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# Vulnerability of Super Extra Edge-Connected Graphs<sup>\*†</sup>

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## Abstract

Edge connectivity is a crucial measure of the robustness of a network. Several edge connectivity variants have been proposed for measuring the reliability and fault tolerance of networks under various conditions. Let  $G$  be a connected graph,  $S$  be a subset of edges in  $G$ , and  $k$  be a positive integer. If  $G - S$  is disconnected and every component has at least  $k$  vertices, then  $S$  is a  $k$ -extra edge-cut of  $G$ . The  $k$ -extra edge-connectivity, denoted by  $\lambda_k(G)$ , is the minimum cardinality over all  $k$ -extra edge-cuts of  $G$ . If  $\lambda_k(G)$  exists and at least one component of  $G - S$  contains exactly  $k$  vertices for any minimum  $k$ -extra edge-cut  $S$ , then  $G$  is super- $\lambda_k$ . Moreover, when  $G$  is super- $\lambda_k$ , the persistence of  $G$ , denoted by  $\rho_k(G)$ , is the maximum integer  $m$  for which  $G - F$  is still super- $\lambda_k$  for any set  $F \subseteq E(G)$  with  $|F| \leq m$ . Previously, bounds of  $\rho_k(G)$  were provided only for  $k \in \{1, 2\}$ . This study provides the bounds of  $\rho_k(G)$  for  $k \geq 2$ .

**Key words:** Edge-connectivity, extra edge-connectivity, fault tolerance, persistence of networks, super extra edge-connectivity.

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# 1 Introduction

Connectivity is a crucial measure of the reliability and fault tolerance of networks. Let  $G$  be the underlying network. A vertex set  $X$  of  $G$  is a *vertex cut* of a connected graph  $G$  if  $G - X$  is disconnected. An edge set  $S$  of  $G$  is an *edge-cut* of  $G$  if  $G - S$  is disconnected. A classic measure for the fault tolerance and reliability of a communication network is the *connectivity* of  $G$ , denoted by  $\kappa(G)$ , which is the minimum cardinality of a vertex set  $S$  such that  $G - S$  is disconnected or has only one vertex. Another measure is the *edge-connectivity* of  $G$ , denoted by  $\lambda(G)$ , which is the minimum cardinality of an edge-cut. The remaining graph is connected when the number of vertices deleted is less than  $\kappa(G)$  or the number of edges deleted is less than  $\lambda(G)$ . Therefore, the higher the  $\kappa(G)$  ( $\lambda(G)$ ) is, the more reliable the network.

Harary [11] discussed the minimum cardinality of a set of vertices (edges) whose deletion disconnects the graph, and every remaining component satisfies a graph-theoretic property. Given a graph-theoretic property  $\mathcal{P}$ , let  $\kappa(G; \mathcal{P})$  be the minimum cardinality of a vertex-cut such that every remaining component satisfies  $\mathcal{P}$ , where the number of vertices is more than two. The notation  $\lambda(G; \mathcal{P})$  for the special edge-cut of  $G$  can be defined similarly. Numerous studies have considered special cases of  $\kappa(G; \mathcal{P})$ . For example, Xu et al. [30, 31] restricted  $\mathcal{P}$  to be “no vertex of a degree less than  $h$ ” for an integer  $h \geq 0$ . Wang and Zhang [28] discussed a case in which  $\mathcal{P}$  is “at least two of components contain cycles.”

Let  $k \geq 1$  be an integer. Fàbrega and Fiol [9, 10] generalized the concept of connectivity and edge-connectivity proposed by Harary [11], to  $k$ -extra connectivity and  $k$ -extra edge-connectivity. A vertex-cut  $X$  of  $G$  is a  *$k$ -extra vertex-cut* of  $G$  if each component of  $G - X$  has at least  $k$  vertices. In other words,  $\kappa(G; P) = \kappa_k(G)$  and  $\lambda(G; P) = \lambda_k(G)$  in which  $P$  express “the number of vertices is more than  $k$ ”. A edge-cut  $X$  of  $G$  is a  *$k$ -extra edge-cut* of  $G$  if each component of  $G - X$  has at least  $k$  vertices. Then, the  *$k$ -extra connectivity* ( *$k$ -extra edge-connectivity*) of  $G$ , denoted by  $\kappa_k(G)$  ( $\lambda_k(G)$ ), is the minimum cardinality of a  $k$ -extra vertex-cut ( $k$ -extra edge-cut) of  $G$ . Extra connectivity and extra edge-connectivity in certain classes of graphs have received considerable attention in recent years [1, 5, 6, 13, 17, 18, 20, 21, 27, 34]. Hsieh and Chang [17]

discussed the 3-extra connectivity of a  $k$ -ary  $n$ -cube for  $k \geq 4$  and  $n \geq 5$ . Chang et al. [6] discussed the 4-extra connectivity and 4-extra edge-connectivity of folded hypercubes. Chang and Hsieh [5] investigated the  $\{3, 4\}$ -extra connectivity of hypercube-like networks. In addition, Meng and Ji [21] showed that regular graphs with an order of more than 5 have at least one 3-extra edge-cut. Hong and Hsieh [13] discussed the 4-extra edge-connectivity of hypercube-like networks. Lü et al. [18] derived the 2-extra edge-connectivity of product graphs. Furthermore, Balbuena and Marcote [1] discussed the  $k$ -extra edge-connectivity of product graphs for  $k \geq 3$ . Zhao and Ou [34] derived the 2-extra edge-connectivity of lexicographic product graphs.

Given a vertex subset  $X \subseteq V(G)$ , the notation  $[X, \overline{X}]$  is used to denote the set of edges having one endpoint in  $X$  and another endpoint in  $\overline{X}$ , where  $\overline{X} = V(G) - X$ . Apparently,  $[X, \overline{X}]$  is an edge-cut of  $G$ . The notation  $G[X]$  is used to denote the subgraph of  $G$  induced by  $X$ . Let  $\omega_G(X) = |[X, \overline{X}]|$  and  $\xi_k(G) = \min\{\omega_G(X) : X \subseteq V(G), |X| = k, \text{ and } G[X] \text{ is connected}\}$ . Obviously,  $\xi_k(G)$  is always an upper bound of  $\kappa_k(G)$  and  $\lambda_k(G)$ . If  $\kappa_k(G) = \xi_k(G)$ , then  $G$  is said to be maximally  $k$ -connected (or optimal- $\kappa_k$ ). If  $\lambda_k(G) = \xi_k(G)$ , then  $G$  is said to be maximally  $k$ -edge-connected (or optimal- $\lambda_k$ ). The properties of optimal- $\kappa_1$  and optimal- $\lambda_k$  for certain classes of graphs have been discussed [1, 12, 18, 22, 24, 25, 29]. Hellwig et al. [12] discussed properties of optimal- $\kappa_1$  and optimal- $\lambda_1$  graphs. Lü et al. [18] stated sufficient conditions for product graphs to be optimal- $\lambda_2$ , and Shang and Zhang [24, 25] provided sufficient conditions for various graphs to be optimal- $\lambda_2$  and optimal- $\lambda_3$ . Wang et al. [29] provided sufficient conditions for graphs with a diameter of 2 to be optimal- $\lambda_k$  for  $k \geq 1$ . In addition, Balbuena and Marcote [1] provided sufficient conditions for product graphs to be optimal- $\lambda_k$  for  $k \geq 3$ .

