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# Polynomial Growth of high Sobolev norms of solutions to the Zakharov-Kuznetsov equation

Raphaël Côte and Frédéric Valet

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## Abstract

We consider the Zakharov-Kuznetsov equation (ZK) in space dimension 2. Solutions  $u$  with initial data  $u_0 \in H^s$  are known to be global if  $s \geq 1$ . We prove that for any integer  $s \geq 2$ ,  $\|u(t)\|_{H^s}$  grows at most polynomially in  $t$  for large times  $t$ . This result is related to wave turbulence and how a solution of (ZK) can move energy to high frequencies.

It is inspired by analogous results by Staffilani [20] on the non linear Schrödinger and Korteweg-de-Vries equation. The main ingredients are adequate bilinear estimates in the context of Bourgain's spaces and a careful study of the variation of the  $H^s$  norm.

We are interested in the 2D Zakharov-Kuznetsov equation

$$\begin{cases} \partial_t u + \partial_x (\Delta u + u^2) = 0, & u : I_t \times \mathbb{R}_{x,y}^2 \rightarrow \mathbb{R} \\ u(0) = u_0 \in H^s(\mathbb{R}^2) \end{cases} \quad (\text{ZK})$$

where  $\Delta = \partial_x^2 + \partial_y^2$  is the laplacian on  $\mathbb{R}^2$ , the time interval  $I_t \ni 0$  is the interval of existence. This equation has been studied to model propagation, in magnetized plasma, of non-linear ion-acoustic waves, see [12]. (ZK) has also been derived from the Euler-Poisson system in dimension  $d = 2$  and  $d = 3$  by Lannes, Linares and Saut in [13], and from the Vlasov-Poisson equation by Han-Kwan [8].

This equation naturally enjoys some conserved quantities (at least formally) like the mass and the energy:

$$M(u) := \frac{1}{2} \int_{\mathbb{R}^2} u^2(t, x, y) dx dy, \quad E(u) := \frac{1}{2} \int \left( |\nabla u(t, x, y)|^2 - \frac{1}{3} u(t, x, y)^3 \right) dx dy.$$

The Cauchy problem of (ZK) has been first studied by Faminskii in [5], who showed local and global well-posedness in  $H^1(\mathbb{R}^2)$ . This local well-posedness has then been improved in [15], and then by Molinet and Pilod in [17] and independently by Grünrock and Herr in [7] who proved the local well-posedness in  $H^s$  for  $s \geq \frac{1}{2}$ . Recently, Kinoshita [11] improved this and prove local well-posedness in  $H^s$  for any  $s > -\frac{1}{4}$ , which is the sharp exponent.

For  $s \geq 1$ , the mass  $M(u)$  and the energy  $E(u)$  are well defined and preserved by the flow. Due to a Gagliardo-Nirenberg inequality, there exist a universal constant  $C$  such that

$$\forall v \in H^1(\mathbb{R}^2), \quad \|v\|_{H^1}^2 \leq C(1 + E(v) + M(v)^2). \quad (1)$$

As a consequence (ZK) is globally well-posed, and the  $H^1$  norm  $\|u(t)\|_{H^1}$  remains bounded for all times.

One can naturally ask what happens for  $\|u(t)\|_{H^s}$ . If for some  $s > 1$ ,  $\|u(t)\|_{H^s} \rightarrow +\infty$ , one speaks of *energy cascade phenomenon*, which means that some energy move from low frequencies to high frequencies: it is an important aspect of out of equilibrium dynamics, predicted and studied on a number of nonlinear dispersive models, under the name of *wave turbulence* in the Physics literature. To the contrary, for integrable systems, one expects that all  $\|u(t)\|_{H^s}$  remain bounded.

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One should note that the proof of local well-posedness often allows to give exponential (or double exponential) bounds on  $\|u(t)\|_{H^s}$ . On the other hand, constructing a solution which displays an energy cascade phenomenon is very delicate, we refer to the work by Hani, Pausader, Tzvetkov and Visciglia [9] for an example in the context of Schrödinger type equations.

In this article, we prove that the  $H^s$ -norm of a solution  $u$  of (ZK) equation grows at most polynomially for large times.

**Theorem 1.** *Let  $s \geq 2$  be an integer and  $u_0 \in H^s(\mathbb{R}^2)$ . Denote  $u$  the solution of (ZK) with initial data  $u_0$  and  $A = \sup_{t \geq 0} \|u(t)\|_{H^1}$ . Then  $u \in \mathcal{C}(\mathbb{R}, H^s)$  and for any  $\beta > \frac{s-1}{2}$ , there exist a constant  $C = C(s, \beta, A)$ , such that*

$$\forall t \in \mathbb{R}, \quad \|u(t)\|_{H^s} \leq C(1 + |t|)^\beta (1 + \|u_0\|_{H^s}), \quad (2)$$

The start of the study of the polynomial growth of norms is due to Staffilani [20] in the context of non linear Schrödinger and Korteweg-de Vries type equations, with ideas of Bourgain [1] and [2]. It was later extended to many situations: let us refer to Sohinger [19] for the Schrödinger and Hartree equations, or Planchon, Tzvetkov, Visciglia [18] for the Schrödinger equation on a manifold, and the references therein.

The method of proof in [20] is to refine the local well-posedness statement. Usually, what is proved is that given  $u_0$ , there exist  $C, T > 0$  such that one can construct a solution  $u$  on  $[-T, T]$  and such that

$$\forall t \in [-T, T], \quad \|u(t)\|_{H^s} \leq C\|u(0)\|_{H^s}. \quad (3)$$

If  $C$  and  $T$  only depend on  $\|u_0\|_{H^1}$ , one can use an iteration argument and obtain global well-posedness in  $H^s$  with an exponential bound on the  $H^s$  norm. The heart of the method lies in the observation that, for  $H^s$ -norms (with  $s > 1$ ), a similar bound can be obtained, but with a slightly better exponent on the  $H^s$  norm on the right-hand side. More precisely, there exists  $\epsilon > 0$  and two functions  $C, T : [0, +\infty) \rightarrow [0, +\infty)$  ( $C$  increasing, and  $T$  decreasing) such that for any solution  $u$  of (ZK), and for all  $t_0 \in \mathbb{R}$ ,

$$\forall t \in [t_0 - T(\|u(t_0)\|_{H^1}), t_0 + T(\|u(t_0)\|_{H^1})], \quad \|u(t)\|_{H^s} \leq \|u(t_0)\|_{H^s} + C(\|u(t_0)\|_{H^1})(1 + \|u(t_0)\|_{H^s}^{1-\epsilon}). \quad (4)$$

The key point is that the time of existence  $T$  and the amplification factor  $C$  do only depend of the  $H^1$  norm of  $u$ , which is known to be uniformly bounded for all times, and so will essentially not depend on  $t_0$ .

With the above improved bound (4) in hand, one can conclude the proof of Theorem 1 by a straightforward iteration argument, by working on the time intervals  $[kT(A), (k+1)T(A)]$  for  $k \in \mathbb{N}$  (see Lemma 13 for details).

To derive estimate (3), one considers the integral formulation of the equation, and proves that the Duhamel term enjoys a better behavior than the linear term; we use Strichartz estimates, a bilinear estimate due to Molinet and Pilod [17], and its tame version. With (3) in hand, one can then look for the improved version (4). For this, [20] makes use of bilinear estimates due to Kenig-Ponce-Vega in [10] in Bourgain spaces  $X^{s,b}$ , [19] relies on the  $I$ -method (first developed in [4]), and [18] considers high order modified functionals of energy type. In this paper, we will follow the method in [20] and work in suitable Bourgain spaces. In order to derive (4), we study the variation of the  $H^s$  norm. Compared with (KdV) for example, one has to be extra careful on how derivatives fall on each factor, because the algebraic structure of (ZK) is not as strong as that of (KdV), and the dispersion effects are weaker for (ZK) than for (KdV): as it can be seen from the bilinear estimates in [17] which require positive index of regularity (and the fact that the local well posedness results are not optimal). To suitably bound the worst terms, we prove a new bilinear estimate (in Proposition 10) for negative regularity, which is the main new technical tool of this paper.

The article is organized as follows. We recall in section 1 Strichartz estimates and embeddings between Bourgain's spaces and Sobolev spaces for (ZK), in order to fix notations. In section 2, we collect and prove the bilinear estimates required to deal with the non linear term. In section 3, we prove a local well posedness result with time of existence depending only on  $\|u(0)\|_{H^1}$ , and then we conclude the proof of Theorem 1.

# 1 Notations and basic facts

## 1.1 Notations

For  $x \in \mathbb{C}$ ,  $|x|$  denotes the module of  $x$ , and for  $(x, y) \in \mathbb{R}^2$ , we use the norm

$$|(x, y)| := \sqrt{3x^2 + y^2},$$

(it is anisotropic but convenient for our purposes). We denote by  $\widehat{u}$  the Fourier transform with respect to the *space and time* variables, and the inverse Fourier transform by  $u^\vee$ . The Fourier transform with respect to  $t$  or the space variables only will be respectively denoted  $\mathcal{F}_t(u)$  or  $\mathcal{F}_{xy}(u)$ . The derivatives will be denoted  $\partial_t$ ,  $\partial_x$  or  $\partial_y$ .

For a general function  $P(t, x, y)$ , we define the associated self-adjoint and positive differential operator by:

$$P(D)f(t, x, y) := \int_{\mathbb{R}^3} |P(i\tau, i\xi, i\eta)| e^{i(\tau t + \xi x + \eta y)} \widehat{f}(\tau, \xi, \eta) d\tau d\xi d\eta.$$

For instance, we will regularly use the differential operators  $D_x$  for  $P = x$ ,  $D_y$  for  $P = y$  and  $D_t$  for  $P = t$  acting on frequencies:

$$\mathcal{F}_{xy}(D_x f)(\xi, \eta) = |\xi| \mathcal{F}_{xy}(f)(\xi, \eta), \quad \mathcal{F}_{xy}(D_y f)(\xi, \eta) = |\eta| \mathcal{F}_{xy}(f)(\xi, \eta), \quad \mathcal{F}_t(D_t f)(\tau) = |\tau| \mathcal{F}_t(f)(\tau).$$

We define the derivation operator  $S(D)$  with  $S(t, x, y) = \langle |(x, y)| \rangle$  (where  $\langle a \rangle := (1 + a^2)^{\frac{1}{2}}$  is the Japanese bracket), to the power  $s$  to define the Sobolev space  $H^s(\mathbb{R}^2)$ :

$$\mathcal{F}_{xy}(S(D)^s f)(x, y) := \langle |(\xi, \eta)| \rangle^s \mathcal{F}_{xy}(f)(\xi, \eta), \quad \|f\|_{H^s} := \|S(D)^s f\|_{L^2}. \quad (5)$$

