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Vulnerability of Super Extra Edge-Connected Graphs^{*†}

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Abstract

Edge connectivity is a crucial measure of the robustness of a network. Several edge connectivity variants have been proposed for measuring the reliability and fault tolerance of networks under various conditions. Let G be a connected graph, S be a subset of edges in G , and k be a positive integer. If $G - S$ is disconnected and every component has at least k vertices, then S is a k -extra edge-cut of G . The k -extra edge-connectivity, denoted by $\lambda_k(G)$, is the minimum cardinality over all k -extra edge-cuts of G . If $\lambda_k(G)$ exists and at least one component of $G - S$ contains exactly k vertices for any minimum k -extra edge-cut S , then G is super- λ_k . Moreover, when G is super- λ_k , the persistence of G , denoted by $\rho_k(G)$, is the maximum integer m for which $G - F$ is still super- λ_k for any set $F \subseteq E(G)$ with $|F| \leq m$. Previously, bounds of $\rho_k(G)$ were provided only for $k \in \{1, 2\}$. This study provides the bounds of $\rho_k(G)$ for $k \geq 2$.

Key words: Edge-connectivity, extra edge-connectivity, fault tolerance, persistence of networks, super extra edge-connectivity.

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1 Introduction

Connectivity is a crucial measure of the reliability and fault tolerance of networks. Let G be the underlying network. A vertex set X of G is a *vertex cut* of a connected graph G if $G - X$ is disconnected. An edge set S of G is an *edge-cut* of G if $G - S$ is disconnected. A classic measure for the fault tolerance and reliability of a communication network is the *connectivity* of G , denoted by $\kappa(G)$, which is the minimum cardinality of a vertex set S such that $G - S$ is disconnected or has only one vertex. Another measure is the *edge-connectivity* of G , denoted by $\lambda(G)$, which is the minimum cardinality of an edge-cut. The remaining graph is connected when the number of vertices deleted is less than $\kappa(G)$ or the number of edges deleted is less than $\lambda(G)$. Therefore, the higher the $\kappa(G)$ ($\lambda(G)$) is, the more reliable the network.

Harary [11] discussed the minimum cardinality of a set of vertices (edges) whose deletion disconnects the graph, and every remaining component satisfies a graph-theoretic property. Given a graph-theoretic property \mathcal{P} , let $\kappa(G; \mathcal{P})$ be the minimum cardinality of a vertex-cut such that every remaining component satisfies \mathcal{P} , where the number of vertices is more than two. The notation $\lambda(G; \mathcal{P})$ for the special edge-cut of G can be defined similarly. Numerous studies have considered special cases of $\kappa(G; \mathcal{P})$. For example, Xu et al. [30,31] restricted \mathcal{P} to be “no vertex of a degree less than h ” for an integer $h \geq 0$. Wang and Zhang [28] discussed a case in which \mathcal{P} is “at least two of components contain cycles.”

Let $k \geq 1$ be an integer. Fàbrega and Fiol [9,10] generalized the concept of connectivity and edge-connectivity proposed by Harary [11], to k -extra connectivity and k -extra edge-connectivity. A vertex-cut X of G is a *k -extra vertex-cut* of G if each component of $G - X$ has at least k vertices. In other words, $\kappa(G; P) = \kappa_k(G)$ and $\lambda(G; P) = \lambda_k(G)$ in which P express “the number of vertices is more than k ”. A edge-cut X of G is a *k -extra edge-cut* of G if each component of $G - X$ has at least k vertices. Then, the *k -extra connectivity* (*k -extra edge-connectivity*) of G , denoted by $\kappa_k(G)$ ($\lambda_k(G)$), is the minimum cardinality of a k -extra vertex-cut (k -extra edge-cut) of G . Extra connectivity and extra edge-connectivity in certain classes of graphs have received considerable attention in recent years [1, 5, 6, 13, 17, 18, 20, 21, 27, 34]. Hsieh and Chang [17]

discussed the 3-extra connectivity of a k -ary n -cube for $k \geq 4$ and $n \geq 5$. Chang et al. [6] discussed the 4-extra connectivity and 4-extra edge-connectivity of folded hypercubes. Chang and Hsieh [5] investigated the $\{3, 4\}$ -extra connectivity of hypercube-like networks. In addition, Meng and Ji [21] showed that regular graphs with an order of more than 5 have at least one 3-extra edge-cut. Hong and Hsieh [13] discussed the 4-extra edge-connectivity of hypercube-like networks. Lü et al. [18] derived the 2-extra edge-connectivity of product graphs. Furthermore, Balbuena and Marcote [1] discussed the k -extra edge-connectivity of product graphs for $k \geq 3$. Zhao and Ou [34] derived the 2-extra edge-connectivity of lexicographic product graphs.

Given a vertex subset $X \subseteq V(G)$, the notation $[X, \bar{X}]$ is used to denote the set of edges having one endpoint in X and another endpoint in \bar{X} , where $\bar{X} = V(G) - X$. Apparently, $[X, \bar{X}]$ is an edge-cut of G . The notation $G[X]$ is used to denote the subgraph of G induced by X . Let $\omega_G(X) = |[X, \bar{X}]|$ and $\xi_k(G) = \min\{\omega_G(X) : X \subseteq V(G), |X| = k, \text{ and } G[X] \text{ is connected}\}$. Obviously, $\xi_k(G)$ is always an upper bound of $\kappa_k(G)$ and $\lambda_k(G)$. If $\kappa_k(G) = \xi_k(G)$, then G is said to be maximally k -connected (or optimal- κ_k). If $\lambda_k(G) = \xi_k(G)$, then G is said to be maximally k -edge-connected (or optimal- λ_k). The properties of optimal- κ_1 and optimal- λ_k for certain classes of graphs have been discussed [1, 12, 18, 22, 24, 25, 29]. Hellwig et al. [12] discussed properties of optimal- κ_1 and optimal- λ_1 graphs. Lü et al. [18] stated sufficient conditions for product graphs to be optimal- λ_2 , and Shang and Zhang [24, 25] provided sufficient conditions for various graphs to be optimal- λ_2 and optimal- λ_3 . Wang et al. [29] provided sufficient conditions for graphs with a diameter of 2 to be optimal- λ_k for $k \geq 1$. In addition, Balbuena and Marcote [1] provided sufficient conditions for product graphs to be optimal- λ_k for $k \geq 3$.

