal{L}(p, \varrho, a, \bar{N})$ we have*

$$2112 \quad \mathcal{E}(\lambda) \leq C \exp(C\lambda^a(1 + \log(\lambda))), \quad \forall \lambda \in \Lambda,$$

2113

$$2114 \quad \mathcal{K}(T) \leq C \exp\left(\frac{C}{T^{1-\frac{a}{a'}}}\right), \quad \forall T \in (0, +\infty);$$

2115 *iii. there exists $C > 0$ depending only on $p, r, \varrho, \eta, a, \bar{N}, a'$ and \tilde{N} such that for*
 2116 *any $\Lambda \in \mathcal{L}(p, \varrho, a, \bar{N}, a', \tilde{N})$ we have*

$$2117 \quad \mathcal{E}(\lambda) \leq C \exp(C\lambda^a), \quad \forall \lambda \in \Lambda,$$

2118

$$2119 \quad \mathcal{K}(T) \leq C \exp\left(\frac{C}{T^{1-\frac{a}{a'}}}\right), \quad \forall T \in (0, +\infty).$$

2120 *Moreover, up to the factor $\mathcal{E}(\lambda_1)$, the estimate (A.3) is sharp: there exists a constant*
 2121 *$C_{p,\eta,\min \Lambda} > 0$ such that any solution v_G of (A.2a) satisfy*

$$2122 \quad (\text{A.4}) \quad \|v_G\|_{L^2(0,T;\mathbb{R})} \geq C_{p,\eta,\min \Lambda} \max_{\substack{\mu \in \mathbb{N}^g \\ \mu \leq \alpha}} \left| \omega \left[\lambda_{\bullet}^{(\mu)} \right] \right|.$$

2123 The statement **i.** is exactly the one given in [10, Theorem 4.1]. Let us detail the
 2124 necessary adjustments to obtain Theorem **A.1 ii.** and Theorem **A.1 iii.**

2125 • Dependency with respect to the variable T

2126 One key point of the strategy developed in [10] is to solve the scalar block moment
 2127 problem in infinite time horizon (see [10, Proposition 4.5]) and then use a uniform
 2128 bound on the inverse of the restriction map

$$2129 \quad R_{\Lambda,T} : f \in A(\Lambda, +\infty) \mapsto f|_{(0,T)} \in A(\Lambda, T),$$

2130 where for any $T \in (0, +\infty]$,

$$2131 \quad A(\Lambda, T) := \overline{\text{Span} \{t \mapsto e^{-\lambda t} ; \lambda \in \Lambda\}}^{L^2(0,T;\mathbb{C})}.$$

2132 This uniform bound (see [10, Proposition 2.9]) was proved by contradiction and thus
 2133 its dependency with respect to T was not explicit.

2134 Actually, one can use instead the following result (which at that time was not
 2135 known to the authors of [10]).

2136 **PROPOSITION A.2.** *Let $a \in (0, 1)$, $\varrho, \bar{N} > 0$ and $\Lambda \in (0, +\infty)^{\mathbb{N}}$ be a family whose*
 2137 *counting function satisfies (1.22). There exists a constant $C_{a,\bar{N}} > 0$ depending only*
 2138 *on a and \bar{N} such that, for any $T > 0$,*

$$2139 \quad \|f\|_{L^2(0,+\infty)} \leq C_{a,\bar{N}} \exp\left(\frac{C_{a,\bar{N}}}{T^{1-\frac{a}{a'}}}\right) \|f\|_{L^2(0,T)}, \quad \forall f \in A(\Lambda, \infty).$$

2140 This general result is a consequence of the study of Remez-type inequalities in Müntz
 2141 spaces that can be found for instance in [13, 14]. A detailed proof is proposed in [16,
 2142 Theorem IV.1.18].

2143 Then, following [10, Section 4], but using Proposition **A.2** instead of [10, Propo-
 2144 sition 2.9] leads to the bounds on $\mathcal{K}(T)$ given in cases **ii.** and **iii.**

2145 • Bounds on $\mathcal{E}(\lambda)$.
 2146 In the estimate (A.3), the term $\mathcal{E}(\lambda)$ comes from the orthogonality condition given
 2147 by the equations (A.2b).
 2148 In the proof of [10, Theorem 4.1], the bound of case i. is given by [10, Proposition
 2149 2.8]. Following exactly the proof given in [10, Section 2.1.6] but using [16, Proposition
 2150 IV.1.14 and Corollary IV.1.16] (see also [16, Theorem IV.1.10] for an explicit formula
 2151 for the function \mathcal{E}) instead of [10, Proposition 2.8] directly yield the bounds on $\mathcal{E}(\lambda)$
 2152 given in Theorem A.1 cases ii. and iii..

2153 Appendix B. An auxiliary optimization argument.

2154 LEMMA B.1. Let Y be a closed subspace of $X_{-\diamond}$. Let $g \in \mathbb{N}^*$ and $\psi_1, \dots, \psi_g \in$
 2155 $P_Y^* X_{\diamond}^*$. For any $y \in Y$, let

$$2156 \quad \xi_y = \begin{pmatrix} \langle y, \psi_1 \rangle_{-\diamond, \diamond} \\ \vdots \\ \langle y, \psi_g \rangle_{-\diamond, \diamond} \end{pmatrix}.$$

2157 Then, for any positive semi-definite symmetric square matrix $M \in \mathcal{M}_g(\mathbb{R})$, we have

$$2158 \quad (\text{B.1}) \quad \sup_{\substack{y \in Y \\ \|y\|_{-\diamond} = 1}} \langle M \xi_y, \xi_y \rangle = \rho(\mathbf{G}_\psi M)$$

2159 with $\mathbf{G}_\psi = \text{Gram}_{X_{\diamond}^*}(\psi_1, \dots, \psi_g)$.

In the course of the proof we will use that there exists an isometric bijection
 $I : X_{-\diamond} \mapsto X_{\diamond}^*$ such that

$$\langle y, \varphi \rangle_{-\diamond, \diamond} = (Iy, \varphi)_{\diamond^*}, \quad \forall y \in X_{-\diamond}, \forall \varphi \in X_{\diamond}^*.$$

Note that it satisfies

$$(Iy, \varphi)_{\diamond^*} = (y, I^{-1}\varphi)_{-\diamond}, \quad \forall y \in X_{-\diamond}, \forall \varphi \in X_{\diamond}^*.$$

Proof. Let S be the value of the supremum in the left-hand side of (B.1). By
 assumption on the $(\psi_i)_i$, we first observe that the supremum can be taken on the
 whole space $X_{-\diamond}$ instead of Y without changing its value. Then, for any $1 \leq i \leq g$,
 we have

$$\langle y, \psi_i \rangle_{-\diamond, \diamond} = (y, I^{-1}\psi_i)_{-\diamond},$$

and therefore the value of S does not change if we take the supremum over the set

$$\tilde{\Psi} = \text{Span}(\tilde{\psi}_1, \dots, \tilde{\psi}_g) \subset X_{-\diamond},$$

