Tensor and Coupled Decompositions in Block Terms: Uniqueness and Irreducibility

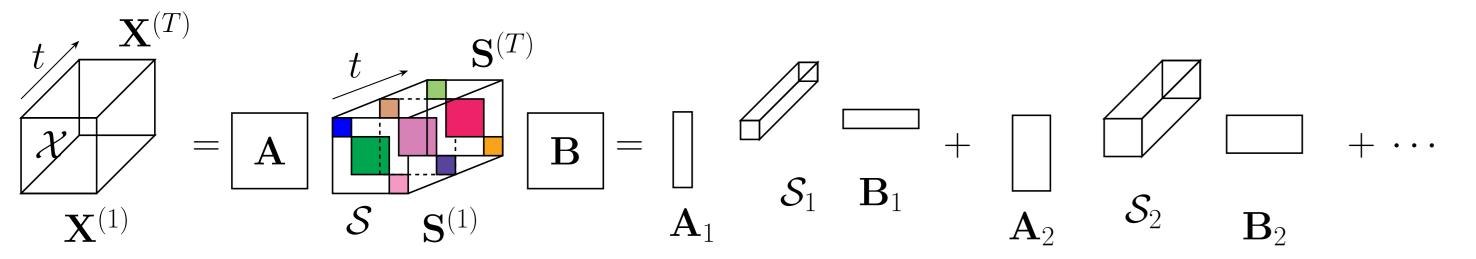
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Decomposition in multilinear rank- (L_r, M_r, \cdot) terms of a 3rd-order tensor



[De Lathauwer, 2008]

$$\mathcal{X} = \mathcal{S} \times_1 \mathbf{A} \times_2 \mathbf{B} = \sum_{r=1}^R \mathcal{S}_r \times_1 \mathbf{A}_r \times_2 \mathbf{B}_r$$
 $\mathbf{X}^{(t)} = \mathbf{A} \mathbf{S}^{(t)} \mathbf{B}^{\top} = \sum_{r=1}^R \mathbf{A}_r \mathbf{S}_{rr}^{(t)} \mathbf{B}_{rr}^{\top} \qquad t = 1, \dots, T$

- \times_n tensor mode-n product
- $\mathcal{X} \in \mathbb{C}^{I \times J \times T}$ third-order tensor
- ullet $\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \cdots & \mathbf{A}_R \end{bmatrix} \in \mathbb{C}^{I \times \sum_{r=1}^R L_r}$, $\mathbf{A}_r \in \mathbb{C}^{I \times L_r}$
- ullet $\mathbf{B} = egin{bmatrix} \mathbf{B}_1 & \cdots & \mathbf{C}_R \end{bmatrix} \in \mathbb{C}^{J imes \sum_{r=1}^R M_r},$ $\mathbf{B} \in \dot{\mathbb{C}}^{J imes M_r}$
- $L_r > 1$, $M_r > 1$
- $\mathcal{S}_r \in \mathbb{C}^{L_r \times M_r \times T}$ "core tensor"
- $\mathcal{S} \in \mathbb{C}^{\sum_{r=1}^{R} L_r \times \sum_{r=1}^{R} M_r \times T}$
- $\mathbf{S}^{(t)} = \text{Diag}\{\mathbf{S}_{11}^{(t)}, \dots, \mathbf{S}_{RR}^{(t)}\} \in \mathbb{C}^{\sum_{r=1}^{R} L_r \times \sum_{r=1}^{R} M_r}$ • $\mathbf{S}_{rr}^{(t)} \in \mathbb{C}^{L_r \times M_r}$

Motivation: more flexibility in representing latent signals, beyond rank-1

Reducibility and Irreducibility

• For any arbitrary nonsingular transformations $\mathbf{Y}_r, \mathbf{Z}_r$, and fixed r,

$$\mathbf{A}_r \mathbf{S}_{rr}^{(t)} \mathbf{B}_r^{\top} = (\mathbf{A}_r \mathbf{Y}_r^{-1}) (\mathbf{Y}_r \mathbf{S}_{rr}^{(t)} \mathbf{Z}_r^{\top}) (\mathbf{Z}_r^{-\top} \mathbf{B}_r^{\top}) \quad \forall t$$

A multilinear rank- (L_r, M_r, \cdot) term is **reducible** by simultaneous $(\mathbf{A}_r = \mathbf{B}_r)$ or generalized $(\mathbf{A}_r
eq \mathbf{B}_r)$ congruence transformation if the core tensor \mathcal{S}_r can be further block-diagonalized by basis transformations in 1st and 2nd modes. Otherwise, it is irreducible by simultaneous or generalized congruence.