Graph G is *super k -extra connected* (*super- κ_k*) if every minimum k -extra vertex-cut of G isolates a component with order k . Graph G is *super k -extra edge-connected* (*super- λ_k*) if every minimum k -extra edge-cut of G isolates a component with order k . The properties of super- κ_1 and super- λ_k of certain classes of graphs have been discussed [1, 7, 19, 24, 25, 29, 34]. Meng [19] provided sufficient conditions for connected vertex-transitive and edge-transitive graphs to be super- κ_1 . Shang and Zhang [24, 25] provided sufficient conditions for various graphs to be super-

Table 1: Summary of Related Studies ($\eta_k(G)$ is the maximum number of edge-disjoint connected subtrees with order k such that each subtree H satisfies $\omega_G(V(H)) = \xi_k(G)$.)

The properties which the graph G satisfies	The elements of F	Lower bound of $ F $	Upper bound of $ F $	Property of $G - F$	Ref.
Optimal- κ_1 , $\eta_1(G) \geq \delta(G)$, and $\kappa_2(G)$ exists	Vertices	$\min\{\kappa_2(G) - \delta(G), \delta(G) - 1\}$	$\delta(G) - 1$	Optimal- κ_1	[15]
Super- κ_1 , $\eta_1(G) \geq \delta(G)$, and $\kappa_2(G)$ exists	Vertices	$\min\{\kappa_2(G) - \delta(G) - 1, \delta(G) - 1\}$	$\delta(G) - 1$	Super- κ_1	[15]
Super- λ_1 and $\lambda_2(G)$ exists	Edges	$\min\{\lambda_2(G) - \delta(G) - 1, \delta(G) - 1\}$	$\delta(G) - 1$	Super- λ_1	[14]
Super- λ_2 , $\eta_2(G) \geq \delta(G)$, and $\lambda_3(G)$ exists	Edges	$\min\{\lambda_3(G) - \xi_2(G) - 1, \delta(G) - 1\}$	$\delta(G) - 1$	Super- λ_2	[16]
Super- λ_k and $\lambda_{k+1}(G)$ does not exist for $2 \leq k \leq \delta(G)$	Edges	$\lambda(G) - 1$	$\lambda(G) - 1$	Super- λ_k	this paper
Super- λ_2 and $\lambda_3(G)$ exists	Edges	$\min\{\lambda_3(G) - \xi_2(G) - 1, \eta_2(G) - 1, \lambda(G) - 1\}$	$\lambda(G) - 1$	Super- λ_2	this paper
Super- λ_3 and $\lambda_4(G)$ exists	Edges	$\min\{\lambda_4(G) - \xi_3(G) - 1, \eta_3(G) - 1, \lambda(G) - 1\}$	$\lambda(G) - 1$	Super- λ_3	this paper
Super- λ_k and $\lambda_{k+1}(G)$ exists for $k \geq 4$	Edges	$\min\{\lambda_{k+1}(G) - \xi_k(G) - 1, \eta_k(G) - 1, \delta(G) - k + 1, \lambda(G) - 1\}$	$\lambda(G) - 1$	Super- λ_k	this paper

λ_2 and super- λ_3 . Furthermore, Zhao and Ou [34] provided sufficient conditions for lexicographic product graphs to be super- λ_2 , and Balbuena and Marcote [1] provided sufficient conditions for product graphs to be super- λ_k for $k \geq 3$.

Link (edge) and process (vertex) faults may occur when a network is activated; therefore, it is crucial to consider faulty networks. Recently, Hong and Zhang [15] discussed the vertex fault tolerance of optimal- κ_1 and super- κ_1 graphs. Hong et al. [14] discussed the edge fault tolerance of super- λ_1 graphs, and Wang and Lu [26] examined the bounds in graphs in detail. Hong and Xu [16] discussed the edge fault tolerance of super- λ_2 graphs under a condition. In this paper, we discuss the edge fault tolerance of super- λ_k graphs for $k \geq 2$. In other words, we determine the number of faulty edges allowed for graph to remain super- λ_k . We illustrate the bounds when $k = 2$, $k = 3$, and $k \geq 4$. (More details about the results are provided in Table 1.)

The remainder of this paper is organized as follows. Section 2 provides definitions and notation. Section 3 shows the edge fault tolerance of super- λ_3 graphs. Section 4 shows the edge fault tolerance of super- λ_k graphs when $k \geq 4$. Section 4 shows the edge fault tolerance of product graphs with some properties. Section 6 provides concluding remarks.

2 Preliminaries

An undirected *graph* $G = (V, E)$ is a pair of the *vertex set* V and *edge set* E , where V is a finite set and E is a subset of $\{(u, v) \mid (u, v) \text{ is an unordered pair of } V\}$. $V(G)$ and $E(G)$ denote the vertex set and edge set of G , respectively. Let $n(G) = |V(G)|$ be the *order* of G . Two vertices u and v are *adjacent* if (u, v) is an edge in G . The edge (u, v) is *incident to* u and v , and u and v are the *endpoints* of (u, v) . The *degree* of vertex v , denoted by $d_G(v)$, is the number of edges incident to it. Let $\delta(G) = \min\{d_G(v) \mid v \in V(G)\}$. A *path* $\langle v_1, v_2, \dots, v_t \rangle$ is a sequence of distinct vertices such that any two consecutive vertices are adjacent. Vertices v_1 and v_t are the *endpoints* of the path. A *cycle* $\langle v_0, v_1, \dots, v_t, v_0 \rangle$ for $t \geq 2$ is a sequence of vertices such that any two consecutive vertices are adjacent, where v_0, v_1, \dots, v_t are all distinct. A *complete graph* is a simple graph whose vertices are pairwise adjacent; a (unlabeled) complete graph with n vertices is denoted K_n .

An *isomorphism* from a simple graph G to a simple graph H is a one-to-one and onto function $\pi : V(G) \rightarrow V(H)$ such that $(u, v) \in E(G)$ if and only if $(\pi(u), \pi(v)) \in E(H)$. We say that “ G is *isomorphic to* H ”, written $G \cong H$, if there is an isomorphism from G to H .

Let $k \geq 1$ be an integer. An edge set $S \subseteq E(G)$ is a *k-extra edge-cut* if $G - S$ is disconnected and each component has at least k vertices. The *k-extra edge-connectivity* of G , denoted by $\lambda_k(G)$, is defined as the minimum cardinality over all k -extra edge-cuts of G . If $\lambda_k(G)$ exists, then G is said to be λ_k -connected and $\lambda(G) = \lambda_1(G) \leq \lambda_2(G) \leq \lambda_3(G) \leq \dots \leq \lambda_k(G)$. The following lemma shows the necessary and sufficient condition for a graph to be λ_k -connected.