2160 with

$$2161 \quad (\text{B.2}) \quad \tilde{\psi}_i = I^{-1}\psi_i.$$

We write any element $y \in \tilde{\Psi}$ as follows $y = \sum_{i=1}^g x_i \tilde{\psi}_i$, with $x = (x_j)_{j \in \llbracket 1, g \rrbracket} \in \mathbb{R}^g$
 so that we can compute

$$\left(y, \tilde{\psi}_i \right)_{-\diamond} = \sum_{j=1}^g x_j \left(\tilde{\psi}_i, \tilde{\psi}_j \right)_{-\diamond} = (\mathbf{G}_{\tilde{\Psi}} x)_i, \quad \forall i \in \llbracket 1, g \rrbracket,$$

$$(y, y)_{-\diamond} = \sum_{i=1}^g \sum_{j=1}^g x_i x_j \left(\tilde{\psi}_i, \tilde{\psi}_j \right)_{-\diamond} = \left\langle \mathbf{G}_{\tilde{\psi}} x, x \right\rangle,$$

where $\mathbf{G}_{\tilde{\psi}}$ is the Gram matrix in $X_{-\diamond}$ of the family $\{\tilde{\psi}_1, \dots, \tilde{\psi}_g\}$. Using that I is an isometry from $X_{-\diamond}$ onto X_{\diamond}^* it actually appears that

$$\mathbf{G}_{\tilde{\psi}} = \mathbf{G}_{\psi}.$$

Finally, we have proved that

$$\xi_y = \mathbf{G}_{\psi} x, \quad \text{and} \quad \|y\|_{-\diamond}^2 = \langle \mathbf{G}_{\psi} x, x \rangle.$$

The supremum we are looking for thus reads

$$S = \sup_{\substack{x \in \mathbb{R}^g \\ \langle \mathbf{G}_{\psi} x, x \rangle = 1}} \langle M \mathbf{G}_{\psi} x, \mathbf{G}_{\psi} x \rangle.$$

- By compactness, we know that this supremum is actually achieved at some point $x_0 \in \mathbb{R}^g$, that is

$$\langle M \mathbf{G}_{\psi} x_0, \mathbf{G}_{\psi} x_0 \rangle = S, \quad \text{and} \quad \langle \mathbf{G}_{\psi} x_0, x_0 \rangle = 1.$$

2162 The Lagrange multiplier theorem gives that there exists $\lambda \in \mathbb{R}$ such that

$$2163 \quad (\text{B.3}) \quad \langle M \mathbf{G}_{\psi} x_0, \mathbf{G}_{\psi} h \rangle = \lambda \langle \mathbf{G}_{\psi} x_0, h \rangle, \quad \forall h \in \mathbb{R}^g.$$

We get

$$\mathbf{G}_{\psi} M \mathbf{G}_{\psi} x_0 = \lambda \mathbf{G}_{\psi} x_0,$$

and since $\mathbf{G}_{\psi} x_0 \neq 0$ (we recall that $\langle \mathbf{G}_{\psi} x_0, x_0 \rangle = 1$), we deduce that λ is an eigenvalue of $\mathbf{G}_{\psi} M$ and therefore

$$\lambda \leq \rho(\mathbf{G}_{\psi} M).$$

Moreover, taking $h = x_0$ in (B.3), we get

$$\langle M \mathbf{G}_{\psi} x_0, \mathbf{G}_{\psi} x_0 \rangle = \lambda \langle \mathbf{G}_{\psi} x_0, x_0 \rangle = \lambda,$$

and thus $\lambda = S$. We have thus proved that

$$S \leq \rho(\mathbf{G}_{\psi} M).$$

- If $\rho(\mathbf{G}_{\psi} M) = 0$, the claim is proved. If not, we set

$$\lambda = \rho(\mathbf{G}_{\psi} M) = \rho(M \mathbf{G}_{\psi}) = \rho\left(\mathbf{G}_{\psi}^{\frac{1}{2}} M \mathbf{G}_{\psi}^{\frac{1}{2}}\right),$$

which is positive and which is an eigenvalue of the three matrices above. In particular, there exists $x_0 \in \mathbb{R}^g \setminus \{0\}$ such that

$$M \mathbf{G}_{\psi} x_0 = \lambda x_0.$$

Taking the inner product with $\mathbf{G}_{\psi} x_0$ we obtain

$$\langle M \mathbf{G}_{\psi} x_0, \mathbf{G}_{\psi} x_0 \rangle = \lambda \langle x_0, \mathbf{G}_{\psi} x_0 \rangle,$$

and since $\langle x_0, \mathbf{G}_{\psi} x_0 \rangle = \left\| \mathbf{G}_{\psi}^{\frac{1}{2}} x_0 \right\|^2$ cannot be equal to zero, we deduce that

$$\lambda \leq S,$$

2164 and the proof is complete. □

2165 **Appendix C. Solving general block moment problems.**

2166 As this paper is oriented towards control theory we do not deal with the most
 2167 general block moment problems. Indeed, in Theorem 1.2, the considered block mo-
 2168 ment problems have a specific right-hand side which is a linear form. This formalism
 2169 is chosen in order to avoid exhibiting a particular basis of the generalized eigenspaces.
 2170 The price to pay is this restriction on the considered right-hand sides. However the
 2171 proofs detailed in Sections 2 and 4 directly lead to the following more general results.

2172 The study with a group composed of geometrically simple eigenvalues (see Sec-
 2173 tions 4.1 and 4.2) leads to the following theorem.

2174 **THEOREM C.1.** *Let $p \in \mathbb{N}^*$, $r, \varrho, \bar{N}, \tilde{N} > 0$, $a \in (0, 1)$, $a' \in [0, a)$ and $\mathcal{N} :$
 2175 $(0, +\infty) \rightarrow \mathbb{R}$. Assume that*

$$2176 \quad \Lambda \in \mathcal{L}(p, \varrho, \mathcal{N}) \cup \mathcal{L}(p, \varrho, a, \bar{N}) \cup \mathcal{L}(p, \varrho, a, \bar{N}, a', \tilde{N})$$

2177 and let $(G_k)_{k \geq 1} \in \mathcal{G}(\Lambda, p, r, \varrho)$ be an associated grouping. Recall that these classes are
 2178 defined in (A.1), (1.23) and (1.25). Let $\eta \in \mathbb{N}^*$ and $T \in (0, +\infty)$.

2179 For any $G = \{\lambda_1, \dots, \lambda_g\} \in (G_k)_k$, for any multi-index $\alpha \in \mathbb{N}^g$ with $|\alpha|_\infty \leq \eta$,
 2180 any $\omega \in \mathbb{R}^{|\alpha|}$ and any $b \in U^{|\alpha|}$ with

$$2181 \quad b_j^0 \neq 0, \quad \forall j \in \llbracket 1, g \rrbracket,$$

2182 there exists $v \in L^2(0, T; U)$ satisfying

$$2183 \quad (\text{C.1a}) \quad \int_0^T \left\langle v(t), (e_t b) \left[\lambda_j^{(l+1)} \right] \right\rangle_U dt = \omega_j^l, \quad \forall j \in \llbracket 1, g \rrbracket, \forall l \in \llbracket 0, \alpha_j - 1 \rrbracket,$$

$$2184 \quad (\text{C.1b}) \quad \int_0^T v(t) t^l e^{-\lambda t} dt = 0, \quad \forall \lambda \in \Lambda \setminus G, \forall l \in \llbracket 0, \eta - 1 \rrbracket.$$

2185 Moreover, we have the following estimate

$$2187 \quad \|v\|_{L^2(0, T; U)}^2 \leq \mathcal{E}(\lambda_1) \mathcal{K}(T) \langle M^{-1} \xi, \xi \rangle, \quad \text{where } \xi := \begin{pmatrix} \omega \left[\lambda_{\bullet}^{(\mu^1)} \right] \\ \vdots \\ \omega \left[\lambda_{\bullet}^{(\mu^{|\alpha|})} \right] \end{pmatrix},$$

2188 the sequence $(\mu^p)_{p \in \llbracket 0, |\alpha_k| \rrbracket}$ is defined in (1.37), the associated matrix M is defined
 2189 in (1.47) and the functions \mathcal{E} and \mathcal{K} satisfy the bounds given in Theorem A.1.