We will manipulate multi-indices at many places: they will always be denoted by a bold letter  $\mathbf{i} = (i_1, i_2)$ , with  $i_1$  and  $i_2$  in  $\mathbb{N}$ . Then we denote  $D^{\mathbf{i}}$  the differential operator with  $i_1$  derivatives on the  $x$  variable, and  $i_2$  on the  $y$  variable:

$$D^{\mathbf{i}} f(x, y) := D_x^{i_1} D_y^{i_2} f(x, y) = \int_{\mathbb{R}^2} |\xi|^{i_1} |\eta|^{i_2} e^{i(\xi x + \eta y)} \mathcal{F}_{xy}(f)(\xi, \eta) d\xi d\eta.$$

The length of a multi-index  $\mathbf{i} = (i_1, i_2)$  will be denoted by  $|\mathbf{i}| := i_1 + i_2$ . We also use a partial order on multi-indices:

$$\mathbf{j} \leq \mathbf{i} \quad \text{if} \quad j_1 \leq i_1 \text{ and } j_2 \leq i_2, \quad \text{and} \quad \mathbf{j} < \mathbf{i} \quad \text{if} \quad (j_1 + 1, j_2) \leq \mathbf{i} \text{ or } (j_1, j_2 + 1) \leq \mathbf{i}.$$

The linear part of the equation (ZK)  $\partial_t u + \partial_x \Delta u = 0$  induces a semigroup denoted  $W : \mathbb{R}_t \rightarrow \mathcal{L}(L^2(\mathbb{R}^2))$ , defined by :

$$\mathcal{F}_{xy}(W(t)u)(\xi, \mu) := e^{it\omega(\xi, \mu)} \mathcal{F}_{xy}(u)(t, \xi, \mu) \quad \text{with} \quad \omega(\xi, \mu) = \xi(\xi^2 + \mu^2).$$

Similarly, this linear flow can be represented in space-time, which motivates the introduction of the function  $F(t, x, y) := \langle |t + x(x^2 + y^2)| \rangle$  and the operator  $F(D)$  to the power  $b \in \mathbb{R}$ :

$$\widehat{F(D)^b f}(\tau, \xi, \eta) := \langle \tau - \omega(\xi, \eta) \rangle^b \widehat{f}(\tau, \xi, \eta). \quad (6)$$

We then introduce the adapted Bourgain spaces. Given two indices  $s, b \in \mathbb{R}$ , the Bourgain space  $X^{s,b}$  takes into account first the norm  $H^s$  in space of a function  $u(t, x, y)$ , and secondly how far away ( $H^b$  in time) the solution is from the linear flow: the associated norm is

$$\|u\|_{X^{s,b}}^2 := \|S(D)^s F(D)^b u\|_{L^2}^2 = \int_{\mathbb{R}^3} \langle |(\xi, \mu)| \rangle^{2s} \langle \tau - \omega(\xi, \mu) \rangle^{2b} |\widehat{u}(\tau, \xi, \mu)|^2 d\tau d\xi d\mu,$$

where  $S$  and  $F$  are defined above in (5) and (6).  $X^{s,b}$  is the completion of the Schwartz functions of  $\mathbb{R}_t \times \mathbb{R}_{x,y}^2$  for this norm.

Because of the conserved quantities, all the solutions  $u$  of (ZK) (even small) will not have finite  $X^{s,b}$ -norm. This is why we will consider solution to a version of the integral formulation of (ZK) which is truncated in time, by the use of a smooth cut-off function. Let the non-negative smooth function  $\phi$  equal to 1 on  $[0, 1]$ , and 0 outside  $[-1, 2]$ , and the  $T$ -dilated functions  $\phi_T(t) = \phi\left(\frac{t}{T}\right)$ . We will always assume in the following that  $T \leq 1$ . The truncated Duhamel operator is given by the following formula:

$$\Psi(w)(t) := \phi_1(t)W(t)u_0 + \phi_T(t) \int_0^t W(t-t')\phi_T(t')^2 \partial_x (w^2)(t') dt'. \quad (7)$$

Observe that if we can construct a fixed point  $w$  such that  $\Psi(w) = w$ , then  $w$  is a solution to (ZK) on the interval  $[-T, T]$  (with initial data  $w(0) = u_0$ ).

In order to prove the various (bilinear) estimates, we will need to work with a Littlewood-Paley decomposition. Let a positive and non increasing function  $\chi_0$  in  $C_0^\infty([0, +\infty))$ , such that

$$\chi_0(r) = 1 \quad \text{if } r \leq \frac{5}{4}, \quad \text{and} \quad \chi_0(r) = 0 \quad \text{if } r \geq \frac{8}{5}.$$

Let

$$\chi(r) = \chi_0\left(\frac{r}{2}\right) - \chi_0(r) \quad \text{and} \quad \forall j \in \mathbb{N}, \quad \chi_j(r) := \chi\left(\frac{r}{2^j}\right).$$

We will use the following truncation operators, related to space and to the linear flow, recalling that  $|(\xi, \mu)| = \sqrt{3\xi^2 + \mu^2}$ :

$$P_N(u) := (\chi_N(|(\xi, \mu)|)) \widehat{u}(\tau, \xi, \mu)^\vee \quad \text{and} \quad Q_L u := (\chi_L(|\tau - \omega(\xi, \mu)|)) \widehat{u}(\tau, \xi, \mu)^\vee. \quad (8)$$

For  $b \in \mathbb{R}$ , we will sometimes denote  $b^+$  for any number  $b + \epsilon$  with  $\epsilon > 0$  small. Finally, given two functions  $f$  and  $g$  (depending on space and/or time), we denote

$$f \lesssim g \quad \text{if} \quad \exists C > 0, \quad f \leq Cg,$$

for an absolute implied constant  $C$ , independent of  $f$  and  $g$  (unless otherwise specified), which can change from one line to the next. Sometimes  $\lesssim$  is used for quantities involving a  $b^+$ : in that case, the implied constant may depend on  $\epsilon > 0$ .

## 1.2 Strichartz estimates and Bourgain spaces

Now we recall some lemmas which are the heart the following proofs, to begin with the Strichartz estimates for the Zakharov-Kuznetsov equation in dimension 2:

**Proposition 2** ([16, Proposition 3.1] and [14, Proposition 2.4]). *Let  $\epsilon \in [0, \frac{1}{2}]$ ,  $\theta \in [0, 1]$  and define*

$$p = \frac{2}{1-\theta}, \quad \text{and} \quad q = \frac{6}{\theta(2+\epsilon)}.$$

*Let  $f$  be a function defined on  $\mathbb{R}^2$ , and  $g$  be defined on  $\mathbb{R}_t \times \mathbb{R}^2$ . Then the following estimates hold:*

$$\begin{aligned} \left\| D_x^{\frac{\epsilon\theta}{2}} W(t)f \right\|_{L_t^q L^p} &\lesssim \|f\|_{L^2}, \\ \left\| D_x^{\theta\epsilon} \int W(t-t')g(t')dt' \right\|_{L_t^q L^p} &\lesssim \|g\|_{L_t^{q'} L^{p'}}, \\ \left\| D_x^{\theta\epsilon} \int W(t)g(t)dt \right\|_{L^2} &\lesssim \|g\|_{L_t^{q'} L^{p'}}. \end{aligned} \quad (9)$$

In particular, the first estimate yields an embedding in the context of Bourgain spaces:

**Corollary 3** ([17, Corollary 3.2]). *There hold*

$$\|u\|_{L_{txy}^4} \lesssim \|u\|_{X^{0, \frac{5}{12}+}}, \quad (10)$$

Furthermore, (9) (up to the use of Cauchy-Schwarz inequality) gives us an other embedding: if  $u \in X^{s, \frac{1}{2}+}$ , then uniformly in  $t \in \mathbb{R}$ ,

$$\|u(t)\|_{H^s} \lesssim \|u\|_{X^{s, \frac{1}{2}+}}. \quad (11)$$

We now focus on the terms of the Duhamel formula (7). We give the following two linear estimates in Bourgain spaces: the first estimate is useful when dealing with the linear term; the second for the Duhamel term.

**Lemma 4** ([6, Lemmas 3.1 and 3.2]). *Fix  $T \leq 1$ .*

*Let  $b \geq 0$  and  $f \in H^s(\mathbb{R}^2)$ . Then:*

$$\|\phi_T(t)W(t)f\|_{X^{s,b}} \lesssim T^{\frac{1}{2}-b}\|f\|_{H^s}. \quad (12)$$

*Let  $-\frac{1}{2} < b' < 0 < b \leq 1 + b'$ , and  $g \in X^{s,b'}$ . Then*

$$\left\| \phi_T(t) \int_0^t W(t-t')g(t')dt' \right\|_{X^{s,b}} \lesssim T^{1-b+b'}\|g\|_{X^{s,b'}}. \quad (13)$$

*Proof.* A proof of these estimates in the context of the Korteweg-de Vries equation is done in the review article by Ginibre [6, Lemma 3.1 and 3.2]. For the convenience of the reader, we provide below a full proof for (ZK).

The idea for estimate (12) is to separate the variables. In particular, we detect well the need of a fixed time interval.

$$\begin{aligned} \|\phi_T W(t)f\|_{X^{s,b}}^2 &= \int_{\mathbb{R}^3} \langle |(\xi, \mu)| \rangle^{2s} \langle \tau - \omega(\xi, \mu) \rangle^{2b} \left| \int e^{-i(t\tau - t\omega + x\xi + y\mu)} (\phi_T f) dx dy dt \right|^2 d\xi d\mu d\tau \\ &= \int \langle |(\xi, \mu)| \rangle^{2s} \langle \tau' \rangle^{2b} |\mathcal{F}_t(\phi_T)(\tau') \mathcal{F}_{xy}(f)(\xi, \mu)|^2 d\xi d\mu d\tau = \|\phi_T\|_{H_t^b}^2 \|f\|_{H_{x,y}^s}^2. \end{aligned}$$

Recalling that  $\langle a \rangle \lesssim 1 + |a|$ , we obtain  $\|\phi_T\|_{H_t^b}^2 \leq T^{1-2b}\|\phi\|_{H_t^b}^2$ , and the first estimate is proved.