Lemma 1. [23] *Let $k \geq 1$ be an integer and G be a connected graph. If G has an order of at least $3k - 2$, then G is a λ_k -connected graph if and only if G contains no vertex u such that every component of $G - \{u\}$ has an order of at most $k - 1$.*

The terms $[X, \overline{X}]$, $\omega_G(X)$, and $\xi_k(G)$ have been defined in Section 1. Studies have shown that $\lambda_k(G) \leq \xi_k(G)$ holds for any λ_k -connected graph, where $1 \leq k \leq 3$ [3, 4, 8, 21]. For $k \geq 4$, Bonsma et al. [4] observed that the inequality $\lambda_k(G) \leq \xi_k(G)$ is no longer true in general.

Let G_1, G_2, \dots, G_m be m copies of K_t , where $m \geq 1$ and $t \geq 1$. Let v be a new vertex such that v is adjacent to every vertex in $\cup_{i=1}^m V(G_i)$. Then, the resulting graph is denoted by $G_{m,t}^*$. When $t = 1$, $G_{m,t}^*$ is a star. The following lemma shows a sufficient condition for a graph G to be λ_k -connected and $\lambda_k(G) \leq \xi_k(G)$ for $1 \leq k \leq \delta(G) + 1$.

Lemma 2. [33] *Let G be a connected graph with an order of at least $2(\delta(G) + 1)$. If G is not isomorphic to $G_{m,\delta(G)}^*$ for any positive integer m , then $\lambda_k(G)$ exists, and $\lambda_k(G) \leq \xi_k(G)$ for any k with $1 \leq k \leq \delta(G) + 1$.*

A graph G is said to be λ_k -optimal if it satisfies $\lambda_k(G) = \xi_k(G)$. Some properties of λ_k -optimal graphs were investigated in [32]. Moreover, if $\lambda_k(G)$ exists and at least one component of $G - S$ contains exactly k vertices for any minimum k -extra edge-cut S , then G is said to be super- λ_k ; $\lambda_k(G) = \xi_k(G)$ if G is super- λ_k . The following lemma shows the necessary and sufficient condition for a λ_k -connected graph to be super- λ_k .

Lemma 3. *Let G be a λ_k -connected graph with $\lambda_k(G) \leq \xi_k(G)$ for some $k \geq 1$. Then, G is super- λ_k if and only if G is not λ_{k+1} -connected or $\omega_G(X) > \xi_k(G)$ holds for any vertex set $X \subseteq V(G)$ with $k + 1 \leq |X| \leq \lfloor n(G)/2 \rfloor$ and $G[X], G[\overline{X}]$ being connected.*

Proof. According to Lemma 1.4 in [16], G is super- λ_k if and only if G is not λ_{k+1} -connected or $\lambda_{k+1}(G) > \xi_k(G)$ for any $k \geq 1$. Because $\omega_G(X) > \xi_k(G)$ holds for any vertex set $X \subseteq V(G)$ with $k + 1 \leq |X| \leq \lfloor |V(G)|/2 \rfloor$ and $G[X]$ and $G[\overline{X}]$ being connected, $\lambda_{k+1}(G) > \xi_k(G)$. Therefore, the result holds. \square

Let $\eta_k(G)$ denote the maximum number of edge-disjoint connected subtrees with order k such that each subtree H satisfies $\omega_G(V(H)) = \xi_k(G)$. For example, consider the graph G shown in Fig. 1. Because vertices a and b are the only two vertices with a minimum degree in this graph, $\eta_1(G) = 2$. Moreover, because $\omega_G(\{(a,b)\}) = \xi_2(G) = 4$ and each of the other edges e does not satisfy the condition (i.e., $\omega_G(\{\text{the endpoints of } e\}) \neq \xi_2(G) = 4$), $\eta_2(G) = 1$. The equality $\eta_3(G) = 2$ holds because we can find two edge-disjoint paths $\langle a, c, b \rangle$ and $\langle a, d, b \rangle$ such

that $\omega_G(\{a, c, b\}) = \omega_G(\{a, d, b\}) = \xi_3(G) = 4$. Finally, the equality $\eta_4(G) = 2$ holds because we can find two edge-disjoint paths $\langle a, c, d, b \rangle$ and $\langle d, a, b, c \rangle$ such that $\omega_G(\{a, b, c, d\}) = \xi_4(G) = 2$.

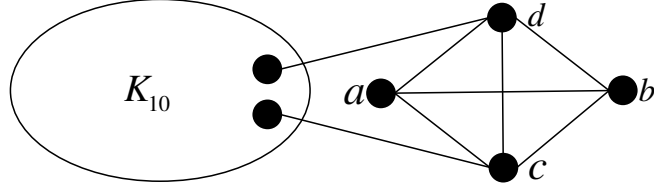


Figure 1: Illustration of $\eta_k(G)$ for $k = 1, 2, 3, 4$

Definition 1. Let $k \geq 1$ be an integer. The *persistence* of the super- λ_k graph G , denoted by $\rho_k(G)$, is the maximum integer m for which $G - F$ is still super- λ_k for any set $F \subseteq E(G)$ with $|F| \leq m$.

Hong et al. [14] showed that $\min\{\lambda_2(G) - \delta(G) - 1, \delta(G) - 1\} \leq \rho_1(G) \leq \delta(G) - 1$ for any super- λ_1 and λ_2 -connected graph G . In addition, Hong and Xu [16] showed that $\min\{\lambda_3(G) - \xi_2(G) - 1, \delta(G) - 1\} \leq \rho_2(G) \leq \delta(G) - 1$ for any super- λ_2 and λ_3 -connected graph G with $\eta_2(G) \geq \delta(G)$. This study determines the following bounds of $\rho_k(G)$ for $k \geq 2$:

1. $\rho_k(G) = \lambda_1(G) - 1$ if graph G is super- λ_k , but not λ_{k+1} -connected,
2. $\min\{\lambda_3(G) - \xi_2(G) - 1, \eta_2(G) - 1, \lambda_1(G) - 1\} \leq \rho_2(G) \leq \lambda_1(G) - 1$ for any super- λ_2 and λ_3 -connected graph G ,
3. $\min\{\lambda_4(G) - \xi_3(G) - 1, \eta_3(G) - 1, \lambda_1(G) - 1\} \leq \rho_3(G) \leq \lambda_1(G) - 1$ for any super- λ_3 and λ_4 -connected graph G , and
4. $\min\{\lambda_{k+1}(G) - \xi_k(G) - 1, \eta_k(G) - 1, \delta(G) - k + 1, \lambda_1(G) - 1\} \leq \rho_k(G) \leq \lambda_1(G) - 1$ for any super- λ_k and λ_{k+1} -connected graph G , where $k \geq 4$.

3 Bounds on the persistence of super- λ_3 graphs

If a graph $G = (V, E)$ has a path with endpoints u and v , then the *distance* between u and v , denoted by $d_G(u, v)$ or simply $d(u, v)$, is the shortest length of a path between u and v . If

G has no such path, then $d(u, v) = \infty$. The *eccentricity* of a vertex u , denoted by $\epsilon(u)$, is $\max_{v \in V} d(u, v)$.

