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Functional representations produced and used by students during their introduction to the concept of derivative: a window on their understanding processes

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This article is about student understanding processes during their introduction to the concept of derivative that were constructed as part of a doctoral dissertation. The observation of these understanding processes is made from the standpoint of representations in the sense of Duval (1993) and Hitt (2006). More specifically, two aspects of the models of the comprehension process are underlined. First, the fact that students can sometimes work well in different registers, but that they can not necessarily do a proper coordination of these representations is put forward. In addition, the particular role played by the verbal register is also discussed.

Keywords: Understanding process, derivatives, representations, teaching experiment.

Introduction

The concept of derivative is the subject of many scientific studies in the field of teaching and learning mathematics. The work of Biza and Zacharides (2010) on the concept of tangent is an example of documented difficulties that students in the calculus course may encounter. Or the work of Sierpinska (1985) around the concept of limit is also a landmark for all those who are interested in the learning and teaching of concepts in differential calculus. Other researchers shed light on factors that may explain these difficulties and have proposed possible solutions to help students overcome these difficulties. For example, Zerr (2010), Haciomerogolu, Aspinwall, and Presmeg (2010), among others, provide insights into the elements that lead to a so-called “conceptual” understanding of the concepts of differential calculus. Among these authors, Zandieh, at the turn of the 2000s, proposes a theoretical framework for the analysis of students’ understanding of the concept of derivatives. But, as Zandieh (2000) mentions, this schema does not comment on how and why students are approaching the concept of derivative in a certain way. In addition, Hähkiöniemi (2006) also proposes a diagram that identifies elements that can allow students to deepen their understanding of the derivative. However, this schema is based on the observation of the students after they have seen the concept in class. The observation of students in action, engaged into a process of understanding the concept of derivative, which is new for them, could allow to add a new dimension to these models.

This article, based on Dufour’s PhD (2018), reports on a study that aims to model the students’ understanding of derivative with regard to the process aspect of this understanding and the implementation of an analytical framework to observe and describe these processes. It is not possible to detail the models obtained in the thesis in this paper. However, this article exemplifies two elements in relation with the representations used by students, that seems for us, key moments in the understanding processes of students of the derivative.

Theoretical Framework

An orientation on the concept of understanding must be taken in order to accurately model one or more understanding processes. Among the studies about differential calculus, some mention the different representations of mathematical concepts as part of a solution or as a possible explanation to the students' difficulties (Eisenberg and Dreyfus, 1991; Biza and Zachariades, 2010). By insisting on this particular aspect of representations, it was possible to detail the model in depth in this sense. Two visions on the representations, one in continuity with the other, are gathered. First, Duval's theory of registers of semiotic representations (1993, 2006) puts forward important cognitive activities related to representations in different semiotic registers. Actions on and with representations, which are essentials, especially in the case where we are interested in introducing a new concept to students, are described by Duval. These actions are *recognition*, *treatment* (to process or manipulate), *production*, *conversion* and *articulation* (coordination). *Recognition* makes it possible to recognize a concept by a given representation. *Processing* makes it possible to modify a representation within the same register in order to obtain a new representation. *Production* produces a representation related to a certain concept. The *conversion* makes it possible, from a given representation, to produce a new representation in a different register from the one of departure. This action is more complex than the others since it requires to recognize the rules of two registers. Finally, the *articulation (coordination)* between representations in different registers is, for Duval, a cognitive activity related to the conceptual apprehension of a mathematical object. Beyond being able to produce a representation in a register or to convert a representation to a new register, it is a question of being able to go back and forth in different registers according to what is recognized as necessary (Duval, 2006) to carry out a mathematical activity.

The representations described by Duval are part of different semiotic registers which are described by a set of rules of conformity (Duval, 1993), and which are the subject of a certain consensus in the mathematical community. By this rather strict definition, we place Duval's representations in the category of institutional representations. However, as the objective of the research presented here is to learn about the "process" aspect of student understanding, a vision of representations that particularly considers intuitive representations of students is necessary. The concept of functional representations of Hitt (2003) describes intuitive representations in construction. Functional representations make it possible to associate representations produced or manipulated by students to a certain register, although the latter do not completely respect the rules established in this register.