For estimate (13): notice that the sum over  $t$  in time is in fact a division by  $\tau$  in frequency. Denote  $\omega := \omega(\xi, \mu)$ , we split the domain whether  $T|\tau - \omega| \leq 1$ , and obtain:

$$\begin{aligned} \mathcal{F}_{xy} \left( \phi_T(t) \int_0^t W(t-t')g(t')dt' \right) &= \phi_T(t) e^{it\omega} \int \frac{e^{it(\tau-\omega)} - 1}{i(\tau-\omega)} \widehat{g}(\tau) d\tau \\ &= \phi_T(t) e^{it\omega} \int_{T|\tau-\omega| \geq 1} \frac{i}{\tau-\omega} \widehat{g}(\tau) d\tau + \phi_T(t) e^{it\omega} \sum_{k \geq 1} \frac{t^k}{k!} \int_{T|\tau-\omega| \leq 1} (i(\tau-\omega))^{k-1} \widehat{g}(\tau) d\tau \\ &\quad + \phi_T(t) e^{it\omega} \int_{T|\tau-\omega| \geq 1} \frac{e^{it(\tau-\omega)}}{i(\tau-\omega)} \widehat{g}(\tau) d\tau \\ &= I + II + III. \end{aligned}$$

For  $I$ , we proceed as in the proof of (12). First, by separation of variables:

$$\mathcal{F}_t(I)(\theta) = \mathcal{F}_t(\phi_T)(\theta - \omega) \int_{T|\tau-\omega| \geq 1} \frac{i}{\tau-\omega} \widehat{g}(\tau) d\tau.$$

Now, observe that  $b' > -\frac{1}{2}$ , and the change of variable  $\theta \rightarrow \theta + \omega$  gives:

$$\begin{aligned} &\int \langle |(\xi, \mu)| \rangle^{2s} \langle \theta - \omega \rangle^{2b} |\mathcal{F}_t(I)(\theta, \xi, \mu)|^2 d\theta d\xi d\mu \\ &\leq \|\phi_T\|_{H_t^b}^2 \int \langle |(\xi, \mu)| \rangle^{2s} \left( \int \langle \tau - \omega \rangle^{2b'} |\widehat{g}|^2 d\tau \right) \left( \int_{T|\tau-\omega| \geq 1} \frac{1}{(\tau-\omega)^{2+2b'}} d\tau \right) d\xi d\mu \\ &\lesssim T^{1-2b} \|g\|_{X^{s,b'}}^2 T^{1+2b'} = \left( T^{1-b+b'} \|g\|_{X^{s,b'}} \right)^2. \end{aligned}$$

Similarly, we compute the time Fourier transform of the term  $II$ :

$$\mathcal{F}_t(II)(\theta) = \sum_{k \geq 1} \frac{1}{k!} \mathcal{F}_t(t^k \phi_T)(\theta - \omega) \int_{T|\tau - \omega| \leq 1} (i(\tau - \omega))^{k-1} \widehat{g}(\tau) d\tau.$$

With similar techniques we used for the term  $I$ :

$$\begin{aligned} & \int \langle |(\xi, \mu)| \rangle^{2s} \langle \theta - \omega \rangle^{2b} |\mathcal{F}_t(II)(\theta, \xi, \mu)|^2 d\theta d\xi d\mu \\ & \leq \sum_{k \geq 1} \frac{1}{k!} \|t^k \phi_T(t)\|_{H_t^b}^2 \int \langle |(\xi, \mu)| \rangle^{2s} \left( \int \langle \tau - \omega \rangle^{2b'} |\widehat{g}(\tau)|^2 d\tau \right) \left( \int_{T|\tau - \omega| \leq 1} \frac{|\tau - \omega|^{2k-2}}{\langle \tau - \omega \rangle^{2b'}} d\tau \right) d\xi d\mu \\ & \lesssim \sum_{k \geq 1} \frac{1}{k!} T^{1-2b+2k} \|g\|_{X^{s,b'}}^2 T^{1-2k+2b'} \leq \left( T^{1-b+b'} \|g\|_{X^{s,b'}} \right)^2. \end{aligned}$$

We finally need to bound  $III$ , in which the time variable  $t$  appears inside the integral on  $\tau$ . The integral :

$$J(t) := \int_{T|\tau - \omega| \geq 1} \frac{e^{it(\tau - \omega)}}{i(\tau - \omega)} \widehat{g}(\tau) d\tau$$

can be seen as the inverse of a Fourier transform:

$$\int \langle \theta - \omega \rangle^{2b} \left| \int e^{-it\theta} J(t) dt \right|^2 d\theta = \int_{T|\theta - \omega| \geq 1} \frac{\langle \theta - \omega \rangle^{2b}}{(\theta - \omega)^2} |\widehat{g}(\theta - \omega)|^2 d\theta \lesssim \left( T^{1-b+b'} \|\widehat{g}(\xi, \mu)\|_{H_t^{b'}} \right)^2.$$

Similarly, for the  $L^2$ -norm:

$$\int \left| \int e^{-it\theta} J(t) dt \right|^2 d\theta \lesssim \left( T^{1+b'} \|\widehat{g}(\xi, \mu)\|_{H_t^{b'}} \right)^2.$$

We can now compute the norm of the term  $III$ :

$$\begin{aligned} & \int \langle |(\xi, \mu)| \rangle^{2s} \langle \theta - \omega \rangle^{2b} |\mathcal{F}_t(III)(\theta, \xi, \mu)|^2 d\theta d\xi d\mu \\ & = \int \langle |(\xi, \mu)| \rangle^{2s} \left( \int \langle \theta - \omega \rangle^{2b} |\mathcal{F}_t(e^{it\omega} \phi_T(t)) * \mathcal{F}_t(J)(\theta)|^2 d\theta \right) d\xi d\mu \\ & \lesssim \|\langle \tau \rangle^{2b} \mathcal{F}_t(\phi_T)\|_{L_t^1} \|J\|_{L_t^2 H^s} + \|\mathcal{F}_t(\phi_T)\|_{L_t^1} \|J\|_{X^{s,b}} \lesssim \left( T^{1-b+b'} \|g\|_{X^{s,b'}} \right)^2. \quad \square \end{aligned}$$

Observe that  $X^{s,b}$  are embedded in one another as  $b$  decreases, even after truncation in time.

**Lemma 5.** For all  $b' < b$  satisfying  $0 < b - b' < \frac{1}{2}$ , and any function  $u$  in  $X^{s,b}$ :

$$\|\phi_T(t)u\|_{X^{s,b'}} \lesssim T^{b-b'} \|\phi_T(t)u\|_{X^{s,b}}. \quad (14)$$

We emphasize that the implicit constants above do not depend on the parameters  $b$ ,  $b'$  and  $T$ .

*Proof.* By choosing  $p := \frac{1}{1+2(b'-b)}$  and  $p' := \frac{1}{2(b-b')}$ :

$$\begin{aligned} \|\phi_T u\|_{X^{s,b'}}^2 & = \int \langle |(\xi, \mu)| \rangle^{2s} \|\langle \tau - \omega \rangle^{b'} \widehat{\phi_T u}\|_{L^2}^2 d(\xi, \mu) \\ & = \int \langle |(\xi, \mu)| \rangle^{2s} \int \langle \tau \rangle^{2b'} |W(t) \widehat{\phi_T u}|^2(\tau) d\tau d(\xi, \mu) \\ & = \int \langle |(\xi, \mu)| \rangle^{2s} \int |\langle D_t \rangle^{2b'} W(t) \mathcal{F}_{xy}(\phi_T u)|^2 dt d(\xi, \mu) \\ & \leq \int \langle |(\xi, \mu)| \rangle^{2s} \left( \int |\langle D_t \rangle^{b'} W(t) \mathcal{F}_{xy}(\phi_T u)|^{2p} dt \right)^{\frac{1}{p}} \left( \int_{-T \leq t \leq 2T} dt \right)^{\frac{1}{p'}} d(\xi, \mu). \end{aligned}$$

Then by a Gagliardo-Nirenberg-Sobolev embedding  $\|v(t)\|_{L_t^{2p}} \leq \|D_t^{\frac{1}{2p}} v(t)\|_{L_t^2}$ :

$$\|\phi_T u\|_{X^{s,b'}}^2 \lesssim T^{2(b-b')} \int \langle |(\xi, \mu)| \rangle^{2s} \int |\langle D_t \rangle^{2b} W(t) \mathcal{F}_{xy}(\phi_T u)|^2 dt d(\xi, \mu) \lesssim T^{2(b-b')} \|\phi_T u\|_{X^{s,b}}^2. \quad \square$$

In order to prove the bilinear estimates in the next section, we will make use of two preliminary lemmas already stated in Molinet Pilod [17]. The first one is the following:

**Lemma 6** ([17, Proposition 3.5]). *Consider the polynomial  $K(x, y) := 3x^2 - y^2$ . For all  $\phi \in L^2(\mathbb{R}^2)$ ,*

$$\left\| K(D)^{1/8} e^{-t\partial_x \Delta} \phi \right\|_{L_{txy}^4} \lesssim \|\phi\|_{L_{xy}^2}, \quad (15)$$

and for all  $u \in X^{0, \frac{1}{2}+}$ ,

$$\left\| K(D)^{1/8} e^{t\partial_x \Delta} u \right\|_{L^4} \lesssim \|u\|_{X^{0, \frac{1}{2}+}}. \quad (16)$$

*Proof.* This result is in fact a direct consequence of the following optimal  $L^4$  restriction estimate for homogenous polynomial hypersurface of degree  $d \geq 2$  in  $\mathbb{R}^3$  proved by Carbery, Kenig and Zisler [3], which goes as follows.

Let  $\Omega \in \mathbb{R}[X, Y]$  be a homogeneous polynomial of degree  $d \geq 2$  and let  $\Gamma(\xi, \mu) = (\xi, \mu\Omega(\xi, \mu))$ . Denote  $K_\Omega(\xi, \mu) = |\det \text{Hess } \Omega(\xi, \mu)|$ . Then there exist a constant  $C > 0$  such that

$$\forall f \in L^{4/3}(\mathbb{R}^3), \quad \left( \int_{\mathbb{R}^2} |\hat{f}(\Gamma(\xi, \mu))|^2 K_\Omega(\xi, \mu)^{1/4} d\xi d\mu \right)^{1/2} \leq C \|f\|_{L^{4/3}}. \quad (17)$$

The symbol associated to  $e^{-t\partial_x \Delta}$  is  $\omega(\xi, \mu) = \xi(\xi^2 + \mu^2)$ , whose Hessian is

$$\det \text{Hess } \omega(\xi, \mu) = 12\xi^2 - 4\mu^2 = 4K(\xi, \mu).$$

We can then apply (17) to  $K$ , and by a direct duality argument, we derive (15) and (16).  $\square$

One can view this result as a  $1/4$  gain of derivative when compared to the Strichartz estimate (9) (we refer to [17, Remark 3.1] for further details).

The second lemma reveals where the gain of regularity occurs in the dispersion relation of (ZK). It is specific to dimension 3, and so well suited to the study in Bourgain spaces of (ZK) in 2 space dimensions. We recall that the truncation operators  $P_N$  and  $Q_L$  were defined in (8).

