

# Tensor and Coupled Decompositions in Block Terms: Uniqueness and Irreducibility

Dana Lahat<sup>1</sup> Christian Jutten<sup>2</sup>

<sup>1</sup>IRIT, Université de Toulouse, CNRS, 31000 Toulouse, France

<sup>2</sup>Univ. Grenoble Alpes, CNRS, Grenoble INP, GIPSA-lab, 38000 Grenoble, France



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## Decomposition in multilinear rank- $(L_r, M_r, \cdot)$ terms of a 3rd-order tensor

$$\begin{array}{c} \mathbf{X}^{(T)} \\ \mathcal{X} \\ \mathbf{X}^{(1)} \end{array} = \mathbf{A} \begin{array}{c} \mathcal{S} \\ \mathbf{S}^{(1)} \end{array} \mathbf{B} = \mathbf{A}_1 \mathcal{S}_1 \mathbf{B}_1 + \mathbf{A}_2 \mathcal{S}_2 \mathbf{B}_2 + \dots$$

[De Lathauwer, 2008]

$$\mathcal{X} = \mathcal{S} \times_1 \mathbf{A} \times_2 \mathbf{B} = \sum_{r=1}^R \mathcal{S}_r \times_1 \mathbf{A}_r \times_2 \mathbf{B}_r$$

$$\mathbf{X}^{(t)} = \mathbf{A} \mathbf{S}^{(t)} \mathbf{B}^\top = \sum_{r=1}^R \mathbf{A}_r \mathbf{S}_{rr}^{(t)} \mathbf{B}_{rr}^\top \quad t = 1, \dots, T$$

- $\times_n$  tensor mode- $n$  product
- $\mathcal{X} \in \mathbb{C}^{I \times J \times T}$  third-order tensor
- $\mathbf{A} = [\mathbf{A}_1 \dots \mathbf{A}_R] \in \mathbb{C}^{I \times \sum_{r=1}^R L_r}$ ,  $\mathbf{A}_r \in \mathbb{C}^{I \times L_r}$
- $\mathbf{B} = [\mathbf{B}_1 \dots \mathbf{B}_R] \in \mathbb{C}^{J \times \sum_{r=1}^R M_r}$ ,  $\mathbf{B} \in \mathbb{C}^{J \times M_r}$
- $L_r \geq 1, M_r \geq 1$
- $\mathcal{S}_r \in \mathbb{C}^{L_r \times M_r \times T}$  “core tensor”
- $\mathcal{S} \in \mathbb{C}^{\sum_{r=1}^R L_r \times \sum_{r=1}^R M_r \times T}$
- $\mathbf{S}^{(t)} = \text{Diag}\{\mathbf{S}_{11}^{(t)}, \dots, \mathbf{S}_{RR}^{(t)}\} \in \mathbb{C}^{\sum_{r=1}^R L_r \times \sum_{r=1}^R M_r}$
- $\mathbf{S}_{rr}^{(t)} \in \mathbb{C}^{L_r \times M_r}$

**Motivation:** more flexibility in representing latent signals, beyond rank-1

## Reducibility and Irreducibility

- For any arbitrary nonsingular transformations  $\mathbf{Y}_r, \mathbf{Z}_r$ , and fixed  $r$ ,

$$\mathbf{A}_r \mathbf{S}_{rr}^{(t)} \mathbf{B}_r^\top = (\mathbf{A}_r \mathbf{Y}_r^{-1}) (\mathbf{Y}_r \mathbf{S}_{rr}^{(t)} \mathbf{Z}_r^\top) (\mathbf{Z}_r^{-\top} \mathbf{B}_r^\top) \quad \forall t$$

A multilinear rank- $(L_r, M_r, \cdot)$  term is **reducible** by simultaneous ( $\mathbf{A}_r = \mathbf{B}_r$ ) or generalized ( $\mathbf{A}_r \neq \mathbf{B}_r$ ) congruence transformation if the core tensor  $\mathcal{S}_r$  can be further block-diagonalized by basis transformations in 1st and 2nd modes. Otherwise, it is **irreducible** by simultaneous or generalized congruence.

