MODULATED FREE ENERGY AND MEAN FIELD LIMIT

Didier Bresch, Pierre-Emmanuel Jabin, Zhenfu Wang

MODULATED FREE ENERGY AND MEAN FIELD LIMIT

DIDIER BRESCH, PIERRE-EMMANUEL JABIN, AND ZHENFU WANG

Abstract. This is the document corresponding to the talk the first author gave at IHÉS for the Laurent Schwartz seminar on November 19, 2019. It concerns our recent introduction of a modulated free energy in mean-field theory in [4]. This physical object may be seen as a combination of the modulated potential energy introduced by S. Serfaty [See *Proc. Int. Cong. Math.* (2018)] and of the relative entropy introduced in mean field limit theory by P.–E. Jabin, Z. Wang [See *Inventiones* 2018]. It allows to obtain, for the first time, a convergence rate in the mean field limit for Riesz and Coulomb repulsive kernels in the presence of viscosity using the estimates in [8] and [20]. The main objective in this paper is to explain how it is possible to cover more general repulsive kernels through a Fourier transform approach as announced in [4] first in the case $\sigma_N \to 0$ when $N \to +\infty$ and then if $\sigma > 0$ is fixed. Then we end the paper with comments on the particle approximation of the Patlak-Keller-Segel system which is associated to an attractive kernel and refer to [*C.R. Acad Science Paris* 357, Issue 9, (2019), 708–720] by the authors for more details.

1. Introduction

This paper concerns interaction particles system and quantitative estimates in mean field limit theory in the spirit of [20], [8] and [15]. Namely we consider N particles, identical and interacting two by two through a kernel K. For simplicity, we consider periodic boundary conditions and we assume the position of the i-th particle $X_i(t) \in \Pi^d$ evolves through as follows

$$dX_i = \frac{1}{N} \sum_{j \neq i} K(X_i - X_j)dt + \sqrt{2\sigma} dW_i$$

for N independent Brownian motions W_i with a gradient flow hypothesis $K = -\nabla V$ where $V \in L^1(\Pi^d)$ will be discussed later-on: For possible vanishing viscosity with respect to N namely $\sigma_N \to 0$ when $N \to +\infty$ we consider singular repulsive kernels with some pointwise and Fourier controls (including Riesz and Coulomb Kernels) and for $\sigma > 0$ we consider more general singular repulsive kernels with Fourier control and then we conclude with some comments on the Patlak-Keller-Segel attractive kernel. The main objective in this document is to give some details in the repulsive case to complete the note written by the authors [*C.R. Acad Sciences* 357, Issue 9, (2019), 708–720] which focused on the attractive Patlak-Keller-Segel kernel. A full paper is still in progress to propose a single complete document, see [5]. Readers interested by some reviews on mean field limit for stochastic particle systems are referred to [16], [18] and on mean field limit for deterministic particle systems are referred to [11], [14], [20].

Date: December 6, 2019.

2010 *Mathematics Subject Classification.* 35Q35, 76N10.

As usually, we introduce ρ_N the joint law of the process $(X_1, \cdots, X_N)$ which satisfies the Liouville equation

$$\partial_t \rho_N + \sum_{i=1}^N \mathrm{div}_{x_i} \left(\rho_N \frac{1}{N} \sum_{j \neq i}^N K(x_i - x_j) \right) = \sigma \sum_{i=1}^N \Delta_{x_i} \rho_N.$$

The main objective is to obtain a rate of convergence with respect to the number of particles N from ρ_N to $\bar\rho_N = \bar\rho^{\otimes N} = \Pi_{i=1}^N \bar\rho$ with

$$\partial_t \bar\rho + \mathrm{div}(\bar\rho u) = \sigma \Delta \bar\rho, \qquad u = -\nabla V \star_x \bar\rho$$

where $\bar\rho$ is a 1-particle distribution namely $\bar\rho \geq 0$ and $\int \bar\rho = 1$. More precisely, the main objective is to prove (for $\sigma > 0$ fixed) that there exists constants $C_{T,\bar\rho,k} > 0$ and $\theta > 0$ such that

$$\| \rho_{N,k} - \Pi_{i=1}^k \bar\rho(t, x_i) \|_{L^1(\Pi^{kd})} \leq C_{T,\bar\rho,k} N^{-\theta}$$

where $\rho_{N,k}$ is the marginal of the system at a fixed rank k,

$$\rho_{N,k}(t, x_1, \ldots, x_k) = \int_{\mathcal{T}^{(N-k)\,d}} \rho_N(t, x_1, \ldots, x_N)\, dx_{k+1} \ldots x_N.$$

under assumptions of global existence of entropy-weak solutions ρ_N of the Liouville equation and global existence of classical solution $\bar\rho$ of the limiting system. Due to the Csiszár-Kullback-Pinsker Inequality, it suffices to control the quantity

$$\frac{1}{k} \int_{\Pi^{kd}} \rho_{N,k} \log \frac{\rho_{N,k}}{\bar\rho^{\otimes k}}$$

and therefore

$$\frac{1}{N} \int_{\Pi^{Nd}} \rho_N \log \frac{\rho_N}{\bar\rho_N}$$

due to the inequality

$$\frac{1}{k} \int_{\Pi^{kd}} \rho_{N,k} \log \frac{\rho_{N,k}}{\bar\rho^{\otimes k}} \leq \frac{1}{N} \int_{\Pi^{Nd}} \rho_N \log \frac{\rho_N}{\bar\rho_N}.$$

In the case $\sigma_N \to 0$ when $N \to +\infty$, the information will be related to a rescaled modulated potential energy similar to the one introduced by S. Serfaty.

As explained in [19] (and commented by F. Golse during the talk) the keys of the paper by S. Serfaty is to introduce a clever truncation of the kernel at a length-scale r_i depending on the point i and equal to a minimal distance from x_i to its nearest neighbors. Using such truncation it is possible to prove that the truncated energy can be controlled by the full energy and conversely. The next crucial result is that, even though positivity is lost, when using the r_i as truncation parameters she can still control for each time slice the expression by the modulated energy itself, up to a small error , and provided the limiting solution is regular enough. This, which is the most difficult part of her proof, uses two ingredients: the first is to re-express as a single integral in terms of a stress-energy tensor, and the second is to show that the expression is in fact close to the same expression with truncated fields.

In this document, we will explain how to construct an appropriate regularized kernel (Lemma 4.1) which in some sense play the role of the appropriate truncation in [20]. In the case $\sigma_N \to 0$, it will be used to control the contribution to the potential energy for close particles switching between the kernel and its regularization. It will also be used coupled

with a Fourier transform property to get rid of the re-expression as a single integral in terms of a stress-energy tensor. In the case $\sigma > 0$ fixed the Fourier transform hypothesis may be relaxed playing with the classical convexity inequality (6.5) and large deviation type estimates.

The paper is divided in seven sections. Section 1 is the present introduction. Section 2 is dedicated to the modulated free energy (2.1) (with G_N and $G_{\bar\rho_N}$ defined respectively by (2.2) and (2.3)) introduced by the authors in [5] which allows to make the link between [20] and [15]. The analysis is based on the important inequality (2.6) with (2.7) with Gronwall Lemma and some appropriate estimates. Section 3 concerns the assumptions (3.1)–(3.7) on the kernels and the main results Theorem 3.1 and Theorem 3.2. We focus on repulsive kernels and provide Theorems first in the case $\sigma_N \to 0$ when $N \to +\infty$ and secondly when σ is fixed. We provide simple comments for the attractive kernel corresponding to the Patlak-Keller-Segel system and refer to [4] for more details. Section 4 is dedicated to an important regularization lemma which will helpful to switch between the kernel and an appropriate regularized one. Section 5 and Section 6 concern the proofs of different controls from below and above coupled with inequality (2.6) with (2.7) firstly when $\sigma_N \to 0$ when $N \to +\infty$ and secondly when σ is fixed. Section 7 concludes the proofs of Theorem 3.1 and Theorem 3.2 from Gronwall arguments. In Section 8, we provide comments on the interesting attractive case corresponding to the particle approximation of the Patlak-Keller-Segel system. We end the paper by Section 9 with some comments and open problems.

2. Physics provides the right mathematical Object

As firstly introduced by the authors in [4], keeping advantage of the idea to introduce appropriate weights from [3], we define the following modulated free energy

$$E\Big(\frac{\rho_N}{G_N}|\frac{\bar\rho_N}{G_{\bar\rho_N}}\Big) = \frac{1}{N}\int_{\Pi^{dN}} \rho_N(t,X^N)\log\left(\frac{\rho_N(t,X^N)}{G_N(X^N)}\frac{G_{\bar\rho_N}(t,X^N)}{\bar\rho_N(t,X^N)}\right)dX^N. \tag{2.1}$$ freeenergy

where the Gibbs equilibrium G_N of the system and $G_{\bar\rho_N}$ the corresponding distribution where the exact field is replaced by the mean field limit according to the law $\bar\rho$ are given by

$$G_N(t,X^N) = \exp\left(-\frac{1}{2N\sigma}\sum_{i\neq j} V(x_i-x_j)\right) \tag{2.2}$$ GN

and

$$G_{\bar\rho_N}(t,X^N) = \exp\left(-\frac{1}{\sigma}\sum_{i=1}^N V\star\bar\rho(x_i) + \frac{N}{2\sigma}\int_{\Pi^d} V\star\bar\rho\bar\rho\right) \tag{2.3}$$ G