- Reducibility can cause non-uniqueness
- As soon as $T \geq 3$, we generally have irreducibility \Rightarrow blocks exist

Non-uniqueness of multilinear rank- (L_r, M_r, \cdot) decomposition

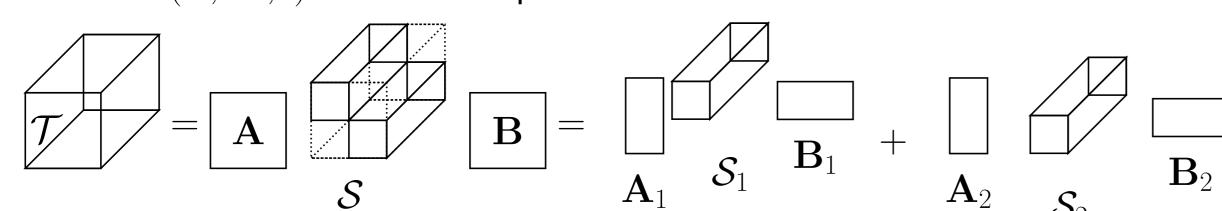
A decomposition in multilinear rank- (L_r, M_r, \cdot) terms with **irreducible** core tensors is not unique if, for some $i \neq j \in \{1, \dots, R\}$, $\exists \ \Psi \in \mathbb{C}^{L_i \times L_i}$, $\Phi \in \mathbb{C}^{M_i \times M_i}$ nonsingular s.t.

 $S_i = S_i \times_1 \Psi \times_2 \Phi$

[Lahat, Cardoso, Messer, 2012], [Lahat and Jutten, 2018]

Example of non-uniqueness

Consider the rank- (L, M, \cdot) block decomposition with R=2



where $\mathcal{T} \in \mathbb{C}^{I \times J \times K}$, $\mathcal{S}_r \in \mathbb{C}^{L \times M \times K}$, $\mathbf{A}_r \in \mathbb{C}^{I \times L}$, $\mathbf{B}_r \in \mathbb{C}^{J \times M}$, r = 1, 2.

Consider an alternative rank- (L, M, \cdot) BTD of $\mathcal T$ with

$$\overline{\mathbf{A}} = \frac{\sqrt{2}}{2} \mathbf{A} \begin{bmatrix} \mathbf{I} - \mathbf{\Psi}^{-1} \\ \mathbf{I} \mathbf{\Psi}^{-1} \end{bmatrix}^{-1}, \ \overline{\mathbf{B}} = \frac{\sqrt{2}}{2} \mathbf{B} \begin{bmatrix} \mathbf{I} - \mathbf{\Phi}^{-1} \\ \mathbf{I} \mathbf{\Phi}^{-1} \end{bmatrix}^{-1}$$

where $\mathbf{\Psi} \in \mathbb{C}^{L \times L}$ and $\mathbf{\Phi} \in \mathbb{C}^{M \times M}$ are nonsingular matrices.

If $S_2 = S_1 \times_1 \Psi \times_2 \Phi$, then

$$\mathcal{S} \times_1 \begin{bmatrix} \mathbf{I} & -\mathbf{\Psi}^{-1} \\ \mathbf{I} & \mathbf{\Psi}^{-1} \end{bmatrix} \times_2 \begin{bmatrix} \mathbf{I} & -\mathbf{\Phi}^{-1} \\ \mathbf{I} & \mathbf{\Phi}^{-1} \end{bmatrix} = \mathcal{S}_1$$

- Ψ, Φ nonsingular $\Rightarrow L_i = L_i$, $M_i = M_i$
- Holds also for not-full-column-rank A, B
- Necessary and sufficient when ${f A}={f B}$ real nonsingular, ${f S}^{(t)}$ symmetric positive definite orall t
- Reduces to Kruskal rank [Kruskal, 1977] equal to 2 when Ψ,Φ are scalars: generalizes collinearity / "proportional columns"