**Lemma 7** ([17, Proposition 3.6]). *Let  $N_1, N_2, L_1$  and  $L_2$  be four integers involved in the operators  $P_N$  and  $Q_L$ , and two functions  $u_1$  and  $v_2$ . Then*

$$\|(P_{N_1} Q_{L_1} u_1)(P_{N_2} Q_{L_2} v_2)\|_{L^2} \lesssim \max(L_1, L_2)^{\frac{1}{2}} \max(N_1, N_2) \|P_{N_1} Q_{L_1} u_1\|_{L^2} \|P_{N_2} Q_{L_2} v_2\|_{L^2}. \quad (18)$$

If furthermore  $N_2 \geq 4N_1$  or  $N_1 \geq 4N_2$ , then

$$\|(P_{N_1} Q_{L_1} u_1)(P_{N_2} Q_{L_2} v_2)\|_{L^2} \lesssim \frac{\max(N_1, N_2)^{\frac{1}{2}}}{\min(N_1, N_2)} (L_1 L_2)^{\frac{1}{2}} \|P_{N_1} Q_{L_1} u_1\|_{L^2} \|P_{N_2} Q_{L_2} v_2\|_{L^2}. \quad (19)$$

Observe in inequalities (19) the  $-\frac{1}{2}$  gain in the quotient  $\frac{\max(N_1, N_2)^{\frac{1}{2}}}{\min(N_1, N_2)}$ .

## 2 Bilinear estimates

The key bilinear estimate in [17] is the following:

**Proposition 8** ([17, Proposition 4.1]). *Let  $s > \frac{1}{2}$ . Then there exists  $0 < \delta < \frac{1}{4}$  such that for all  $u, v$ , two functions of  $X^{s, \frac{1}{2}+\delta}$  :*

$$\|\partial_x(uv)\|_{X^{s, -\frac{1}{2}+2\delta}} \lesssim \|u\|_{X^{s, \frac{1}{2}+\delta}} \|v\|_{X^{s, \frac{1}{2}+\delta}}. \quad (20)$$



This estimates allows to gain one derivative, as required by the non linear term in (ZK), in the context of Bourgain spaces. When  $s$  is large, we need to slightly improve this estimate when  $s > 1$ , to derive a tame bilinear estimate where one  $s$  is exchanged for a regularity index 1: this is important so as to make a full use of  $H^1$  bounds.

**Corollary 9.** *Let  $s > 1$ . Then there exists a constant  $0 < \delta < \frac{1}{4}$ , and a constant  $C(s)$  such that:*

$$\forall u, v \in X^{s, \frac{1}{2} + \delta}, \quad \|\partial_x(uv)\|_{X^{s, -\frac{1}{2} + 2\delta}} \leq C(s) \left( \|u\|_{X^{s, \frac{1}{2} + \delta}} \|v\|_{X^{1, \frac{1}{2} + \delta}} + \|u\|_{X^{1, \frac{1}{2} + \delta}} \|v\|_{X^{s, \frac{1}{2} + \delta}} \right). \quad (21)$$

*Proof.* First, by definition of the bracket,

$$\langle |(\xi_0, \mu_0)| \rangle^{2(s-1)} \leq C(s) \left( \langle |(\xi_1, \mu_1)| \rangle^{2(s-1)} + \langle |(\xi_1 - \xi_0, \mu_1 - \mu_0)| \rangle^{2(s-1)} \right),$$

which implies, by using the previous proposition :

$$\begin{aligned} \|\partial_x(uv)\|_{X^{s, -\frac{1}{2} + 2\delta}} &\lesssim \|\partial_x(\langle |(\overline{D_x, \overline{D_y}})| \rangle^{s-1}(uv))\|_{X^{1, -\frac{1}{2} + 2\delta}} + \|\partial_x(u\langle |(\overline{D_x, \overline{D_y}})| \rangle^{s-1}(v))\|_{X^{1, -\frac{1}{2} + 2\delta}} \\ &\lesssim \|\langle |(\overline{D_x, \overline{D_y}})| \rangle^{s-1}u\|_{X^{1, \frac{1}{2} + \delta}} \|v\|_{X^{1, \frac{1}{2} + \delta}} + \|u\|_{X^{1, \frac{1}{2} + \delta}} \|\langle |(\overline{D_x, \overline{D_y}})| \rangle^{s-1}v\|_{X^{1, \frac{1}{2} + \delta}} \quad \square \end{aligned}$$

We will use both estimates (20) and (21) in the next section (in particular in the fixed point result). In order to prove of Theorem 1, we will also need another bilinear estimate in  $X^{-\rho, b}$  spaces, with *negative* regularity index  $-\rho < 0$ : this gain of space derivative is crucial, and possible because we won't need the space derivative on  $uv$  there.

**Proposition 10.** *Let  $\delta > 0$  small ( $\delta < \frac{1}{12}$ ),  $b' = -\frac{1}{2} + 2\delta$  and  $b = \frac{1}{2} + \delta$ . There exist a constant  $C$ , independent of  $\delta$ , such that for all  $0 < \rho < \frac{1}{2} - 6\delta$  the following estimate holds:*

$$\forall u, v \in X^{-\rho, b}, \quad \|uv\|_{X^{-\rho, b'}} \leq C \|u\|_{X^{-\rho, b}} \|v\|_{X^{-\rho, b}}. \quad (22)$$

This proposition is the main new technical result of the paper.

*Proof.* The idea of proof is similar to the one for (20) in [17], working in frequencies, and splitting in various domains. The main difference is that we will take advantage of the absence of derivative  $\partial_x$  in the estimate (22) compared to (20) to work with a lower space regularity  $-\rho < 0$  instead of  $s > \frac{1}{2}$ .

The desired estimate (22) is equivalent by duality to prove

$$\int_{\mathbb{R}^6} \widehat{w_0} \widehat{u_1} \widehat{v_2} \frac{\langle |(\xi_1, \eta_1)| \rangle^\rho \langle |(\xi_2, \eta_2)| \rangle^\rho}{\langle |(\xi_0, \eta_0)| \rangle^\rho} \frac{d(\tau_0, \xi_0, \mu_0) d(\tau_1, \xi_1, \mu_1)}{\langle \tau_0 - \omega_0 \rangle^{-b'} \langle \tau_1 - \omega_1 \rangle^b \langle \tau_2 - \omega_2 \rangle^b} \lesssim \|w_0\|_{L^2} \|u_1\|_{L^2} \|v_2\|_{L^2},$$

where  $\xi_2 := \xi_0 - \xi_1$ ,  $\mu_2 := \mu_0 - \mu_1$ ,  $\tau_2 := \tau_0 - \tau_1$ ,  $\widehat{w_0} := \widehat{w}(\tau_0, \xi_0, \mu_0)$ ,  $\widehat{u_1} := \widehat{u}(\tau_1, \xi_1, \mu_1)$ ,  $\widehat{v_2} := \widehat{v}(\tau_2, \xi_2, \mu_2)$ , and  $\omega_i := \xi_i (\xi_i^2 + \mu_i^2)$ .

We decompose along different frequencies: given integers  $N_0, N_1, N_2$ , let

$$I_{N_0, N_1, N_2} := \int_{\mathbb{R}^6} \widehat{P_{N_0} w_0} \widehat{P_{N_1} u_1} \widehat{P_{N_2} v_2} \frac{\langle |(\xi_1, \eta_1)| \rangle^\rho \langle |(\xi_2, \eta_2)| \rangle^\rho}{\langle |(\xi_0, \eta_0)| \rangle^\rho} \frac{d(\tau_0, \xi_0, \mu_0) d(\tau_1, \xi_1, \mu_1)}{\langle \tau_0 - \omega_0 \rangle^{-b'} \langle \tau_1 - \omega_1 \rangle^b \langle \tau_2 - \omega_2 \rangle^b}.$$

Analogously, for the operators  $Q_L$ , given furthermore integers  $L_0, L_1, L_2$ , denote

$$I_{N_0, N_1, N_2}^{L_0, L_1, L_2} := \int_{\mathbb{R}^6} (Q_{L_0} \widehat{P_{N_0} w_0}) (Q_{L_1} \widehat{P_{N_1} u_1}) (Q_{L_2} \widehat{P_{N_2} v_2}) \frac{\langle |(\xi_1, \eta_1)| \rangle^\rho \langle |(\xi_2, \eta_2)| \rangle^\rho d(\tau_0, \xi_0, \mu_0) d(\tau_1, \xi_1, \mu_1)}{\langle |(\xi_0, \eta_0)| \rangle^\rho \langle \tau_0 - \omega_0 \rangle^{-b'} \langle \tau_1 - \omega_1 \rangle^b \langle \tau_2 - \omega_2 \rangle^b}.$$

We split the frequencies into five main domains.

**First domain.**  $N_1 \leq 2$ ,  $N_2 \leq 2$  and  $N_0 \leq 2$ . We use the Plancherel equality and Hölder inequality:

$$|I_{N_0, N_1, N_2}| \lesssim \left\| \left( \frac{\widehat{u_1}}{\langle \tau_1 - \omega_1 \rangle^b} \right)^\vee \right\|_{L^4} \left\| \left( \frac{\widehat{v_2}}{\langle \tau_2 - \omega_2 \rangle^b} \right)^\vee \right\|_{L^4} \|w_0\|_{L^2},$$

and we conclude by the adequate embedding (10) :  $X^{0, \frac{5}{12}+} \hookrightarrow L^4$ .

**Second domain.**  $4 \leq N_1, N_2 \leq \frac{N_1}{4}$  so  $\frac{N_1}{2} \leq N_0 \leq 2N_1$ . We use the upper bound of the localized frequencies (19):

$$\begin{aligned} I_{N_0, N_1, N_2}^{L_0, L_1, L_2} &\lesssim \frac{N_1^\rho N_2^\rho}{N_0^\rho L_0^{-b'} L_1^b L_2^b} \|(P_{N_1} Q_{L_1} u_1)(P_{N_2} Q_{L_2} v_2)\|_{L^2} \|P_{N_0} Q_{L_0} w_0\|_{L^2} \\ &\lesssim \frac{N_1^\rho}{L_0^{-b'} L_1^b L_2^b} \frac{N_2^{\frac{1}{2}}}{N_1} L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} \|P_{N_1} Q_{L_1} u_1\|_{L^2} \|P_{N_2} Q_{L_2} v_2\|_{L^2} \|P_{N_0} Q_{L_0} w_0\|_{L^2}. \end{aligned}$$

For  $\rho < \frac{1}{2}$ , the sum over  $L_0, L_1, L_2, N_1, N_2$  and  $N_0$  gives:

$$\sum_{N_1 \leq 2, N_2 \leq \frac{N_1}{4}, N_0} I_{N_0, N_1, N_2} \lesssim \|w_0\|_{L^2} \|u_1\|_{L^2} \|v_2\|_{L^2}.$$

**Third domain.**  $4 \leq N_2, N_1 \leq \frac{N_2}{4}$  so  $\frac{N_2}{2} \leq N_0 \leq 2N_2$ . By symmetry between  $N_1$  and  $N_2$ , the third domain is dealt with as for the second domain.