It may be written

$$E\Big(\frac{\rho_N}{G_N}|\frac{\bar\rho_N}{G_{\bar\rho_N}}\Big) = \mathcal{H}_N(\rho_N|\bar\rho_N) + \mathcal{K}_N(G_N|G_{\bar\rho_N})$$

where

$$\mathcal{H}_N(\rho_N|\bar\rho_N) = \frac{1}{N}\int_{\Pi^{dN}} \rho_N(t,X^N)\log\left(\frac{\rho_N(t,X^N)}{\bar\rho_N(t,X^N)}\right)dX^N \tag{2.4}$$ relativeentr

is exactly the relative entropy introduced by Jabin-Wang and

$$\mathcal{K}_N(G_N|G_{\bar\rho_N}) = -\frac{1}{N}\int_{\Pi^{dN}} \rho_N(t,X^N)\log\left(\frac{G_N(X^N)}{G_{\bar\rho_N}(t,X^N)}\right)dX^N \tag{2.5}$$

modulatedene

is the expectation of the modulated potential energy introduced by S. Serfaty multiplied by $1/\sigma$. It is then possible to show that the free energy has the right algebraic structure for any V even. More precisely, using smoothing properties and definition of global entropy-weak solution of the Liouville system and global classical solution of the limit system, we get the inequality

$$\begin{aligned} E_N\left(\frac{\rho_N}{G_N}\mid\frac{\bar\rho_N}{G_{\bar\rho_N}}\right)(t) + \frac{\sigma}{N}\int_0^t\int_{\Pi^{d\,N}} d\rho_N\left|\nabla\log\frac{\rho_N}{\bar\rho_N} - \nabla\log\frac{G_N}{G_{\bar\rho_N}}\right|^2 \\ \leq E_N\left(\frac{\rho_N}{G_N}\mid\frac{\bar\rho_N}{G_{\bar\rho_N}}\right)(0) + I_N \end{aligned} \tag{2.6}$$

INEG

where

$$\begin{aligned} I_N = -\frac{1}{2}\int_0^t\int_{\Pi^{dN}}\int_{\Pi^{2\,d}\cap\{x\neq y\}} \nabla V(x-y)\cdot\left(\nabla\log\frac{\bar\rho}{G_{\bar\rho}}(x) - \nabla\log\frac{\bar\rho}{G_{\bar\rho}}(y)\right) \\ (d\mu_N - d\bar\rho)^{\otimes 2} d\rho_N, \end{aligned} \tag{2.7}$$

IN

where $\mu_N = \sum_{i=1}^N \delta(x-x_i)/N$ is the empirical measure. It is important to note that the right-hand side is exactly the expectation of the quantity obtained by S. Serfaty with the modulated potential energy when $\sigma = 0$ and to remark that the parasite terms involving $\mathrm{div}K$ in the work by P.–E. Jabin and Z. Wang have disappeared. In order to use Gronwall Lemma, the previous expression leaves two main points in the proof namely the existence $\theta > 0$ and $C > 0$ such that we have

I) *An upper bound of I_N given by* (2.7). If the viscosity $\sigma > 0$ is fixed namely

$$I_N \leq \int_0^t E_N\left(\frac{\rho_N}{G_N}\mid\frac{\bar\rho_N}{G_{\bar\rho_N}}\right)(s)\,ds + \frac{C}{N^\theta}$$

and if $\sigma_N \to 0$ when $N \to +\infty$

$$\sigma_N I_N \leq C\sigma_N\int_0^t \mathcal{K}_N\left(\frac{\rho_N}{G_N}\mid\frac{\bar\rho_N}{G_{\bar\rho_N}}\right)(s)\,ds + \frac{C}{N^\theta}$$

II) *A control from below on modulated quantities.* For $\sigma > 0$ fixed, the modulated free energy E_N is almost positive or more specifically that for some constant

$$E_N\left(\frac{\rho_N}{G_N}|\frac{\bar\rho_N}{G_{\bar\rho_N}}\right)(t) \geq \frac{1}{C}\mathcal{H}_N(\rho_N|\bar\rho_N)(t) - \frac{C}{N^\theta}$$

and, for $\sigma_N \to 0$ when $N \to +\infty$, the rescaled modulated potential energy is almost positive namely

$$\sigma_N\mathcal{K}_N \geq -\frac{C}{N^\theta}.$$

Remark 2.1. Combining the relative entropy with a modulated energy has bee successively used for various limit in kinetic theory such as quasi neutral limit (see [12], [17]) or Vlasov-Maxwell-Boltzmann to incompressible viscous EMHD (see [1]).

Kw PuSa ArSa

Remark 2.2. Note the presence of modulated Fisher type information

$$\mathcal{D} = \frac{\sigma}{N}\int_0^t \int_{\Pi^{d\,N}} d\rho_N \left| \nabla \log \frac{\rho_N}{\bar\rho_N} - \nabla \log \frac{G_N}{G_{\bar\rho_N}} \right|^2 \tag{2.8}$$ diffusive

in the inequality (IN 2.7).

3. Assumptions and main results.

We will split the discussions in three parts. The first part concerns the case with viscosity $\sigma \to 0$ when $N \to +\infty$. We show how to consider more general singular kernels than in [Se 19], [Du 8] extending their methods. The second part concerns a fixed viscosity $\sigma > 0$ using the modulated free energy where we indicate kernels that may be considered. In the last part, we give comments regarding the Patlak-Keller-Segel system which concerns an attractive kernels and viscosity $2d\sigma > \lambda$ where λ measures the intensity of the kernel.

I) **Repulsive Kernels.** In this part, the first assumptions on V are

$$V(-x) = V(x), \quad V \in L^p(\Pi^d) \text{ for some } p > 1 \tag{3.1}$$ hyp01

with following Fourier sign

$$\hat V(\xi) \geq 0 \text{ for all } \xi \in \mathbb{R}^d \tag{3.2}$$ hyp03

and the following pointwise controls for all $x \in \mathbb{T}^d$: There exists constants k and $C > 0$ such that

$$|\nabla V(x)| \leq \frac{C}{|x|^k}, \quad |\nabla^2 V(x)| \leq \frac{C}{|x|^{k'}} \qquad \text{for some } k, k' > 1/2 \tag{3.3}$$ hyp001

and

$$|\nabla V(x)| \leq C\,\frac{V(x)}{|x|}. \tag{3.4}$$ hyp02

We also assume that

$$\lim_{|x|\to 0} V(x) = +\infty, \qquad V(x) \leq C\,V(y) \text{ for all } |y| \leq 2|x| \tag{3.5}$$ hyp04

and

$$|\nabla_\xi \hat V(\xi)| \leq \frac{C}{1+|\xi|}\left(\hat V(\xi) + f(\sigma)\frac{1}{1+|\xi|^{d-\alpha}}\right) \text{ with } 0 < \alpha < d \text{ for all } \xi \in \mathbb{R}^d \tag{3.6}$$ hyp05

where

$$f(\sigma) = 0 \text{ if } \sigma \to 0 \text{ and } f(\sigma) = 1 \text{ if } \sigma \text{ is fixed.} \tag{3.7}$$ hyp06

Remark 3.1. Remark that Riesz and Coulomb kernels perturbed to get periodic kernels satisfy hypothesis (hyp01–hyp06). Note that constraints $k, k' > 1/2$ are chosen for simplicity in the argument: the results cover any k and k'.

I-1) *Case $\sigma_N \to 0$ and repulsive kernels.* The convergence rate theorem reads as follows

Conv **Theorem 3.1.** *Assume $K = -\nabla V$ with V satisfying (hyp01–hyp06) with $f(\sigma) = 0$. Consider $\bar\rho$ a smooth enough solution with $\inf \bar\rho > 0$. There exists constants $C > 0$ and a function $\eta(N)$ with $\eta(N) \to 0$ as $N \to \infty$ s.t. for $\bar\rho_N = \Pi_{i=1}^N \bar\rho(t, x_i)$, and for the joint law ρ_N on Π^{dN} of any entropy-weak solution to the SDE system,*

$$\sigma_N\,\mathcal{K}_N(t) \leq e^{C_{\bar\rho}\,\|K\|\,t}\,\left(\sigma_N\,\mathcal{K}_N(t=0) + \sigma_N\,\mathcal{H}_N(t=0) + \eta(N)\right),$$

Hence if $\sigma_N \mathcal{K}_N^0 + \sigma_N \mathcal{H}_N^0 \leq \eta(N)$*, for any fixed marginal* $\rho_{N,k}$

$$W_1(\rho_{N,k},\ \Pi_{i=1}^k \bar{\rho}(t,x_i)) \leq C_{T,\bar{\rho},k}\, \eta(N),$$

where W_1 *is the Wasserstein distance.*

I-2) *Case* $\sigma > 0$ *fixed with respect to* N *and repulsive kernels.* The convergence rate theorem reads as follows

Convgene

Theorem 3.2. *Assume* $K = -\nabla V$ *with* V *satisfying* (3.1)–(3.7) *with* $f(\sigma) = 1$*. Consider* $\bar{\rho}$ *a smooth enough solution with* $\inf \bar{\rho} > 0$*. There exists constants* $C > 0$ *and* $\theta > 0$ *s.t. for* $\bar{\rho}_N = \Pi_{i=1}^N \bar{\rho}(t,x_i)$*, and for the joint law* ρ_N *on* Π^{dN} *of any entropy-weak solution to the SDE system,*

$$H_N(t) + |\mathcal{K}_N(t)| \leq e^{C_{\bar{\rho}}\,\|K\|\,t} \left(H_N(t=0) + |\mathcal{K}_N(t=0)| + \frac{C}{N^\theta} \right),$$

Hence if $H_N^0 + |\mathcal{K}_N^0| \leq C\, N^{-\theta}$*, for any fixed marginal* $\rho_{N,k}$

$$\|\rho_{N,k} - \Pi_{i=1}^k \bar{\rho}(t,x_i)\|_{L^1(\Pi^{k\,d})} \leq C_{T,\bar{\rho},k}\, N^{-\theta}.$$

Remark 3.2. In the case $\sigma > 0$ fixed, it is possible to enlarge the class of kernels for instance assuming only $V \geq -C$ with $C > 0$ and choosing $\alpha = 0$ in (3.6). To allow $\alpha = 0$ requires a novel large deviation inequality, similar in spirit to Prop. 6.1 but for singular attractive potentials. More precisely, we can use the following proposition for which the proof is more complex and we refer to our upcoming article [5] for it. There exists $\eta_0 > 0$, $\theta > 0$ s.t. if $G(x) \leq C \log \frac{1}{|x|} + C$ with $C > 0$ and $\eta \leq \eta_0$ then

$$\int_{\Pi^{d\,N}} \int_{\{x \neq y\} \cap \{|x-y| \leq \eta\}} G(x-y)\, (d\mu_N - d\bar{\rho})^{\otimes 2}\, d\rho_N \leq C\, \mathcal{H}_N + \frac{C}{N^\theta}.$$

Note that such control is also central in the proof for the attractive Patlak-Keller-Segel kernel, see [4] for a sketch.