Thus, along with the theoretical framework, an analytical framework is drawn up to describe the students' processes of understanding when they are introduced to a new mathematical concept into the classroom. Indeed, we can now describe these processes by the actions taken by the students on and with representations belonging to different registers of representations and having a certain nature: institutional or functional. On the other hand, it is advisable to specify the different registers likely to be encountered during the observation of the mathematical activity of the students. Due to the nature of the targeted concept, the derivative, which belongs to the mathematical domain of analysis, the registers of representations are: graphic, tabular, verbal (whether written or oral), algebraic and numerical.

Methodology

Two key elements were taken into account in the choice of a methodology for this study. First, the observation of students understanding processes will be in the form of constructing one or more models of these processes. The chosen methodology must therefore be consistent with this central objective. In addition, the goal also installs the research in a particular context of teaching sessions designed to encourage the use of different representations. The Teaching Experiment (TE) was ideal for this project. Indeed, this methodology aims to document, through the production of a model, the mathematical development of students by observing, among other things, their learning process and their conceptions in a teaching context (Steffe and Thompson, 2000).

The theoretical position on understanding of this research, in particular Hitt's (2003) perspective suggesting these student's representations evolve through interactions with the teacher and other students, sets a particular context for the TE. It is therefore inspired by the position of Cobb (2000), among others, that the TE took its shape in this project. Cobb supports the need to conduct a TE in the classroom through the individual and social aspects of learning, which is consistent with the position taken in this research. However, a completely natural classroom context would have made the fine observation of the different representations used difficult. A "hybrid" form that lies between the individual interview outside the classroom and the natural classroom context is used. A TE with a small group of six students, which allows for individual, team or large groups work with the teacher-researcher and spans over five sessions, has been put in place.

The teaching sessions were videotaped and a journal was written before and after each session. The videotapes were translated into transcripts that were analysed with specific regard on the representations produced and used by students. The analysis of the transcripts and students' productions took place in four steps, or iterations, inspired by Powell, Francisco and Maher (2003)' analysis model. Table 1 resumes the different layers of analysis.

Layer of analysis	Description
First	<ul style="list-style-type: none"> • Identify key moments in the student's understanding process in order to use these moments as a basis for building the other sessions. • Record the privileged or used registers of representations in the session.
Second	From the transcripts (coding) : <ul style="list-style-type: none"> • Identify the different representations used, produced or processed by the students. • Identify some actions on these representations by the students. • Add comments on the transcripts.
Third	From the coded and commented transcripts: <ul style="list-style-type: none"> • Write, as a story, the development of the session by dividing it into key moments. • For each of these moments, identify and interpret, from the perspective of the theoretical framework, the representations used or produced by the students and the actions taken on these representations. • Support these interpretations with excerpts from transcripts of the sessions or figures from students' productions.
Fourth	From the story written before: <ul style="list-style-type: none"> • Identify the moments that are directly related to the concept of derivative. • Highlight from these moments the elements that can be part of the description of the process of understanding students. • Support this analysis with excerpts of transcripts of the sessions or figures from students' productions.

Table 1: Layers of analysis

Results and Discussion

The TE put in place and the many layers of analysis allowed us to build two collective models (for two teams of three students) of understanding processes of the concept of derivative during its introduction to students around 18 years old. Since the models take the form of a follow-up text revealing the different representations produced and used by the students and especially the way in which they have used and modified them during the different interactions between them and with the teacher-researcher, it is impossible to report the whole of them in this article. We will instead offer some excerpts that seem particularly rich to better understand the understanding processes of students with a particular focus on the necessity of a coordination between different registers. The first excerpt illustrates the ability of students to produce and process representations of a concept in different registers without being able to demonstrate a real coordination between the registers. The second emphasizes the particular role of the verbal register in the understanding processes of students.

Production of Representations in Different Registers Without Achieving Coordination

First, during the fourth session, a problem of bacterial proliferation was proposed to students through a scenario in words and a table of values. The question posed in this problem was to identify when an antibiotic administered to a patient allows the reduction of the bacterial population present in the patient. We are particularly interested here in the work of Guillaume, Jérémie and Antoine's team (see figure 1 showing Jeremie's work).