2190 Moreover, any $v \in L^2(0, T; U)$ such that (C.1a) holds satisfy

$$2191 \quad \|v\|_{L^2(0, T; U)}^2 \geq C_{p, \eta, \min \Lambda} \langle M^{-1} \xi, \xi \rangle$$

2192 for some $C_{p, \eta, \min \Lambda} > 0$.

2193 **REMARK C.1.** As detailed in Remark 1.10, when the eigenvalues in G are also
 2194 algebraically simple, i.e. $\alpha_\lambda = \gamma_\lambda = 1$ for any $\lambda \in G$, the expression of ξ reduces to

$$2195 \quad \xi := \begin{pmatrix} \omega[\lambda_1] \\ \vdots \\ \omega[\lambda_1, \dots, \lambda_g] \end{pmatrix},$$

2196 and the expression of M reduces to the one given in (1.49).

2197 The study with a group composed of semi-simple eigenvalues (see Section 4.3)
 2198 leads to the following theorem.

2199 **THEOREM C.2.** *Let $p \in \mathbb{N}^*$, $r, \varrho, \bar{N}, \tilde{N} > 0$, $a \in (0, 1)$, $a' \in [0, a)$ and $\mathcal{N} :$
 2200 $(0, +\infty) \rightarrow \mathbb{R}$. Assume that*

$$2201 \quad \Lambda \in \mathcal{L}(p, \varrho, \mathcal{N}) \cup \mathcal{L}(p, \varrho, a, \bar{N}) \cup \mathcal{L}(p, \varrho, a, \bar{N}, a', \tilde{N})$$

2202 and let $(G_k)_{k \geq 1} \in \mathcal{G}(\Lambda, p, r, \varrho)$ be an associated grouping. Recall that these classes are
 2203 defined in (A.1), (1.23) and (1.25). Let $\gamma_1, \dots, \gamma_g \in \mathbb{N}^*$ and $\gamma_G = \gamma_1 + \dots + \gamma_g$. Let
 2204 $\eta \in \mathbb{N}^*$ and $T \in (0, +\infty)$.

2205 For any $G = \{\lambda_1, \dots, \lambda_g\} \in (G_k)_k$, for any $(\omega_{j,i})_{j \in \llbracket 1, g \rrbracket, i \in \llbracket 1, \gamma_j \rrbracket} \in U^{\gamma_G}$ and any
 2206 $(b_{j,i})_{j \in \llbracket 1, g \rrbracket, i \in \llbracket 1, \gamma_j \rrbracket} \in U^{\gamma_G}$ such that $b_{j,1}, \dots, b_{j,\gamma_j}$ are linearly independent for every
 2207 $j \in \llbracket 1, g \rrbracket$, there exists $v \in L^2(0, T; U)$ satisfying

$$2208 \quad (\text{C.2a}) \quad \int_0^T \langle v(t), e^{-\lambda_j t} b_{j,i} \rangle_U dt = \omega_{j,i}, \quad \forall j \in \llbracket 1, g \rrbracket, \forall i \in \llbracket 1, \gamma_j \rrbracket,$$

$$2209 \quad (\text{C.2b}) \quad \int_0^T v(t) t^l e^{-\lambda t} dt = 0, \quad \forall \lambda \in \Lambda \setminus G, \forall l \in \llbracket 0, \eta - 1 \rrbracket.$$

2211 Moreover, we have the following estimate

$$2212 \quad \|v\|_{L^2(0, T; U)}^2 \leq \mathcal{E}(\lambda_1) \mathcal{K}(T) \langle M^{-1} \xi, \xi \rangle,$$

2213 where $\xi \in \mathbb{R}^{\gamma_G}$ is defined by blocks with

$$2214 \quad \xi_j := \begin{pmatrix} \omega_{j,1} \\ \vdots \\ \omega_{j,g} \end{pmatrix},$$

2215 the matrix M is defined in (1.51) and the functions \mathcal{E} and \mathcal{K} satisfy the bounds given
 2216 in Theorem A.1.

2217 Moreover, any $v \in L^2(0, T; U)$ such that (C.2a) holds satisfy

$$2218 \quad \|v\|_{L^2(0, T; U)}^2 \geq C_{p, \min \Lambda} \langle M^{-1} \xi, \xi \rangle$$

2219 for some $C_{p, \min \Lambda} > 0$.

2220 **Appendix D. Post-processing formulas.**

2221 The minimal null control time given in Theorem 1.3, together with the compu-
 2222 tation of the contribution of each group given in Theorems 1.8 and 1.10, allow to
 2223 answer the question of minimal null control time for a wide variety of one dimensional
 2224 parabolic control problems. However, for a given problem, the precise estimate of the
 2225 quantity of interest $\langle M^{-1} \xi, \xi \rangle$ can remain a tricky question.

2226 There is no normalization condition on the eigenvectors and no uniqueness of
 2227 the considered Jordan chains. Thus, it happens that there are choices for which the
 2228 quantity of interest $\langle M^{-1} \xi, \xi \rangle$ is easier to compute (see for instance Remark 1.9). We
 2229 gather here some results that are use in Sections 5 and 6 to estimate such quantities.

2230 We will make an intensive use of the following reformulation. Let $n \in \mathbb{N}^*$ and let
 2231 $T, M \in \text{GL}_n(\mathbb{R})$. For any $\xi \in \mathbb{R}^n$, let $\tilde{\xi} := T\xi$. Then,

$$2232 \quad (\text{D.1}) \quad \langle M^{-1} \xi, \xi \rangle = \langle M^{-1} T^{-1} \tilde{\xi}, T^{-1} \tilde{\xi} \rangle = \langle \tilde{M}^{-1} \tilde{\xi}, \tilde{\xi} \rangle$$

2233 where

$$2234 \quad (D.2) \quad \widetilde{M} := TM^tT.$$

2235 As the matrix M is a sum of Gram matrices we will also use the following lemma.

2236 **LEMMA D.1.** *Let X be an Hilbert space. Let $n \in \mathbb{N}^*$ and $e = (e_1, \dots, e_n) \in X^n$.*
 2237 *Let $T \in \mathcal{M}_n(\mathbb{R})$. Then,*

$$2238 \quad T\text{Gram}_X(e_1, \dots, e_n)^tT = \text{Gram}_X((Te)_1, \dots, (Te)_n)$$