II) **An Attractive case.** This part concerns the first quantitative estimate related to particle approximation of the Patlak-Keller-Segel system. Namely we get the same conclusion than in Theorem 3.2 for

$$V = \lambda \log |x| + \textit{perturbation}$$

if $0 \leq \lambda < 2d\sigma$ with the perturbation being regular kernel to get V periodic. We refer the readers to [4] for the explanation of the steps in the Proof. Some comments are given at the end of the present paper.

4. An important regularization lemma

In our approach an important Lemma will be the construction of an appropriate regularized kernel using hypothesis (3.1)–(3.5). More precisely we prove the following Lemma

Lemma 4.1. *Let V satisfying (3.1)–(3.4). Then there exists a smooth approximation V_ε of V and a function of $\eta(\varepsilon)$ with $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that*

$$\hat{V}_\varepsilon \geq 0,$$

$$\|V_\varepsilon - V\|_{L^1(\Pi^d)} \leq \eta(\varepsilon), \quad \|\mathbb{I}_{|x|\geq\delta}\,(V - V_\varepsilon)\|_{L^1} \leq C\,\frac{\varepsilon}{\delta^k}, \quad \|\mathbb{I}_{|x|\geq\delta}\,(\nabla V - \nabla V_\varepsilon)\|_{L^1} \leq C\,\frac{\varepsilon}{\delta^{k'}},$$

$$V_\varepsilon(x) \leq V(x) + \varepsilon \text{ for all } x.$$

lemmatruncate

This step asks for appropriate regularization which in some sense depend on the total number of particles N: It uses the pointwise properties of the kernels (3.3)–(3.5) which includes a doubling variable property.

Proof. Some preliminary controls. Consider a smooth kernel K^1 with compact support in $B(0,1)$ and s.t. $\hat{K}^1 \geq 0$. Observe that for $|x| \geq 2\,\delta$ and since $|\nabla V(y)| \leq \frac{C}{|x|^k}$ for $|y| \geq |x|/2$,

$$|K^1_\delta \star V(x) - V(x)| \leq \int_{|z|\leq 1} K^1(z)\,|V(x - \delta\,z) - V(x)|\,dz \leq C\,\frac{\delta}{|x|^k}.$$

Therefore

$$K^1_\delta \star V(x) \leq V(x) + C\,\delta^{1/2}, \quad \forall\, |x| \geq \delta^{1/2k}. \tag{4.1}$$

|x|geqdelta

On the other hand, we also have that $K^1_\delta \star V(x) \leq C\,\delta^{-d}\,\|V\|_{L^1}$. Since $V(x) \to \infty$ as $|x| \to 0$, we also have that for some increasing function $f(\delta)$

$$K^1_\delta \star V(x) \leq V(x), \quad \forall\, |x| \leq f(\delta). \tag{4.2}$$

|x|leqdelta

As k is chosen such that $k > 1/2$ then $\delta^{1/2k} \geq 2\delta$, so we need to be more precise where $f(\delta) \leq |x| \leq \delta^{1/2k}$.

Construction of an appropriate regularized kernel W_ε.

Case $|x| \geq 2\delta$. First of all we notice that by the doubling property, we have directly if $|x| \geq 2\,\delta$ that $|x|/2 \leq |x - \delta\,z| \leq 2|x|$ for $|z| \leq 1$ and thus

$$K^1_\delta \star V(x) \leq C\,V(x), \quad \forall\, |x| \geq 2\,\delta. \tag{4.3}$$

|x|sim2delta

Case $|x| \leq 2\delta$. For $|x| \leq 2\,\delta$, the doubling property on its own only gives that

$$K^1_\delta \star V(x) \leq C\,K^1_\delta \star V(0) \leq C\,\delta^{-d} \int_{B(0,\delta)} V(y)\,dy. \tag{4.4}$$

K*V(0)

We now define a sequence $\delta_n \to 0$ s.t.

$$\int_{B(0,\delta_n)} V(y)\,dy \leq C \int_{\delta_n/2\leq|y|\leq\delta_n} V(y)\,dy. \tag{4.5}$$

defdelta_n

The existence of such a sequence is straightforward to show by contradiction, as otherwise, we would have for some $\bar{\delta}$ and all $\delta \leq \bar{\delta}$ that

$$\int_{B(0,\delta/2)} V(y)\,dy \geq (C-1) \int_{\delta/2\leq|y|\leq\delta} V(y)\,dy.$$

By induction, this would imply that

$$\int_{B(0,\bar{\delta}/2^k)} V(y)\,dy \geq (1 - 1/C)^{k-1}(C-1) \int_{\bar{\delta}/2\leq|y|\leq\bar{\delta}} V(y)\,dy. \tag{4.6}$$

contrad

Using the L^p bound on V, we have

$$\int_{B(0,\bar\delta/2^k)} V(y)\,dy \le C'\,2^{-kp/(p-1)}$$

this provides a contradiction with (4.6) if $C-1$ is too large. Note by the way that if we assume some explicit rate on V then we can have explicit bound on how large δ_n can be w.r.t. δ_{n+1}.

If $\delta=\delta_n$, then (4.5) and (4.4) together implies that

$$K^1_\delta \star V(x) \le C\,\delta_n^{-d}\int_{\delta_n/2\le|y|\le\delta_n} V(y)\,dy.$$

This is where we use (3.4) which implies that if $|x|\le 2\,\delta_n$ and $\delta_n/2\le|y|\le\delta_n$ then $V(y)\le C\,V(x)$. Hence eventually for $\delta=\delta_n$, we obtain the counterpart to (4.3) and find

$$K_{\delta_n}\star V(x)\le C\,V(x),\qquad \forall x. \tag{4.7}$$

roughgeneral

Now for every ε, we are going to choose M of the parameters δ_n, with M the integer part of ε large, and define first

$$W_\varepsilon=\frac{1}{M}\sum_{i=1}^M K_{\delta_{n_i}}\star V.$$

We start by taking $\delta_{n_1}\le\varepsilon^2/C$ and we then define the n_i recursively s.t.

$$\delta_{n_{i+1}}\le\min(f(\delta_{n_i}),\ \delta_{n_i}^{2k}).$$

We of course have automatically that $\hat W_\varepsilon\ge 0$. Moreover since $\max\delta_{n_i}\to 0$ as $\varepsilon\to 0$, the standard approximation by convolution shows that $\|V-W_\varepsilon\|_{L^1}\to 0$. By using (3.3), we also directly have that

$$\|\mathbb{I}_{|x|\ge\delta}\,(V-W_\varepsilon)\|_{L^1}\le C\,\frac{\max\delta_{n_i}}{\delta^k}\le C\,\frac{\varepsilon}{\delta^k},$$

and similarly

$$\|\mathbb{I}_{|x|\ge\delta}\,(\nabla V-\nabla W_\varepsilon)\|_{L^1}\le C\,\frac{\varepsilon}{\delta^{k'}}.$$

It only remains to compare W_ε and V. For this consider any x, if $|x|\ge\delta_{n_1}^{1/2k}$ then (4.1) directly implies that $W_\varepsilon(x)\le V(x)+\varepsilon$. If $|x|\le f(\delta_{n_M})$ then (4.2) also directly implies that $W_\varepsilon(x)\le V(x)$.

This only leaves the case where $|x|$ is somewhere between δ_{n_M} and δ_{n_1}. In that case, there exists i s.t. $\delta_{n_{i+1}}\le|x|\le\delta_{n_i}$. By the definition of the δ_{n_j}, one has that $|x|\le f(\delta_{n_j})$ if $j<i$ and $|x|\ge\delta_{n_j}^{1/2k}$ if $j>i+1$. Using again (4.1) and (4.2), we then have

$$K_{\delta_{n_j}}\star V(x)\le V(x)+\varepsilon\quad\text{if } j>i+1,\qquad K_{\delta_{n_j}}\star V(x)\le V(x)\quad\text{if } j<i.$$

Using (4.7) for $j=i$ and $j=i+1$, we get

$$W_\varepsilon(x)\le(1+2\,C/M)\,V(x)+\varepsilon=(1+2\,C\,\varepsilon)\,V(x)+\varepsilon.$$

Definition of V_ϵ and conclusion. This leads to the final definition $V_\varepsilon=W_\varepsilon/(1+2\,C\,\varepsilon)$ which indeed satisfies $V_\varepsilon\le V+\varepsilon$, $\hat V_\varepsilon\ge 0$ and still $\|V_\varepsilon-V\|_{L^1}\to 0$ together with the other convergences since obviously $\|V_\varepsilon-W_\varepsilon\|_{L^1}\to 0$. □

5. Control of terms for Repulsive Kernels with vanishing viscosity $\sigma_N \to 0$

As a corollary of Lemma 4.1, we obtain a straightforward control on the contribution to the potential energy from close particles

Lemma 5.1. *Under the assumptions of Lemma 4.1, there exists a function $\eta(N)$ with $\eta(N) \to 0$ as $N \to \infty$, one has that*

$$\sigma_N \mathcal{K}_N(t) \geq -\eta(N).$$

Furthermore there exists $\eta(\delta)$ with $\eta(\delta) \to 0$ as $\eta \to 0$, such that for any δ

$$\mathbb{E}\Big(\frac{1}{N^2}\sum_{i\neq j} V(x_i - x_j)\,\mathbb{I}_{|x_i-x_j|\leq\delta}\Big) \leq 4\,\sigma_N\,\mathcal{K}_N(t) + \eta(N) + \eta(\delta).$$

KNgeq0

Proof. We start by noticing that

$$\sigma_N\,\mathcal{K}_N(t) = \frac{1}{2}\int_{|x-y|\geq\delta} V(x-y)\,(\mu_N - \bar\rho)^{\otimes 2} + \frac{1}{2}\int_{|x-y|<\delta,\,x\neq y} V(x-y)\,(\mu_N-\bar\rho)^{\otimes 2}.$$