Graph  $G$  is *super  $k$ -extra connected* (*super- $\kappa_k$* ) if every minimum  $k$ -extra vertex-cut of  $G$  isolates a component with order  $k$ . Graph  $G$  is *super  $k$ -extra edge-connected* (*super- $\lambda_k$* ) if every minimum  $k$ -extra edge-cut of  $G$  isolates a component with order  $k$ . The properties of super- $\kappa_1$  and super- $\lambda_k$  of certain classes of graphs have been discussed [1, 7, 19, 24, 25, 29, 34]. Meng [19] provided sufficient conditions for connected vertex-transitive and edge-transitive graphs to be super- $\kappa_1$ . Shang and Zhang [24, 25] provided sufficient conditions for various graphs to be super-

Table 1: Summary of Related Studies ( $\eta_k(G)$  is the maximum number of edge-disjoint connected subtrees with order  $k$  such that each subtree  $H$  satisfies  $\omega_G(V(H)) = \xi_k(G)$ .)

| The properties which the graph $G$ satisfies   | The elements of $F$ | Lower bound of $ F $  | Upper bound of $ F $ | Property of $G - F$ | Ref.       |
|--|---------------------|---|----------------------|---------------------|------------|
| Optimal- $\kappa_1$ , $\eta_1(G) \geq \delta(G)$ , and $\kappa_2(G)$ exists            | Vertices            | $\min\{\kappa_2(G) - \delta(G), \delta(G) - 1\}$  | $\delta(G) - 1$      | Optimal- $\kappa_1$ | [15]       |
| Super- $\kappa_1$ , $\eta_1(G) \geq \delta(G)$ , and $\kappa_2(G)$ exists              | Vertices            | $\min\{\kappa_2(G) - \delta(G) - 1, \delta(G) - 1\}$  | $\delta(G) - 1$      | Super- $\kappa_1$   | [15]       |
| Super- $\lambda_1$ and $\lambda_2(G)$ exists   | Edges               | $\min\{\lambda_2(G) - \delta(G) - 1, \delta(G) - 1\}$                                       | $\delta(G) - 1$      | Super- $\lambda_1$  | [14]       |
| Super- $\lambda_2$ , $\eta_2(G) \geq \delta(G)$ , and $\lambda_3(G)$ exists            | Edges               | $\min\{\lambda_3(G) - \xi_2(G) - 1, \delta(G) - 1\}$  | $\delta(G) - 1$      | Super- $\lambda_2$  | [16]       |
| Super- $\lambda_k$ and $\lambda_{k+1}(G)$ does not exist for $2 \leq k \leq \delta(G)$ | Edges               | $\lambda(G) - 1$  | $\lambda(G) - 1$     | Super- $\lambda_k$  | this paper |
| Super- $\lambda_2$ and $\lambda_3(G)$ exists   | Edges               | $\min\{\lambda_3(G) - \xi_2(G) - 1, \eta_2(G) - 1, \lambda(G) - 1\}$                        | $\lambda(G) - 1$     | Super- $\lambda_2$  | this paper |
| Super- $\lambda_3$ and $\lambda_4(G)$ exists   | Edges               | $\min\{\lambda_4(G) - \xi_3(G) - 1, \eta_3(G) - 1, \lambda(G) - 1\}$                        | $\lambda(G) - 1$     | Super- $\lambda_3$  | this paper |
| Super- $\lambda_k$ and $\lambda_{k+1}(G)$ exists for $k \geq 4$                        | Edges               | $\min\{\lambda_{k+1}(G) - \xi_k(G) - 1, \eta_k(G) - 1, \delta(G) - k + 1, \lambda(G) - 1\}$ | $\lambda(G) - 1$     | Super- $\lambda_k$  | this paper |

$\lambda_2$  and super- $\lambda_3$ . Furthermore, Zhao and Ou [34] provided sufficient conditions for lexicographic product graphs to be super- $\lambda_2$ , and Balbuena and Marcote [1] provided sufficient conditions for product graphs to be super- $\lambda_k$  for  $k \geq 3$ .

Link (edge) and process (vertex) faults may occur when a network is activated; therefore, it is crucial to consider faulty networks. Recently, Hong and Zhang [15] discussed the vertex fault tolerance of optimal- $\kappa_1$  and super- $\kappa_1$  graphs. Hong et al. [14] discussed the edge fault tolerance of super- $\lambda_1$  graphs, and Wang and Lu [26] examined the bounds in graphs in detail. Hong and Xu [16] discussed the edge fault tolerance of super- $\lambda_2$  graphs under a condition. In this paper, we discuss the edge fault tolerance of super- $\lambda_k$  graphs for  $k \geq 2$ . In other words, we determine the number of faulty edges allowed for graph to remain super- $\lambda_k$ . We illustrate the bounds when  $k = 2$ ,  $k = 3$ , and  $k \geq 4$ . (More details about the results are provided in Table 1.)

The remainder of this paper is organized as follows. Section 2 provides definitions and notation. Section 3 shows the edge fault tolerance of super- $\lambda_3$  graphs. Section 4 shows the edge fault tolerance of super- $\lambda_k$  graphs when  $k \geq 4$ . Section 4 shows the edge fault tolerance of product graphs with some properties. Section 6 provides concluding remarks.

## 2 Preliminaries

An undirected *graph*  $G = (V, E)$  is a pair of the *vertex set*  $V$  and *edge set*  $E$ , where  $V$  is a finite set and  $E$  is a subset of  $\{(u, v) \mid (u, v) \text{ is an unordered pair of } V\}$ .  $V(G)$  and  $E(G)$  denote the vertex set and edge set of  $G$ , respectively. Let  $n(G) = |V(G)|$  be the *order* of  $G$ . Two vertices  $u$  and  $v$  are *adjacent* if  $(u, v)$  is an edge in  $G$ . The edge  $(u, v)$  is *incident to*  $u$  and  $v$ , and  $u$  and  $v$  are the *endpoints* of  $(u, v)$ . The *degree* of vertex  $v$ , denoted by  $d_G(v)$ , is the number of edges incident to it. Let  $\delta(G) = \min\{d_G(v) \mid v \in V(G)\}$ . A *path*  $\langle v_1, v_2, \dots, v_t \rangle$  is a sequence of distinct vertices such that any two consecutive vertices are adjacent. Vertices  $v_1$  and  $v_t$  are the *endpoints* of the path. A *cycle*  $\langle v_0, v_1, \dots, v_t, v_0 \rangle$  for  $t \geq 2$  is a sequence of vertices such that any two consecutive vertices are adjacent, where  $v_0, v_1, \dots, v_t$  are all distinct. A *complete graph* is a simple graph whose vertices are pairwise adjacent; a (unlabeled) complete graph with  $n$  vertices is denoted  $K_n$ .

An *isomorphism* from a simple graph  $G$  to a simple graph  $H$  is a one-to-one and onto function  $\pi : V(G) \rightarrow V(H)$  such that  $(u, v) \in E(G)$  if and only if  $(\pi(u), \pi(v)) \in E(H)$ . We say that “ $G$  is *isomorphic to*  $H$ ”, written  $G \cong H$ , if there is an isomorphism from  $G$  to  $H$ .

Let  $k \geq 1$  be an integer. An edge set  $S \subseteq E(G)$  is a *k-extra edge-cut* if  $G - S$  is disconnected and each component has at least  $k$  vertices. The *k-extra edge-connectivity* of  $G$ , denoted by  $\lambda_k(G)$ , is defined as the minimum cardinality over all  $k$ -extra edge-cuts of  $G$ . If  $\lambda_k(G)$  exists, then  $G$  is said to be  $\lambda_k$ -connected and  $\lambda(G) = \lambda_1(G) \leq \lambda_2(G) \leq \lambda_3(G) \leq \dots \leq \lambda_k(G)$ . The following lemma shows the necessary and sufficient condition for a graph to be  $\lambda_k$ -connected.