2239 where, for any $i \in \llbracket 1, n \rrbracket$, $(Te)_i$ is defined by

$$2240 \quad (Te)_i := \sum_{j=1}^n T_{i,j}e_j.$$

2241 *Proof.* For any $\omega \in \mathbb{R}^n$, it comes that

$$2242 \quad (D.3) \quad \langle T\text{Gram}_X(e_1, \dots, e_n)^tT\omega, \omega \rangle = \langle \text{Gram}_X(e_1, \dots, e_n)({}^tT\omega), ({}^tT\omega) \rangle$$

$$2243 \quad (D.4) \quad = \left\| \sum_{i=1}^n ({}^tT\omega)_i e_i \right\|^2$$

$$2244 \quad (D.5) \quad = \left\| \sum_{i=1}^n \sum_{j=1}^n T_{j,i} \omega_j e_i \right\|^2$$

$$2245 \quad (D.6) \quad = \left\| \sum_{j=1}^n \omega_j (Te)_j \right\|^2$$

$$2246 \quad (D.7) \quad = \langle \text{Gram}_X((Te)_1, \dots, (Te)_n)\omega, \omega \rangle. \quad \square$$

2248 Depending on the phenomenon at stake on actual examples, with a suitable choice of
 2249 $\tilde{\xi}$ (i.e. of T), the quantity $\langle \widetilde{M}^{-1}\tilde{\xi}, \tilde{\xi} \rangle$ can be easier to estimate than $\langle M^{-1}\xi, \xi \rangle$.

2250 **D.1. Dilatations.**

2251 Notice that

$$2252 \quad \langle \widetilde{M}^{-1}\tilde{\xi}, \tilde{\xi} \rangle \leq \|\widetilde{M}^{-1}\| \|\tilde{\xi}\|^2.$$

2253 When the minimal null control time can be estimated with rough estimates (this can
 2254 only characterize the minimal time when $T_0 = 0$), it can simplify the computations
 2255 to have a bounded $\|\tilde{\xi}\|$. To do so, it is convenient to consider dilatations of ξ .

2256 Let X be an Hilbert space. Let $n \in \mathbb{N}^*$ and $e_1, \dots, e_n \in X$. Let $\xi \in \mathbb{R}^n$ and
 2257 $\beta \in \mathbb{R}^n$ with non-zero entries. Let

$$2258 \quad T = D_\beta := \text{diag}(\beta) \in \text{GL}_n(\mathbb{R}), \quad \tilde{\xi} = T\xi.$$

2259 Then, from Lemma D.1, it comes that

$$2260 \quad T\text{Gram}_X(e_1, \dots, e_n)^tT = \text{Gram}_X(\beta_1 e_1, \dots, \beta_n e_n).$$

2261 **D.2. Invariance by scale change.**

2262 In our assumptions there is no normalization condition on the eigenvectors (see
 2263 Remark 1.9). This allows to have simpler expressions for these eigenvectors. Actually,
 2264 the computation of $\langle M^{-1}\xi, \xi \rangle$ can be done with a different scale change on every
 2265 generalized eigenvector as detailed in the following proposition.

2266 PROPOSITION D.2. *Let M and ξ be as defined in Theorem 1.8. Let $\beta \in \mathbb{R}^{|\alpha|}$ be
 2267 such that $\beta_j^0 \neq 0$ for all $j \in \llbracket 1, g \rrbracket$. Set*

$$2268 \quad \tilde{\xi} = \begin{pmatrix} \langle y_0, (\beta\phi) [\lambda^{\mu^1}] \rangle_{-\diamond, \diamond} \\ \vdots \\ \langle y_0, (\beta\phi) [\lambda^{\mu^{|\alpha|}}] \rangle_{-\diamond, \diamond} \end{pmatrix}$$

2269 Then,

$$2270 \quad \langle M^{-1}\xi, \xi \rangle = \langle \widetilde{M}^{-1}\tilde{\xi}, \tilde{\xi} \rangle$$

2271 where

$$2272 \quad (\text{D.8}) \quad \widetilde{M} := \sum_{l=1}^{|\alpha|} \text{Gram}_U \left(\underbrace{0, \dots, 0}_{l-1}, (\beta b) [\lambda^{\mu^l - \mu^{l-1}}], \dots, (\beta b) [\lambda^{\mu^{|\alpha|} - \mu^{l-1}}] \right).$$

2273 *Proof.* From Leibniz formula [10, Proposition 7.13], it comes that for any $p \in$
 2274 $\llbracket 1, |\alpha| \rrbracket$,

$$2275 \quad (\beta\phi) [\lambda^{\mu^p}] = \sum_{q=1}^{\mu^p} \beta [\lambda^{\mu^p - \mu^{q-1}}] \phi [\lambda^{\mu^q}].$$

2276 Thus, $\tilde{\xi} = T\xi$ where T is the following lower triangular matrix

$$2277 \quad T = \left(\mathbf{1}_{q \leq p} \beta [\lambda^{\mu^p - \mu^{q-1}}] \right)_{p, q \in \llbracket 1, |\alpha| \rrbracket}.$$

2278 The diagonal entries of this lower triangular matrix are β_j^0 and thus $T \in \text{GL}_{|\alpha|}(\mathbb{C})$.
 2279 From (D.2),

$$2280 \quad \widetilde{M} := \sum_{l=1}^{|\alpha|} T \text{Gram}_U \left(\underbrace{0, \dots, 0}_{l-1}, b [\lambda^{\mu^l - \mu^{l-1}}], \dots, b [\lambda^{\mu^{|\alpha|} - \mu^{l-1}}] \right) {}^t T.$$

2281 Let $l \in \llbracket 1, |\alpha| \rrbracket$ and

$$2282 \quad e_1 = \dots = e_{l-1} = 0,$$

$$2283 \quad e_p = b [\lambda^{\mu^p - \mu^{l-1}}], \quad \forall p \in \llbracket l, |\alpha| \rrbracket.$$

2284 Then, for any $p \in \llbracket 1, |\alpha| \rrbracket$,

$$2285 \quad (Te)_p = \sum_{q=1}^{|\alpha|} \mathbf{1}_{q \leq p} \beta [\lambda^{\mu^p - \mu^{q-1}}] e_q.$$

2287 Thus, $(Te)_1 = \dots = (Te)_{l-1} = 0$ and, for any $p \in \llbracket l, |\alpha| \rrbracket$,

$$2288 \quad (Te)_p = \sum_{q=1}^{|\alpha|} \mathbf{1}_{q \leq p} \beta \left[\lambda^{(\mu^p - \mu^{q-1})} \right] e_q = \sum_{q=l}^p \beta \left[\lambda^{(\mu^p - \mu^{q-1})} \right] b \left[\lambda^{(\mu^q - \mu^{l-1})} \right].$$

2289 Let

$$2290 \quad \kappa^j := \mu^{l-1+j} - \mu^{l-1}, \quad \forall j \in \llbracket 1, p-l+1 \rrbracket.$$

2291 Then, using again Leibniz formula [10, Proposition 7.13], we obtain

$$2292 \quad (\beta b) \left[\lambda^{(\mu^p - \mu^{l-1})} \right] = (\beta b) \left[\lambda^{(\kappa^{p-l+1})} \right]$$

$$2293 \quad = \sum_{r=1}^{p-l+1} \beta \left[\lambda^{(\kappa^{p-l+1} - \kappa^{r-1})} \right] b \left[\lambda^{(\kappa^r)} \right]$$

$$2294 \quad = \sum_{r=1}^{p-l+1} \beta \left[\lambda^{(\mu^p - \mu^{l-1+r-1})} \right] b \left[\lambda^{(\mu^{l-1+r} - \mu^{l-1})} \right]$$

$$2295 \quad = \sum_{q=l}^p \beta \left[\lambda^{(\mu^p - \mu^{q-1})} \right] b \left[\lambda^{(\mu^q - \mu^{l-1})} \right]$$

$$2296 \quad = (Te)_p.$$

2298 Finally, applying (D.1) and Lemma D.1 ends the proof of Proposition D.2. \square

2299 **REMARK D.1.** *As there is no normalization condition on the eigenvectors a similar statement automatically holds with M and ξ defined in Theorem 1.10.*