**Fourth domain.**  $4 \leq N_1, N_0 \leq \frac{N_1}{4}$  so  $\frac{N_1}{2} \leq N_2 \leq 2N_1$ . (or equivalently,  $4 \leq N_2, N_0 \leq \frac{N_2}{4}$  so  $\frac{N_2}{2} \leq N_1 \leq 2N_2$ ). We use the same decomposition as the second domain, an interpolation between (18) and (19) by a coefficient  $\theta \in (0, 1)$ , with  $\tilde{f}(x, y, z) := f(-x, -y, -z)$ :

$$\begin{aligned} I_{N_0, N_1, N_2}^{L_0, L_1, L_2} &\lesssim \frac{N_1^\rho N_2^\rho}{N_0^\rho L_0^{-b'} L_1^b L_2^b} \left\| \left( \widehat{P_{N_1} Q_{L_1} u_1} \right) P_{N_0} Q_{L_0} w_0 \right\|_{L^2} \|P_{N_2} Q_{L_2} v_2\|_{L^2} \\ &\lesssim \frac{N_1^\rho}{L_0^{-b'} L_1^b L_2^b} \frac{N_0^{\frac{1}{2}(1+\theta)}}{N_1^{1-\theta}} L_0^{\frac{1}{2}(1-\theta)} L_1^{\frac{1}{2}} \|P_{N_0} Q_{L_0} w_0\|_{L^2} \|P_{N_1} Q_{L_1} u_1\|_{L^2} \|P_{N_2} Q_{L_2} v_2\|_{L^2} \\ &\lesssim \frac{N_1^{-\frac{1}{2} + \frac{3}{2}\theta + \rho}}{L_0^{-b' + \frac{\theta}{2} - \frac{1}{2}} L_1^{b - \frac{1}{2}} L_2^b} \|P_{N_0} Q_{L_0} w_0\|_{L^2} \|P_{N_1} Q_{L_1} u_1\|_{L^2} \|P_{N_2} Q_{L_2} v_2\|_{L^2}. \end{aligned}$$

By choosing  $\theta$  such that  $4\delta < \theta < \frac{1}{3}(1 - 2\rho)$  (this is one of the points giving the bounds on  $\delta$  and  $\rho$ ), the sum over  $L_0, L_1, L_2, N_0, N_1$  and  $N_2$  is bounded by  $\|w_0\|_{L^2} \|u_1\|_{L^2} \|v_2\|_{L^2}$ .

**Fifth domain.**  $4 \leq N_1, 4 \leq N_2, N_0 \geq \frac{N_1}{2}$  and  $N_0 \geq \frac{N_2}{2}$  (so  $\frac{N_2}{2} \leq N_1 \leq 2N_2, \frac{N_1}{2} \leq N_0 \leq 2N_1$  and  $\frac{N_2}{2} \leq N_0 \leq 2N_2$ ).

We divide this domain depending on the values of  $\xi_i$  and  $\mu_i$ , with a coefficient  $\alpha$  to define later:

$$\begin{aligned} \mathcal{D}_1 &:= \left\{ (\tau_0, \xi_0, \mu_0, \tau_1, \xi_1, \mu_1); (1 - \alpha)^{\frac{1}{2}} \sqrt{3} |\xi_i| \leq |\mu_i| \leq (1 - \alpha)^{-\frac{1}{2}} \sqrt{3} |\xi_i|, i = 1, 2 \right\}, \\ \mathcal{D}_2 &:= \left\{ (\tau_0, \xi_0, \mu_0, \tau_1, \xi_1, \mu_1); (1 - \alpha)^{\frac{1}{2}} \sqrt{3} |\xi_i| \leq |\mu_i| \leq (1 - \alpha)^{-\frac{1}{2}} \sqrt{3} |\xi_i|, i = 0, 1 \right\}, \\ \mathcal{D}_3 &:= \left\{ (\tau_0, \xi_0, \mu_0, \tau_1, \xi_1, \mu_1); (1 - \alpha)^{\frac{1}{2}} \sqrt{3} |\xi_i| \leq |\mu_i| \leq (1 - \alpha)^{-\frac{1}{2}} \sqrt{3} |\xi_i|, i = 0, 2 \right\}, \\ \mathcal{D}_4 &:= \mathbb{R}_{\mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3}^6. \end{aligned}$$

First, we work on  $\mathcal{D}_1$ , in a subregion where  $\xi_1 \xi_2 > 0$  and  $\mu_1 \mu_2 > 0$ . We claim that:

$$N_1^3 \lesssim \max \{ |\tau_0 - \omega_0|, |\tau_1 - \omega_1|, |\tau_2 - \omega_2| \}.$$

In fact, if  $|\tau_1 - \omega_1| + |\tau_2 - \omega_2| \lesssim N_1^3$ , then:

$$\tau_0 - \omega_0 = \tau_1 - \omega_1 + \tau_2 - \omega_2 - \xi_1 (\xi_2^2 + 2\xi_1 \xi_2 + \mu_2^2 + 2\mu_1 \mu_2) - \xi_2 (\xi_1^2 + 2\xi_1 \xi_2 + \mu_1^2 + 2\mu_1 \mu_2) \simeq \pm N_1^3,$$

which proves the claim if  $\xi_1 \xi_2 > 0$  and  $\mu_1 \mu_2 > 0$ . The other cases are similar. We can thus argue as in the first domain, and obtain that for  $\rho \leq \frac{3}{2} - 6\delta$ ,

$$\begin{aligned} \sum_{N_0, N_1, N_2} I_{N_0, N_1, N_2} &\lesssim \sum_{N_0, N_1, N_2} N_1^{\rho+3b'} \left\| \left( \frac{\widehat{P_{N_1} u_1}}{\langle \tau_1 - \omega_1 \rangle^b} \right)^\vee \right\|_{X^{0, \frac{5}{8}+}} \left\| \left( \frac{\widehat{P_{N_2} u_2}}{\langle \tau_2 - \omega_2 \rangle^b} \right)^\vee \right\|_{X^{0, \frac{5}{8}+}} \|P_{N_0} w_0\|_{L^2} \\ &\lesssim \|w_0\|_{L^2} \|u_1\|_{L^2} \|v_2\|_{L^2}. \end{aligned}$$

Next, the subregions  $\xi_1 \xi_2 > 0$  and  $\mu_1 \mu_2 < 0$ , or  $\xi_1 \xi_2 < 0$  and  $\mu_1 \mu_2 > 0$ , we use the dyadic decomposition along

the flow with the operators  $Q_L$  and by a Cauchy-Schwarz inequality:

$$I_{N_0, N_1, N_2}^{L_0, L_1, L_2} \lesssim \frac{N_1^\rho}{L_0^{-b'} L_1^b L_2^b} \|(P_{N_1} Q_{L_1} u_1) (P_{N_2} Q_{L_2} v_2)\|_{L^2} \|P_{N_0} Q_{L_0} w_0\|_{L^2}.$$

To deal with the first  $L^2$ -norm, we need to bound the interaction of the high/high frequencies. This is the purpose of the following lemma.

**Lemma 11.** *Let  $N_1, N_2, L_1$  and  $L_2$  be four integers involved in the operators  $P_N$  and  $Q_L$ , and two functions  $u_1$  and  $v_2$ . In the case  $\frac{N_1}{2} \leq N_2 \leq 2N_1$ , define the subsets  $S_1$  and  $S_2$  of  $\mathbb{R}_{\xi_1, \mu_1}^2 \times \mathbb{R}_{\xi_2, \mu_2}^2$  by*

$$\begin{aligned} S_1(\xi_1, \xi_2, \mu_1, \mu_2) &:= \{\xi_1 \xi_2 > 0, \mu_1 \mu_2 < 0\} \cup \{\xi_1 \xi_2 < 0, \mu_1 \mu_2 > 0\}, \\ S_2(\xi_1, \xi_2, \mu_1, \mu_2) &:= \{\xi_1 \xi_2 < 0, \mu_1 \mu_2 < 0\}, \end{aligned}$$

and the two Fourier multipliers  $(J_i)_{1 \leq i \leq 2}$ , which take into account only some interaction of frequencies, by:

$$J_i(\widehat{u_1, v_2})(\tau, \xi, \mu) := \int_{\mathbb{R}^3} \mathbf{1}_{S_i(\xi_1, \xi - \xi_1, \mu_1, \mu - \mu_1)} \widehat{u_1}(\tau_1, \xi_1, \mu_1) \widehat{v_2}(\tau - \tau_1, \xi - \xi_1, \mu - \mu_1) d\tau_1 d\xi_1 d\mu_1.$$

Then, for  $\alpha$  small enough, one has

$$\|J_i(P_{N_1} Q_{L_1} u_1, P_{N_2} Q_{L_2} v_2)\|_{L^2} \lesssim N_1^{-\frac{1}{2}} (L_1 L_2)^{\frac{1}{2}} \|P_{N_1} Q_{L_1} u_1\|_{L^2} \|P_{N_2} Q_{L_2} v_2\|_{L^2}. \quad (23)$$

Observe in (23) the  $-\frac{1}{2}$  gain for high/high interaction in  $N_1^{-1/2}$ ; the smallness requirement on  $\alpha$  comes from this lemma. Let us postpone its proof to the appendix, and conclude now the proof of Proposition 10. We thus use inequality (23) with  $J_1$  to obtain:

$$I_{N_0, N_1, N_2}^{L_0, L_1, L_2} \lesssim \frac{1}{N_1^{\frac{1}{2}-\rho} L_0^{-b'} L_1^{b-\frac{1}{2}} L_2^{b-\frac{1}{2}}} \|P_{N_1} Q_{L_1} u_1\|_{L^2} \|P_{N_2} Q_{L_2} v_2\|_{L^2} \|P_{N_0} Q_{L_0} w_0\|_{L^2},$$

and conclude summing over  $N_0, N_1, N_2, L_0, L_1$  and  $L_2$ . In the subregion  $\xi_1 \xi_2 < 0, \mu_1 \mu_2 < 0$ , we use the same process with  $J_2$  in (23).

In the regions  $\mathcal{D}_2$  and  $\mathcal{D}_3$ , the estimates are similar: it suffices to establish a change of variable which brings us exactly to the case of  $\mathcal{D}_1$ . Observe that there is no influence from neither  $b$  nor  $b'$ .

Finally, in  $\mathcal{D}_4$ , we use the Fourier multiplier operator  $K(\xi, \mu) := |3\xi^2 - \mu^2| \gtrsim |(\xi, \mu)|$ , and the estimate (16), so that for  $\rho < \frac{1}{2}$ :

$$\begin{aligned} I_{N_0, N_1, N_2} &\lesssim N_1^{\rho-\frac{1}{2}} \left\| \left\| K(D)^{\frac{1}{8}} \left( \frac{\widehat{P_{N_1} u_1}}{\langle \tau_1 - \omega_1 \rangle^b} \right)^\vee \right\|_{L^4} \left\| K(D)^{\frac{1}{8}} \left( \frac{\widehat{P_{N_2} v_2}}{\langle \tau_2 - \omega_2 \rangle^b} \right)^\vee \right\|_{L^4} \right\|_{L^4} \|P_{N_0} w_0\|_{L^2} \\ &\lesssim \|P_{N_0} w_0\|_{L^2} \|P_{N_1} u_1\|_{L^2} \|P_{N_2} v_2\|_{L^2}. \quad \square \end{aligned}$$

### 3 Growth of Sobolev norms

We start this section with a local well posedness result where we carefully track the dependency of the existence time: it is crucial that it only depends on  $\|u_0\|_{H^1}$ .