Temps écoulé en heures depuis la prise de l'antibiotique (t)	0	2	4	10	12	14
Nombre de bactéries dans la population (b(t))	1 000 000	1 025 200	1 044 800	1 067 200	1 067 200	1 058 800
		$\frac{12600}{2}$	$\frac{19000}{2}$	$\frac{22400}{6}$	$\frac{2000}{2}$	$\frac{4700}{2}$ (8400)

Figure 1: Treatment of the Table of Values and Numerical Representations (Dufour, 2018, p. 152)

The three students were able to produce, from a few treatments on the table of values, verbal representations of the situation that are related to the concept of derivative (see Table 2, translated from Dufour, 2018). Table 2 shows the representations and actions made by the three students during their work on this problem. These representations and actions are also interpreted in this table.

Although these functional representations are incomplete or even erroneous, the students associated them with the concept of derivative. They could thus had used the recognition of this concept (derivative) to produce algebraic representations implying the concept of derivative at a point which would be null and which could had allowed them to solve the problem. Unfortunately, this conversion did not take place and the students finally found another way to answer the question, but this solution is not what interests us here.

Type of representations and actions on these representations	Student	Interpretations
<p>Conversion from TR → NR (Tabular representation) → Numerical Representation</p> <p>From a table of value, calculations of different average rates of change. (see figure 1)</p>	Jérémie	Jérémie makes a good use of the data in the given table to calculate the rate of change on different intervals. However, he doesn't go further on the interpretation of what he could do with this new information.
<p>Conversion NR → VR (verbal representation)</p> <p>Because I asked, he produces a VR for his calculations (average rate of change) which is “means” (VR).</p>	Jérémie	<p>Although this VR is <u>incomplete</u>, Jeremy is not wrong. Indeed, his various calculations can be associated with the concept of mean in the sense that he obtains a number of bacteria produced/dead for each hour in this interval, that is to say, a number of bacteria produced each hour if the same number of bacteria was produced every hour over this interval.</p> <p>This is an incomplete VR in the sense that it does not identify NRs as “rates of change” (VR), which is an important conversion for the rest of the problem and especially for the process of understanding the derivative.</p>
<p>Conversion NR + VR → VR + VR</p> <p>Conversion from the average rate of change calculated (NR) and the verbal representation “mean” (VR) to the verbal representations “slope” (VR) and “variation” (VR).</p>	Guillaume	Guillaume uses the representation “slope” to talk about what Jérémie calculated (average rate of change). These two VRs can indeed designate the same concept. It can be emphasized that the term “slope” is more often used with reference to the graphic register which is not necessarily the case here, although some links with the graphical register are formed later.
<p>Conversion NR + VRs → VR</p> <p>Conversion from the average rate of change calculated (NR) and the verbal representations “mean” (VR), “slope” (VR) and “variation” (VR) to the verbal representation “derivative” (VR)</p>	Antoine	Antoine continues the discussion by introducing the term “derivative”. It is true that what Jérémie calculates (average rate of change) is not very far from the concept of derivative (instantaneous rate of change). Recall that what distinguishes these two concepts is that the average rate of change is related to a given interval or secant line to the function involved. Whereas the instantaneous rate of change (derivative at a point) is related to the rate of change for a value of the independent variable in particular or the rate of change of a tangent line at a point of a function. Therefore, the use of the RV “derivative” is erroneous in this case.
<p>Conversion NR + VR → VR + VR</p> <p>Conversion from the average rate of change calculated (NR) and the verbal representation “derivative” to two verbal representations: “The derivative between 12 and 14” (VR) and “the tangent between 12 and 14” (VR)</p>	Antoine	Antoine goes further by producing the VR “the derivative between 12 and 14” and the VR “the tangent between 12 and 14”. For the same reasons as those raised above, these two VRs are erroneous. Indeed, a derivative or a tangent line can't be associated with an interval of this kind. In this case, Antoine should have used the representations “average rate of change” or “secant line between 12 and 14”, for example, so that the different representations in this discussion are coordinated coherently.