By using the regularity of $\bar\rho$ and the L^p integrability of V, we may bound from below the second term in the right-hand side by

$$\int_{|x-y|<\delta,\,x\neq y} V(x-y)\,(\mu_N-\bar\rho)^{\otimes 2} \geq \frac{1}{N^2}\sum_{i\neq j} V(x_i-x_j)\,\mathbb{I}_{|x_i-x_j|\leq\delta} - C\,\delta^\alpha,$$

for some positive exponent α. Using Lemma 4.1 and more precisely the inequality $V_\varepsilon \leq V + \varepsilon$, we obtain that

$$\int_{|x-y|<\delta,\,x\neq y} V(x-y)\,(\mu_N-\bar\rho)^{\otimes 2} \geq \frac{1}{N^2}\sum_{i\neq j} V_\varepsilon(x_i-x_j)\,\mathbb{I}_{|x_i-x_j|\leq\delta} - C\,\delta^\alpha - \varepsilon.$$

Observe now that by the second point in Lemma 4.1, we have that

$$\int_{|x-y|\geq\delta} V(x-y)\,(\mu_N-\bar\rho)^{\otimes 2} \geq \int_{|x-y|\geq\delta} V_\varepsilon(x-y)\,(\mu_N-\bar\rho)^{\otimes 2} - C\,\frac{\varepsilon}{\delta^k}. \tag{5.1}$$

intermed0

Therefore by summing, we obtain that

$$\int_{x\neq y} V(x-y)\,(\mu_N-\bar\rho)^{\otimes 2} \geq \int_{x\neq y} V_\varepsilon(x-y)\,(\mu_N-\bar\rho)^{\otimes 2} - C\,\delta^\alpha - \varepsilon - C\,\frac{\varepsilon}{\delta^k}.$$

We may simply add the diagonal to find

$$\int_{x\neq y} V(x-y)\,(\mu_N-\bar\rho)^{\otimes 2} \geq \int V_\varepsilon(x-y)\,(\mu_N-\bar\rho)^{\otimes 2} - \frac{C\,\|V_\varepsilon\|_{L^\infty}}{N} - C\,\delta^\alpha - \varepsilon - C\,\frac{\varepsilon}{\delta^k}. \tag{5.2}$$

intermed1

Since $\hat V_\varepsilon \geq 0$, this yields

$$\int_{x\neq y} V(x-y)\,(\mu_N-\bar\rho)^{\otimes 2} \geq -\frac{C\,\|V_\varepsilon\|_{L^\infty}}{N} - C\,\delta^\alpha - \varepsilon - C\,\frac{\varepsilon}{\delta^k},$$

and conclude the first point by optimizing in ε and δ.

To prove the second point we first remind that

$$\frac{1}{N^2}\sum_{i\neq j} V(x_i-x_j)\,\mathbb{I}_{|x_i-x_j|\leq\delta} \leq \int_{|x-y|<\delta,\, x\neq y} V(x-y)\,(\mu_N-\bar\rho)^{\otimes 2} + C\,\delta^\alpha,$$

and thus

$$\frac{1}{N^2}\sum_{i\neq j} V(x_i-x_j)\,\mathbb{I}_{|x_i-x_j|\leq\delta} \leq \int_{x\neq y} V(x-y)\,(\mu_N-\bar\rho)^{\otimes 2} + C\,\delta^\alpha - \int_{|x-y|\geq\delta,\, x\neq y} V(x-y)\,(\mu_N-\bar\rho)^{\otimes 2}.$$

Using again (5.1), we get

$$\frac{1}{N^2}\sum_{i\neq j} V(x_i-x_j)\,\mathbb{I}_{|x_i-x_j|\leq\delta} \leq \int_{x\neq y} V(x-y)\,(\mu_N-\bar\rho)^{\otimes 2} + C\,\delta^\alpha - \int_{|x-y|\geq\delta,\, x\neq y} V_\varepsilon(x-y)\,(\mu_N-\bar\rho)^{\otimes 2} + C\,\frac{\varepsilon}{\delta^k}.$$

Remark that since we are on the torus, $\mathbb{I}_{|x|\geq\delta}$ has L^1 norm less than 1. Therefore the Fourier transform of $V_\varepsilon\,\mathbb{I}_{|x|\geq\delta}$ is dominated by $\hat V_\varepsilon$ and

$$-\int_{|x-y|\geq\delta,\, x\neq y} V_\varepsilon(x-y)\,(\mu_N-\bar\rho)^{\otimes 2} \leq -\int V_\varepsilon(x-y)\,(\mu_N-\bar\rho)^{\otimes 2}.$$

Appealing to (5.2), we hence finally get

$$\frac{1}{N^2}\sum_{i\neq j} V(x_i-x_j)\,\mathbb{I}_{|x_i-x_j|\leq\delta} \leq 2\int_{x\neq y} V(x-y)\,(\mu_N-\bar\rho)^{\otimes 2} + C\,\delta^\alpha + C\,\frac{\varepsilon}{\delta^k} + \varepsilon + C\,\frac{\|V_\varepsilon\|_{L^\infty}}{N},$$

which concludes the second point, again by optimizing in ε. □

We need to control terms from above like

$$I_N = -\int_{\Pi^{dN}} d\rho_N \int_{\Pi^{2d}\cap\{x\neq y\}} \nabla V(x-y)\cdot(\psi(x)-\psi(y))(d\mu_N-d\bar\rho)^{\otimes 2},$$

for ψ regular enough in terms of the potential energy. We use Fourier transform for the regularized kernel that not use explicit formula of the kernel as in [19], [8]. This procedure allows to treat more general kernels because it is not based on the reformulation of the energy in terms of potential or extension representation (for the fractional laplacian) by Caffarelli-Silvestre as in [19], [20], [8]. More precisely, we prove that

Lemma 5.2. *Let $\psi\in W^{1,\infty}(\Pi^d)$ and if V satisfies (3.1)–(3.2), then for any measure ν, we have that*

$$-\int \nabla V(x-y)(\psi(x)-\psi(y))\nu^{\otimes 2} \leq C\int |\hat\nu(\xi)|^2\,\hat V(\xi)\,d\xi.$$

lemfourier

Remark 5.1. Remark that Riesz and Coulomb Kernel satisfy Hypothesis (3.1)–(3.6).

Proof. This is where we need (3.6) which gives studying $|\xi| \sim |\zeta|$

$$|\hat{V}(\xi) - \hat{V}(\zeta)| \leq \frac{C|\xi - \zeta|}{1 + |\zeta|}\hat{V}(\zeta)$$

using Gronwall Lemma. For ψ regular enough, we use the following calculation

$$\begin{aligned} -\int \nabla V(x-y)(\psi(x) - \psi(y))\nu^{\otimes 2} &= -\mathrm{Re}\int i(\xi\hat{V}(\xi) - \zeta\hat{V}(\zeta))\hat{\psi}(\xi-\zeta)\overline{\hat{\nu}(\xi)}\hat{\nu}(\zeta)d\xi d\zeta \\ &= -\mathrm{Re}\int i\left(\xi(\hat{V}(\xi) - \hat{V}(\zeta)) + (\xi-\zeta)\hat{V}(\zeta)\right)\hat{\psi}(\xi-\zeta)\overline{\hat{\nu}(\xi)}\hat{\nu}(\zeta)d\xi d\zeta \\ &\leq C\int |\xi-\zeta||\hat{\psi}(\xi-\zeta)|\sqrt{\hat{V}(\xi)}\sqrt{\hat{V}(\zeta)}|\hat{\nu}(\zeta)||\hat{\nu}(\xi)|d\xi d\zeta \end{aligned}$$

and therefore, using some regularity on ψ, by Cauchy-Schwartz

$$-\int \nabla V(x-y)(\psi(x) - \psi(y))\nu^{\otimes 2} \leq C\int |\hat{\nu}(\zeta)|^2\hat{V}(\zeta)\,d\zeta.$$

□

Of course we cannot directly use Lemma 6.7 with V on I_N as

$$\int V(x-y)\,(d\mu_N - d\bar{\rho})^{\otimes 2}$$

will in general be infinite as the diagonal is not removed. But we can now easily combine Lemma 5.2 with Lemma 5.1 to obtain

Corollary 5.1. *Assume that $\psi \in W^{s,\infty}$ and that V satisfies (3.1)–(3.6). Then there exists a function $\eta(N)$ with $\eta(N) \to 0$ as $N \to \infty$ and such that*

$$I_N = -\int_{\Pi^{dN}} d\rho_N \int_{\Pi^{2d}\cap\{x\neq y\}} \nabla V(x-y)\cdot(\psi(x)-\psi(y))(d\mu_N - d\bar{\rho})^{\otimes 2} \leq C\,\sigma_N\,\mathcal{K}_N(t) + \eta(N).$$

Proof. The basic strategy is again to split I_N into two parts

$$\begin{aligned} I_N = &-\int_{\Pi^{dN}} d\rho_N \int_{\{|x-y|\leq\delta\}\cap\{x\neq y\}} \nabla V(x-y)\cdot(\psi(x)-\psi(y))(d\mu_N - d\bar{\rho})^{\otimes 2} \\ &-\int_{\Pi^{dN}} d\rho_N \int_{\{|x-y|>\delta\}} \nabla V(x-y)\cdot(\psi(x)-\psi(y))(d\mu_N - d\bar{\rho})^{\otimes 2}. \end{aligned}$$

For the second term in the right-hand side, we want to replace ∇V by ∇V_ε. We simply use the second point of Lemma 4.1 again (similarly to the obtention of (5.1) in Lemma 5.1) to get that

$$\begin{aligned} &-\int_{\{|x-y|>\delta\}} \nabla V(x-y)\cdot(\psi(x)-\psi(y))(d\mu_N - d\bar{\rho})^{\otimes 2} \\ &\qquad \leq -\int_{\{|x-y|>\delta\}} \nabla V_\varepsilon(x-y)\cdot(\psi(x)-\psi(y))(d\mu_N - d\bar{\rho})^{\otimes 2} + C\,\frac{\varepsilon}{\delta^{k'}}. \end{aligned} \tag{5.3}$$