**Lemma 1.** [23] *Let  $k \geq 1$  be an integer and  $G$  be a connected graph. If  $G$  has an order of at least  $3k - 2$ , then  $G$  is a  $\lambda_k$ -connected graph if and only if  $G$  contains no vertex  $u$  such that every component of  $G - \{u\}$  has an order of at most  $k - 1$ .*

The terms  $[X, \overline{X}]$ ,  $\omega_G(X)$ , and  $\xi_k(G)$  have been defined in Section 1. Studies have shown that  $\lambda_k(G) \leq \xi_k(G)$  holds for any  $\lambda_k$ -connected graph, where  $1 \leq k \leq 3$  [3, 4, 8, 21]. For  $k \geq 4$ , Bonsma et al. [4] observed that the inequality  $\lambda_k(G) \leq \xi_k(G)$  is no longer true in general.

Let  $G_1, G_2, \dots, G_m$  be  $m$  copies of  $K_t$ , where  $m \geq 1$  and  $t \geq 1$ . Let  $v$  be a new vertex such that  $v$  is adjacent to every vertex in  $\cup_{i=1}^m V(G_i)$ . Then, the resulting graph is denoted by  $G_{m,t}^*$ . When  $t = 1$ ,  $G_{m,t}^*$  is a star. The following lemma shows a sufficient condition for a graph  $G$  to be  $\lambda_k$ -connected and  $\lambda_k(G) \leq \xi_k(G)$  for  $1 \leq k \leq \delta(G) + 1$ .

**Lemma 2.** [33] *Let  $G$  be a connected graph with an order of at least  $2(\delta(G) + 1)$ . If  $G$  is not isomorphic to  $G_{m,\delta(G)}^*$  for any positive integer  $m$ , then  $\lambda_k(G)$  exists, and  $\lambda_k(G) \leq \xi_k(G)$  for any  $k$  with  $1 \leq k \leq \delta(G) + 1$ .*

A graph  $G$  is said to be  $\lambda_k$ -optimal if it satisfies  $\lambda_k(G) = \xi_k(G)$ . Some properties of  $\lambda_k$ -optimal graphs were investigated in [32]. Moreover, if  $\lambda_k(G)$  exists and at least one component of  $G - S$  contains exactly  $k$  vertices for any minimum  $k$ -extra edge-cut  $S$ , then  $G$  is said to be super- $\lambda_k$ ;  $\lambda_k(G) = \xi_k(G)$  if  $G$  is super- $\lambda_k$ . The following lemma shows the necessary and sufficient condition for a  $\lambda_k$ -connected graph to be super- $\lambda_k$ .

**Lemma 3.** *Let  $G$  be a  $\lambda_k$ -connected graph with  $\lambda_k(G) \leq \xi_k(G)$  for some  $k \geq 1$ . Then,  $G$  is super- $\lambda_k$  if and only if  $G$  is not  $\lambda_{k+1}$ -connected or  $\omega_G(X) > \xi_k(G)$  holds for any vertex set  $X \subseteq V(G)$  with  $k + 1 \leq |X| \leq \lfloor n(G)/2 \rfloor$  and  $G[X]$ ,  $G[\bar{X}]$  being connected.*

*Proof.* According to Lemma 1.4 in [16],  $G$  is super- $\lambda_k$  if and only if  $G$  is not  $\lambda_{k+1}$ -connected or  $\lambda_{k+1}(G) > \xi_k(G)$  for any  $k \geq 1$ . Because  $\omega_G(X) > \xi_k(G)$  holds for any vertex set  $X \subseteq V(G)$  with  $k + 1 \leq |X| \leq \lfloor |V(G)|/2 \rfloor$  and  $G[X]$  and  $G[\bar{X}]$  being connected,  $\lambda_{k+1}(G) > \xi_k(G)$ . Therefore, the result holds.  $\square$

Let  $\eta_k(G)$  denote the maximum number of edge-disjoint connected subtrees with order  $k$  such that each subtree  $H$  satisfies  $\omega_G(V(H)) = \xi_k(G)$ . For example, consider the graph  $G$  shown in Fig. 1. Because vertices  $a$  and  $b$  are the only two vertices with a minimum degree in this graph,  $\eta_1(G) = 2$ . Moreover, because  $\omega_G(\{(a, b)\}) = \xi_2(G) = 4$  and each of the other edges  $e$  does not satisfy the condition (i.e.,  $\omega_G(\{\text{the endpoints of } e\}) \neq \xi_2(G) = 4$ ),  $\eta_2(G) = 1$ . The equality  $\eta_3(G) = 2$  holds because we can find two edge-disjoint paths  $\langle a, c, b \rangle$  and  $\langle a, d, b \rangle$  such

that  $\omega_G(\{a, c, b\}) = \omega_G(\{a, d, b\}) = \xi_3(G) = 4$ . Finally, the equality  $\eta_4(G) = 2$  holds because we can find two edge-disjoint paths  $\langle a, c, d, b \rangle$  and  $\langle d, a, b, c \rangle$  such that  $\omega_G(\{a, b, c, d\}) = \xi_4(G) = 2$ .

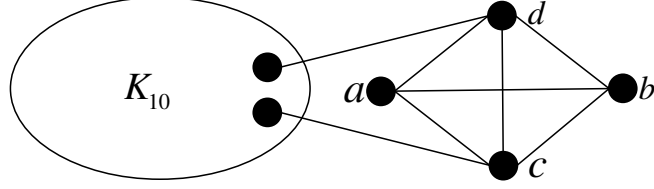


Figure 1: Illustration of  $\eta_k(G)$  for  $k = 1, 2, 3, 4$

**Definition 1.** Let  $k \geq 1$  be an integer. The *persistence* of the super- $\lambda_k$  graph  $G$ , denoted by  $\rho_k(G)$ , is the maximum integer  $m$  for which  $G - F$  is still super- $\lambda_k$  for any set  $F \subseteq E(G)$  with  $|F| \leq m$ .

Hong et al. [14] showed that  $\min\{\lambda_2(G) - \delta(G) - 1, \delta(G) - 1\} \leq \rho_1(G) \leq \delta(G) - 1$  for any super- $\lambda_1$  and  $\lambda_2$ -connected graph  $G$ . In addition, Hong and Xu [16] showed that  $\min\{\lambda_3(G) - \xi_2(G) - 1, \delta(G) - 1\} \leq \rho_2(G) \leq \delta(G) - 1$  for any super- $\lambda_2$  and  $\lambda_3$ -connected graph  $G$  with  $\eta_2(G) \geq \delta(G)$ . This study determines the following bounds of  $\rho_k(G)$  for  $k \geq 2$ :

1.  $\rho_k(G) = \lambda_1(G) - 1$  if graph  $G$  is super- $\lambda_k$ , but not  $\lambda_{k+1}$ -connected,
2.  $\min\{\lambda_3(G) - \xi_2(G) - 1, \eta_2(G) - 1, \lambda_1(G) - 1\} \leq \rho_2(G) \leq \lambda_1(G) - 1$  for any super- $\lambda_2$  and  $\lambda_3$ -connected graph  $G$ ,
3.  $\min\{\lambda_4(G) - \xi_3(G) - 1, \eta_3(G) - 1, \lambda_1(G) - 1\} \leq \rho_3(G) \leq \lambda_1(G) - 1$  for any super- $\lambda_3$  and  $\lambda_4$ -connected graph  $G$ , and
4.  $\min\{\lambda_{k+1}(G) - \xi_k(G) - 1, \eta_k(G) - 1, \delta(G) - k + 1, \lambda_1(G) - 1\} \leq \rho_k(G) \leq \lambda_1(G) - 1$  for any super- $\lambda_k$  and  $\lambda_{k+1}$ -connected graph  $G$ , where  $k \geq 4$ .