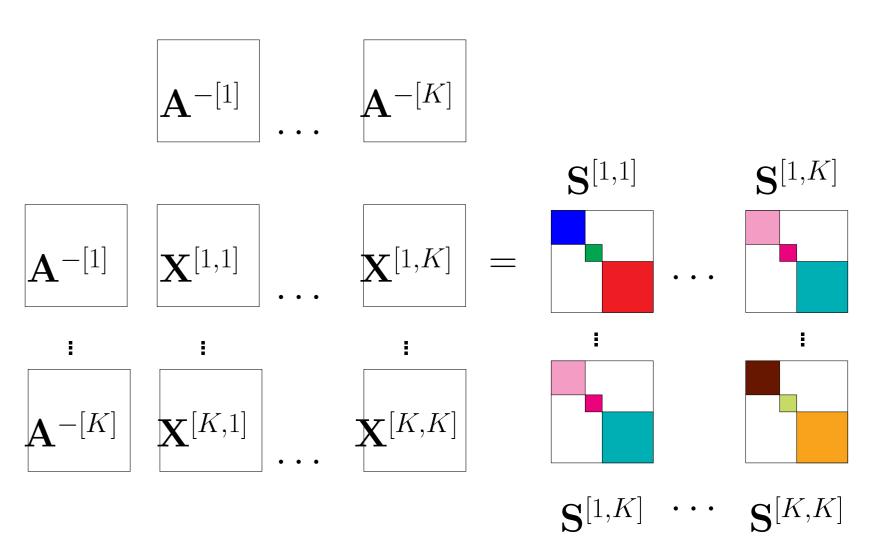
Conjecture: Kruskal-like results for more than two irreducible core tensors? e.g.,

- $k'_{A} + k'_{B} + k^{\text{core}}_{S} \ge 2R + 2$ (k'_{A} is "Kruskal rank" for partitioned matrices [De Lathauwer, 2008])
- $k_{\mathcal{S}}^{\text{core}} = 1$ when our non-uniqueness condition holds
- $k_{\mathcal{S}}^{\mathsf{core}} = 2$ if $k_{\mathcal{S}}^{\mathsf{core}} \neq 1$, and \exists a triplet (i, j, k), $i \neq j \neq k$ satisfying

$$S_k = S_i \times_1 \mathbf{\Psi} \times_2 \mathbf{\Phi} + S_i \times_1 \mathbf{\Psi}' \times_2 \mathbf{\Phi}'$$

with Ψ , Ψ' , Φ , and Φ' nonsingular matrices, or similarly

Coupled block decomposition (CBD) of K matrices



[Lahat and Jutten, 2015]

$$\mathbf{X}^{[k,\ell]} = \mathbf{A}^{[k]} \mathbf{S}^{[k,l]} \mathbf{A}^{[\ell]\top} = \sum_{r=1}^{R} \mathbf{A}_r^{[k]} \mathbf{S}_{rr}^{[k,l]} \mathbf{A}_r^{[\ell]\top} \quad k, \ell = 1, \dots, K$$

$$\mathbf{A}^{-[k]} \mathbf{X}^{[k,\ell]} \mathbf{A}^{-[\ell]\top} = \mathbf{S}^{[k,l]}$$

- Not a (reasonable) tensor decomposition
- $\bullet \ \mathbf{A}^{[k]} = \left[\mathbf{A}_1^{[k]} \cdots \mathbf{A}_R^{[k]}\right] \in \mathbb{C}^{I^{[k]} \times \sum_{r=1}^{k} L_r^{[k]}}$
- $\mathbf{S}^{[k,\ell]} = \mathrm{Diag}\{\mathbf{S}_{11}^{[k,\ell]},\ldots,\mathbf{S}_{RR}^{[k,\ell]}\}$ $oldsymbol{f S}_{rr}^{[k,\ell]} \in \mathbb{C}^{L_r^{[k]} imes L_r^{[\ell]}}$

- $oldsymbol{oldsymbol{A}}_r \in \mathbb{C}^{I^{[k]} imes L^{[k]}_r}$
- CBD and multilinear rank- (L_r, M_r, \cdot) are related
- Extendable to coupled tensor decompositions [Lahat and Jutten, 2016]

Motivation: uniqueness and identifiability of latent variables in multimodal data fusion

Reducibility and Irreducibility (CBD)