2301 **D.3. An equivalent formula for simple eigenvalues.**

2302 In this section, we consider the case of a group of simple eigenvalues *i.e.* $\alpha_\lambda =$
 2303 $\gamma_\lambda = 1$ for every $\lambda \in G$. In that case, the cost of the group G can be computed either
 2304 using the formula of Theorem 1.8 for geometrically simple eigenvalues or the formula
 2305 of Theorem 1.10 for semi-simple eigenvalues. Even though these theorems imply that
 2306 those two formulas coincide (as they are both the cost of the group) we give a direct
 2307 proof of this statement.

2308 **PROPOSITION D.3.** *Let M and ξ be the matrix and the vector given in Theorem 1.8*
 2309 *i.e.*

$$2310 \quad M := \sum_{l=1}^g \text{Gram}_U \left(\underbrace{0, \dots, 0}_{l-1}, b[\lambda_l], \dots, b[\lambda_l, \dots, \lambda_g] \right)$$

2311 and

$$2312 \quad \xi = \begin{pmatrix} \langle y_0, \phi[\lambda_1] \rangle_{-\diamond, \diamond} \\ \vdots \\ \langle y_0, \phi[\lambda_1, \dots, \lambda_g] \rangle_{-\diamond, \diamond} \end{pmatrix}.$$

2313 Let \widetilde{M} and $\widetilde{\xi}$ be the matrix and the vector given in Theorem 1.10 *i.e.*

$$2314 \quad (\text{D.9}) \quad \widetilde{M} := \sum_{l=1}^g \text{Gram}_U (\delta_l^1 b[\lambda_1], \dots, \delta_l^g b[\lambda_g]) \quad \text{and} \quad \widetilde{\xi} := \begin{pmatrix} \langle y_0, \phi[\lambda_1] \rangle_{-\diamond, \diamond} \\ \vdots \\ \langle y_0, \phi[\lambda_g] \rangle_{-\diamond, \diamond} \end{pmatrix}.$$

2315 Then,

$$2316 \quad \langle M^{-1}\xi, \xi \rangle = \langle \widetilde{M}^{-1}\tilde{\xi}, \tilde{\xi} \rangle$$

2317 *Proof.* The usual interpolation formula [10, Proposition 7.6] gives

$$2318 \quad (\text{D.10}) \quad \phi[\lambda_i] = \sum_{j=1}^i \left(\prod_{k=1}^{j-1} \lambda_i - \lambda_k \right) \phi[\lambda_1, \dots, \lambda_j].$$

2319 Recall that the notation δ_j^i has been introduced in (1.50). With these notations,
2320 $\tilde{\xi} = T\xi$ where T is the following lower triangular matrix

$$2321 \quad T = (\delta_j^i)_{i,j \in \llbracket 1, g \rrbracket} \in \text{GL}_g(\mathbb{C}).$$

2322 From (D.2), we define

$$2323 \quad \widehat{M} := \sum_{l=1}^g T \text{Gram}_U \left(\underbrace{0, \dots, 0}_{l-1}, b[\lambda_l], \dots, b[\lambda_l, \dots, \lambda_g] \right) {}^t T.$$

2324 Let $l \in \llbracket 1, g \rrbracket$ and

$$2325 \quad e_1 = \dots = e_{l-1} = 0,$$

$$2326 \quad e_j = b[\lambda_l, \dots, \lambda_j], \quad \forall j \in \llbracket l, g \rrbracket.$$

2328 Then, $(Te)_1 = \dots = (Te)_{l-1} = 0$ and for $i \in \llbracket l, g \rrbracket$, using again the interpolation
2329 property [10, Proposition 7.6], we obtain

$$2330 \quad (Te)_i = \sum_{j=l}^g \delta_j^i b[\lambda_l, \dots, \lambda_j]$$

$$2331 \quad = \sum_{j=l}^i \delta_j^i b[\lambda_l, \dots, \lambda_j]$$

$$2332 \quad = \delta_l^i \sum_{j=l}^i \left(\prod_{k=l}^{j-1} \lambda_i - \lambda_k \right) b[\lambda_l, \dots, \lambda_j]$$

$$2333 \quad = \delta_l^i b[\lambda_i].$$

2335 Recalling that $\delta_l^1 = \dots = \delta_l^{l-1} = 0$, we thus obtain

$$2336 \quad (Te)_i = \delta_l^i b[\lambda_i], \quad \forall i \in \llbracket 1, g \rrbracket.$$

2337 Finally, from Lemma D.1, we deduce that $\widehat{M} = \widetilde{M}$ which ends the proof of Proposi-
2338 tion D.3. \square

2339 Appendix E. Technical estimates for Sturm-Liouville operators.

2340 In this appendix, we prove Lemma 6.5.

2341 To do so, we start with the following corollary of Lemma 5.1.

2342 COROLLARY E.1. Let A be the Sturm-Liouville operator defined in (5.1). There
 2343 exists $C > 0$ such that for any $0 < a < b < 1$ there exists $\lambda_0 > 0$ (depending on $b - a$)
 2344 such that for any $\lambda \geq \lambda_0$, for any $F \in L^2(0, 1; \mathbb{R})$, for any u satisfying

$$2345 \quad (A - \lambda)u = F \quad \text{on } [a, b],$$

2346 and for any $x \in [a, b]$, we have

$$2347 \quad |u(x)|^2 + \frac{\gamma(x)}{\lambda} |u'(x)|^2 \leq \frac{C}{b-a} \left(\|u\|_{(a,b)}^2 + \frac{(b-a)^2}{\lambda} \|F\|_{(a,b)}^2 \right).$$

2348 *Proof.* We start with the proof in the case $F = 0$.

2349 Let $\chi_0 \in C^\infty(\mathbb{R}; \mathbb{R})$ be a cut-off function such that $0 \leq \chi_0 \leq 1$ and

- 2350 • $\chi_0(x) = 1$ for every $x \in [1/4, 3/4]$,
- 2351 • $\chi_0(x) = 0$ for every $x \notin (0, 1)$.

We then set

$$\chi(x) = \chi_0 \left(\frac{x-a}{b-a} \right),$$

2352 in such a way that, if we set $\alpha = a + \frac{b-a}{4}$ and $\beta = b - \frac{b-a}{4}$, we have

- 2353 • $\chi(x) = 1$ for every $x \in [\alpha, \beta]$,
- 2354 • $\chi(x) = 0$ for every $x \notin (a, b)$.

2355 Let $C_1 > 0$ be the constant given by Lemma 5.1 with $\lambda_0 = 1$. In the rest of this
 2356 proof we set

$$2357 \quad (\text{E.1}) \quad \lambda_0 = \max \left\{ 1, 2C_1^2 \|\chi'\|_{L^\infty}^2 \right\}$$

2358 and assume that $\lambda \geq \lambda_0$. Note that λ_0 only depends on $b - a$ since $\|\chi'\|_{L^\infty} =$
 2359 $\|\chi_0'\|_{L^\infty} (b-a)^{-1}$.