In previous results set in the context of Bourgain spaces, the proofs were made with dependency on  $\|u_0\|_{H^s}$ : for instance, in Kenig-Ponce-Vega [10] (where they assume  $\|u_0\|_{H^s} \leq 1$  and then use a scaling argument) or Molinet Pilod [17]. This is very good for low  $s < 1$ , but does not quite give a suitable result in our perspective of large  $s > 1$ . This is why we provide a full proof below, using in particular the tame bilinear estimate (21) (which is only relevant for  $s > 1$ ).

Throughout the remainder of this section we fix

$$s > 1, \quad \text{and} \quad \delta \in \left(0, \frac{1}{12}\right).$$

(In particular, the constant  $C_0$  and the function  $T$  below depend on  $s$  and  $\delta$ ).

**Proposition 12.** *There exists a universal constant  $C_0 > 0$  such that if we define*

$$\forall A > 0, \quad T(A) := \frac{1}{(8C_0^2 A)^{\frac{1}{8}}}, \quad (24)$$

there exists a unique solution  $u \in \mathcal{C}([-T(\|u_0\|_{H^1}), T(\|u_0\|_{H^1})], H^s)$  of (ZK) with initial condition  $u(0) = u_0$ . Furthermore,  $u$  satisfies the estimates :

$$\|u\|_{\mathcal{C}([-T(\|u_0\|_{H^1}), T(\|u_0\|_{H^1})], H^s)} + \|\phi_{T(\|u_0\|_{H^1})} u\|_{X^{s, \frac{1}{2} + \delta}} \leq C_0 \|u_0\|_{H^s} \text{ and } \|\phi_{T(\|u_0\|_{H^1})} u\|_{X^{1, \frac{1}{2} + \delta}} \leq C_0 \|u_0\|_{H^1}. \quad (25)$$

*Proof.* We consider the initial condition as  $u_0$ , with a finite norm in  $H^s$ ,  $s \geq 2$ . We work on

$$B := \{w \in X^{s, b}; \|w\|_{X^{s, b}} \leq 2C_0 \|u_0\|_{H^s} \text{ and } \|w\|_{X^{1, b}} \leq 2C_0 \|u_0\|_{H^1}\},$$

with the distance  $d(v, w) := \|v - w\|_{X^{1, b}}$ .  $(B, d)$  is a complete metric space.

Let  $T > 0$  to be chosen later. We want to apply the fixed point theorem to the function  $\Psi$  defined in (7) on the space  $B$ . Let  $\delta$  be given as in the bilinear estimates (20) and (21),  $b := \frac{1}{2} + \delta$  and  $b' := 2b - \frac{3}{2} = -\frac{1}{2} + 2\delta$ . Using lemma 4 and proposition 9 :

$$\begin{aligned} \|\Psi(w)\|_{X^{s, b}} &\leq \|\phi_1(t)W(t)u_0\|_{X^{s, b}} + \left\| \phi_T(t) \int_0^t W(t-t') \phi_T(t')^2 \partial_x (w^2)(t') dt' \right\|_{X^{s, b}} \\ &\leq C_0 \left( \|u_0\|_{H^s} + T^{1-b+b'} \left\| \partial_x \left( (\phi_T(t)w)^2 \right) \right\|_{X^{s, b'}} \right) \\ &\leq C_0 \|u_0\|_{H^s} (1 + 4C_0^2 T^\delta \|u_0\|_{H^1}), \end{aligned}$$

and similarly, (increasing the constant  $C_0$  if necessary),

$$\|\Psi(w)\|_{X^{1, b}} \leq C_0 \|u_0\|_{H^1} (1 + 4T^\delta C_0^2 \|u_0\|_{H^1}).$$

Furthermore, the function  $\Psi$  is a contraction on  $(B, d)$ :

$$\|\Psi(\omega) - \Psi(\tilde{\omega})\|_{X^{1, b}} \leq C_0 T^\delta \|\omega - \tilde{\omega}\|_{X^{1, b}} \|\omega + \tilde{\omega}\|_{X^{1, b}} \leq 4C_0^2 T^\delta \|u_0\|_{H^1} \|\omega - \tilde{\omega}\|_{X^{1, b}}.$$

Hence, by choosing  $T > 0$  such that  $T^\delta = \frac{1}{8C_0^2 \|u_0\|_{H^1}}$ , we can apply the fixed point theorem. Hence we obtain

$$\|\phi_T u\|_{X^{s, b}} \leq C_0 \|u_0\|_{H^s}.$$

The  $\|u\|_{\mathcal{C}([-T(\|u_0\|_{H^1}), T(\|u_0\|_{H^1})], H^s)}$  bound is immediate from (11).

Finally, the proof of continuity of the unique solution  $u \in \mathcal{C}([0, T], H^s)$  is inspired by Kenig-Ponce-Vega [10]. We consider continuity at 0, the other points are similar. Fix  $t > 0$ . Then the Duhamel formula can be rewritten

$$u(t) = W(t)u_0 + \int_0^t W(t-t') \partial_x ((\phi_t(t')u(t'))^2) dt'.$$

We use successively embedding (11), estimate (13) and estimate (20):

$$\begin{aligned} \|u(t) - u(0)\|_{H^s} &\leq \|W(t)u_0 - u_0\|_{H^s} + \left\| \int_0^t W(t-t') \partial_x ((\phi_t(t')u(t'))^2) dt' \right\|_{H^s} \\ &\leq \|W(t)u_0 - u_0\|_{H^s} + C(b) \left\| \int_0^t W(t-t') \partial_x ((\phi_t(t')u(t'))^2) dt' \right\|_{X^{s, b}} \\ &\leq \|W(t)u_0 - u_0\|_{H^s} + C(b) t^{1-b+b'} \left\| \partial_x ((\phi_t u)^2) \right\|_{X^{s, b'}} \\ &\leq \|W(t)u_0 - u_0\|_{H^s} + C(b, s) t^\delta \|\phi_t u\|_{X^{s, b}}^2, \end{aligned}$$

which tends to 0 as  $t \rightarrow 0$ . This proves continuity at 0 in  $H^s$ .  $\square$

We will now prove Theorem 1, and for this assume that  $s > 1$  is an integer.

*Proof of Theorem 1.* We denote  $u \in \mathcal{C}(I, H^s)$  to be the maximal  $H^s$  development of  $u_0$  under the (ZK) flow. Denote

$$A := \sup_{t \in I} \|u(t)\|_{H^1} \leq \sqrt{C(E(u_0) + M(u_0)^2)}.$$

(See (1)). The equation (ZK) is reversible, it suffices to prove (2) for positive times  $t \geq 0$ .

Proposition 12 yields that for all  $t_0 \in I$ ,  $[t_0 - T(A), t_0 + T(A)] \subset I$ , and

$$\forall t \in [t_0 - T(A), t_0 + T(A)], \quad \|u(t)\|_{H^s} \leq C_0 \|u(t_0)\|_{H^s}.$$

This shows by iteration that  $I = \mathbb{R}$ , and that for all  $k \in \mathbb{N}$ ,

$$\forall t \in [kT(A), (k+1)T(A)], \quad \|u(t)\|_{H^s} \leq C_0^{k+1} \|u_0\|_{H^s}.$$

From this, we infer the exponential bound  $\|u(t)\|_{H^s} \leq \exp(Ct)$  for some large constant  $C$  and all  $t \geq 0$ .

To improve this to a polynomial bound, we will follow the idea of Staffilani [20] to obtain more refined inequality, namely

$$\forall t \in [0, T(\|u_0\|_{H^1})], \quad \|u(t)\|_{H^s}^2 - \|u(0)\|_{H^s}^2 \leq C(\|u_0\|_{H^1})(1 + \|u_0\|_{H^s}^{2-\epsilon}), \quad (26)$$

for some  $\epsilon$  close to 0 and some function  $C : [0, +\infty) \rightarrow [0, +\infty)$ .

This will imply that for all  $t_0 \in \mathbb{R}$ , and for some uniform constant  $C_1 \geq 1$  (depending on  $A$  and  $C$ )

$$\forall t \in [t_0 - T(A), t_0 + T(A)], \quad \|u(t)\|_{H^s}^2 \leq \|u(t_0)\|_{H^s}^2 + C_1(1 + \|u(t_0)\|_{H^s}^{2-\epsilon}).$$

Therefore, denoting  $u_k = \sup_{t \in [kT(A), (k+1)T(A)]} \|u(t)\|_{H^s}$ , then (with  $t_0 = kT$ )

$$\forall k \in \mathbb{N}, \quad u_{k+1} \leq (u_k^2 + C_1(1 + u_k^{2-\epsilon}))^{1/2} \leq u_k + C_1(1 + u_k^{1-\epsilon}).$$

As observed in [20], an iteration argument then shows that  $u_k$  has polynomial growth: this is the content of the following lemma.

**Lemma 13.** *Let  $(u_k)_{k \in \mathbb{N}}$  be a sequence of non negative numbers,  $K_1 > 0$ ,  $\epsilon \in (0, 1)$  such that*

$$\forall k \in \mathbb{N}, \quad u_{k+1} \leq u_k + K_1(1 + u_k^{1-\epsilon}).$$

*Then for any  $d > \frac{1}{\epsilon}$ , there exist a constant  $K_2 = K_2(K_1, d)$  such that*

$$\forall k \in \mathbb{N}, \quad u_{k+1} \leq K_2(1+k)^d(1+u_0).$$

*Proof.* Observe that due to convexity of  $x \mapsto |x|^d$ , for all  $k > 0$ ,

$$(2+k)^d = (1+k)^d \left(1 + \frac{1}{1+k}\right)^d \geq (1+k)^d + d(1+k)^{d-1}. \quad (27)$$

Also, as  $1 + u_{k+1} \leq 1 + u_k + K_1(1 + u_k^{1-\epsilon}) \leq (2K_1 + 1)(1 + u_k)$ , we infer that  $u_k \leq (2K_1 + 1)^k(1 + u_0)$  for all  $k \in \mathbb{N}$ .