Let H_1 be a connected graph that has an order of at least 6 and satisfies the following conditions: (a) H_1 contains no cycles of a length greater than 3 and (b) there exists exactly one vertex $v_0 \in V(H_1)$ with a degree greater than 2, and v_0 has eccentricity equal to or less than 2. Fig. 2 illustrates graphs H_1 and H_2 . The following lemma shows the necessary and sufficient condition for a graph to be λ_3 -connected.

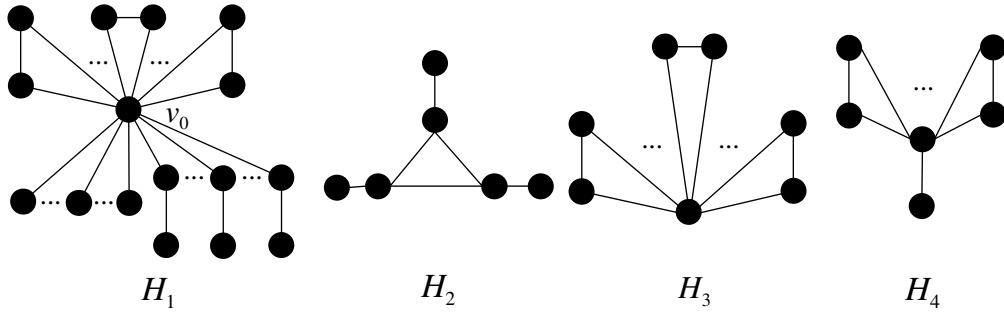


Figure 2: Illustration of Lemmas 4 and 5

Lemma 4. [4] *A connected graph with an order of 6 is not λ_3 -connected if and only if G is isomorphic to H_1 or H_2 (Fig. 2). Furthermore, if G is λ_3 -connected, then $\lambda_3(G) \leq \xi_3(G)$.*

According to Lemma 4, we can determine the edge fault tolerance of a λ_3 -connected graph.

Lemma 5. *Let G be a λ_3 -connected graph with $\delta(G) \geq 3$. Then, $G - F$ is a λ_3 -connected graph and $\lambda_3(G - F) \leq \xi_3(G - F)$ for any $F \subseteq E(G)$ with $|F| \leq \lambda_1(G) - 1$.*

Proof. Assume that $G - F$ is not λ_3 -connected. According to Lemma 4, $G - F \cong H_1$ or $G - F \cong H_2$. Consider the following scenarios.

Case 1: $\delta(G - F) \geq 2$. In this case, $G - F \cong H_3$ (Fig. 2) and there are at least six vertices with a degree of 2 in $G - F$. Therefore,

$$|F| = |E(G)| - |E(G - F)| \geq \lceil \frac{(6\delta(G) - 6 \cdot 2)}{2} \rceil = 3\delta(G) - 6 \geq \delta(G) \geq \lambda_1(G),$$

which leads to a contradiction.

Case 2: $\delta(G - F) = 1$. Because $|F| \leq \lambda_1(G) - 1 \leq \delta(G) - 1$, exactly one vertex in $G - F$ has a degree of one, and all edges of F are incident to this vertex v . Therefore, $G - F$ is not isomorphic to H_2 . According to Lemma 4, $G - F \cong H_4$ (Fig. 2), and at least four vertices with a degree of 2. All edges of F are incident to the vertex with a degree of 1; therefore,

$$|F| = |E(G)| - |E(G - F)| \geq 4\delta(G) - 4 \cdot 2 \geq \delta(G) + 1 \geq \lambda_1(G),$$

which leads to a contradiction.

Combining the two cases completes the proof. □

Next, we show the bounds of $\rho_3(G)$ for the super- λ_3 graph G in Theorem 1.

Theorem 1. *Let G be a super- λ_3 graph with $\delta(G) \geq 3$. Then the following statements hold: (a) If G is not λ_4 -connected, then $\rho_3(G) = \lambda_1(G) - 1$. (b) If G is λ_4 -connected, then $\min\{\lambda_4(G) - \xi_3(G) - 1, \eta_3(G) - 1, \lambda_1(G) - 1\} \leq \rho_3(G) \leq \lambda_1(G) - 1$.*

Proof. There exists one edge set F with size $\lambda_1(G)$ such that $G - F$ is disconnected. Then, $G - F$ is not super- λ_3 , implying that $\rho_3(G) \leq |F| - 1 = \lambda_1(G) - 1$. Next, we determine the lower bound of $\rho_3(G)$ according to whether G is λ_4 -connected.

(a) To prove that $\rho_3(G) \geq \lambda_1(G) - 1$, it suffices to show that for any $F \subseteq E(G)$ with $|F| \leq \lambda_1(G) - 1$, $G - F$ is super- λ_3 . According to Lemma 5, $G - F$ is λ_3 -connected with $\lambda_3(G - F) \leq \xi_3(G - F)$. Because G is not λ_4 -connected, $G - F$ is also not λ_4 -connected. According to Lemma 3, $G - F$ is super- λ_3 and (a) is proved.

(b) Let $m = \min\{\lambda_4(G) - \xi_3(G) - 1, \eta_3(G) - 1, \lambda_1(G) - 1\}$. To prove that $\rho_3(G) \geq m$, it suffices to show that for any $F \subseteq E(G)$ with $|F| \leq m$, $G' = G - F$ is super- λ_3 .

Note that $|F| \leq m \leq \lambda_1(G) - 1$. According to Lemma 5, $G - F$ is λ_3 -connected with $\lambda_3(G - F) \leq \xi_3(G - F)$. If $G - F$ is not λ_4 -connected, then according to Lemma 3, $G - F$ is super- λ_3 and (b) is proved. If $G - F$ is λ_4 -connected, then let X be any subset of $V(G')$ with $|X| \geq 4$ and $|\bar{X}| \geq 4$ such that $G'[X]$ and $G'[\bar{X}]$ are connected. Therefore,

$$\omega_{G-F}(X) \geq \omega_G(X) - |F| \geq \lambda_4(G) - (\lambda_4(G) - \xi_3(G) - 1) = \xi_3(G) + 1. \quad (1)$$

Because $|F| \leq m \leq \eta_3(G) - 1$, there exists one subtree A with an order of 3 such that $\omega_G(A) = \xi_3(G)$ and $A = A - F$. In other words,

$$\xi_3(G) = \omega_G(A) \geq \omega_{G-F}(A) \geq \xi_3(G - F). \quad (2)$$

According to equations (1) and (2), $\omega_{G-F}(X) > \xi_3(G - F)$. According to Lemma 3, $G - F$ is super- λ_3 and (b) is proved. \square

Evidently, $\lambda_1(G) - 1$ is a necessary term in the proposed lower bound formula because the value is also used as an upper bound in Theorem 1. Examples 1 and 2 demonstrate that the other values are also necessary.