2360 Let $x \in [a, b]$. We apply Lemma 5.1 and integrate in the variable $y \in (\alpha, \beta)$ to
 2361 obtain

$$2362 \quad (\text{E.2}) \quad \frac{b-a}{2} \left(|u(x)|^2 + \frac{\gamma(x)}{\lambda} |u'(x)|^2 \right) \leq C_1 \left(\|u\|_{(a,b)}^2 + \frac{1}{\lambda} \int_\alpha^\beta \gamma(y) |u'(y)|^2 dy \right).$$

2363 Then integrating by parts, using $(A - \lambda)u = 0$ and Cauchy-Schwarz inequality yield

$$\begin{aligned} 2364 \quad \frac{1}{\lambda} \int_\alpha^\beta \gamma(y) |u'(y)|^2 dy &\leq \frac{1}{\lambda} \int_a^b \chi(y) \gamma(y) |u'(y)|^2 dy \\ 2365 &= -\frac{1}{\lambda} \int_a^b \chi'(y) (\gamma u')(y) u(y) dy + \frac{1}{\lambda} \int_a^b \chi(y) (\lambda - c(y)) |u(y)|^2 dy \\ 2366 &\leq \left(\frac{\|\chi'\|_{L^\infty}}{\lambda} \|\sqrt{\gamma} u'\|_{(a,b)} \right) (\|\sqrt{\gamma}\|_{L^\infty} \|u\|_{(a,b)}) \\ 2367 &\quad + \int_a^b \chi(y) \left| 1 - \frac{c(y)}{\lambda} \right| |u(y)|^2 dy \\ 2368 &\leq \left(1 + \|c\|_{L^\infty} + \frac{\|\gamma\|_{L^\infty}}{2} \right) \|u\|_{(a,b)}^2 + \frac{\|\chi'\|_{L^\infty}^2}{2\lambda^2} \|\sqrt{\gamma} u'\|_{(a,b)}^2. \end{aligned}$$

2370 Plugging it into estimate (E.4) we obtain

$$2371 \quad (\text{E.3}) \quad \frac{b-a}{2} \left(|u(x)|^2 + \frac{\gamma(x)}{\lambda} |u'(x)|^2 \right) \leq C \|u\|_{(a,b)}^2 + \frac{C_1 \|\chi'\|_{L^\infty}^2}{2\lambda^2} \|\sqrt{\gamma} u'\|_{(a,b)}^2.$$

2372 Applying again Lemma 5.1 gives that, for any $y \in (a, b)$,

$$2373 \quad \frac{\gamma(y)}{\lambda} |u'(y)|^2 \leq C_1 \left(|u(x)|^2 + \frac{\gamma(x)}{\lambda} |u'(x)|^2 \right).$$

2374 Integrating in the variable $y \in (a, b)$ and using the definition of λ_0 given in (E.1) we
2375 obtain

$$2376 \quad \frac{C_1 \|\chi'\|_{L^\infty}^2}{2\lambda^2} \|\sqrt{\gamma}u'\|_{(a,b)}^2 \leq \frac{C_1^2 \|\chi'\|_{L^\infty}^2}{2\lambda} (b-a) \left(|u(x)|^2 + \frac{\gamma(x)}{\lambda} |u'(x)|^2 \right) \\ 2377 \quad \leq \frac{b-a}{4} \left(|u(x)|^2 + \frac{\gamma(x)}{\lambda} |u'(x)|^2 \right). \\ 2378$$

2379 Plugging it into (E.3) ends the proof of Corollary E.1 in the case $F = 0$.

2380 Assume now that $F \neq 0$. Let $x \in [a, b]$ and define \tilde{u} as the solution of the Cauchy
2381 problem

$$2382 \quad \begin{cases} (A - \lambda)\tilde{u} = 0, \\ \tilde{u}(x) = u(x), \\ \tilde{u}'(x) = u'(x). \end{cases}$$

2383 From the case $F = 0$ we deduce that

$$2384 \quad (\text{E.4}) \quad |u(x)|^2 + \frac{\gamma(x)}{\lambda} |u'(x)|^2 = |\tilde{u}(x)|^2 + \frac{\gamma(x)}{\lambda} |\tilde{u}'(x)|^2 \leq \frac{C}{b-a} \|\tilde{u}\|_{(a,b)}^2.$$

2385 Notice that

$$2386 \quad \begin{cases} (A - \lambda)(u - \tilde{u}) = F, \\ (u - \tilde{u})(x) = (u - \tilde{u})'(x) = 0. \end{cases}$$

2387 Thus, from Lemma 5.1 we obtain that for any $y \in [a, b]$

$$2388 \quad |(u - \tilde{u})(y)|^2 \leq \frac{C}{\lambda} \left| \int_x^y |F(s)| ds \right|^2 \leq C \frac{b-a}{\lambda} \|F\|_{(a,b)}^2$$

2389 which implies

$$2390 \quad \|u - \tilde{u}\|_{(a,b)}^2 \leq C \frac{(b-a)^2}{\lambda} \|F\|_{(a,b)}^2.$$

2391 Thus, we deduce that

$$2392 \quad \|\tilde{u}\|_{(a,b)}^2 \leq C \left(\|u\|_{(a,b)}^2 + \frac{(b-a)^2}{\lambda} \|F\|_{(a,b)}^2 \right).$$

2393 Together with (E.4) this ends the proof of Corollary E.1. \square

2394 **REMARK E.1.** Notice that, when $\gamma \in C^2([0, 1]; \mathbb{R})$, for λ large enough, for any u
2395 satisfying $(A - \lambda)u = 0$, there exist α and β in (a, b) such that

$$2396 \quad u(\alpha) = u(\beta) = 0, \quad \beta - \alpha \geq \frac{b-a}{2}.$$

2397 This is a consequence of the Sturm comparison theorem (see for instance [16, Corollary
2398 A.5.16]). With this choice one does not need to introduce the cut-off function χ in
2399 the first step of the proof of Corollary E.1 which simplifies the proof.

2400 We now have all the ingredients to prove Lemma 6.5.

2401 *Proof (of Lemma 6.5).* We denote by $\omega_1, \dots, \omega_N$ the connected components of ω
 2402 labeled such that

$$2403 \quad \sup \omega_j \leq \inf \omega_{j+1}, \quad \forall j \in \llbracket 1, N-1 \rrbracket.$$

2404 For any $j \in \llbracket 1, N \rrbracket$, let $\lambda_0(\omega_j) > 0$ be the constant given by Corollary E.1 applied to
 2405 the interval $\overline{\omega_j}$. Let

$$2406 \quad \lambda_0 = \max_{j \in \llbracket 1, N \rrbracket} \lambda_0(\omega_j)$$

2407 and, let $K > 0$ be such that

$$2408 \quad k \geq K \implies \nu_k \geq \lambda_0.$$

2409 *We start with the proof of item i.*

2410 Let $\mathfrak{C} = [a, b]$ be a connected component of $\overline{(0, 1) \setminus \omega}$. Integrating by parts we
 2411 obtain

$$2412 \quad \int_{\mathfrak{C}} F(x) \varphi_k(x) dx = -(\gamma u' \varphi_k)(b) + (\gamma u' \varphi_k)(a) + (u \gamma \varphi_k)'(b) - (u \gamma \varphi_k)'(a).$$