### 3 Bounds on the persistence of super- $\lambda_3$ graphs

If a graph  $G = (V, E)$  has a path with endpoints  $u$  and  $v$ , then the *distance* between  $u$  and  $v$ , denoted by  $d_G(u, v)$  or simply  $d(u, v)$ , is the shortest length of a path between  $u$  and  $v$ . If



$G$  has no such path, then  $d(u, v) = \infty$ . The *eccentricity* of a vertex  $u$ , denoted by  $\epsilon(u)$ , is  $\max_{v \in V} d(u, v)$ .

Let  $H_1$  be a connected graph that has an order of at least 6 and satisfies the following conditions: (a)  $H_1$  contains no cycles of a length greater than 3 and (b) there exists exactly one vertex  $v_0 \in V(H_1)$  with a degree greater than 2, and  $v_0$  has eccentricity equal to or less than 2. Fig. 2 illustrates graphs  $H_1$  and  $H_2$ . The following lemma shows the necessary and sufficient condition for a graph to be  $\lambda_3$ -connected.

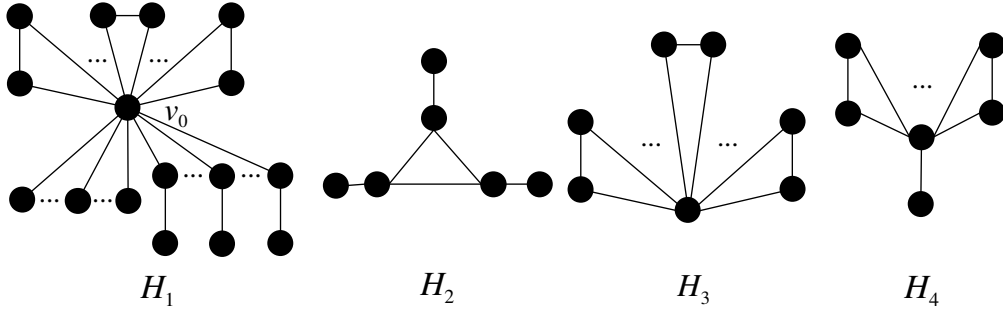


Figure 2: Illustration of Lemmas 4 and 5

**Lemma 4.** [4] *A connected graph with an order of 6 is not  $\lambda_3$ -connected if and only if  $G$  is isomorphic to  $H_1$  or  $H_2$  (Fig. 2). Furthermore, if  $G$  is  $\lambda_3$ -connected, then  $\lambda_3(G) \leq \xi_3(G)$ .*

According to Lemma 4, we can determine the edge fault tolerance of a  $\lambda_3$ -connected graph.

**Lemma 5.** *Let  $G$  be a  $\lambda_3$ -connected graph with  $\delta(G) \geq 3$ . Then,  $G - F$  is a  $\lambda_3$ -connected graph and  $\lambda_3(G - F) \leq \xi_3(G - F)$  for any  $F \subseteq E(G)$  with  $|F| \leq \lambda_1(G) - 1$ .*

*Proof.* Assume that  $G - F$  is not  $\lambda_3$ -connected. According to Lemma 4,  $G - F \cong H_1$  or  $G - F \cong H_2$ . Consider the following scenarios.

**Case 1:**  $\delta(G - F) \geq 2$ . In this case,  $G - F \cong H_3$  (Fig. 2) and there are at least six vertices with a degree of 2 in  $G - F$ . Therefore,

$$|F| = |E(G)| - |E(G - F)| \geq \lceil \frac{(6\delta(G) - 6 \cdot 2)}{2} \rceil = 3\delta(G) - 6 \geq \delta(G) \geq \lambda_1(G),$$

which leads to a contradiction.

**Case 2:**  $\delta(G - F) = 1$ . Because  $|F| \leq \lambda_1(G) - 1 \leq \delta(G) - 1$ , exactly one vertex in  $G - F$  has a degree of one, and all edges of  $F$  are incident to this vertex  $v$ . Therefore,  $G - F$  is not isomorphic to  $H_2$ . According to Lemma 4,  $G - F \cong H_4$  (Fig. 2), and at least four vertices with a degree of 2. All edges of  $F$  are incident to the vertex with a degree of 1; therefore,

$$|F| = |E(G)| - |E(G - F)| \geq 4\delta(G) - 4 \cdot 2 \geq \delta(G) + 1 \geq \lambda_1(G),$$

which leads to a contradiction.

Combining the two cases completes the proof.  $\square$

Next, we show the bounds of  $\rho_3(G)$  for the super- $\lambda_3$  graph  $G$  in Theorem 1.

**Theorem 1.** *Let  $G$  be a super- $\lambda_3$  graph with  $\delta(G) \geq 3$ . Then the following statements hold: (a) If  $G$  is not  $\lambda_4$ -connected, then  $\rho_3(G) = \lambda_1(G) - 1$ . (b) If  $G$  is  $\lambda_4$ -connected, then  $\min\{\lambda_4(G) - \xi_3(G) - 1, \eta_3(G) - 1, \lambda_1(G) - 1\} \leq \rho_3(G) \leq \lambda_1(G) - 1$ .*

*Proof.* There exists one edge set  $F$  with size  $\lambda_1(G)$  such that  $G - F$  is disconnected. Then,  $G - F$  is not super- $\lambda_3$ , implying that  $\rho_3(G) \leq |F| - 1 = \lambda_1(G) - 1$ . Next, we determine the lower bound of  $\rho_3(G)$  according to whether  $G$  is  $\lambda_4$ -connected.

(a) To prove that  $\rho_3(G) \geq \lambda_1(G) - 1$ , it suffices to show that for any  $F \subseteq E(G)$  with  $|F| \leq \lambda_1(G) - 1$ ,  $G - F$  is super- $\lambda_3$ . According to Lemma 5,  $G - F$  is  $\lambda_3$ -connected with  $\lambda_3(G - F) \leq \xi_3(G - F)$ . Because  $G$  is not  $\lambda_4$ -connected,  $G - F$  is also not  $\lambda_4$ -connected. According to Lemma 3,  $G - F$  is super- $\lambda_3$  and (a) is proved.

(b) Let  $m = \min\{\lambda_4(G) - \xi_3(G) - 1, \eta_3(G) - 1, \lambda_1(G) - 1\}$ . To prove that  $\rho_3(G) \geq m$ , it suffices to show that for any  $F \subseteq E(G)$  with  $|F| \leq m$ ,  $G' = G - F$  is super- $\lambda_3$ .