- Reducibility can cause non-uniqueness
- As soon as  $T \geq 3$ , we generally have irreducibility  $\Rightarrow$  blocks exist

## Non-uniqueness of multilinear rank- $(L_r, M_r, \cdot)$ decomposition

A decomposition in multilinear rank- $(L_r, M_r, \cdot)$  terms with **irreducible** core tensors is not unique if, for some  $i \neq j \in \{1, \dots, R\}$ ,  $\exists \mathbf{\Psi} \in \mathbb{C}^{L_i \times L_i}$ ,  $\mathbf{\Phi} \in \mathbb{C}^{M_i \times M_i}$  nonsingular s.t.

$$\mathcal{S}_j = \mathcal{S}_i \times_1 \mathbf{\Psi} \times_2 \mathbf{\Phi}$$

[Lahat, Cardoso, Messer, 2012], [Lahat and Jutten, 2018]

## Example of non-uniqueness

Consider the rank- $(L, M, \cdot)$  block decomposition with  $R = 2$

$$\begin{array}{c} \mathcal{T} \\ \mathbf{T} \end{array} = \mathbf{A} \begin{array}{c} \mathcal{S} \\ \mathbf{S} \end{array} \mathbf{B} = \mathbf{A}_1 \mathcal{S}_1 \mathbf{B}_1 + \mathbf{A}_2 \mathcal{S}_2 \mathbf{B}_2$$

where  $\mathcal{T} \in \mathbb{C}^{I \times J \times K}$ ,  $\mathcal{S}_r \in \mathbb{C}^{L \times M \times K}$ ,  $\mathbf{A}_r \in \mathbb{C}^{I \times L}$ ,  $\mathbf{B}_r \in \mathbb{C}^{J \times M}$ ,  $r = 1, 2$ .

Consider an alternative rank- $(L, M, \cdot)$  BTD of  $\mathcal{T}$  with

$$\overline{\mathbf{A}} = \frac{\sqrt{2}}{2} \mathbf{A} \begin{bmatrix} \mathbf{I} & -\mathbf{\Psi}^{-1} \\ \mathbf{I} & \mathbf{\Psi}^{-1} \end{bmatrix}^{-1}, \quad \overline{\mathbf{B}} = \frac{\sqrt{2}}{2} \mathbf{B} \begin{bmatrix} \mathbf{I} & -\mathbf{\Phi}^{-1} \\ \mathbf{I} & \mathbf{\Phi}^{-1} \end{bmatrix}^{-1}$$

where  $\mathbf{\Psi} \in \mathbb{C}^{L \times L}$  and  $\mathbf{\Phi} \in \mathbb{C}^{M \times M}$  are nonsingular matrices.

If  $\mathcal{S}_2 = \mathcal{S}_1 \times_1 \mathbf{\Psi} \times_2 \mathbf{\Phi}$ , then

$$\mathcal{S} \times_1 \begin{bmatrix} \mathbf{I} & -\mathbf{\Psi}^{-1} \\ \mathbf{I} & \mathbf{\Psi}^{-1} \end{bmatrix} \times_2 \begin{bmatrix} \mathbf{I} & -\mathbf{\Phi}^{-1} \\ \mathbf{I} & \mathbf{\Phi}^{-1} \end{bmatrix} = \mathcal{S}_1 \mathbf{S}_1$$

- $\mathbf{\Psi}, \mathbf{\Phi}$  nonsingular  $\Rightarrow L_i = L_i, M_i = M_j$
- Holds also for not-full-column-rank  $\mathbf{A}, \mathbf{B}$
- Necessary and sufficient when  $\mathbf{A} = \mathbf{B}$  real nonsingular,  $\mathbf{S}^{(t)}$  symmetric positive definite  $\forall t$
- Reduces to Kruskal rank [Kruskal, 1977] equal to 2 when  $\mathbf{\Psi}, \mathbf{\Phi}$  are scalars: generalizes collinearity / “proportional columns”