2413 Recall that from (5.7),

$$2414 \quad |\varphi_k(x)| + \frac{1}{\sqrt{\nu_k}} |\varphi_k'(x)| \leq C, \quad \forall x \in (0, 1), \forall k \geq 1.$$

2415 Similarly, applying Lemma 5.1 with $y = 0$ we obtain

$$2416 \quad (\text{E.5}) \quad |\tilde{\varphi}_k(x)| + \frac{1}{\sqrt{\nu_k}} |\tilde{\varphi}_k'(x)| \leq C, \quad \forall x \in (0, 1), \forall k \geq 1.$$

2417 Thus,

$$2418 \quad \frac{1}{\sqrt{\nu_k}} \left| \int_{\mathfrak{C}} F(x) \varphi_k(x) dx \right| \leq C \left(|u(a)| + \frac{\sqrt{\gamma(a)}}{\sqrt{\nu_k}} |u'(a)| \right) + C \left(|u(b)| + \frac{\sqrt{\gamma(b)}}{\sqrt{\nu_k}} |u'(b)| \right).$$

2419 • If $\mathfrak{C} \cap \{0, 1\} = \emptyset$, then there exists $j \in \llbracket 2, N \rrbracket$ such that $a \in \overline{\omega_{j-1}}$ and $b \in \overline{\omega_j}$.
 2420 Applying twice Corollary E.1 we obtain

$$2421 \quad |u(a)| + \frac{\sqrt{\gamma(a)}}{\sqrt{\nu_k}} |u'(a)| \leq C (\sqrt{\nu_k} \|u\|_{\omega_{j-1}} + \|F\|_{\omega_{j-1}}),$$

2422 and

$$2423 \quad |u(b)| + \frac{\sqrt{\gamma(b)}}{\sqrt{\nu_k}} |u'(b)| \leq C (\sqrt{\nu_k} \|u\|_{\omega_j} + \|F\|_{\omega_j})$$

2424 where C now also depends on ω . This implies

$$2425 \quad \left| \int_{\mathfrak{C}} F(x) \varphi_k(x) dx \right| \leq C (\sqrt{\nu_k} \|u\|_{\omega} + \|F\|_{\omega})$$

2426 The same computations hold for $|\int_{\mathfrak{C}} F(x) \tilde{\varphi}_k(x) dx|$.

2427 • Now, if $a = 0$, taking into account the boundary condition $\varphi_k(a) = 0$, the
 2428 same computations yields

$$2429 \quad \frac{1}{\sqrt{\nu_k}} \left| \int_{\mathfrak{e}} F(x) \varphi_k(x) dx \right| \leq C |u(0)| + C \left(|u(b)| + \frac{\sqrt{\gamma(b)}}{\sqrt{\nu_k}} |u'(b)| \right).$$

2430 As $b \in \bar{\omega}$, applying Corollary E.1 we obtain

$$2431 \quad |u(b)| + \frac{\sqrt{\gamma(b)}}{\sqrt{\nu_k}} |u'(b)| \leq C (\sqrt{\nu_k} \|u\|_{\omega} + \|F\|_{\omega})$$

2432 which implies

$$2433 \quad \left| \int_{\mathfrak{e}} F(x) \varphi_k(x) dx \right| \leq C (\sqrt{\nu_k} \|u\|_{\omega} + \sqrt{\nu_k} |u(0)| + \|F\|_{\omega}).$$

2434 • Similarly, if $b = 1$, we prove that

$$2435 \quad \left| \int_{\mathfrak{e}} F(x) \varphi_k(x) dx \right| \leq C (\sqrt{\nu_k} \|u\|_{\omega} + \sqrt{\nu_k} |u(1)| + \|F\|_{\omega}).$$

2436 Gathering these results proves item i.

2437 *We now turn to the proof of item ii.*

2438 • We start designing u a solution of

$$2439 \quad \begin{cases} (A - \nu_k)u = F, \\ u(0) = u(1) = 0, \end{cases}$$

2440 such that

$$2441 \quad (\text{E.6}) \quad |u(x)| + \frac{\sqrt{\gamma(x)}}{\sqrt{\nu_k}} |u'(x)| \leq \frac{C}{\sqrt{\nu_k}} (\mathcal{M}_k(F, \omega) + \|F\|_{\omega_1}), \quad \forall x \in \bar{\omega}_1.$$

2442 Let \bar{u} be any solution of

$$2443 \quad \begin{cases} (A - \nu_k)\bar{u} = F, \\ \bar{u}(0) = \bar{u}(1) = 0. \end{cases}$$

2444 If $0 \notin \bar{\omega}_1$ we set $b = \inf \omega_1$ whereas if $0 \in \bar{\omega}_1$ we set $b \in \omega_1$. Notice that in
 2445 both cases

$$2446 \quad \int_0^b F(x) \varphi_k(x) dx = \bar{u}(b) \gamma(b) \varphi_k'(b) - \gamma(b) \bar{u}'(b) \varphi_k(b).$$

2447 Applying Lemma 5.1 with $y = b$, integrating with respect to the variable
 2448 $x \in (0, 1)$ and using $\|\varphi_k\|_{(0,1)} = 1$ we obtain that there exists $C > 0$ such
 2449 that

$$2450 \quad |\varphi_k(b)| + \frac{\sqrt{\gamma(b)}}{\sqrt{\nu_k}} |\varphi_k'(b)| \geq C.$$

2451 – If $|\varphi_k(b)| \geq \frac{C}{2}$, we set $u = \bar{u} - \frac{\bar{u}(b)}{\varphi_k(b)} \varphi_k$.

2452 Thus, we have $u(b) = 0$ which implies

$$2453 \quad \sqrt{\gamma(b)}u'(b) = \frac{-1}{\sqrt{\gamma(b)\varphi_k(b)}} \int_0^b F(x)\varphi_k(x)dx.$$

2454 Thus,

$$2455 \quad (\text{E.7}) \quad |u(b)| + \frac{\sqrt{\gamma(b)}}{\sqrt{\nu_k}}|u'(b)| \leq \frac{C}{\sqrt{\nu_k}} \left| \int_0^b F(x)\varphi_k(x)dx \right|.$$

2456 – Otherwise, we have $\frac{\sqrt{\gamma(b)}}{\sqrt{\nu_k}}|\varphi'_k(b)| \geq \frac{C}{2}$. Setting $u = \bar{u} - \frac{\bar{u}'(b)}{\varphi'_k(b)}\varphi_k$, the
2457 same computations also imply (E.7).