Note that  $|F| \leq m \leq \lambda_1(G) - 1$ . According to Lemma 5,  $G - F$  is  $\lambda_3$ -connected with  $\lambda_3(G - F) \leq \xi_3(G - F)$ . If  $G - F$  is not  $\lambda_4$ -connected, then according to Lemma 3,  $G - F$  is super- $\lambda_3$  and (b) is proved. If  $G - F$  is  $\lambda_4$ -connected, then let  $X$  be any subset of  $V(G')$  with  $|X| \geq 4$  and  $|\overline{X}| \geq 4$  such that  $G'[X]$  and  $G'[\overline{X}]$  are connected. Therefore,

$$\omega_{G-F}(X) \geq \omega_G(X) - |F| \geq \lambda_4(G) - (\lambda_4(G) - \xi_3(G) - 1) = \xi_3(G) + 1. \quad (1)$$

Because  $|F| \leq m \leq \eta_3(G) - 1$ , there exists one subtree  $A$  with an order of 3 such that  $\omega_G(A) = \xi_3(G)$  and  $A = A - F$ . In other words,

$$\xi_3(G) = \omega_G(A) \geq \omega_{G-F}(A) \geq \xi_3(G - F). \quad (2)$$

According to equations (1) and (2),  $\omega_{G-F}(X) > \xi_3(G - F)$ . According to Lemma 3,  $G - F$  is super- $\lambda_3$  and (b) is proved.  $\square$

Evidently,  $\lambda_1(G) - 1$  is a necessary term in the proposed lower bound formula because the value is also used as an upper bound in Theorem 1. Examples 1 and 2 demonstrate that the other values are also necessary.

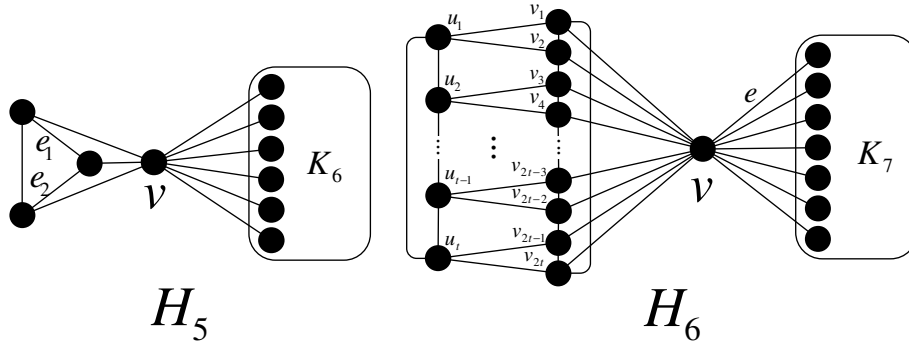


Figure 3: Illustration of Examples 1 and 2

**Example 1.** Consider the graph  $G \cong H_5$  shown in Fig. 3. A vertex  $v \in V(H)$  is adjacent to all the other vertices in  $G$ . Because  $\lambda_1(G) = \lambda_3(G) = \xi_3(G) = 3$  and  $\lambda_4(G) = 6$ , according to Lemma 3,  $G$  is super- $\lambda_3$ . Moreover,  $\min\{\lambda_4(G) - \xi_3(G) - 1, \lambda_1(G) - 1\} = 2$  and  $G' = G - \{e_1, e_2\}$  is not super- $\lambda_3$  because  $\lambda_3(G') = \lambda_4(G') = 6$ . Therefore,  $\eta_3(G) - 1$  is a necessary term in the proposed lower bound formula (in this case,  $\eta_3(G) - 1 = 0$ ).

**Example 2.** Consider the graph  $G \cong H_6$  shown in Fig. 3;  $\{u_1, u_2, \dots, u_t\}$  and  $\{v_1, v_2, \dots, v_{2t}\}$  form two separate cycles. The vertices  $u_i$  and  $v_j$  are adjacent when  $j \in \{2i - 1, 2i\}$ . Because  $\lambda_1(G) = 4$ ,  $\lambda_3(G) = \xi_3(G) = 6$ , and  $\lambda_4(G) = 7$ , according to Lemma 3,  $G$  is super- $\lambda_3$ . Moreover,  $\min\{\eta_3(G) - 1, \lambda_1(G) - 1\} = 3$  and  $G' = G - \{e\}$  is not super- $\lambda_3$  because  $\lambda_3(G') = \lambda_4(G') = 6$ . Therefore,  $\lambda_4(G) - \xi_3(G) - 1$  is a necessary term in the proposed lower bound formula (in this case,  $\lambda_4(G) - \xi_3(G) - 1 = 0$ ).

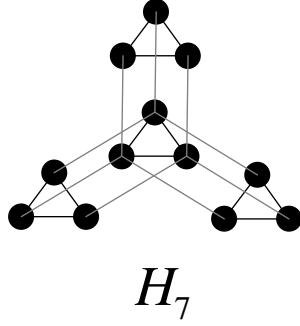


Figure 4: Illustration of Example 3

**Example 3.** Consider the graph  $G \cong H_7$  shown in Fig. 4;  $\rho_3(G) = 2$ . Because  $\lambda_1(G) = \lambda_3(G) = \xi_3(G) = 3$  and  $\lambda_4(G) = 6$ , according to Lemma 3,  $G$  is super- $\lambda_3$ . According to Theorem 1,  $\min\{\lambda_4(G) - \xi_3(G) - 1, \eta_3(G) - 1, \lambda_1(G) - 1\} = 2 = \lambda_1(G) - 1$ . Therefore, the lower bound given in Theorem 1 is sharp.

**Corollary 1.** *Let  $G$  be a super- $\lambda_2$  and  $\lambda_3$ -connected graph. Then,  $\min\{\lambda_3(G) - \xi_2(G) - 1, \eta_2(G) - 1, \lambda_1(G) - 1\} \leq \rho_2(G) \leq \lambda_1(G) - 1$ .*

*Proof.* Note that  $n(G) \geq 6$  because  $G$  is  $\lambda_3$ -connected. Let  $m = \min\{\lambda_3(G) - \xi_2(G) - 1, \eta_2(G) - 1, \lambda_1(G) - 1\}$ . We first prove that  $G - F$  is  $\lambda_2$ -connected with  $\lambda_2(G - F) \leq \xi_2(G - F)$  for any  $F \subseteq E(G)$  with  $|F| \leq m$ .

When  $\delta(G) = 1$ ,  $|F| \leq \lambda_1(G) - 1 \leq \delta(G) - 1 \leq 0$  and  $G = G - F$ . Therefore,  $G = G - F$  is not a star because  $G$  is  $\lambda_3$ -connected. When  $\delta(G) \geq 2$ , assume that  $G - F$  is a star, and there exists  $n(G) - 1$  vertices with a degree of 1 in  $G - F$ . Because  $\delta(G) \geq 2$ , at least  $\delta(G) - 1$  edges of  $F$  are incident to each of the aforementioned  $n(G) - 1$  vertices. Consequently,

$$\lceil \frac{([n(G) - 1][\delta(G) - 1])}{2} \rceil \leq |E(G)| - |E(G - F)| = |F| \leq \lambda_1(G) - 1 \leq \delta(G) - 1.$$

In other words,  $n(G) \leq 3$ , which leads to a contradiction. Therefore,  $G - F$  is not a star. According to the results of [3, 21],  $G - F$  is  $\lambda_2$ -connected with  $\lambda_2(G - F) \leq \xi_2(G - F)$  if  $G - F$  is not a star; therefore,  $G - F$  is  $\lambda_2$ -connected with  $\lambda_2(G - F) \leq \xi_2(G - F)$ . Consequently, this result of Corollary 1 can be proved using a method similar to that used in Theorem 1.  $\square$

Figure 1 of [16] shows that the lower bound given in Corollary 1 is sharp.

## 4 Bounds on the persistence of super- $\lambda_k$ graphs for $k \geq 4$

First, we determine the edge fault tolerance of  $\lambda_k$ -connected graphs for a positive integer  $k \geq 2$ .