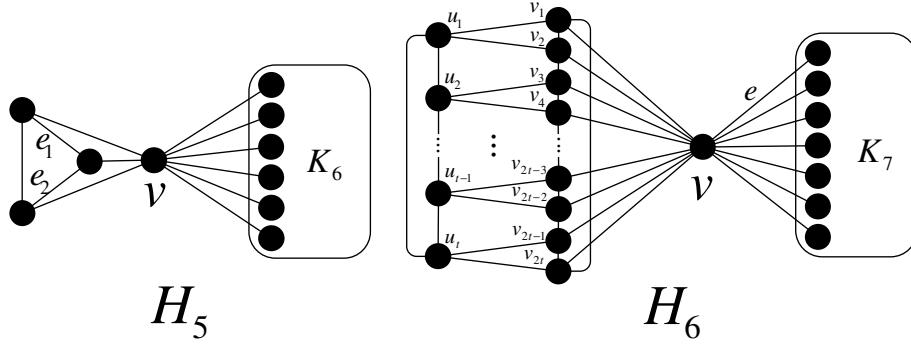


Figure 3: Illustration of Examples 1 and 2

Example 1. Consider the graph $G \cong H_5$ shown in Fig. 3. A vertex $v \in V(H)$ is adjacent to all the other vertices in G . Because $\lambda_1(G) = \lambda_3(G) = \xi_3(G) = 3$ and $\lambda_4(G) = 6$, according to Lemma 3, G is super- λ_3 . Moreover, $\min\{\lambda_4(G) - \xi_3(G) - 1, \lambda_1(G) - 1\} = 2$ and $G' = G - \{e_1, e_2\}$ is not super- λ_3 because $\lambda_3(G') = \lambda_4(G') = 6$. Therefore, $\eta_3(G) - 1$ is a necessary term in the proposed lower bound formula (in this case, $\eta_3(G) - 1 = 0$).

Example 2. Consider the graph $G \cong H_6$ shown in Fig. 3; $\{u_1, u_2, \dots, u_t\}$ and $\{v_1, v_2, \dots, v_{2t}\}$ form two separate cycles. The vertices u_i and v_j are adjacent when $j \in \{2i - 1, 2i\}$. Because $\lambda_1(G) = 4$, $\lambda_3(G) = \xi_3(G) = 6$, and $\lambda_4(G) = 7$, according to Lemma 3, G is super- λ_3 . Moreover, $\min\{\eta_3(G) - 1, \lambda_1(G) - 1\} = 3$ and $G' = G - \{e\}$ is not super- λ_3 because $\lambda_3(G') = \lambda_4(G') = 6$. Therefore, $\lambda_4(G) - \xi_3(G) - 1$ is a necessary term in the proposed lower bound formula (in this case, $\lambda_4(G) - \xi_3(G) - 1 = 0$).

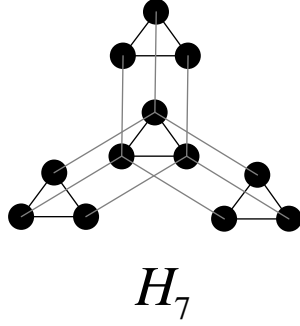


Figure 4: Illustration of Example 3

Example 3. Consider the graph $G \cong H_7$ shown in Fig. 4; $\rho_3(G) = 2$. Because $\lambda_1(G) = \lambda_3(G) = \xi_3(G) = 3$ and $\lambda_4(G) = 6$, according to Lemma 3, G is super- λ_3 . According to Theorem 1, $\min\{\lambda_4(G) - \xi_3(G) - 1, \eta_3(G) - 1, \lambda_1(G) - 1\} = 2 = \lambda_1(G) - 1$. Therefore, the lower bound given in Theorem 1 is sharp.

Corollary 1. *Let G be a super- λ_2 and λ_3 -connected graph. Then, $\min\{\lambda_3(G) - \xi_2(G) - 1, \eta_2(G) - 1, \lambda_1(G) - 1\} \leq \rho_2(G) \leq \lambda_1(G) - 1$.*

Proof. Note that $n(G) \geq 6$ because G is λ_3 -connected. Let $m = \min\{\lambda_3(G) - \xi_2(G) - 1, \eta_2(G) - 1, \lambda_1(G) - 1\}$. We first prove that $G - F$ is λ_2 -connected with $\lambda_2(G - F) \leq \xi_2(G - F)$ for any $F \subseteq E(G)$ with $|F| \leq m$.

When $\delta(G) = 1$, $|F| \leq \lambda_1(G) - 1 \leq \delta(G) - 1 \leq 0$ and $G = G - F$. Therefore, $G = G - F$ is not a star because G is λ_3 -connected. When $\delta(G) \geq 2$, assume that $G - F$ is a star, and there exists $n(G) - 1$ vertices with a degree of 1 in $G - F$. Because $\delta(G) \geq 2$, at least $\delta(G) - 1$ edges of F are incident to each of the aforementioned $n(G) - 1$ vertices. Consequently,

$$\lceil \frac{[(n(G) - 1)[\delta(G) - 1]]}{2} \rceil \leq |E(G)| - |E(G - F)| = |F| \leq \lambda_1(G) - 1 \leq \delta(G) - 1.$$

In other words, $n(G) \leq 3$, which leads to a contradiction. Therefore, $G - F$ is not a star. According to the results of [3, 21], $G - F$ is λ_2 -connected with $\lambda_2(G - F) \leq \xi_2(G - F)$ if $G - F$ is not a star; therefore, $G - F$ is λ_2 -connected with $\lambda_2(G - F) \leq \xi_2(G - F)$. Consequently, this result of Corollary 1 can be proved using a method similar to that used in Theorem 1. \square

Figure 1 of [16] shows that the lower bound given in Corollary 1 is sharp.

4 Bounds on the persistence of super- λ_k graphs for $k \geq 4$

First, we determine the edge fault tolerance of λ_k -connected graphs for a positive integer $k \geq 2$.