2458 We now prove that (E.7) implies (E.6).

2459 As $b \in \overline{\omega_1}$, applying Lemma 5.1 and (E.7) we obtain for any $x \in \overline{\omega_1}$,

$$2460 \quad |u(x)| + \frac{\sqrt{\gamma(x)}}{\sqrt{\nu_k}}|u'(x)| \leq C \left(|u(b)| + \frac{\sqrt{\gamma(b)}}{\sqrt{\nu_k}}|u'(b)| + \frac{1}{\sqrt{\nu_k}}\|F\|_{\omega_1} \right)$$

$$2461 \quad \leq \frac{C}{\sqrt{\nu_k}} \left(\left| \int_0^b F(x)\varphi_k(x)dx \right| + \|F\|_{\omega_1} \right).$$

2462

2463 – Assume first that $0 \notin \overline{\omega_1}$ and recall that $b = \inf \omega_1$. Then, by definition
2464 of $\mathcal{M}_k(F, \omega)$ (see (6.8)), we have

$$2465 \quad \left| \int_0^b F(x)\varphi_k(x)dx \right| \leq \mathcal{M}_k(F, \omega).$$

2466 Thus, for any $x \in \overline{\omega_1}$,

$$2467 \quad |u(x)| + \frac{\sqrt{\gamma(x)}}{\sqrt{\nu_k}}|u'(x)| \leq \frac{C}{\sqrt{\nu_k}}(\mathcal{M}_k(F, \omega) + \|F\|_{\omega_1}).$$

2468 – Otherwise, $0 \in \overline{\omega_1}$ and we have set $b \in \omega_1$. Then, since $(0, b) \subset \omega_1$ and
2469 $\|\varphi_k\|_{(0,1)} = 1$, we have

$$2470 \quad \left| \int_0^b F(x)\varphi_k(x)dx \right| \leq \|F\|_{\omega_1}.$$

2471 Thus, for any $x \in \overline{\omega_1}$,

$$2472 \quad |u(x)| + \frac{\sqrt{\gamma(x)}}{\sqrt{\nu_k}}|u'(x)| \leq \frac{C}{\sqrt{\nu_k}}\|F\|_{\omega_1}.$$

2473 Gathering these two cases proves (E.6).

2474 • We prove by induction that the function u designed at the previous step
2475 satisfies

$$2476 \quad (\text{E.8}) \quad |u(x)| + \frac{\sqrt{\gamma(x)}}{\sqrt{\nu_k}}|u'(x)| \leq \frac{C}{\sqrt{\nu_k}}(\mathcal{M}_k(F, \omega) + \|F\|_{\omega}), \quad \forall x \in \overline{\omega_j}.$$

2477 The case $j = 1$ was proved in the previous step. Let $j \in \llbracket 2, N \rrbracket$ be such that

$$2478 \quad |u(x)| + \frac{\sqrt{\gamma(x)}}{\sqrt{\nu_k}} |u'(x)| \leq \frac{C}{\sqrt{\nu_k}} (\mathcal{M}_k(F, \omega) + \|F\|_\omega), \quad \forall x \in \overline{\omega_{j-1}}.$$

2479 Let $a_j = \sup \omega_{j-1}$ and $b_j = \inf \omega_j$. Integrating by parts we obtain

$$2480 \quad \frac{1}{\sqrt{\nu_k}} \int_{a_j}^{b_j} F(x) \varphi_k(x) dx = u(b_j) \frac{\gamma(b_j) \varphi'_k(b_j)}{\sqrt{\nu_k}} - \frac{\gamma(b_j) u'(b_j)}{\sqrt{\nu_k}} \varphi_k(b_j) \\ 2481 \quad - u(a_j) \frac{\gamma(a_j) \varphi'_k(a_j)}{\sqrt{\nu_k}} - \frac{\gamma(a_j) u'(a_j)}{\sqrt{\nu_k}} \varphi_k(a_j). \\ 2482$$

2483 The same computations hold replacing φ_k by $\tilde{\varphi}_k$. This can be rewritten in matrix form as

$$2484 \quad (\text{E.9}) \\ 2485 \quad \begin{pmatrix} \frac{\gamma(b_j) \varphi'_k(b_j)}{\sqrt{\nu_k}} & -\varphi_k(b_j) \\ \frac{\gamma(b_j) \tilde{\varphi}'_k(b_j)}{\sqrt{\nu_k}} & -\tilde{\varphi}_k(b_j) \end{pmatrix} \begin{pmatrix} u(b_j) \\ \frac{\gamma(b_j) u'(b_j)}{\sqrt{\nu_k}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{\nu_k}} \int_{a_j}^{b_j} F(x) \varphi_k(x) dx \\ \frac{1}{\sqrt{\nu_k}} \int_{a_j}^{b_j} F(x) \tilde{\varphi}_k(x) dx \end{pmatrix} + R_j$$

2486 with

$$2487 \quad \|R_j\| \leq C \left(|u(a_j)| + \frac{\sqrt{\gamma(a_j)}}{\sqrt{\nu_k}} |u'(a_j)| \right).$$

2488 Notice that the determinant of the matrix appearing in (E.9) is a wronskian which is constant. Thus,

$$2490 \quad \det \begin{pmatrix} \frac{\gamma(b_j) \varphi'_k(b_j)}{\sqrt{\nu_k}} & -\varphi_k(b_j) \\ \frac{\gamma(b_j) \tilde{\varphi}'_k(b_j)}{\sqrt{\nu_k}} & -\tilde{\varphi}_k(b_j) \end{pmatrix} = -\frac{\gamma(0) \varphi'_k(0)}{\sqrt{\nu_k}}$$

2491 which is bounded from below. From (5.7) and (E.5), all the coefficients of this matrix are also bounded. Thus, we obtain

$$2493 \quad |u(b_j)| + \frac{\sqrt{\gamma(b_j)}}{\sqrt{\nu_k}} |u'(b_j)| \leq \frac{C}{\sqrt{\nu_k}} \mathcal{M}_k(F, \omega) + C \|R_j\|.$$

2494 As $a_j \in \overline{\omega_{j-1}}$ the induction hypothesis imply

$$2495 \quad \|R_j\| \leq \frac{C}{\sqrt{\nu_k}} (\mathcal{M}_k(F, \omega) + \|F\|_\omega)$$

2496 and thus

$$2497 \quad |u(b_j)| + \frac{\sqrt{\gamma(b_j)}}{\sqrt{\nu_k}} |u'(b_j)| \leq \frac{C}{\sqrt{\nu_k}} (\mathcal{M}_k(F, \omega) + \|F\|_\omega).$$

2498 As $b_j \in \overline{\omega_j}$, applying Lemma 5.1 we obtain for any $x \in \overline{\omega_j}$

$$2499 \quad |u(x)| + \frac{\sqrt{\gamma(x)}}{\sqrt{\nu_k}} |u'(x)| \leq C \left(|u(b_j)| + \frac{\sqrt{\gamma(b_j)}}{\sqrt{\nu_k}} |u'(b_j)| + \frac{1}{\sqrt{\nu_k}} \|F\|_{\omega_j} \right) \\ 2500 \quad \leq \frac{C}{\sqrt{\nu_k}} (\mathcal{M}_k(F, \omega) + \|F\|_\omega) \\ 2501$$

2502 This proves (E.8).

2503 • Conclusion. From (E.8) we obtain

$$2504 \quad |u(x)| \leq \frac{C}{\sqrt{\nu_k}} (\mathcal{M}_k(F, \omega) + \|F\|_\omega), \quad \forall x \in \omega_j, \forall j \in \llbracket 1, N \rrbracket.$$

2505 This leads to

$$2506 \quad \|u\|_{\omega_j} \leq \frac{C}{\sqrt{\nu_k}} (\mathcal{M}_k(F, \omega) + \|F\|_\omega), \quad \forall j \in \llbracket 1, N \rrbracket$$

2507 and ends the proof of item ii. \square

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