For the first term in the right-hand side, we first use the regularity of $\bar\rho$ together with the L^p bound on V from (3.1) to deduce that

$$
\begin{aligned}
&-\int_{\{|x-y|\le\delta\}\cap\{x\neq y\}} \nabla V(x-y)\cdot(\psi(x)-\psi(y))(d\mu_N-d\bar\rho)^{\otimes 2}\\
&\quad\le -\int_{\{|x-y|\le\delta\}\cap\{x\neq y\}} \nabla V(x-y)\cdot(\psi(x)-\psi(y))d\mu_N^{\otimes 2}+C\,\delta^\alpha,
\end{aligned}
$$

for some $\alpha>0$. We know use the pointwise bound on ∇V from (3.4) and the Lipschitz bound on ψ to obtain that

$$
\begin{aligned}
&-\int_{\{|x-y|\le\delta\}\cap\{x\neq y\}} \nabla V(x-y)\cdot(\psi(x)-\psi(y))(d\mu_N-d\bar\rho)^{\otimes 2}\\
&\quad\le C\int_{\{|x-y|\le\delta\}\cap\{x\neq y\}} V(x-y)\,d\mu_N^{\otimes 2}+C\,\delta^\alpha.
\end{aligned}
$$

Using the second point in Lemma 5.1, we hence have that

$$
\begin{aligned}
-\int_{\Pi^{dN}} d\rho_N \int_{\{|x-y|\le\delta\}\cap\{x\neq y\}} &\nabla V(x-y)\cdot(\psi(x)-\psi(y))(d\mu_N-d\bar\rho)^{\otimes 2}\\
&\le C\,\sigma_N\,\mathcal{K}_N+\eta(N)+\eta(\delta).
\end{aligned}
\tag{5.4}
$$

intermed1'

By the construction of V_ε the same estimate applies

$$
\begin{aligned}
-\int_{\Pi^{dN}} d\rho_N \int_{\{|x-y|\le\delta\}\cap\{x\neq y\}} &\nabla V_\varepsilon(x-y)\cdot(\psi(x)-\psi(y))(d\mu_N-d\bar\rho)^{\otimes 2}\\
&\le C\,\sigma_N\,\mathcal{K}_N+\eta(N)+\eta(\delta).
\end{aligned}
\tag{5.5}
$$

intermed2'

We may combine (5.5) with (5.3) to obtain that

$$
\begin{aligned}
&-\int_{\Pi^{dN}} d\rho_N \int_{\{|x-y|>\delta\}\cap\{x\neq y\}} \nabla V(x-y)\cdot(\psi(x)-\psi(y))(d\mu_N-d\bar\rho)^{\otimes 2}\\
&\quad\le -\int_{\Pi^{dN}} d\rho_N \int_{\{x\neq y\}} \nabla V_\varepsilon(x-y)\cdot(\psi(x)-\psi(y))(d\mu_N-d\bar\rho)^{\otimes 2}\\
&\quad + C\,\frac{\varepsilon}{\delta^{k'}}+C\,\sigma_N\,\mathcal{K}_N+\eta(N)+\eta(\delta).
\end{aligned}
$$

Together with (5.4), this finally concludes that

$$
\begin{aligned}
&-\int_{\Pi^{dN}} d\rho_N \int_{\{x\neq y\}} \nabla V(x-y)\cdot(\psi(x)-\psi(y))(d\mu_N-d\bar\rho)^{\otimes 2}\\
&\quad\le -\int_{\Pi^{dN}} d\rho_N \int \nabla V_\varepsilon(x-y)\cdot(\psi(x)-\psi(y))(d\mu_N-d\bar\rho)^{\otimes 2}\\
&\quad + C\,\frac{\varepsilon}{\delta^{k'}}+C\,\sigma_N\,\mathcal{K}_N+\eta(N)+\eta(\delta).
\end{aligned}
\tag{5.6}
$$

intermed3'

We know apply Lemma lemfourier 5.2 on V_ε which by construction still satisfies (hyp01)-(hyp05) and this yields

$$
\begin{aligned}
&-\int_{\Pi^{dN}} d\rho_N \int_{\{x\neq y\}} \nabla V(x-y)\cdot(\psi(x)-\psi(y))(d\mu_N-d\bar\rho)^{\otimes 2}\\
&\qquad\le C\int_{\Pi^{dN}} d\rho_N \int V_\varepsilon(x-y)\,(d\mu_N-d\bar\rho)^{\otimes 2}\\
&\qquad + C\,\frac{\varepsilon}{\delta^{k'}} + C\,\sigma_N\,\mathcal{K}_N + \eta(N)+\eta(\delta).
\end{aligned}
$$

We may now remove the diagonal

$$
\begin{aligned}
&-\int_{\Pi^{dN}} d\rho_N \int_{\{x\neq y\}} \nabla V(x-y)\cdot(\psi(x)-\psi(y))(d\mu_N-d\bar\rho)^{\otimes 2}\\
&\qquad\le C\int_{\Pi^{dN}} d\rho_N \int_{\{x\neq y\}} V_\varepsilon(x-y)\,(d\mu_N-d\bar\rho)^{\otimes 2} + C\,\frac{\|V_\varepsilon\|_{L^\infty}}{N}\\
&\qquad + C\,\frac{\varepsilon}{\delta^{k'}} + C\,\sigma_N\,\mathcal{K}_N + \eta(N)+\eta(\delta),
\end{aligned}
$$

and using the third point in Lemma lemmatruncate 4.1,

$$
\begin{aligned}
&-\int_{\Pi^{dN}} d\rho_N \int_{\{x\neq y\}} \nabla V(x-y)\cdot(\psi(x)-\psi(y))(d\mu_N-d\bar\rho)^{\otimes 2}\\
&\qquad\le C\int_{\Pi^{dN}} d\rho_N \int_{\{x\neq y\}} V(x-y)\,(d\mu_N-d\bar\rho)^{\otimes 2} + C\,\varepsilon + C\,\frac{\|V_\varepsilon\|_{L^\infty}}{N}\\
&\qquad + C\,\frac{\varepsilon}{\delta^{k'}} + C\,\sigma_N\,\mathcal{K}_N + \eta(N)+\eta(\delta).
\end{aligned}
$$

The conclusion follows by optimizing in ε and δ. □

6. Control of terms for Repulsive Kernels with fixed viscosity $\sigma > 0$

6.1. **A large deviation result.**

LargeDeviation

Proposition 6.1. *There exists $\delta_0 > 0$ and some exponent $\theta > 0$, s.t. for any $\bar\rho \in L^\infty \cap \mathcal{P}(\Pi^d)$ with $\log\bar\rho \in W^{1,\infty}$, for any $W \in L^p(\Pi^{2d})$ for some $p > 1$ with $W(-x) = W(x)$, $W \ge 0$, $|\nabla W(x)| \le \frac{C}{|x|^k}$ and finally $\|W\|_{L^1} \le \delta_0$ then*

$$
\frac{1}{N}\log\int_{\Pi^{dN}} e^{-N\int_{\{x\neq y\}} W(x-y)\,(d\mu_N-d\bar\rho)^{\otimes^2}}\,\bar\rho^{\otimes^N}\,dX^N \le \frac{C}{N^\theta}.
$$

Proof. We denote

$$
F(\mu_N) = \int_{\{x\neq y\}} W(x-y)\,(d\mu_N-d\bar\rho)^{\otimes^2},
$$

and introduce a simple truncation W_ε of W by

$$
W_\varepsilon(x) = W(x)\,\mathbb{I}_{|x|\ge\varepsilon}.
$$

We define as well

$$
F_\varepsilon(\mu) = \int_{\Pi^{2d}\cap\{x\neq y\}} W_\varepsilon(x-y)\,(d\mu-d\bar\rho)^{\otimes 2}.
$$

We may expand F to find

$$\begin{aligned} F(\mu_N) &= \int_{\Pi^{2d}\cap\{x\neq y\}} W(x-y)\,(d\mu_N - d\bar\rho)^{\otimes 2} \\ &= \frac{1}{N^2}\sum_{i\neq j} W(x_i-x_j) - 2\frac{1}{N}\sum_i W\star\bar\rho(x_i) \\ &\quad + \int_{\Pi^{2d}\cap\{x\neq y\}} W(x-y)\,\bar\rho(x)\,\bar\rho(y)\,dx\,dy. \end{aligned}$$

Now note that by the L^p integrability on W and the L^∞ integrability on $\bar\rho$,

$$\begin{aligned} |W_\varepsilon\star\bar\rho(x) - W\star\bar\rho(x)| &\leq \|W_\varepsilon - W\|_{L^1}\,\|\bar\rho\|_{L^\infty} \leq \|W\,\mathbb{I}_{|z|\leq\varepsilon}\|_{L^1}\,\|\bar\rho\|_{L^\infty} \\ &\leq C\,\varepsilon^{1/2p^*}\,\|\bar\rho\|_{L^\infty}. \end{aligned}$$

Further note that since $W\geq 0$, we have that $W_\varepsilon(x)\leq W(x)$ so this directly implies that

$$-F(\mu_N) \leq -F_\varepsilon(\mu_N) + C\,\|\bar\rho\|_{L^\infty}\,\varepsilon^{1/2p^*}.$$

We are hence led to bounding

$$Z_{N,\varepsilon} = \frac{1}{N}\log\int_{\Pi^{dN}} e^{-N\int_{\Pi^{2d}} W_\varepsilon(x-y)\,(d\mu_N-d\bar\rho)^{\otimes 2}}\,\bar\rho^{\otimes^N}\,dX^N,$$

since

$$Z_N = \frac{1}{N}\log\int_{\Pi^{dN}} e^{-N\int_{\{x\neq y\}} W(x-y)\,(d\mu_N-d\bar\rho)^{\otimes 2}}\,\bar\rho^{\otimes^N}\,dX^N \leq Z_{N,\varepsilon} + C\,\varepsilon^{1/2p^*}. \tag{6.1}$$ boundZN