**Conjecture:** Kruskal-like results for more than two irreducible core tensors? e.g.,

- $k'_A + k'_B + k_S^{\text{core}} \geq 2R + 2$  ( $k'_A$  is “Kruskal rank” for partitioned matrices [De Lathauwer, 2008])
- $k_S^{\text{core}} = 1$  when our non-uniqueness condition holds
- $k_S^{\text{core}} = 2$  if  $k_S^{\text{core}} \neq 1$ , and  $\exists$  a triplet  $(i, j, k)$ ,  $i \neq j \neq k$  satisfying

$$\mathcal{S}_k = \mathcal{S}_j \times_1 \mathbf{\Psi} \times_2 \mathbf{\Phi} + \mathcal{S}_i \times_1 \mathbf{\Psi}' \times_2 \mathbf{\Phi}'$$

with  $\mathbf{\Psi}, \mathbf{\Psi}', \mathbf{\Phi}$ , and  $\mathbf{\Phi}'$  nonsingular matrices, or similarly

## Coupled block decomposition (CBD) of $K$ matrices

$$\begin{array}{c} \mathbf{A}^{-[1]} \dots \mathbf{A}^{-[K]} \\ \mathbf{A}^{-[1]} \mathbf{X}^{[1,1]} \dots \mathbf{X}^{[1,K]} \\ \vdots \\ \mathbf{A}^{-[K]} \mathbf{X}^{[K,1]} \dots \mathbf{X}^{[K,K]} \end{array} = \begin{array}{c} \mathbf{S}^{[1,1]} \dots \mathbf{S}^{[1,K]} \\ \vdots \\ \mathbf{S}^{[K,1]} \dots \mathbf{S}^{[K,K]} \end{array}$$

[Lahat and Jutten, 2015]

$$\mathbf{X}^{[k,\ell]} = \mathbf{A}^{[k]} \mathbf{S}^{[k,\ell]} \mathbf{A}^{[\ell]\top} = \sum_{r=1}^R \mathbf{A}_r^{[k]} \mathbf{S}_{rr}^{[k,\ell]} \mathbf{A}_r^{[\ell]\top} \quad k, \ell = 1, \dots, K$$

$$\mathbf{A}^{-[k]} \mathbf{X}^{[k,\ell]} \mathbf{A}^{-[\ell]\top} = \mathbf{S}^{[k,\ell]}$$

- Not a (reasonable) tensor decomposition
- $\mathbf{A}^{[k]} = [\mathbf{A}_1^{[k]} \dots \mathbf{A}_R^{[k]}] \in \mathbb{C}^{I^{[k]} \times \sum_{r=1}^R L_r^{[k]}}$
- $\mathbf{A}_r \in \mathbb{C}^{I^{[k]} \times L_r^{[k]}}$
- $L_r^{[k]} \geq 1$
- $\mathbf{S}^{[k,\ell]} = \text{Diag}\{\mathbf{S}_{11}^{[k,\ell]}, \dots, \mathbf{S}_{RR}^{[k,\ell]}\}$
- $\mathbf{S}_{rr}^{[k,\ell]} \in \mathbb{C}^{L_r^{[k]} \times L_r^{[\ell]}}$
- CBD and multilinear rank- $(L_r, M_r, \cdot)$  are related
- Extendable to coupled tensor decompositions [Lahat and Jutten, 2016]

**Motivation:** uniqueness and identifiability of latent variables in multimodal data fusion

## Reducibility and Irreducibility (CBD)

For any arbitrary nonsingular  $\mathbf{Z}_r^{[k]}$  of size  $L_r^{[k]} \times L_r^{[k]} \forall r, k$ ,

$$\mathbf{A}_r^{[k]} \mathbf{S}_{rr}^{[k,\ell]} \mathbf{A}_r^{[\ell]\top} = (\mathbf{A}_r^{[k]} \mathbf{Z}_r^{-[k]}) (\mathbf{Z}_r^{[k]} \mathbf{S}_{rr}^{[k,\ell]} \mathbf{Z}_r^{[\ell]\top}) (\mathbf{Z}_r^{-[\ell]\top} \mathbf{A}_r^{[\ell]\top})$$