For any arbitrary nonsingular $\mathbf{Z}_r^{[k]}$ of size $L_r^{[k]} imes L_r^{[k]} \ orall r, k$,

$$\mathbf{A}_r^{[k]} \mathbf{S}_{rr}^{[k,\ell]} \mathbf{A}_r^{[\ell]\top} = (\mathbf{A}_r^{[k]} \mathbf{Z}_r^{-[k]}) (\mathbf{Z}_r^{[k]} \mathbf{S}_{rr}^{[k,\ell]} \mathbf{Z}_r^{[\ell]\top}) (\mathbf{Z}_r^{-[\ell]\top} \mathbf{A}_r^{[\ell]\top})$$

 $\{\mathbf{S}_{rr}^{[k,\ell]}\}_{k,\ell=1}^{K,K}$ is reducible in the coupled sense if for fixed r, each $\mathbf{S}_{rr}^{[k,\ell]}$ can be brought, by **coupled basis transformations**, to the block form:

$$L_r^{[k]}\left\{ \begin{array}{|c|c|} \hline x & * \\ \hline \mathbf{0} & * \end{array} \right\} \alpha^{[k]} \qquad \qquad \text{where } 0 \leq \alpha^{[k]} \leq L_r^{[k]} \text{, at least one } \alpha^{[k]} \neq 0 \text{,} \\ \text{and at least one } \alpha^{[k]} \neq L_r^{[k]}.$$

- $L_r^{[k]} = L_r \ \forall k \Rightarrow$ Generally irreducible as soon as $K \geq 3 \Rightarrow$ blocks exist
- $L_r^{[k]} \neq L_r^{[\ell]}$ for at least one $k \neq \ell \Rightarrow$ for certain block sizes, always reducible

Non-uniqueness of CBD

CBD with **irreducible** blocks is not unique if \exists at least one pair (i,j), $i \neq j$, for which $L_i^{[k]} = L_i^{[k]} \; orall k$ and

$$\mathbf{S}_{ii}^{[k,\ell]} = \mathbf{\Psi}^{[k]} \mathbf{S}_{ii}^{[k,\ell]} \mathbf{\Psi}^{[\ell] op} \quad orall k, \ell$$

where $\{\mathbf{\Psi}^{[k]}\}_{k=1}^K$ are nonsingular $L_i^{[k]} imes L_i^{[k]}$ matrices.

[Lahat and Jutten, 2015, 2019]

$$egin{bmatrix} oldsymbol{\Psi}^{[1]} & oldsymbol{0} \ oldsymbol{\Psi}^{[K]} \end{bmatrix} = egin{bmatrix} oldsymbol{\Psi}^{[1]} & oldsymbol{0} \ oldsymbol{0} \ oldsymbol{\Psi}^{[K]} \end{bmatrix}^ op egin{bmatrix} oldsymbol{\Psi}^{[1]} & oldsymbol{0} \ oldsymbol{0} \ oldsymbol{\Psi}^{[K]} \end{bmatrix}^ op egin{bmatrix} oldsymbol{\Psi}^{[1]} & oldsymbol{0} \ oldsymbol{0} \ oldsymbol{\Psi}^{[K]} \end{bmatrix}^ op egin{bmatrix} oldsymbol{\Psi}^{[1]} & oldsymbol{0} \ oldsymbol{0}$$

Example of non-uniqueness

Use $\overline{\mathbf{A}}^{[\kappa]}$ similarly to the rank- (L_r, M_r, \cdot) example

- Holds also for not-full-column-rank ${f A}^{[k]}$
- Necessary and sufficient when $\mathbf{A}^{[k]} = \mathbf{A}$ real nonsingular $\forall k$, $[\mathbf{S}_{rr}]_{k,\ell} = \mathbf{S}_{rr}^{[k,\ell]}$ symmetric positive definite $\forall r$
- Generalizes Kruskal rank to coupled -block- decompositions (beyond tensors)
- Our proofs rely on irreducibility, Schur's lemma [Schur, 1905], [Lahat, Jutten, Shapiro, 2018, and a statistical model for the data
- Related results, generic uniqueness: e.g., [De Lathauwer, 2008], [Domanov et al., 2018] (asymmetric); [Cai and Liu, 2017], [Cai, Cheng, Shi, 2019] (symmetric)