Lemma 6. *Let G be a λ_k -connected graph with $2 \leq k \leq \delta(G)$, and $n(G) \geq 3\delta(G) - 2$. Then, $G - F$ is λ_k -connected graph for any $F \subseteq E(G)$ with $|F| \leq \lambda_1(G) - 1$.*

Proof. Apparently, $G - F$ is connected because $|F| \leq \lambda_1(G) - 1$. If $G - F$ is not λ_k -connected, according to Lemma 1, there exists $n(G) - 1$ vertices with a degree of at most $k - 1$ in $G - F$. Because $\delta(G) \geq k$, at least one edge of F is incident to each of the aforementioned $n(G) - 1$ vertices. Thus,

$$|F| \geq \lceil (n(G) - 1)/2 \rceil \geq \lceil (3\delta(G) - 3)/2 \rceil > \delta(G) - 1 \geq \lambda_1(G) - 1,$$

which leads to a contradiction. Therefore, $G - F$ is a λ_k -connected graph. \square

Now, let G be a super- λ_k graph, but not λ_{k+1} -connected for $k \geq 2$. We show the bounds of $\rho_k(G)$ in Theorem 2.

Theorem 2. *Let G be a super- λ_k graph with $2 \leq k \leq \delta(G)$, and $n(G) \geq 3\delta(G) - 2$. If G is not λ_{k+1} -connected, then $\rho_k(G) = \lambda_1(G) - 1$.*

Proof. There exists one edge set F with size $\lambda_1(G)$ such that $G - F$ is disconnected. Then, $G - F$ is not super- λ_k . We have $\rho_k(G) \leq |F| - 1 = \lambda_1(G) - 1$.

To prove that $\rho_k(G) \geq \lambda_1(G) - 1$, it suffices to show that for any $F \subseteq E(G)$ with $|F| \leq \lambda_1(G) - 1$, $G - F$ is super- λ_k . According to Lemma 6, $G - F$ is λ_k -connected. Because G is not λ_{k+1} -connected, $G - F$ is also not λ_{k+1} -connected. Thus, every minimum k -extra edge-cut of $G - F$ isolates at least one connected subgraph of order k . Therefore, $G - F$ is super- λ_k and this theorem is proved. \square

Next, we determine the bounds of $\rho_k(G)$ for a super- λ_k and λ_{k+1} -connected graph G .

Lemma 7. *Let G be a λ_k -connected graph with $\lambda_k(G) \leq \xi_k(G)$, $k \leq \delta(G) + 1$, and $n(G) \geq 2\delta(G) + 2$. Then, $G - F$ is a λ_k -connected graph and $\lambda_k(G - F) \leq \xi_k(G - F)$ for any $F \subseteq E(G)$ with $|F| \leq \min\{\delta(G) - k + 1, \lambda_1(G) - 1\}$.*

Proof. Because $|F| \leq \lambda_1(G) - 1$, $G - F$ is connected. If $G - F$ is not isomorphic to $G_{m,k-1}^*$ for any positive integer m , according to Lemma 2, $G - F$ is λ_k -connected and $\lambda_k(G - F) \leq \xi_k(G - F)$ because $n(G) \geq 2\delta(G) + 2 \geq 2\delta(G - F) + 2$.

Therefore, we consider only the case when $G - F$ is isomorphic to $G_{m,k-1}^*$ for a positive integer m . Consider the following scenarios.

Case 1: $\delta(G) \geq k$. Because $G - F \cong G_{m,k-1}^*$, there exists $n(G) - 1$ vertices with degree $k - 1$ in $G - F$. Because $\delta(G) \geq k$, at least $\delta(G) - k + 1$ edges of F are incident to each of the aforementioned $n(G) - 1$ vertices. Therefore,

$$\lceil \frac{([n(G) - 1][\delta(G) - k + 1])}{2} \rceil \leq |E(G)| - |E(G - F)| = |F| \leq \delta(G) - k + 1,$$

which implies that $n(G) \leq 3$ and leads to a contradiction.

Case 2: $\delta(G) = k - 1$. In this case, $|F| = \delta(G) - k + 1 = 0$ and $G - F = G$. Evidently, $G - F$ is a λ_k -connected graph and $\lambda_k(G - F) \leq \xi_k(G - F)$. \square

Theorem 3. *Let G be a super- λ_k graph with $k \geq 4$ and $n(G) \geq 2\delta(G) + 2$. If G is λ_{k+1} -connected, then $\min\{\lambda_{k+1}(G) - \xi_k(G) - 1, \eta_k(G) - 1, \delta(G) - k + 1, \lambda_1(G) - 1\} \leq \rho_k(G) \leq \lambda_1(G) - 1$.*

Proof. There exists one edge set F with size $\lambda_1(G)$ such that $G - F$ is disconnected. Then, $G - F$ is not super- λ_k . We have $\rho_k(G) \leq |F| - 1 = \lambda_1(G) - 1$.

Let $m = \min\{\lambda_{k+1}(G) - \xi_k(G) - 1, \eta_k(G) - 1, \delta(G) - k + 1, \lambda_1(G) - 1\}$. To prove that $\rho_k(G) \geq m$, it suffices to show that for any $F \subseteq E(G)$ with $|F| \leq m$, $G' = G - F$ is super- λ_k . Note that $|F| \leq m \leq \min\{\delta(G) - k + 1, \lambda_1(G) - 1\}$; therefore, according to Lemma 7, $G - F$ is λ_k -connected with $\lambda_k(G - F) \leq \xi_k(G - F)$.

If $G - F$ is not λ_{k+1} -connected, according to Lemma 3, $G - F$ is super- λ_k and this theorem is proved. If $G - F$ is λ_{k+1} -connected, then let X be any subset of $V(G')$ with $|X| \geq k + 1$ and $|\overline{X}| \geq k + 1$ such that $G'[X]$ and $G'[\overline{X}]$ are connected. Therefore,

$$\omega_{G-F}(X) \geq \omega_G(X) - |F| \geq \lambda_{k+1}(G) - (\lambda_{k+1}(G) - \xi_k(G) - 1) = \xi_k(G) + 1. \quad (3)$$

Because $|F| \leq m \leq \eta_k(G) - 1$, there exists one subtree A with order k such that $\omega_G(A) = \xi_k(G)$ and $A = A - F$. In other words,

$$\xi_k(G) = \omega_G(A) \geq \omega_{G-F}(A) \geq \xi_k(G - F). \quad (4)$$

According to equations (3) and (4), $\omega_{G-F}(X) > \xi_k(G - F)$. According to Lemma 3, $G - F$ is super- λ_k and this theorem is proved. \square

5 Bounds on the persistence of product graphs

In this section, the results obtained in Sections 3 and 4 are applied to product graphs. The product graph $G * H$ of two graphs G and H was introduced in [2].

Definition 2. [2] Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two graphs. Let π_{xy} be a permutation of $V(H)$ such that $\pi_{xy}^{-1} = \pi_{yx}$ for each edge $xy \in E(G)$. Then, the product graph $G * H$ is the graph with $V(G) \times V(H)$ as its vertex set, and two vertices, (x, x') and (y, y') , are adjacent in $G * H$ if and only if either $x = y$ and $x'y' \in E(H)$ or $xy \in E(G)$ and $y' = \pi_{xy}(x')$.