**Lemma 6.** *Let  $G$  be a  $\lambda_k$ -connected graph with  $2 \leq k \leq \delta(G)$ , and  $n(G) \geq 3\delta(G) - 2$ . Then,  $G - F$  is  $\lambda_k$ -connected graph for any  $F \subseteq E(G)$  with  $|F| \leq \lambda_1(G) - 1$ .*

*Proof.* Apparently,  $G - F$  is connected because  $|F| \leq \lambda_1(G) - 1$ . If  $G - F$  is not  $\lambda_k$ -connected, according to Lemma 1, there exists  $n(G) - 1$  vertices with a degree of at most  $k - 1$  in  $G - F$ . Because  $\delta(G) \geq k$ , at least one edge of  $F$  is incident to each of the aforementioned  $n(G) - 1$  vertices. Thus,

$$|F| \geq \lceil (n(G) - 1)/2 \rceil \geq \lceil (3\delta(G) - 3)/2 \rceil > \delta(G) - 1 \geq \lambda_1(G) - 1,$$

which leads to a contradiction. Therefore,  $G - F$  is a  $\lambda_k$ -connected graph.  $\square$

Now, let  $G$  be a super- $\lambda_k$  graph, but not  $\lambda_{k+1}$ -connected for  $k \geq 2$ . We show the bounds of  $\rho_k(G)$  in Theorem 2.

**Theorem 2.** *Let  $G$  be a super- $\lambda_k$  graph with  $2 \leq k \leq \delta(G)$ , and  $n(G) \geq 3\delta(G) - 2$ . If  $G$  is not  $\lambda_{k+1}$ -connected, then  $\rho_k(G) = \lambda_1(G) - 1$ .*

*Proof.* There exists one edge set  $F$  with size  $\lambda_1(G)$  such that  $G - F$  is disconnected. Then,  $G - F$  is not super- $\lambda_k$ . We have  $\rho_k(G) \leq |F| - 1 = \lambda_1(G) - 1$ .

To prove that  $\rho_k(G) \geq \lambda_1(G) - 1$ , it suffices to show that for any  $F \subseteq E(G)$  with  $|F| \leq \lambda_1(G) - 1$ ,  $G - F$  is super- $\lambda_k$ . According to Lemma 6,  $G - F$  is  $\lambda_k$ -connected. Because  $G$  is not  $\lambda_{k+1}$ -connected,  $G - F$  is also not  $\lambda_{k+1}$ -connected. Thus, every minimum  $k$ -extra edge-cut of  $G - F$  isolates at least one connected subgraph of order  $k$ . Therefore,  $G - F$  is super- $\lambda_k$  and this theorem is proved.  $\square$

Next, we determine the bounds of  $\rho_k(G)$  for a super- $\lambda_k$  and  $\lambda_{k+1}$ -connected graph  $G$ .

**Lemma 7.** *Let  $G$  be a  $\lambda_k$ -connected graph with  $\lambda_k(G) \leq \xi_k(G)$ ,  $k \leq \delta(G) + 1$ , and  $n(G) \geq 2\delta(G) + 2$ . Then,  $G - F$  is a  $\lambda_k$ -connected graph and  $\lambda_k(G - F) \leq \xi_k(G - F)$  for any  $F \subseteq E(G)$  with  $|F| \leq \min\{\delta(G) - k + 1, \lambda_1(G) - 1\}$ .*

*Proof.* Because  $|F| \leq \lambda_1(G) - 1$ ,  $G - F$  is connected. If  $G - F$  is not isomorphic to  $G_{m,k-1}^*$  for any positive integer  $m$ , according to Lemma 2,  $G - F$  is  $\lambda_k$ -connected and  $\lambda_k(G - F) \leq \xi_k(G - F)$  because  $n(G) \geq 2\delta(G) + 2 \geq 2\delta(G - F) + 2$ .

Therefore, we consider only the case when  $G - F$  is isomorphic to  $G_{m,k-1}^*$  for a positive integer  $m$ . Consider the following scenarios.

**Case 1:**  $\delta(G) \geq k$ . Because  $G - F \cong G_{m,k-1}^*$ , there exists  $n(G) - 1$  vertices with degree  $k - 1$  in  $G - F$ . Because  $\delta(G) \geq k$ , at least  $\delta(G) - k + 1$  edges of  $F$  are incident to each of the aforementioned  $n(G) - 1$  vertices. Therefore,

$$\lceil \frac{([n(G) - 1][\delta(G) - k + 1])}{2} \rceil \leq |E(G)| - |E(G - F)| = |F| \leq \delta(G) - k + 1,$$

which implies that  $n(G) \leq 3$  and leads to a contradiction.

**Case 2:**  $\delta(G) = k - 1$ . In this case,  $|F| = \delta(G) - k + 1 = 0$  and  $G - F = G$ . Evidently,  $G - F$  is a  $\lambda_k$ -connected graph and  $\lambda_k(G - F) \leq \xi_k(G - F)$ .  $\square$

**Theorem 3.** Let  $G$  be a super- $\lambda_k$  graph with  $k \geq 4$  and  $n(G) \geq 2\delta(G) + 2$ . If  $G$  is  $\lambda_{k+1}$ -connected, then  $\min\{\lambda_{k+1}(G) - \xi_k(G) - 1, \eta_k(G) - 1, \delta(G) - k + 1, \lambda_1(G) - 1\} \leq \rho_k(G) \leq \lambda_1(G) - 1$ .

*Proof.* There exists one edge set  $F$  with size  $\lambda_1(G)$  such that  $G - F$  is disconnected. Then,  $G - F$  is not super- $\lambda_k$ . We have  $\rho_k(G) \leq |F| - 1 = \lambda_1(G) - 1$ .

Let  $m = \min\{\lambda_{k+1}(G) - \xi_k(G) - 1, \eta_k(G) - 1, \delta(G) - k + 1, \lambda_1(G) - 1\}$ . To prove that  $\rho_k(G) \geq m$ , it suffices to show that for any  $F \subseteq E(G)$  with  $|F| \leq m$ ,  $G' = G - F$  is super- $\lambda_k$ . Note that  $|F| \leq m \leq \min\{\delta(G) - k + 1, \lambda_1(G) - 1\}$ ; therefore, according to Lemma 7,  $G - F$  is  $\lambda_k$ -connected with  $\lambda_k(G - F) \leq \xi_k(G - F)$ .

If  $G - F$  is not  $\lambda_{k+1}$ -connected, according to Lemma 3,  $G - F$  is super- $\lambda_k$  and this theorem is proved. If  $G - F$  is  $\lambda_{k+1}$ -connected, then let  $X$  be any subset of  $V(G')$  with  $|X| \geq k + 1$  and  $|\overline{X}| \geq k + 1$  such that  $G'[X]$  and  $G'[\overline{X}]$  are connected. Therefore,

$$\omega_{G-F}(X) \geq \omega_G(X) - |F| \geq \lambda_{k+1}(G) - (\lambda_{k+1}(G) - \xi_k(G) - 1) = \xi_k(G) + 1. \quad (3)$$

Because  $|F| \leq m \leq \eta_k(G) - 1$ , there exists one subtree  $A$  with order  $k$  such that  $\omega_G(A) = \xi_k(G)$  and  $A = A - F$ . In other words,

$$\xi_k(G) = \omega_G(A) \geq \omega_{G-F}(A) \geq \xi_k(G - F). \quad (4)$$

According to equations (3) and (4),  $\omega_{G-F}(X) > \xi_k(G - F)$ . According to Lemma 3,  $G - F$  is super- $\lambda_k$  and this theorem is proved.  $\square$

## 5 Bounds on the persistence of product graphs

In this section, the results obtained in Sections 3 and 4 are applied to product graphs. The product graph  $G * H$  of two graphs  $G$  and  $H$  was introduced in [2].