We now rely on a quantitative variant of a classical large deviation result

Theorem 6.1. *Assume that $\log\bar\rho\in W^{1,\infty}$ and that L is a standard convolution kernel. Then there exists a constant C depending only on d, L, s.t. for any $F:\ \mathcal{P}(\Pi^d)\to\mathbb{R}$, continuous on continuous functions, one has that*

$$\begin{aligned} \frac{1}{N}\log\int_{\Pi^{dN}} e^{-N\,F(L_\delta\star\mu_N)}\,\bar\rho_N\,dX^N &\leq I(F) \\ &+ \frac{C}{N^{1/(d+1)}\,\delta^{d/(d+1)}}\,(\log N + |\log\delta| + \|\log\bar\rho\|_{L^\infty}) + C\,\delta\,\|\log\bar\rho\|_{W^{1,\infty}}, \end{aligned}$$

where

$$I(F) = \max_{\mu\in\mathcal{P}(\Pi^d)} -\Big[F(\mu) + \int_{\Pi^d}\mu\,\log\frac{\mu}{\bar\rho}\,dx\Big]. \tag{6.2}$$ largedeviati

ntlargedeviation

Proof. The proof of Theorem variantlargedeviation 6.1 relies on classical arguments and we refer to our upcoming article for more details. Since $|\nabla W|\leq C/|x|^k$, the potential W_ε is smooth and hence

$$-F_\varepsilon(\mu_N) \leq -F_\varepsilon(L_\delta\star\mu_N) + C\,\frac{\delta}{\varepsilon^k}.$$

Using Theorem variantlargedeviation 6.1, we thus have that

$$\begin{aligned} Z_{N,\varepsilon} \leq & I(F_\varepsilon) + C\,\frac{\delta}{\varepsilon^k} + \frac{C}{N^{1/(d+1)}\,\delta^{d/(d+1)}}\,(\log N + |\log\delta| + \|\log\bar\rho\|_{L^\infty}) \\ & + C\,\delta\,\|\log\bar\rho\|_{W^{1,\infty}}. \end{aligned} \tag{6.3}$$ boundZNeps

The last step of the proof is hence to estimate $I(F_\varepsilon)$ for which we appeal to

Lemma 6.1. *For any $\bar\rho \in L^\infty$, there exists a truncation δ_0 s.t. for any $\tilde W$ with $\|\tilde W\|_{L^1} \le \delta_0$ and $W(x) \ge 0$ then, defining*

$$F_{\tilde W}(\mu) = \int_{\{x\neq y\}} \tilde W(x-y)\,(\mu(dx) - \bar\rho(x)\,dx)\,(\mu(dy) - \bar\rho(y)\,dy),$$

I(F)log *one then has that $I(F_{\tilde W}) = 0$.*

Assuming that Lemma 6.1 holds, we may apply it to $\tilde W = W_\varepsilon$ which implies $I(F_\varepsilon) = 0$ and finally combining (6.1) and (6.3)

$$\begin{aligned} Z_N \le & C\,\varepsilon^{1/2p^*} + C\,\frac{\delta}{\varepsilon^k} + \frac{C}{N^{1/(d+1)}\,\delta^{d/(d+1)}}\,(\log N + |\log\delta| + \|\log\bar\rho\|_{L^\infty}) \\ & + C\,\delta\,\|\log\bar\rho\|_{W^{1,\infty}}. \end{aligned}$$

We may immediately conclude by optimizing in ε and δ. □

Sketch of the proof of Lemma 6.1. Since $\tilde W \ge 0$, $I(F_{\tilde W})$ is coercive and by considering a maximizing sequence, we may find a maximum $\bar\mu$ which is bounded in $L\log L$. Such a maximum must satisfy that

$$1 + \log\frac{\bar\mu}{\bar\rho} + 2\,\tilde W \star (\bar\mu - \bar\rho) = \kappa,$$

where the constant κ is chosen so that $\int \bar\mu = 1$. This may be rewritten as

$$\bar\mu = \frac{\bar\rho}{M}\,e^{-2\,\tilde W\star(\bar\mu-\bar\rho)}, \quad M = \int \bar\rho\,e^{-2\,\tilde W\star(\bar\mu-\bar\rho)}\,dx.$$

Let us denote $u = -\tilde W \star (\bar\mu - \bar\rho)$ and to emphasize the dependence on u in M

$$M = M_u = \int \bar\rho\,e^{2\,u(x)}\,dx.$$

We observe that u is a solution to

$$u = -\tilde W \star \left(\bar\rho\,\left(\frac{e^{2\,u(x)}}{M_u} - 1\right)\right), \tag{6.4}$$ eulerlagrang

which is in fact a sort of non-linear elliptic equation. It is straightforward to show that the unique solution to (6.4) is $u = 0$ provided that $\|\tilde W\|_{L^1}$ is small enough. □

□

6.2. **Control on E_N.** The second ingredient to bound E_N from below is the following classical convexity inequality

$$\int_{\Pi^{dN}} \psi(X^N)\,d\rho_N \le \frac{1}{\alpha}\,\frac{1}{N}\int d\rho_N\,\log\frac{\rho_N}{\bar\rho_N} + \frac{1}{\alpha}\,\frac{1}{N}\log\int_{\Pi^{dN}} e^{\alpha N\,\psi(X^N)}\,d\bar\rho_N. \tag{6.5}$$ ineg

The proper control of E_N however requires truncating interactions after some distance so that we define

$$\mathcal{K}_N^\eta = \frac{1}{2\,\sigma}\int_{\Pi^{d\,N}}\int_{\{x\neq y\}} V(x-y)\,\chi(|x-y|/\eta)\,(d\mu_N - d\bar\rho)^{\otimes 2}\,\rho_N\,dX^N,$$

where χ is some smooth non-negative function with χ=1 on $[0,1]$ and $\chi = 0$ on $[2,\infty)$, together with $\hat{\chi} \geq 0$. Finally we define $E_N^\eta = \mathcal{H}_N + \mathcal{K}_N^\eta$. Combined with Prop. 6.1 this inequality lets us bound from below E_N^η as per

boundEN **Proposition 6.2.** *Assume that V satisfies (3.1)-(3.4) and (3.7) then there exists $\eta_0 > 0$ and $\theta > 0$ so that for any $\eta \leq \eta_0$*

$$E_N^\eta \geq -\frac{C}{N^\theta}.$$

Moreover for any $\delta \leq \eta$ with $\eta \leq \eta_0$

$$\mathbb{E}\left(\frac{1}{N^2}\sum_{i\neq j} V(x_i - x_j)\,\mathbb{I}_{|x_i-x_j|\leq\delta}\right) \leq E_N^\eta + \frac{C}{N^\theta} + C\,\delta^\theta.$$

Proof. For the first point, using (6.5) on $\mathcal{K}_N^\eta$, we find that

$$-\mathcal{K}_N^\eta \leq H_N + \frac{1}{N}\log\int_{\Pi^{dN}} e^{-\frac{N}{2\sigma}\int_{\{x\neq y\}} V(x-y)\,\chi(|x-y|/\eta)\,(d\mu_N - d\bar{\rho})^{\otimes 2}}\,d\bar{\rho}_N.$$

We now apply Prop. 6.1 to $W = \frac{1}{2\sigma}V(x)\,\chi(|x|/\eta)$. From (3.1) and (3.3), W trivially satisfies all assumptions of Prop. 6.1 with the exception of $\|W\|_{L^1} \leq \delta_0$. For this, we remark that

$$\|V(x)\,\chi(|x|/\eta)\|_{L^1} \leq \int_{|x|\leq 2\,\eta} V(x)\,dx \leq C\,\|V\|_{L^p}\,\eta^{1/p^*},$$

so that $\|W\|_{L^1} \leq \delta_0$ is ensured by choosing $\eta \leq \eta_0$ with η_0 small enough. Therefore Prop. 6.1 implies that

$$-\mathcal{K}_N^\eta \leq H_N + \frac{C}{N^\theta}.$$

For the second point, observe first that

$$\begin{aligned}\frac{1}{N^2}\sum_{i\neq j} V(x_i - x_j)\,\mathbb{I}_{|x_i-x_j|\leq\delta} &\leq \frac{1}{N^2}\sum_{i\neq j} V(x_i - x_j)\,\chi(|x_i-x_j|/\delta)\\ &\leq \int_{\{x\neq y\}} V(x-y)\,\chi(|x-y|/\delta)\,(d\mu_N - d\bar{\rho}_N)^{\otimes 2} + C\,\delta^\theta.\end{aligned}$$

Therefore

$$\begin{aligned}&\mathbb{E}\left(\frac{1}{2\,\sigma\,N^2}\sum_{i\neq j} V(x_i - x_j)\,\mathbb{I}_{|x_i-x_j|\leq\delta}\right) - E_N^\eta\\ &\quad\leq -H_N - \frac{1}{2\,\sigma}\int_{\{x\neq y\}} V(x-y)\,(\chi(|x-y|/\eta) - \chi(|x-y|/\delta))\,(d\mu_N - d\bar{\rho}_N)^{\otimes 2} + C\,\delta^\theta.\end{aligned}$$

Using (6.5)

$$\begin{aligned}&\mathbb{E}\left(\frac{1}{2\,\sigma\,N^2}\sum_{i\neq j} V(x_i - x_j)\,\mathbb{I}_{|x_i-x_j|\leq\delta}\right) - E_N^\eta\\ &\quad\leq \frac{1}{N}\log\int_{\Pi^{d\,N}} e^{-\frac{N}{2\,\sigma}\int_{\{x\neq y\}} V(x-y)\,(\chi(|x-y|/\eta) - \chi(|x-y|/\delta))\,(d\mu_N - d\bar{\rho}_N)^{\otimes 2}}\,d\bar{\rho}_N + C\,\delta^\theta.\end{aligned}$$

Of course $\chi(|x-y|/\eta)-\chi(|x-y|/\delta)\geq 0$ if $\delta\leq\eta$ so that we may again apply Prop. 6.1 to $W(x)=V(x)\,(\chi(|x|/\eta)-\chi(|x|/\delta))$ and find as claimed

$$\mathbb{E}\left(\frac{1}{2\,\sigma\,N^2}\sum_{i\neq j}V(x_i-x_j)\,\mathbb{I}_{|x_i-x_j|\leq\delta}\right)-E_N^\eta\leq C\,\delta^\theta+\frac{C}{N^\theta}.$$