Lemma 8. [1] *Let $k \geq 1$ be an integer. Let G be an r -regular connected graph and let H be an s -regular connected graph, with $r \geq 2k - 1$ and $s \geq r + 2k - 1$, respectively. Assume that $\lambda_1(G)|V(H)| \geq 2k(s - k) + 2$ and H is λ_{k+1} -connected with $\lambda_{k+1}(H) \geq 2(s - k)$. Then, the $(r + s)$ -regular graph $G * H$ is super- λ_k .*

According to Lemma 8, Corollary 1, and $\xi_2(G) = 2r + 2s - 2$ for any $(r + s)$ -regular graph G , the following result is true.

Corollary 2. *Let G be an r -regular connected graph and let H be an s -regular connected graph, with $r \geq 3$, $s \geq r + 3$. Suppose that $\lambda_1(G)|V(H)| \geq 4s - 6$, and also that H is λ_3 -connected with $\lambda_3(H) \geq 2s - 4$. Then the $(r + s)$ -regular graph $G * H$ is super- λ_2 and $\min\{\lambda_3(G * H) - 2r - 2s + 1, \eta_2(G * H) - 1, \lambda_1(G * H) - 1\} \leq \rho_2(G * H) \leq \lambda_1(G * H) - 1$.*

If G is an $(r + s)$ -regular graph, then $\xi_3(G) = 3r + 3s - 6$ if G contains a triangle or $\xi_3(G) = 3r + 3s - 4$ if G contains no triangle. According to Lemma 8 and Theorem 1, the

following results are true.

Corollary 3. *Let G be an r -regular connected graph and let H be an s -regular connected graph, with $r \geq 5$ and $s \geq r+5$, respectively. Assume that $\lambda_1(G)|V(H)| \geq 6s-16$ and H is λ_4 -connected with $\lambda_4(H) \geq 2s-6$. If H has a triangle subgraph, then the $(r+s)$ -regular graph $G*H$ is super- λ_3 and $\min\{\lambda_4(G*H) - 3r - 3s + 5, \eta_3(G*H) - 1, \lambda_1(G*H) - 1\} \leq \rho_3(G*H) \leq \lambda_1(G*H) - 1$.*

Corollary 4. *Let G be an r -regular connected graph and let H be an s -regular connected graph, with $r \geq 5$ and $s \geq r+5$, respectively. Assume that $\lambda_1(G)|V(H)| \geq 6s-16$ and H is λ_4 -connected with $\lambda_4(H) \geq 2s-6$. If G and H have no triangle subgraph, then the $(r+s)$ -regular graph $G*H$ is super- λ_3 and $\min\{\lambda_4(G*H) - 3r - 3s + 3, \eta_3(G*H) - 1, \lambda_1(G*H) - 1\} \leq \rho_3(G*H) \leq \lambda_1(G*H) - 1$.*

According to Lemma 8 and Theorem 3, the following result is true.

Corollary 5. *Let $k \geq 4$ be an integer. Let G be an r -regular connected graph and let H be an s -regular connected graph, with $r \geq 2k - 1$ and $s \geq r + 2k - 1$, respectively. Assume that $\lambda_1(G)|V(H)| \geq 2k(s - k) + 2$ and H is λ_{k+1} -connected with $\lambda_{k+1}(H) \geq 2(s - k)$. Then, the $(r + s)$ -regular graph $G * H$ is super- λ_k and $\min\{\lambda_{k+1}(G * H) - \xi_k(G * H) - 1, \eta_k(G * H) - 1, r + s - k + 1, \lambda_1(G * H) - 1\} \leq \rho_k(G * H) \leq \lambda_1(G * H) - 1$.*

In [1], the super- λ_k property of a product of two complete graphs was proved.

Lemma 9. [1] *For all integers $k \geq 1$, $r \geq 2k + 1$, and $s \geq r + 2k - 1$, the graph $K_r * K_s$ is super- λ_j for every $1 \leq j \leq k$.*

According to Lemma 9, Corollary 1, and $\eta_2(G) = |E(G)|$, for any regular graph G , the following result is true.

Corollary 6. *For all integers $r \geq 5$ and $s \geq r + 3$, the graph $G = K_r * K_s$ is super- λ_2 and $\min\{\lambda_3(G) - 2r - 2s + 5, \lambda_1(G) - 1\} \leq \rho_2(G) \leq \lambda_1(G) - 1$.*

For any complete graph K_s for $s \geq 4$, we can prove that $\eta_3(K_s) \geq s - 1$ by induction on s . When $s = 4$, there are three edge-disjoint paths with an order of 3. Therefore, $\eta_3(K_4) = 3 \geq s - 1$. Assume that the result holds when $s \leq t$. Consider $s = t + 1 > 4$, and let v be a vertex of K_s .

According to the induction hypothesis, there are at least $(s - 2)$ edge-disjoint paths with an order of 3 in $K_s - \{v\}$. The vertex v and two other vertices in $K_s - \{v\}$ can form additional edge-disjoint paths with an order of 3. Then, $\eta_3(K_s) \geq s - 1$. Moreover, $\eta_3(K_r * K_s) \geq r(s - 1) \geq r + s - 2 \geq \lambda_1(K_r * K_s)$ for $r \geq 1$ and $s \geq 4$. According to Lemma 9 and Theorem 1, the following result is true.

Corollary 7. *For all integers $r \geq 7$ and $s \geq r + 5$, the graph $G = K_r * K_s$ is super- λ_3 and $\min\{\lambda_4(G) - 3r - 3s + 11, \lambda_1(G) - 1\} \leq \rho_3(G) \leq \lambda_1(G) - 1$.*

We have

$$\xi_k(K_r * K_s) = k(r + s - 2) - k(k - 1) = k(r + s - 1) - k^2.$$

According to Lemma 9 and Theorem 3, the following result is true.