Let  $N \in \mathbb{N}$  such that  $N \geq \left(\frac{K_1}{d}\right)^{1/(d\epsilon-1)}$ , so that for  $k \geq N$  (as  $1 - d\epsilon < 0$ )

$$d - K_1(1+k)^{1-d\epsilon} \geq 1.$$

Let  $K_2 := \max\left((2K_1 + 1)^N, \frac{K_1}{N^{d-1}}\right)$ , so that

1. for  $k \leq N$ ,  $u_k \leq K_2(1 + u_0) \leq K_2(1+k)^d(1 + u_0)$  and
2. for  $k \geq N$ ,

$$K_2(1+k)^{d-1} (d - K_1(1+k)^{1-d\epsilon}) \geq K_2(1+k)^{d-1} \geq K_2 N^{d-1} \geq K_1,$$

and so

$$K_2 d(1+k)^{d-1} \geq K_1 + K_1 K_2 (1+k)^{d-1+1-d\epsilon} \geq K_1 + K_1 K_2 (1+k)^{d(1-\epsilon)}. \quad (28)$$

We prove now the statement by induction on  $k$ . For  $k = 0, \dots, N$ , the statement hold due to 1. On the other side, if for some  $k \geq N$ , there hold  $u_k \leq K_2(1+k)^d$ , then using (27) and (28),

$$u_{k+1} \leq K_2(1+k)^d + K_1(1+K_2(1+k)^{d(1-\epsilon)}) \leq K_2(1+k)^d + K_2 d(1+k)^{d-1} \leq K_2(2+k)^d.$$

This proves the induction step for  $k \geq N$ , and the proof is complete.  $\square$

From the above lemma,  $u_k \lesssim (1+k)^{\frac{1}{\epsilon}+} (1+\|u_0\|_{H^s})$  (with implicit constant depending on  $A$ ), and from there the polynomial bound (2) follows immediately with  $\beta = \frac{1}{\epsilon}+$ . We can now focus on proving (26).

We recall that  $T$  has been defined in the previous proposition, and  $s \geq 2$  is an integer. Let  $t \in [0, T]$ ,  $b$  and  $b'$  as in the proposition 10, satisfying  $-b' \leq b$ . We control the evolution of the  $H^s$ -norm:

$$\begin{aligned} \|u(t)\|_{H^s}^2 - \|u(0)\|_{H^s}^2 &= \int_0^t \int \langle |(\xi, \mu)| \rangle^{2s} \frac{d}{dt} |\mathcal{F}_{xy}(u)(t')|^2 d\xi d\mu dt' \\ &= \int_0^t \int \langle |(\xi, \mu)| \rangle^{2s} \phi_T(t')^3 (\mathcal{F}_{xy}(u) \mathcal{F}_{xy}(\partial_x(u^2)))(t') d\xi d\mu dt' \\ &\simeq \sum_{|\mathbf{i}| \leq s} \int_0^t \int \mathcal{F}_{xy}(D^{\mathbf{i}} \phi_T u) \mathcal{F}_{xy}(D^{\mathbf{i}} \partial_x(\phi_T^2 u^2))(t') d\xi d\mu dt' = I_0 + I_{1\dots s-1} + I_s, \end{aligned}$$

where  $I_0$  counts the term with none derivative,  $I_s$  the terms with  $s$  derivatives, and  $I_{1\dots s-1}$  the others.

Let first deal with  $I_0$ , thus the case  $|\mathbf{i}| = 0$ . We obtain:

$$I_0 = \int_0^t \int \phi_T u \partial_x(\phi_T^2 u^2)(t') dx dy dt = 0.$$

Second, let us find an upper bound for  $I_{1\dots s-1}$ , the term for which the sum of the derivatives does not reach  $s$ . By a Cauchy-Schwarz inequality, recalling the function  $F$  to follow the flow in (6), we find the flow of our solution, up to a power  $b'$ :

$$\begin{aligned} I_{1\dots s-1} &= \sum_{0 < |\mathbf{i}| < s} \int_0^t \int \mathcal{F}_{xy}(D^{\mathbf{i}} \phi_T u) \mathcal{F}_{xy}(D^{\mathbf{i}} \partial_x(\phi_T^2 u^2))(t') d\xi d\mu dt' \\ &\leq \sum_{0 < |\mathbf{i}| < s} \int_0^t \left\| F(D)^{-b'} D^{\mathbf{i}}(\phi_T u)(t') \right\|_{L^2} \left\| F(D)^{b'} \partial_x D^{\mathbf{i}}(\phi_T^2 u^2)(t') \right\|_{L^2} dt'. \end{aligned}$$

We can apply estimate (20) to the term on the right because  $|\mathbf{i}| \geq 1$ . We use the estimate (20), a Gagliardo-Nirenberg inequality, and twice estimate (14):

$$\begin{aligned} I_{1\dots s-1} &\leq \sum_{0 < |\mathbf{i}| < s} \|D^{\mathbf{i}}(\phi_T u)\|_{X^{0,-b'}} \sum_{(0,0) \leq \mathbf{j} \leq \mathbf{i}} \|\phi_T u\|_{X^{|\mathbf{j}|,b}} \|\phi_T u\|_{X^{|\mathbf{i}-\mathbf{j}|,b}} \\ &\lesssim \sum_{0 < |\mathbf{i}| < s} \|\phi_T u\|_{X^{s,-b'}}^{|\mathbf{i}|/s} \|\phi_T u\|_{X^{0,-b'}}^{1-|\mathbf{i}|/s} \sum_{(0,0) < \mathbf{j} < \mathbf{i}} \|\phi_T u\|_{X^{s,b}}^{|\mathbf{i}|/s} \max\left(\|\phi_T u\|_{X^{0,b}}^{1-|\mathbf{j}|/s}, \|\phi_T u\|_{X^{0,b}}\right) \\ &\leq C \max(1, \|\phi_T u\|_{X^{0,b}}^2) \max\left(\|\phi_T u\|_{X^{s,b}}^{\frac{2}{s}}, \|\phi_T u\|_{X^{s,b}}^{2(1-\frac{1}{s})}\right). \end{aligned}$$

Third, the terms  $I_s$  with  $|\mathbf{i}| = s$ . We separate  $I_s = I_s^1 + I_s^2$ , where  $I_s^2$  contains the two terms where all the derivatives are distributed on the same term, and  $I_s^1$  all the others. Let  $0 < \rho < \frac{1}{2}$ , and  $\alpha_j := \frac{|j|-1+\rho}{s-1}$ . We deal

first with  $I_s^1$ , the terms where all the derivatives are not on the same term. By using the estimate (20):

$$\begin{aligned}
I_s^1 &:= \sum_{|\mathbf{i}|=s} \sum_{(0,0) < \mathbf{j} < \mathbf{i}} \iint |D^{\mathbf{i}}(\phi_T u) \partial_x (D^{\mathbf{j}}(\phi_T u) D^{\mathbf{i}-\mathbf{j}}(\phi_T u))| (t') dx dy dt' \\
&\lesssim \sum_{|\mathbf{i}|=s} \sum_{(0,0) < \mathbf{j} < \mathbf{i}} \|D^{\mathbf{i}}(\phi_T u)\|_{X^{-\rho, -b'}} \|D^{\mathbf{j}}(\phi_T u)\|_{X^{\rho, b}} \|D^{\mathbf{i}-\mathbf{j}}(\phi_T u)\|_{X^{\rho, b}} \\
&\lesssim \|\phi_T u\|_{X^{s-\rho, -b'}} \sum_{(0,0) < \mathbf{j} < \mathbf{i}} \|\phi_T u\|_{X^{s, b}}^{\alpha_{\mathbf{j}}} \|\phi_T u\|_{X^{1, b}}^{1-\alpha_{\mathbf{j}}} \|\phi_T u\|_{X^{s, b}}^{\alpha_{\mathbf{i}-\mathbf{j}}} \|\phi_T u\|_{X^{1, b}}^{1-\alpha_{\mathbf{i}-\mathbf{j}}} \\
&\lesssim \|\phi_T u\|_{X^{s-\rho, -b'}} \|\phi_T u\|_{X^{s, b}}^{1+\frac{2\rho-1}{s-1}} \max(1, \|\phi_T u\|_{X^{1, b}}^{\frac{s-2\rho}{s-1}}) \\
&\leq C \max(1, \|\phi_T u\|_{X^{1, b}}^2) \|\phi_T u\|_{X^{s, b}}^{2-\frac{1-\rho}{s-1}},
\end{aligned}$$

where the last line comes from a final interpolation. One can notice that the last estimate is sharp, and that  $\frac{1}{2(s-1)} < \frac{1-\rho}{s-1}$ .

We can not directly use the bilinear estimate (20) for the term  $I_s^2$  where all the derivatives are distributed on one term, due to the lower limit of  $s > \frac{1}{2}$ . This term is dealt with using estimate (22) as it authorizes a negative index in  $X^{-\rho, b}$ . Let  $\rho > 0$  be given by Proposition 10. We recall the definition of  $S$  in (5) and the Gagliardo-Nirenberg inequality  $\|D^\rho v\|_{L^2_{x, y}} \lesssim \|D^{s-1} v\|_{L^2_{x, y}}^{\frac{\rho}{s-1}} \|v\|_{L^2_{x, y}}^{1-\frac{\rho}{s-1}}$ . Using an integration by parts when needed, we can bound

$$\begin{aligned}
I_s^2 &:= \sum_{|\mathbf{i}|=s} \iint (D^{\mathbf{i}}(\phi_T u) \partial_x (\phi_T u D^{\mathbf{i}}(\phi_T u))) (t') dx dy dt' \\
&\leq \sum_{|\mathbf{i}|=s} \iint |\partial_x(\phi_T u)| |D^{\mathbf{i}}(\phi_T u)|^2 dx dy dt' \leq \|\partial_x \phi_T u\|_{X^{\rho, -b'}} \left\| (D^{\mathbf{i}}(\phi_T u))^2 \right\|_{X^{-\rho, b'}} \\
&\lesssim \|S(D)^\rho \partial_x \phi_T u\|_{X^{0, b}} \|S(D)^{s-\rho}(\phi_T u)\|_{X^{0, b}}^2 \lesssim \left( \|\phi_T u\|_{X^{1, b}}^{1-\frac{\rho}{s-1}} \|\phi_T u\|_{X^{s, b}}^{\frac{\rho}{s-1}} \right) \|\phi_T u\|_{X^{s, b}}^{2\frac{s-\rho-1}{s-1}} \|\phi_T u\|_{X^{1, b}}^{2\frac{\rho}{s-1}} \\
&\leq C \|\phi_T u\|_{X^{1, b}}^{1+\frac{\rho}{s-1}} \|\phi_T u\|_{X^{s, b}}^{2-\frac{\rho}{s-1}}.
\end{aligned}$$

We thus notice that this time,  $0 < \frac{\rho}{s-1} < \frac{1}{2(s-1)}$ .