□

6.3. **Control of the right-hand side.** As for the case $\sigma\to 0$, the goal is to control

$$I_N=-\int_{\Pi^{d\,N}}\int_{\{x\neq y\}}\nabla V(x-y)\cdot(\psi(x)-\psi(y))\,(d\mu_N-d\bar\rho)^{\otimes 2}\,d\rho_N.$$

The Fourier assumption (3.6) for $\sigma>0$ is more general that assumption (3.6) in the case $\sigma=0$ because we can use Inequality (6.5) and Theorem 6.1 to control the new term in the Fourier procedure. More precisely, we have

Lemma 6.2. *Assume that $\psi\in W^{k,+\infty}$ with k large enough and that V satisfies (3.1)-(3.4) and*

$$|\hat V(\xi)-\hat V(\zeta)|\leq C\frac{|\xi-\zeta|}{1+|\zeta|}\Big(\hat V(\zeta)+\frac{\chi(\zeta)}{1+|\zeta|^{d-\alpha}}\Big)\tag{6.6}$$ diffhatV

with $0<\alpha<d$ and $\chi\in L^\infty_\zeta$. Then for any measure ν, we have that

$$-\int\nabla V(x-y)(\psi(x)-\psi(y))\nu^{\otimes 2}\leq C\int|\hat\nu(\xi)|^2\,(\hat V(\xi)+\chi(\xi)/(1+|\xi|^{d-\alpha})\,d\xi.\tag{6.7}$$ fourier

lemfourier2

Proof. The proof follows the same lines as for Lemma 5.2. □

Note that the extra term in the right-hand side of inequality (6.7) is controlled by the relative entropy because going back to the physical space, we will get a term written as

$$\int_{x\neq y}G(x-y)(\mu_N-\bar\rho)^{\otimes 2}(dxdy)$$

with $G(x)\underset{\sim}{}|x|^{-\alpha}\in L^p$ for some $p>1$. We can now prove the equivalent of Corollary 5.1.

olINnonvanishing

Corollary 6.1. *Assume that $\psi\in W^{s,\infty}$ and that V satisfies all assumptions (3.1)–(3.7). Then there exists η_0 and $\theta>0$ s.t. if $\eta\leq\eta_0$*

$$I_N=-\int_{\Pi^{dN}}d\rho_N\int_{\Pi^{2d}\cap\{x\neq y\}}\nabla V(x-y)\cdot(\psi(x)-\psi(y))(d\mu_N-d\bar\rho)^{\otimes 2}\leq C\,E_N^\eta(t)+\frac{C}{N^\theta}.$$

Proof. The proof closely follows the one for Corollary (5.1). More precisely all arguments up to obtaining (5.6) are identical and hence we have still

$$\begin{aligned}I_N&=-\int_{\Pi^{dN}}d\rho_N\int_{\{x\neq y\}}\nabla V(x-y)\cdot(\psi(x)-\psi(y))(d\mu_N-d\bar\rho)^{\otimes 2}\\&\leq-\int_{\Pi^{dN}}d\rho_N\int_{\{x\neq y\}}\nabla V_\varepsilon(x-y)\cdot(\psi(x)-\psi(y))(d\mu_N-d\bar\rho)^{\otimes 2}\\&\quad+C\,\frac{\varepsilon}{\delta^{k'}}+C\,\mathcal{K}_N+\frac{C}{N^\theta}+C\,\eta^\theta.\end{aligned}$$

From this point, instead of Lemma 5.2, we apply Lemma 6.2. Let us take V_ε as constructed in Lemma 4.1. From Assumptions (3.6)-(3.7), we have that

$$|\hat V(\xi)-\hat V(\zeta)|\le C\frac{|\xi-\zeta|}{1+|\zeta|}\Big(\hat V(\zeta)+\frac{\hat K_\varepsilon(\zeta)}{1+|\zeta|^{d-\alpha}}\Big),$$

so that (6.6) is satisfied with $\chi=\hat K_\varepsilon$ and Lemma 6.2 yields

$$\begin{aligned} I_N &\le C\int_{\Pi^{dN}} d\rho_N \int_{\{x\neq y\}} V_\varepsilon(x-y)\,(d\mu_N-d\bar\rho)^{\otimes 2}\\ &\quad + C\int_{\Pi^{dN}} d\rho_N \int \frac{|\hat\mu_N-\hat{\bar\rho}|^2}{1+|\xi|^{d-\alpha}}\,\hat K_\varepsilon(\xi)\,d\xi\\ &\quad + C\,\frac{\varepsilon}{\delta^{k'}} + C\,\mathcal{K}_N + \frac{C}{N^\theta} + C\,\eta^\theta. \end{aligned}$$

Because we have that $V_\varepsilon\le C\,V+\varepsilon$, by Lemma 4.1, we deduce that

$$\begin{aligned} I_N &\le C\int_{\Pi^{dN}} d\rho_N \int_{\{x\neq y\}} V(x-y)\,(d\mu_N-d\bar\rho)^{\otimes 2}\\ &\quad + C\int_{\Pi^{dN}} d\rho_N \int \frac{|\hat\mu_N-\hat{\bar\rho}|^2}{1+|\xi|^{d-\alpha}}\,\hat K_\varepsilon(\xi)\,d\xi\\ &\quad + C\,\frac{\varepsilon}{\delta^{k'}} + C\,\mathcal{K}_N + \frac{C}{N^\theta} + C\,\eta^\theta. \end{aligned}$$

Using Prop. 6.2, we then have that

$$\begin{aligned} I_N &\le C\,E_N^\eta + C\int_{\Pi^{dN}} d\rho_N \int \frac{|\hat\mu_N-\hat{\bar\rho}|^2}{1+|\xi|^{d-\alpha}}\,\hat K_\varepsilon(\xi)\,d\xi\\ &\quad + C\,\frac{\varepsilon}{\delta^{k'}} + C\,\mathcal{K}_N + \frac{C}{N^\theta} + C\,\eta^\theta. \end{aligned} \tag{6.8}$$

intermed0''

By taking the inverse Fourier transform, we can write

$$\int_{\Pi^{dN}} d\rho_N \int \frac{|\hat\mu_N-\hat{\bar\rho}|^2}{1+|\xi|^{d}}\,K_\varepsilon(\xi)\,d\xi = \int_{\Pi^{dN}} d\rho_N \int_{\Pi^{2d}} K_\varepsilon\star G(x-y)\,(d\mu_N-d\bar\rho)^{\otimes 2},$$

where $\hat G(\xi)=(1+|\xi|)^{-d+\alpha}$ and hence $G(x)\le C\,\frac{1}{|x|^\alpha}$. We may remove the diagonal

$$\begin{aligned} &\int_{\Pi^{dN}} d\rho_N \int \frac{|\hat\mu_N-\hat{\bar\rho}|^2}{1+|\xi|^{d}}\,K_\varepsilon(\xi)\,d\xi\\ &\qquad\le \int_{\Pi^{dN}} d\rho_N \int_{\{x\neq y\}} K_\varepsilon\star G(x-y)\,(d\mu_N-d\bar\rho)^{\otimes 2} + C\,\frac{1}{\varepsilon^\alpha N}, \end{aligned}$$

and then simply bound $K_\varepsilon\star G$ by $1/|x|^\alpha$ to get

$$\begin{aligned} &\int_{\Pi^{dN}} d\rho_N \int \frac{|\hat\mu_N-\hat{\bar\rho}|^2}{1+|\xi|^{d}}\,K_\varepsilon(\xi)\,d\xi\\ &\qquad\le C\int_{\Pi^{dN}} d\rho_N \int_{\{x\neq y\}} \frac{1}{|x-y|^\alpha}\,(d\mu_N-d\bar\rho)^{\otimes 2} + C\,\frac{1}{\varepsilon^\alpha N}. \end{aligned}$$

By using the convexity inequality and Prop. 6.1, we finally obtain that

$$\int_{\Pi^{dN}} d\rho_N \int \frac{|\hat{\mu}_N - \hat{\bar{\rho}}|^2}{1+|\xi|^d} K_\varepsilon(\xi)\, d\xi \leq C\, \mathcal{H}_N + C\, \frac{1}{\varepsilon^\alpha N},$$

and inserting this into (6.8)

$$I_N \leq C\, E_N^\eta + C\, \mathcal{H}_N + C\, \frac{1}{\varepsilon^\alpha N} + C\, \frac{\varepsilon}{\delta^{k'}} + C\, \mathcal{K}_N + \frac{C}{N^\theta} + C\, \eta^\theta,$$

concluding by optimizing on δ and ε. □

7. Concluding the proofs of Theorems 3.1 and 3.2

The proofs of Theorems 3.1 and 3.2 now follow from a straightforward Gronwall argument starting from (2.6). For the proof of Theorem 3.1, we rescale (2.6) to deduce

$$\sigma_N\, E_N(t) \leq \sigma_N E_N(t=0) + \int_0^t I_N(s)\, ds,$$

with

$$I_N = \int_{\Pi^{d\,N}} \int_{\{x\neq y\}} \nabla V(x-y)\cdot(\psi(x)-\psi(y))\,(d\mu_N - d\bar{\rho})^{\otimes 2}\, d\rho_N,$$

and

$$\psi(x) = \frac{\sigma_N}{2}\, \nabla \log \frac{\bar{\rho}}{G_{\bar{\rho}}}(x).$$

Note that $\psi \in W^{s,\infty}$ uniformly in N since $\sigma_N \log G_{\bar{\rho}}$ is smooth uniformly in N (but the scaling by σ_N is needed here). We may hence apply Corollary 5.1 to obtain that

$$I_N \leq C\, \sigma_N\, \mathcal{K}_N(t) + \eta(N),$$

and hence recalling that $E_N = \mathcal{K}_N(t) + \mathcal{H}_N(t)$,

$$\sigma_N\, \mathcal{K}_N(t) + \sigma_N\, \mathcal{H}_N(t) \leq \sigma_N\, \mathcal{K}_N(t=0) + \sigma_N\, \mathcal{H}_N(t=0) + C \int_0^t \sigma_N\, \mathcal{K}_N(s)\, ds + \eta(N). \quad (7.1)$$

We recall from Lemma 5.1 that $\sigma_N\, \mathcal{K}_N \geq -\eta(N)$ and hence applying Gronwall lemma, we conclude from (7.1) that

$$\sigma_N\, \mathcal{K}_N(t) \leq e^{C\,t}\, \left(\sigma_N\, \mathcal{K}_N(t=0) + \sigma_N\, \mathcal{H}_N(t=0) + \eta(N)\right),$$

finishing the proof of Theorem 3.1. For the proof of Theorem 3.2, as mentioned just after (6.5), the control of E_N required truncating interactions after some distance and it remains now to control the long range part in V which we need to deal with on it own. The procedure is well explained in [5]. First we recall that this long-range part reads

$$W(x) = V(x)(1-\chi(|x|/\eta))$$

and we define $\mathcal{K}_N^W$ given by (2.5) with G_N^W and $G_{\bar{\rho}_N}^W$ replacing V by W in (2.2) and (2.3). The different results in Section 6 actually concern E_N^η and we need to evaluate the contribution of $\mathcal{K}_N^W$ which is the complement. Calculating its evolution in time and using the fact that the kernel V is $W^{2,\infty}$ far from 0, we can prove that

$$\frac{d}{dt}\mathcal{K}_N(G_N^W|G_{\bar{\rho}_N}^W) \leq C\mathcal{H}_N(\rho_N|\bar{\rho}_N) + \frac{C}{N}.$$

The interested readers are referred to [4] for more explanations and to [5] for the forthcoming single complete document. Plugging everybody together allows to get Theorem 3.2.