Corollary 8. *For all integers $k \geq 4$, $r \geq 2k + 1$, and $s \geq r + 2k - 1$, the graph $K_r * K_s$ is super- λ_k and $\min\{\lambda_{k+1}(G) + k^2 - k(r + s - 1) - 1, \eta_k(G) - 1, r + s - k - 1, \lambda_1(G) - 1\} \leq \rho_2(G) \leq \lambda_1(G) - 1$.*

6 Conclusion

Fault tolerance is critical for retaining a system's reliability. This study investigates the edge fault tolerance of super- λ_k graphs. The bounds of this result for $k \in \{1, 2\}$ have recently been presented. In this paper, we show that (1) $\rho_k(G) = \lambda_1(G) - 1$ if the graph G is super- λ_k , but not λ_{k+1} -connected for $k \geq 2$, (2) $\min\{\lambda_3(G) - \xi_2(G) - 1, \eta_2(G) - 1, \lambda_1(G) - 1\} \leq \rho_2(G) \leq \lambda_1(G) - 1$ for any super- λ_2 and λ_3 -connected graph G , (3) $\min\{\lambda_4(G) - \xi_3(G) - 1, \eta_3(G) - 1, \lambda_1(G) - 1\} \leq \rho_3(G) \leq \lambda_1(G) - 1$ for any super- λ_3 and λ_4 -connected graph G , and (4) $\min\{\lambda_{k+1}(G) - \xi_k(G) - 1, \eta_k(G) - 1, \delta(G) - k + 1, \lambda_1(G) - 1\} \leq \rho_k(G) \leq \lambda_1(G) - 1$ for any super- λ_k and λ_{k+1} -connected graph G , where $k \geq 4$. Future studies should evaluate tighter bounds of $\rho_k(G)$ for graphs that satisfy specific conditions.

References

- [1] C. Balbuena and X. Marcote : The k-restricted edge-connectivity of a product of graphs. Discrete Applied Mathematics 161(1), 52-59 (2013)

- [2] J. C. Bermond, C. Delorme, and G. Farhi : Large graphs with given degree and diameter II. *Journal of Combinatorial Theory, Series B*, 36(1), 32V48 (1984)
- [3] F. T. Boesch and J. F. Wang : Super line-connectivity properties of circulant graphs. *SIAM Journal on Algebraic Discrete Methods* 7(1), 89-98 (1986)
- [4] P. Bonsma, N Ueffing, and L. Volkmann : Edge-cuts leaving components of order at least three. *Discrete Mathematics* 256(1), 431-439 (2002)
- [5] N. W. Chang and S. Y. Hsieh : (2,3)-Extraconnectivities of hypercube-like networks. *Journal of Computer and System Sciences* 79(5), 669-688 (2013)
- [6] N. W. Chang, C. Y. Tsai, and S. Y. Hsieh : On 3-extra connectivity and 3-extra edge connectivity of folded hypercubes. *IEEE Transactions on Computers* 63(6), 1594-1600 (2014)
- [7] E. Cheng, M. J. Lipman, and H. Park : Super connectivity of star graphs, alternating group graphs and split-stars. *Ars Combinatoria* 59, 107-116 (2001)
- [8] A. H. Esfahanian and S. L. Hakimi : On computing a conditional edge-connectivity of a graph. *Information Processing Letters* 27(4), 195-199 (1988)
- [9] J. Fàbrega and M. A. Fiol : Extraconnectivity of graphs with large girth. *Discrete Mathematics* 127(1), 163-170 (1994)
- [10] J. Fàbrega and M. A. Fiol : On the extraconnectivity of graphs. *Discrete Mathematics* 155(1), 49-57 (1996)
- [11] F. Harary : Conditional connectivity. *Networks* 13, 347-357 (1983)
- [12] A. Hellwig, L. Volkmann : Maximally edge-connected and vertex-connected graphs and digraphs: a survey. *Discrete Mathematics* 308(15), 3265-3296 (2008)
- [13] W. S. Hong and S. Y. Hsieh : Extra edge connectivity of hypercube-like networks. *International Journal of Parallel, Emergent and Distributed Systems* 28(2), 123-133 (2013)
- [14] Y. Hong, J. Meng, and Z. Zhang : Edge fault tolerance of graphs with respect to super edge connectivity. *Discrete Applied Mathematics* 160(4), 579-587 (2012)
- [15] Y. Hong and Z. Zhang : Vertex fault tolerance of optimal- κ graphs and super- κ graphs. *Information Processing Letters* 109(20), 1151-1155 (2009)
- [16] Z. M. Hong and J. M. Xu : Vulnerability of super edge-connected networks. *Theoretical Computer Science* 520, 75-86 (2014)
- [17] S. Y. Hsieh and Y. H. Chang : Extraconnectivity of k-ary n-cube networks. *Theoretical Computer Science* 443(20), 63-69 (2012)
- [18] M. Ĺu, G. L. Chen, and X. R. Xu : On super edge-connectivity of product graphs. *Applied Mathematics and Computation* 207(2), 300-306 (2009)
- [19] J. Meng : Connectivity of vertex and edge transitive graphs. *Discrete applied mathematics* 127(3), 601-613 (2003)
- [20] J. X. Meng : Optimally super-edge-connected transitive graphs. *Discrete Mathematics* 260(1), 239-248 (2003)
- [21] J. X. Meng and Y. H. Ji : On a kind of restricted edge connectivity of graphs. *Discrete applied mathematics* 117, 183-193 (2002)
- [22] O. R. Oellermann : Connectivity and edge-connectivity in graphs: a survey. *Congressus Numerantium* 116, 231-252 (1996)
- [23] J. Ou : Edge cuts leaving components of order at least m. *Discrete mathematics* 305(1), 365-371 (2005)
- [24] L. Shang and H. Zhang : Sufficient conditions for graphs to be λ' -optimal and super- λ' . *Networks* 49(3), 234-242 (2007)
- [25] L. Shang and H. Zhang : Degree conditions for graphs to be λ_3 -optimal and super- λ_3 . *Discrete Mathematics* 309(10), 3336-3345 (2009)

- [26] D. Wang and M. Lu : Edge fault tolerance of super edge connectivity for three families of interconnection networks. *Information Sciences* 188, 260-268 (2012)
- [27] M. Wang and Q. Li : Conditional edge connectivity properties, reliability comparison and transitivity of graphs. *Discrete Mathematics* 258, 205-214 (2002)
- [28] B. Wang and Z. Zhang : On cyclic edge-connectivity of transitive graphs. *Discrete Mathematics* 309(13), 4555-4563 (2009)
- [29] S. Wang, S. Lin, and C. Li : Sufficient conditions for super k -restricted edge connectivity in graphs of diameter 2. *Discrete Mathematics*, 309(4), 908-919 (2009)
- [30] J. M. Xu : On conditional edge-connectivity of graphs. *Acta Mathematicae Applicatae Sinica* 16(4), 414V419 (2000)
- [31] J. M. Xu and Q. Liu : 2-restricted edge connectivity of vertex-transitive graphs. *Australasian Journal of Combinatorics* 30, 41-50 (2004)
- [32] J. Xu : Topological structure and analysis of interconnection networks. Springer Publishing Company, Incorporated (2010)
- [33] Z. Zhang and J. Yuan : A proof of an inequality concerning k -restricted edge connectivity. *Discrete mathematics* 304(1), 128-134 (2005)
- [34] W. Zhao and J. Ou : On restricted edge-connectivity of lexicographic product graphs. *International Journal of Computer Mathematics* 91(8), 1618-1626 (2014)