Table 2: Representations Produced by Students and Our Interpretations (Dufour, 2018, p. 157)

Later, the teacher-researcher directly suggested to the team to use the concept of derivative to solve this problem. At this point, the three students produced algebraic representations of the derivative at a point such that the derivative at this point would be zero (see Figure 2 for an example). They even managed to manipulate these representations to find that famous moment when one can observe a change of growth of the function.

$$\lim_{h \rightarrow 0} \frac{f(t+h) - f(t)}{h}$$

$$\frac{-700(t+h)^2 + 14000(t+h) + 10^6 - (-700t^2 + 14000t + 10^6)}{h}$$

$$\frac{-700(t^2 + 2th + h^2) + 14000t + 14000h + 10^6 + 700^2 - 14000t - 10^6}{h}$$

$$\frac{-700t^2 - 1400th + 700h^2 + 14000t + 14000h + 700^2 - 14000t - 10^6}{h}$$

$$\lim_{h \rightarrow 0} \frac{-1400th - 700h^2 + 14000h}{h}$$

$$\lim_{h \rightarrow 0} (-1400t - 700h + 14000) = 0$$

$a = 14000$
 $b =$

$Y = ax + b$

$Y = 14000x + b$ $- 14000(14000)$

$f(t) = 14000t + 14000 \Rightarrow 0$ 14

Figure 2: Treatment of an algebraic representation of the derivative of a function by Guillaume (Dufour, 2018, p. 166)

What is interesting to observe here is the completely parallel use of the different registers. Indeed, their work in “closed vases” did not allow them to coordinate coherently the different representations they can produce and thus achieve a better understanding of the concept of derivative. However, their use of the derivative in the algebraic register to solve the problem might suggested that once the concept was identified, which they had done verbally before (see Table 2), they could had completed the problem by processing an algebraic representation. Although their functional verbal representations were not completely adequate, the students identified them as the derivative. They could had allowed them to move to an algebraic representation as they did later at the request of the teacher.

The Particular Role of the Verbal Register in the Coordination of Different Representations

According to the position on the understanding adopted in this article, to understand, one must coordinate different representations of various registers. However, beyond this necessity, the models constructed make it possible to identify the verbal register as being a central element of this coordination. In fact, the register is at least a mirror that can reflect the understanding of students. Indeed, it was often when the students were brought to produce such type of representations that we can observed if the coordination between the representations was coherent or not. The representations in this register sometimes revealed to students themselves that they cannot articulated representations in different registers.

As an example, some of the functional verbal representations showed in table 1 are not coherent with the concept of derivation such as “the derivative **between** [an interval]”. Other functional verbal representations produced by the same student later such as “the slope of a derivative is a tangent” and “the derivative **line** is the **slope** of a tangent” are also examples of incoherent

representations of the derivative. This kind of functional verbal representations let us realized that, even if some representations were coherent and complete in their own register (algebraic for example), the concept of derivative still provoked a confusion to the observed students. Even more important, when the students were asked, by the teacher-researcher or by their colleagues to explain what they were doing in a certain register (often algebraic), they, themselves, realized that they often cannot explained or justified the representations they produced or used. This was, then, a key moment for them in their understanding process.

Another example appeared when the students had to identify in different registers the concept of rate of change to be able to go through their idea of using derivative to solve a problem. Some examples of this difficulty were observed during the TE sessions. One of them was when we proposed a problem from Selden, Mason and Selden (1989) (see figure 3).

Find values of a and b so that the line $2x+3y=a$ is tangent to the graph of $f(x) = bx^2$ at the point where $x=3$.

Figure 3: Problem proposed to the students during the fifth session

The students, in both team of three participants, were able to identify in the verbal register the need to equal the rate of change of the given linear function and the derivative of the given function f for $x=3$. However, they were not able to articulate the verbal representation to a coherent algebraic representation. It needed some interventions from the teacher-researcher to finally be able to coordinate these representations. In this case too, the verbal representation was important for two reasons. First, it sheds light on the fact that the students were able to recognize the usefulness of the derivative. Second, it helped to determine that the problem was not directly the concept of derivative, in that case, but that it was the concept of rate of change. This coordination of different representations of the concept of rate of change seemed to be confusing for students. We must precise that if the concept of derivative is new for them, they work with the concept of rate of change since at least three school years. However, it seemed that this concept in particular still is of high importance in the understanding process of the derivative.

Conclusion

The few examples given above are only some of the important elements that the construction of models of understanding processes of the concept of derivative made it possible to underline. The use of representations in different registers in a rather parallel way than in a coordinated way had been observed several times during the construction of the understanding processes models. Certainly, this step, working in close vases, may be inevitable and part of a deep understanding process of the concept. However, it must certainly be considered in teaching by attempting to provoke, not only representations in a variety of registers, but treating them in a coordinated manner. It's about emphasizing how these various registers are related to each other. The verbal register seems appropriate to allow such disclosure of the links between the registers.

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