**Definition 2.** [2] Let  $G = (V(G), E(G))$  and  $H = (V(H), E(H))$  be two graphs. Let  $\pi_{xy}$  be a permutation of  $V(H)$  such that  $\pi_{xy}^{-1} = \pi_{yx}$  for each edge  $xy \in E(G)$ . Then, the product graph  $G * H$  is the graph with  $V(G) \times V(H)$  as its vertex set, and two vertices,  $(x, x')$  and  $(y, y')$ , are adjacent in  $G * H$  if and only if either  $x = y$  and  $x'y' \in E(H)$  or  $xy \in E(G)$  and  $y' = \pi_{xy}(x')$ .

**Lemma 8.** [1] Let  $k \geq 1$  be an integer. Let  $G$  be an  $r$ -regular connected graph and let  $H$  be an  $s$ -regular connected graph, with  $r \geq 2k - 1$  and  $s \geq r + 2k - 1$ , respectively. Assume that  $\lambda_1(G)|V(H)| \geq 2k(s - k) + 2$  and  $H$  is  $\lambda_{k+1}$ -connected with  $\lambda_{k+1}(H) \geq 2(s - k)$ . Then, the  $(r + s)$ -regular graph  $G * H$  is super- $\lambda_k$ .

According to Lemma 8, Corollary 1, and  $\xi_2(G) = 2r + 2s - 2$  for any  $(r + s)$ -regular graph  $G$ , the following result is true.

**Corollary 2.** Let  $G$  be an  $r$ -regular connected graph and let  $H$  be an  $s$ -regular connected graph, with  $r \geq 3$ ,  $s \geq r + 3$ . Suppose that  $\lambda_1(G)|V(H)| \geq 4s - 6$ , and also that  $H$  is  $\lambda_3$ -connected with  $\lambda_3(H) \geq 2s - 4$ . Then the  $(r + s)$ -regular graph  $G * H$  is super- $\lambda_2$  and  $\min\{\lambda_3(G * H) - 2r - 2s + 1, \eta_2(G * H) - 1, \lambda_1(G * H) - 1\} \leq \rho_2(G * H) \leq \lambda_1(G * H) - 1$ .

If  $G$  is an  $(r + s)$ -regular graph, then  $\xi_3(G) = 3r + 3s - 6$  if  $G$  contains a triangle or  $\xi_3(G) = 3r + 3s - 4$  if  $G$  contains no triangle. According to Lemma 8 and Theorem 1, the

following results are true.

**Corollary 3.** *Let  $G$  be an  $r$ -regular connected graph and let  $H$  be an  $s$ -regular connected graph, with  $r \geq 5$  and  $s \geq r+5$ , respectively. Assume that  $\lambda_1(G)|V(H)| \geq 6s-16$  and  $H$  is  $\lambda_4$ -connected with  $\lambda_4(H) \geq 2s-6$ . If  $H$  has a triangle subgraph, then the  $(r+s)$ -regular graph  $G*H$  is super- $\lambda_3$  and  $\min\{\lambda_4(G*H) - 3r - 3s + 5, \eta_3(G*H) - 1, \lambda_1(G*H) - 1\} \leq \rho_3(G*H) \leq \lambda_1(G*H) - 1$ .*

**Corollary 4.** *Let  $G$  be an  $r$ -regular connected graph and let  $H$  be an  $s$ -regular connected graph, with  $r \geq 5$  and  $s \geq r+5$ , respectively. Assume that  $\lambda_1(G)|V(H)| \geq 6s-16$  and  $H$  is  $\lambda_4$ -connected with  $\lambda_4(H) \geq 2s-6$ . If  $G$  and  $H$  have no triangle subgraph, then the  $(r+s)$ -regular graph  $G*H$  is super- $\lambda_3$  and  $\min\{\lambda_4(G*H) - 3r - 3s + 3, \eta_3(G*H) - 1, \lambda_1(G*H) - 1\} \leq \rho_3(G*H) \leq \lambda_1(G*H) - 1$ .*

According to Lemma 8 and Theorem 3, the following result is true.

**Corollary 5.** *Let  $k \geq 4$  be an integer. Let  $G$  be an  $r$ -regular connected graph and let  $H$  be an  $s$ -regular connected graph, with  $r \geq 2k-1$  and  $s \geq r+2k-1$ , respectively. Assume that  $\lambda_1(G)|V(H)| \geq 2k(s-k)+2$  and  $H$  is  $\lambda_{k+1}$ -connected with  $\lambda_{k+1}(H) \geq 2(s-k)$ . Then, the  $(r+s)$ -regular graph  $G*H$  is super- $\lambda_k$  and  $\min\{\lambda_{k+1}(G*H) - \xi_k(G*H) - 1, \eta_k(G*H) - 1, r + s - k + 1, \lambda_1(G*H) - 1\} \leq \rho_k(G*H) \leq \lambda_1(G*H) - 1$ .*

In [1], the super- $\lambda_k$  property of a product of two complete graphs was proved.

**Lemma 9.** [1] *For all integers  $k \geq 1$ ,  $r \geq 2k+1$ , and  $s \geq r+2k-1$ , the graph  $K_r * K_s$  is super- $\lambda_j$  for every  $1 \leq j \leq k$ .*

According to Lemma 9, Corollary 1, and  $\eta_2(G) = |E(G)|$ , for any regular graph  $G$ , the following result is true.

**Corollary 6.** *For all integers  $r \geq 5$  and  $s \geq r+3$ , the graph  $G = K_r * K_s$  is super- $\lambda_2$  and  $\min\{\lambda_3(G) - 2r - 2s + 5, \lambda_1(G) - 1\} \leq \rho_2(G) \leq \lambda_1(G) - 1$ .*

For any complete graph  $K_s$  for  $s \geq 4$ , we can prove that  $\eta_3(K_s) \geq s-1$  by induction on  $s$ . When  $s = 4$ , there are three edge-disjoint paths with an order of 3. Therefore,  $\eta_3(K_4) = 3 \geq s-1$ . Assume that the result holds when  $s \leq t$ . Consider  $s = t+1 > 4$ , and let  $v$  be a vertex of  $K_s$ .



According to the induction hypothesis, there are at least  $(s - 2)$  edge-disjoint paths with an order of 3 in  $K_s - \{v\}$ . The vertex  $v$  and two other vertices in  $K_s - \{v\}$  can form additional edge-disjoint paths with an order of 3. Then,  $\eta_3(K_s) \geq s - 1$ . Moreover,  $\eta_3(K_r * K_s) \geq r(s - 1) \geq r + s - 2 \geq \lambda_1(K_r * K_s)$  for  $r \geq 1$  and  $s \geq 4$ . According to Lemma 9 and Theorem 1, the following result is true.

**Corollary 7.** *For all integers  $r \geq 7$  and  $s \geq r + 5$ , the graph  $G = K_r * K_s$  is super- $\lambda_3$  and  $\min\{\lambda_4(G) - 3r - 3s + 11, \lambda_1(G) - 1\} \leq \rho_3(G) \leq \lambda_1(G) - 1$ .*

We have

$$\xi_k(K_r * K_s) = k(r + s - 2) - k(k - 1) = k(r + s - 1) - k^2.$$

According to Lemma 9 and Theorem 3, the following result is true.