8. An attractive interesting case: The Patlak-Keller-Segel Kernel.

In space dimension 2, the Patlak-Keller-Segel system reads

$$\begin{cases} \partial_t \bar\rho + \operatorname{div}(\bar\rho\, u) = \sigma\, \Delta\, \bar\rho, \\ u = \nabla\Phi, \quad -\Delta\Phi = 2\,\pi\,\lambda\,\bar\rho. \end{cases}$$

This is an important system in biology for instance. Classical solutions for such system may not exist for all times as the singular attractive interactions can lead to concentration (see J. Dolbeault and B. Perthame [7]):

- Global existence of classical solution if $\lambda \leq 4\,\sigma$ (or $\lambda \leq 2\,d\,\sigma$).
- Always blow-up if $\lambda > 4\,\sigma$.

Based on the free energy of the system

$$\int \bar\rho\, \log \bar\rho\, dx + \frac{\lambda}{2} \int \log|x-y|\, \bar\rho(x)\, \bar\rho(y)\, dx\, dy.$$

Blanchet-Dolbeault-Perthame (see [2]) show existence of global weak satisfying free energy control with subcritical mass. Note that our modulated free energy may be seen as a particle version of such free energy. Particle approximation of the Patlak-Keller-Segel system has been studied by several authors such as [6], [10], [13] for example. Recently N. Fournier and B. Jourdain (see [9]) proved limit for $\lambda < \sigma$ with no quantitative estimates. In all these papers, the particle system is also studied from an existence view point which is an important and difficult question. This is not the objective of our study which focuses on quantitative estimates under assumptions of existence of solutions.

To prove a quantitative estimate between the particle approximation of the Patlak-Keller-Segel system and its formal limit, the upper-bound of (2.7) in the inequality (2.6) encoding the propagation of E_N is more simple than for the repulsive case. It uses that $|\nabla V(x)| \leq C/|x|$ and Theorem [15]. The lower bound control of $\mathcal{K}_N$ is more complicated: We choose to prove an upper bound of the opposite. The method is quite similar that the repulsive case when $\sigma > 0$ is fixed. It uses an appropriate cut-off smooth function close to singularity with a regularization of the kernel and the proof that for an appropriate cut-off size the large deviation function is zero. This uses the Logarithmic Hardy-Littlewood-Sobolev inequality to show the maximum is attained for η small enough for $\mu = \bar\rho$. The interested readers are referred to [4] for more explanations and to [5] for a single complete document.

9. Conclusion.

Using the right physics is the key in [4] to make the link between two important results namely [19] and [15]. This link allows to consider more general singular kernels with possible presence of viscosity. The method provides a statistical control with a large class of attractive-repulsive interactions but some works are needed to obtain the best convergence rate and it is not yet fully clear how general the interactions can be. Note that we have not used the presence of the diffusive term (2.8) in the inequality concerning the free-energy which could perhaps help to improve the rate of convergence when $\sigma > 0$

is fixed. It could be also interesting to study the case with blow-up for attractive kernels namely the super-critical cases. An other important problem could be non-gradient flow systems and hamiltonian systems. It could also be the extension of the work to the Keller-Segel parabolic-parabolic equations, see [21]. An existence result improving the results by [6], [9] for the particle approximation of the Patlak-Keller-Segel system is also a challenging and interesting problem.

Acknowledgments. The first author wants to thank F. Golse and F. Merle for the invitation to present the work [4] at IHÉS for the Laurent Schwartz seminar on November 2019 and for questions which have been the starting point to design the content of the document to help readers to understand the main steps of this new approach. The second author wants to thank S. Serfaty for sharing many insights on her result. The two first authors thank also L. Saint-Raymond for discussions on many-particle systems and their mean field limit. The first author is partially supported by the SingFlows project, grant ANR-18-CE40-0027. The second author is partially supported by NSF DMS Grant 161453, 1908739, and NSF Grant RNMS (Ki-Net) 1107444.

References

ArSa [1] D. Arsenio, L. Saint–Raymond. *From the Vlasov Maxwell Boltzmann System to Incompressible Viscous Electro–magneto–hydrodynamics.* EMS Monographs in Mathematics (2019).

BlDoPe [2] A. Blanchet, J. Dolbeault, B. Perthame. Two-dimensional Keller-Segel model: Optimal critical mass and qualitative properties of the solutions. *Electronic Journal of Differential Equations [EJDE)[electronic only]* (2006).

BrJa [3] D. Bresch, P.–E. Jabin. Global existence of weak solutions for compressible Navier-Stokes equations: Thermodynamically unstable pressure and anisotropic viscous stress tensor. *Annals. of Math,* 577–684, volume 188, (2018).

BrJaWa [4] D. Bresch, P.–E. Jabin, Z. Wang. On mean-field limits and quantitative estimates with a large class of singular kernels: Application to the Patlak-Keller-Segel model. *C.R. Acad. Sciences Mathématiques,* 357, Issue 9, (2019), 708–720.

BrJaWa1 [5] D. Bresch, P.–E. Jabin, Z. Wang. Mean field limit and quantitative estimates with a large class of singular kernels. In preparation (2019).

CaPe [6] P. Cattiaux, L. Pédèches. The 2-D stochastic Keller-Segel particle model: existence and uniqueness. *ALEA Lat. Am. J. Probab Stat,* 13 (1) (2016, 447–463.

DoPe [7] J. Dolbeault, B. Perthame, Optimal critical mass in the two-dimensional Keller-Segel model in R^2. *C. R. Math. Acad. Sci.* Paris 339, no. 9, 611–616, (2004).

Du [8] M. Duerinckx. Mean Field Limit for some Riesz interaction gradient flows. *SIAM J. Math. Anal,* 48, 3, (2016), 2269–2300.

FoJo [9] N. Fournier, B. Jourdain. Stochastic particle approximation of the Keller-Segel equation and two dimensional generalization of Bessel processes. *Ann. Appl. Proba,* 5 (2017), 2807–2861.

GoQu [10] D. Godinh, C. Quininao. Propagation of chaos for a sub-critical Keller-Segel Model. *Ann. Inst. H. Poincaré Probab. Statist.* 51, 965–992 (2015).

Go [11] F. Golse. On the dynamics of large particle systems in the mean field limit. In:Macroscopic and Large Scale Phenomena: Coarse Graining, Mean Field Limits and Ergodicity. Volume 3 of the series Lecture Notes in Applied Mathematics and Mechanics, pp. 1–144. Springer, (2016)

Kw [12] D. Han-Kwan. Quasineutral limit of the Vlasov-Poisson system with massless electrons. *Comm. Partial Diff. Eqs.* 1385–1425, (2010).

HaSc [13] J. Haskovec, C. Schmeiser. Convergence of a stochastic particle approximation for measure solutions of the 2D Keller-Segel system. *Comm. Partial Differential Equations* 36, 940–960 (2011).

Ja [14] P.–.E. Jabin. A review for the mean field limit for Vlasov equations. *Kinet. Relat. Models* 7, 661–711 (2014).

JaWa [15] P.–E. Jabin, Z. Wang. Quantitative estimates of propagation of chaos for Stochastic systems with $W^{-1,\infty}$ kernels. *Inventiones,* 214 (1), (2018), 523–591.

JaWa1 [16] P.–E. Jabin, Z. Wang. Mean field limit for stochastic particle systems. Volume 1: Theory, Models, Applications, Birkhauser-Springer (Boston), series Modelling and Simulation in Science Engineering and Technology (2017).

PuSa [17] M. Puel, L. Saint-Raymond. Quasineutral limit for the relativistic VlasovMaxwell system *Asymptotic Analysis,* vol. 40, no. 3,4, pp. 303–352, (2004).

Sa [18] L. Saint-Raymond. Des Points vortex aux équations de Navier-Stokes (d'après P.–E. Jabin et Z. Wang). In Séminaire Bourbaki, 70ème année, 2017–2018.

Se [19] S. Serfaty Systems of points with Coulomb Interactions, in *Proc. Int. Cong. of Math.* Rio de Janeiro, vol. 1, 2018, 935–978.

Se1 [20] S. Serfaty. Mean Field limit for Coulomb-type flows. Submitted for publication (2018).

To [21] M. Tomasevic. On a probabilistic interpretation of the Keller-Segel parabolic-parabolic equations. PhD Thesis, Université Côte d'Azur, (2018).

LAMA CNRS UMR5127, Univ. Savoie Mont-Blanc, 73376 Le Bourget du Lac, France.
E-mail address: didier.bresch@univ-smb.fr

CSCAMM and departement of Mathematics, Univ. of Maryland, College Park, MD, USA
E-mail address: pjabin@cscamm.umd.edu

Department of Mathematics, Univ. of Pennsylvania, Phyladelphia, PA USA
E-mail address: zwang423@math.upenn.edu