Summing up the three previous inequalities, we obtain :

$$\|u(t)\|_{H^s}^2 - \|u(0)\|_{H^s}^2 \leq C \max(1, \|\phi_T u\|_{X^{1, b}}^2) \max\left(\|\phi_T u\|_{X^{s, b}}^{\frac{2}{s}}, \|\phi_T u\|_{X^{s, b}}^{2-\frac{\rho}{s-1}}\right)$$

and by the embeddings (25):

$$\|u(t)\|_{H^s} \leq \|u(0)\|_{H^s} + C \max(1, C_0 \|u(0)\|_{H^1}) C_0^{1-\frac{\rho}{2(s-1)}} \max\left(\|u(0)\|_{H^s}^{\frac{1}{s}}, \|u(0)\|_{H^s}^{1-\frac{\rho}{2(s-1)}}\right).$$

This concludes the proof of the estimate (26), by choosing  $\epsilon = \frac{\rho}{s-1}$  and  $\rho \in (0, \frac{1}{2})$ .  $\square$

## 4 Appendix : Proof of lemma 11

This lemma is specific to (total) dimension 3. Its proof is almost similar to that of the estimate 5.2 [17, Proposition 3.6], which is slightly different from (23). We recall the assumption on the coefficients:  $N_1, N_2, L_1$  and  $L_2$  be four integers, two functions  $u_1$  and  $v_2$ , and  $\frac{N_1}{2} \leq N_2 \leq 2N_1$ . We define the subsets:

$$\begin{aligned}
S_1(\xi_1, \xi_2, \mu_1, \mu_2) &:= \{\xi_1 \xi_2 > 0, \mu_1 \mu_2 < 0\} \cup \{\xi_1 \xi_2 < 0, \mu_1 \mu_2 > 0\}, \\
S_2(\xi_1, \xi_2, \mu_1, \mu_2) &:= \{\xi_1 \xi_2 < 0, \mu_1 \mu_2 < 0\},
\end{aligned}$$

and the two Fourier multipliers  $(J_i)_{1 \leq i \leq 2}$ , by:

$$\widehat{J_i(u_1, v_2)}(\tau, \xi, \mu) := \int_{\mathbb{R}^3} \mathbf{1}_{S_i(\xi_1, \xi - \xi_1, \mu_1, \mu - \mu_1)} \widehat{u_1}(\tau_1, \xi_1, \mu_1) \widehat{v_2}(\tau - \tau_1, \xi - \xi_1, \mu - \mu_1) d\tau_1 d\xi_1 d\mu_1.$$

We want to prove that:

$$\|J_i(P_{N_1}Q_{L_1}u_1, P_{N_2}Q_{L_2}v_2)\|_{L^2} \lesssim N_1^{-\frac{1}{2}}(L_1L_2)^{\frac{1}{2}} \|P_{N_1}Q_{L_1}u_1\|_{L^2} \|P_{N_2}Q_{L_2}v_2\|_{L^2}. \quad (29)$$

We consider  $J_1$  first, and explain at the end how  $J_2$  will be different. First, the convolution of the functions brings us to define a set on which the functions are not null:

$$A_{(\xi, \mu, \tau)} := \left\{ \begin{array}{l} (\xi_1, \mu_1, \tau_1) \in \mathbb{R}^3; (\xi_1, \xi - \xi_1, \mu_1, \mu - \mu_1) \in S_1, \\ |(\xi_1, \mu_1)| \simeq N_1, |(\xi - \xi_1, \mu - \mu_1)| \simeq N_2, \\ |\tau_1 - \omega(\xi_1, \mu_1)| \simeq L_1, |\tau - \tau_1 - \omega(\xi - \xi_1, \mu - \mu_1)| \simeq L_2 \end{array} \right\}.$$

We thus get, by Minkowski inequality, and Cauchy-Schwarz inequality:

$$\begin{aligned} & \|J_1(P_{N_1}Q_{L_1}u_1, P_{N_2}Q_{L_2}v_2)\|_{L^2} \\ & \leq \int \left( \int \mathbf{1}_{A_{(\xi, \mu, \tau)}}(\xi_1, \mu_1, \tau_1) P_{N_1}\widehat{Q_{L_1}u_1}(\xi_1, \mu_1, \tau_1)^2 P_{N_2}\widehat{Q_{L_2}v_2}(\xi - \xi_1, \mu - \mu_1, \tau - \tau_1)^2 d\xi d\mu d\tau \right)^{\frac{1}{2}} d\xi_1 d\mu_1 d\tau_1 \\ & \leq \|P_{N_1}Q_{L_1}u_1\|_{L^2} \left( \int \int \mathbf{1}_{A_{(\xi, \mu, \tau)}}(\xi_1, \mu_1, \tau_1) |P_{N_2}Q_{L_2}v_2(\xi - \xi_1, \mu - \mu_1, \tau - \tau_1)|^2 d\xi d\mu d\tau d\xi_1 d\mu_1 d\tau_1 \right)^{\frac{1}{2}} \\ & \leq \sup_{\xi, \mu, \tau} |A_{(\xi, \mu, \tau)}|^{\frac{1}{2}} \|P_{N_1}Q_{L_1}u_1\|_{L^2} \|P_{N_2}Q_{L_2}v_2\|_{L^2}. \end{aligned}$$

To prove (29), it suffices to bound:

$$\sup_{\xi, \mu, \tau} |A_{(\xi, \mu, \tau)}| \lesssim \frac{L_1^{\frac{1}{2}} L_2^{\frac{1}{2}}}{N_1^{\frac{1}{2}}}. \quad (30)$$

To do so, define the resonance function  $\mathcal{H}(\xi_1, \xi_2, \mu_1, \mu_2) := \omega(\xi_1 + \xi_2, \mu_1 + \mu_2) - \omega(\xi_1, \mu_1) - \omega(\xi_2, \mu_2)$ . On the set  $A_{(\xi, \mu, \tau)}$ , it writes:

$$\mathcal{H}(\xi_1, \xi - \xi_1, \mu, \mu - \mu_1) = \omega(\xi, \mu) - \omega(\xi_1, \mu_1) - \omega(\xi - \xi_1, \mu - \mu_1).$$

We thus get a bound on the measure of  $A_{(\xi, \mu, \tau)}$ :

$$|A_{(\xi, \mu, \tau)}| \lesssim \min(L_1, L_2) |B_{(\xi, \mu, \tau)}|, \quad (31)$$

with

$$B_{(\xi, \mu, \tau)} := \left\{ \begin{array}{l} (\xi_1, \mu_1) \in \mathbb{R}^2; (\xi_1, \xi - \xi_1, \mu_1, \mu - \mu_1) \in S_1, \\ |(\xi_1, \mu_1)| \simeq N_1, |(\xi - \xi_1, \mu - \mu_1)| \simeq N_2, \\ |\tau - \omega(\xi, \mu) + \mathcal{H}(\xi_1, \xi - \xi_1, \mu_1, \mu - \mu_1)| \lesssim \max(L_1, L_2) \end{array} \right\}.$$

To bound the set  $B_{(\xi, \mu, \tau)}$ , we define first the set of projection on one variable:

$$B_{(\xi, \mu, \tau)}(\mu_1) := \{ \xi_1 \in \mathbb{R}; (\xi_1, \mu_1) \in B_{(\xi, \mu, \tau)} \},$$

study the set by the variation of  $\mathcal{H}$  along  $\mu_1$ , and conclude by a corollary of the intermediate value theorem:

**Lemma 14** ([17, Lemma 3.8]). *Let  $I$  and  $J$  two intervals of  $\mathbb{R}$ , and  $f : I \rightarrow J$  a  $\mathcal{C}^1$  function. Then we get:*

$$|\{x \in I; f(x) \in J\}| \leq \frac{|J|}{\inf_{x \in I} |f'(x)|}. \quad (32)$$

Lets compute the variations of the function of resonance. The derivative of  $\mathcal{H}$  can be bounded from below:

$$\left| \frac{d}{d\mu_1} \mathcal{H}(\xi_1, \xi - \xi_1, \mu_1, \mu - \mu_1) \right| = |2\xi_1\mu_1 - 2(\xi - \xi_1)(\mu - \mu_1)| \gtrsim \max(N_1, N_2)^2 \simeq N_1^2. \quad (33)$$



We thus get a bound on the measure of  $B_{(\xi, \mu, \tau)}$ , by applying (32) to (33):

$$|B_{(\xi, \mu, \tau)}| \leq \int_{|\mu_1| \leq 2N_1, |\mu - \mu_1| \leq 2N_2} |B_{(\xi, \mu, \tau)}(\mu_1)| d\mu_1 \lesssim \max(N_1, N_2) \frac{\max(L_1, L_2)}{N_1^2} \simeq \frac{\max(L_1, L_2)}{N_1}. \quad (34)$$

Gathering the bounds (31) and (34), we conclude the proof (30).

The assumption of working in  $S_1$  was used in (33), where the sign condition was essential.

For  $J_2$ , we work on  $S_2$ : the result is similar, except that we differentiate  $\mathcal{H}$  in the  $\xi_1$  direction:

$$\left| \frac{d}{d\xi_1} \mathcal{H}(\xi_1, \xi - \xi_1, \mu_1, \mu - \mu_1) \right| = |3\xi_1^2 + \mu_1^2 - 3(\xi - \xi_1)^2 - (\mu - \mu_1)|. \quad (35)$$

Without any assumption on the signs of the variables, we need to localize the different frequencies. The result is immediate from the following lemma:

**Lemma 15** ([17, Lemma 4.2]). *Let  $0 < \alpha < 1$ . There exists a continuous function  $f : [0, 1] \rightarrow \mathbb{R}$  satisfying  $f(\alpha) \rightarrow 0$  as  $\alpha \rightarrow 0$ , such that for all  $(\xi_1, \mu_1)$  and  $(\xi_2, \mu_2)$  satisfying:*

$$\xi_1 \xi_2 < 0, \quad \mu_1 \mu_2 < 0, \quad (1 - \alpha)^{\frac{1}{2}} \sqrt{3} |\xi_i| \leq |\mu_i| \leq (1 - \alpha)^{-\frac{1}{2}} \sqrt{3} |\xi_i|,$$

we obtain:

$$|(\xi_1 + \xi_2, \mu_1 + \mu_2)|^2 \leq |(\xi_1, \mu_1)|^2 - |(\xi_2, \mu_2)|^2 + f(\alpha) \max(|(\xi_1, \mu_1)|^2, |(\xi_2, \mu_2)|^2).$$

We recall that  $|(\xi_1, \mu_1)|^2 = 3\xi_1^2 + \mu_1^2 \simeq N_1^2$ .

On  $\mathcal{D}_{1,3}$  and  $S_2$ , the assumptions of the lemma are satisfied, and with  $\alpha > 0$  so small that  $f(\alpha) \leq \frac{1}{10}$ , we deduce:

$$\frac{9}{10} |(\xi, \mu)|^2 \leq |(\xi_1, \mu_1)|^2 - |(\xi - \xi_1, \mu - \mu_1)|^2,$$

and in the area of *high*  $\times$  *high*  $\rightarrow$  *high* interaction, injecting in (35):

$$\left| \frac{d}{d\xi_1} \mathcal{H}(\xi_1, \xi - \xi_1, \mu_1, \mu - \mu_1) \right| \gtrsim N^2 \simeq N_1^2.$$

This estimate is equivalent to (33), and concludes the proof of the bound (30) in  $S_2$ .

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