**Corollary 8.** *For all integers  $k \geq 4$ ,  $r \geq 2k + 1$ , and  $s \geq r + 2k - 1$ , the graph  $K_r * K_s$  is super- $\lambda_k$  and  $\min\{\lambda_{k+1}(G) + k^2 - k(r + s - 1) - 1, \eta_k(G) - 1, r + s - k - 1, \lambda_1(G) - 1\} \leq \rho_2(G) \leq \lambda_1(G) - 1$ .*

## 6 Conclusion

Fault tolerance is critical for retaining a system's reliability. This study investigates the edge fault tolerance of super- $\lambda_k$  graphs. The bounds of this result for  $k \in \{1, 2\}$  have recently been presented. In this paper, we show that (1)  $\rho_k(G) = \lambda_1(G) - 1$  if the graph  $G$  is super- $\lambda_k$ , but not  $\lambda_{k+1}$ -connected for  $k \geq 2$ , (2)  $\min\{\lambda_3(G) - \xi_2(G) - 1, \eta_2(G) - 1, \lambda_1(G) - 1\} \leq \rho_2(G) \leq \lambda_1(G) - 1$  for any super- $\lambda_2$  and  $\lambda_3$ -connected graph  $G$ , (3)  $\min\{\lambda_4(G) - \xi_3(G) - 1, \eta_3(G) - 1, \lambda_1(G) - 1\} \leq \rho_3(G) \leq \lambda_1(G) - 1$  for any super- $\lambda_3$  and  $\lambda_4$ -connected graph  $G$ , and (4)  $\min\{\lambda_{k+1}(G) - \xi_k(G) - 1, \eta_k(G) - 1, \delta(G) - k + 1, \lambda_1(G) - 1\} \leq \rho_k(G) \leq \lambda_1(G) - 1$  for any super- $\lambda_k$  and  $\lambda_{k+1}$ -connected graph  $G$ , where  $k \geq 4$ . Future studies should evaluate tighter bounds of  $\rho_k(G)$  for graphs that satisfy specific conditions.

## References

- [1] C. Balbuena and X. Marcote : The k-restricted edge-connectivity of a product of graphs. Discrete Applied Mathematics 161(1), 52-59 (2013)

- [2] J. C. Bermond, C. Delorme, and G. Farhi : Large graphs with given degree and diameter II. *Journal of Combinatorial Theory, Series B*, 36(1), 32V48 (1984)
- [3] F. T. Boesch and J. F. Wang : Super line-connectivity properties of circulant graphs. *SIAM Journal on Algebraic Discrete Methods* 7(1), 89-98 (1986)
- [4] P. Bonsma, N. Ueffing, and L. Volkmann : Edge-cuts leaving components of order at least three. *Discrete Mathematics* 256(1), 431-439 (2002)
- [5] N. W. Chang and S. Y. Hsieh : (2,3)-Extraconnectivities of hypercube-like networks. *Journal of Computer and System Sciences* 79(5), 669-688 (2013)
- [6] N. W. Chang, C. Y. Tsai, and S. Y. Hsieh : On 3-extra connectivity and 3-extra edge connectivity of folded hypercubes. *IEEE Transactions on Computers* 63(6), 1594-1600 (2014)
- [7] E. Cheng, M. J. Lipman, and H. Park : Super connectivity of star graphs, alternating group graphs and split-stars. *Ars Combinatoria* 59, 107-116 (2001)
- [8] A. H. Esfahanian and S. L. Hakimi : On computing a conditional edge-connectivity of a graph. *Information Processing Letters* 27(4), 195-199 (1988)
- [9] J. Fàbrega and M. A. Fiol : Extraconnectivity of graphs with large girth. *Discrete Mathematics* 127(1), 163-170 (1994)
- [10] J. Fàbrega and M. A. Fiol : On the extraconnectivity of graphs. *Discrete Mathematics* 155(1), 49-57 (1996)
- [11] F. Harary : Conditional connectivity. *Networks* 13, 347-357 (1983)
- [12] A. Hellwig, L. Volkmann : Maximally edge-connected and vertex-connected graphs and digraphs: a survey. *Discrete Mathematics* 308(15), 3265-3296 (2008)
- [13] W. S. Hong and S. Y. Hsieh : Extra edge connectivity of hypercube-like networks. *International Journal of Parallel, Emergent and Distributed Systems* 28(2), 123-133 (2013)
- [14] Y. Hong, J. Meng, and Z. Zhang : Edge fault tolerance of graphs with respect to super edge connectivity. *Discrete Applied Mathematics* 160(4), 579-587 (2012)
- [15] Y. Hong and Z. Zhang : Vertex fault tolerance of optimal- $\kappa$  graphs and super- $\kappa$  graphs. *Information Processing Letters* 109(20), 1151-1155 (2009)
- [16] Z. M. Hong and J. M. Xu : Vulnerability of super edge-connected networks. *Theoretical Computer Science* 520, 75-86 (2014)
- [17] S. Y. Hsieh and Y. H. Chang : Extraconnectivity of k-ary n-cube networks. *Theoretical Computer Science* 443(20), 63-69 (2012)
- [18] M. Ĺu, G. L. Chen, and X. R. Xu : On super edge-connectivity of product graphs. *Applied Mathematics and Computation* 207(2), 300-306 (2009)
- [19] J. Meng : Connectivity of vertex and edge transitive graphs. *Discrete applied mathematics* 127(3), 601-613 (2003)
- [20] J. X. Meng : Optimally super-edge-connected transitive graphs. *Discrete Mathematics* 260(1), 239-248 (2003)
- [21] J. X. Meng and Y. H. Ji : On a kind of restricted edge connectivity of graphs. *Discrete applied mathematics* 117, 183-193 (2002)
- [22] O. R. Oellermann : Connectivity and edge-connectivity in graphs: a survey. *Congressus Numerantium* 116, 231-252 (1996)
- [23] J. Ou : Edge cuts leaving components of order at least m. *Discrete mathematics* 305(1), 365-371 (2005)
- [24] L. Shang and H. Zhang : Sufficient conditions for graphs to be  $\lambda'$ -optimal and super- $\lambda'$ . *Networks* 49(3), 234-242 (2007)
- [25] L. Shang and H. Zhang : Degree conditions for graphs to be  $\lambda_3$ -optimal and super- $\lambda_3$ . *Discrete Mathematics* 309(10), 3336-3345 (2009)

- [26] D. Wang and M. Lu : Edge fault tolerance of super edge connectivity for three families of interconnection networks. *Information Sciences* 188, 260-268 (2012)
- [27] M. Wang and Q. Li : Conditional edge connectivity properties, reliability comparison and transitivity of graphs. *Discrete Mathematics* 258, 205-214 (2002)
- [28] B. Wang and Z. Zhang : On cyclic edge-connectivity of transitive graphs. *Discrete Mathematics* 309(13), 4555-4563 (2009)
- [29] S. Wang, S. Lin, and C. Li : Sufficient conditions for super  $k$ -restricted edge connectivity in graphs of diameter 2. *Discrete Mathematics*, 309(4), 908-919 (2009)
- [30] J. M. Xu : On conditional edge-connectivity of graphs. *Acta Mathematicae Applicatae Sinica* 16(4), 414V419 (2000)
- [31] J. M. Xu and Q. Liu : 2-restricted edge connectivity of vertex-transitive graphs. *Australasian Journal of Combinatorics* 30, 41-50 (2004)
- [32] J. Xu : Topological structure and analysis of interconnection networks. Springer Publishing Company, Incorporated (2010)
- [33] Z. Zhang and J. Yuan : A proof of an inequality concerning  $k$ -restricted edge connectivity. *Discrete mathematics* 304(1), 128-134 (2005)
- [34] W. Zhao and J. Ou : On restricted edge-connectivity of lexicographic product graphs. *International Journal of Computer Mathematics* 91(8), 1618-1626 (2014)