$\{\mathbf{S}_{rr}^{[k,\ell]}\}_{k,\ell=1}^{K,K}$  is **reducible in the coupled sense** if for fixed  $r$ , each  $\mathbf{S}_{rr}^{[k,\ell]}$  can be brought, by **coupled basis transformations**, to the block form:

$$L_r^{[k]} \begin{bmatrix} * & * \\ \mathbf{0} & * \end{bmatrix} \alpha^{[k]} \quad \text{where } 0 \leq \alpha^{[k]} \leq L_r^{[k]}, \text{ at least one } \alpha^{[k]} \neq 0, \text{ and at least one } \alpha^{[k]} \neq L_r^{[k]}.$$

- $L_r^{[k]} = L_r \forall k \Rightarrow$  Generally irreducible as soon as  $K \geq 3 \Rightarrow$  blocks exist
- $L_r^{[k]} \neq L_r^{[\ell]}$  for at least one  $k \neq \ell \Rightarrow$  for certain block sizes, always reducible

## Non-uniqueness of CBD

CBD with **irreducible** blocks is not unique if  $\exists$  at least one pair  $(i, j)$ ,  $i \neq j$ , for which  $L_j^{[k]} = L_i^{[k]} \forall k$  and

$$\mathbf{S}_{jj}^{[k,\ell]} = \mathbf{\Psi}^{[k]} \mathbf{S}_{ii}^{[k,\ell]} \mathbf{\Psi}^{[\ell]\top} \quad \forall k, \ell$$

where  $\{\mathbf{\Psi}^{[k]}\}_{k=1}^K$  are nonsingular  $L_i^{[k]} \times L_i^{[k]}$  matrices.

[Lahat and Jutten, 2015, 2019]

$$\begin{bmatrix} \mathbf{S}_{11}^{[1,1]} & \dots & \mathbf{S}_{11}^{[1,K]} \\ \vdots & \ddots & \vdots \\ \mathbf{S}_{11}^{[K,1]} & \dots & \mathbf{S}_{11}^{[K,K]} \end{bmatrix} = \begin{bmatrix} \mathbf{\Psi}^{[1]} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Psi}^{[K]} \end{bmatrix} \begin{bmatrix} \mathbf{S}_{11}^{[1,1]} & \dots & \mathbf{S}_{11}^{[1,K]} \\ \vdots & \ddots & \vdots \\ \mathbf{S}_{11}^{[K,1]} & \dots & \mathbf{S}_{11}^{[K,K]} \end{bmatrix} \begin{bmatrix} \mathbf{\Psi}^{[1]} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Psi}^{[K]} \end{bmatrix}^\top$$

## Example of non-uniqueness

Use  $\overline{\mathbf{A}}^{[k]}$  similarly to the rank- $(L_r, M_r, \cdot)$  example

- Holds also for not-full-column-rank  $\mathbf{A}^{[k]}$
- Necessary and sufficient when  $\mathbf{A}^{[k]} = \mathbf{A}$  real nonsingular  $\forall k$ ,  $[\mathbf{S}_{rr}]_{k,\ell} = \mathbf{S}_{rr}^{[k,\ell]}$  symmetric positive definite  $\forall r$
- ? Generalizes Kruskal rank to coupled -block- decompositions (beyond tensors)

- Our proofs rely on irreducibility, Schur's lemma [Schur, 1905], [Lahat, Jutten, Shapiro, 2018], and a statistical model for the data
- Related results, generic uniqueness: e.g., [De Lathauwer, 2008], [Domanov et al., 2018] (asymmetric); [Cai and Liu, 2017], [Cai, Cheng, Shi, 2019] (symmetric)

