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# Dominating sets reconfiguration under token sliding 

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#### Abstract

Let $G$ be a graph and $D_{s}$ and $D_{\mathrm{t}}$ be two dominating sets of $G$ of size $k$. Does there exist a sequence $\left\langle D_{0}=D_{s}, D_{1}, \ldots, D_{\ell-1}, D_{\ell}=D_{\mathrm{t}}\right\rangle$ of dominating sets of $G$ such that $D_{i+1}$ can be obtained from $D_{i}$ by replacing one vertex with one of its neighbors? In this paper, we investigate the complexity of this decision problem. We first prove that this problem is PSPACE-complete, even when restricted to split, bipartite or bounded tree-width graphs. On the other hand, we prove that it can be solved in polynomial time on dually chordal graphs (a superclass of both trees and interval graphs) or cographs.


## 1 Introduction

General introduction. Reconfiguration problems arise when, given an instance of a problem, we want to find a step-by-step transformation (called reconfiguration sequence) between two feasible solutions such that all intermediate solutions are also feasible. Unfortunately, such a transformation does not always exist and some solutions may even be frozen ie. they can not be modified at all. In this context, two natural questions arise: (i) when can we ensure that there exists such a transformation? (ii) What is the complexity of finding such a transformation?

Interest in combinatorial reconfiguration steadily increased during the last decade. Reconfiguration of several problems, including Coloring [2, 7, 9], Independent Set [3, 4, 19], Dominating Set [13, 20, 22, 25] or Satisfiability [11, 21] have been studied. For an overview of recent results on reconfiguration problems, the reader is referred to the surveys of van den Heuvel [26] and Nishimura [23]. In this article, we focus on the reconfiguration of dominating sets.

A dominating set is a set of vertices such that every vertex not in the set has a neighbour in it. Visualize a dominating set as materialized by tokens placed on the vertices that are part of it. Then, modifying a dominating set corresponds to shifting the tokens according to some rule, called reconfiguration rule. In the literature, three kinds of operations have been mainly studied:

1. Token Addition and Removal (TAR): one can add or remove a token as long as the total number of tokens does not go beyond a given threshold;
2. Token Jumping (TJ): one can move a token to any vertex of the graph;
3. Token Sliding (TS): one can slide a token along an edge i.e. one moves a token to a neighbour of its current location.

One can observe that in the last two models, the size of each solution remains constant at any time, as opposed to what happens in the TAR model. In this article, we are mostly interested in the Token Sliding model.

We define the reconfiguration graph for domination denoted $\mathcal{R}_{G}(k)$ as follows: the vertices of $\mathcal{R}_{G}(k)$ are the dominating sets of size $k$ and there is an edge between two vertices if and only if one can go from the first to the second thanks to the considered reconfiguration rule (token sliding in our case). Three natural problems can be identified:

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Figure 1: Our results: the frontier between PSPACE-completeness and tractability.

1. The reachability problem: given a graph $G$ and two dominating sets $D_{s}$ and $D_{\mathrm{t}}$ is there a path between $D_{s}$ and $D_{\mathrm{t}}$ in $\mathcal{R}_{G}(k)$ ? In other words, does there exist a reconfiguration sequence between $D_{s}$ and $D_{\mathrm{t}}$ ?
2. The connectivity problem: given a graph $G$, is the reconfiguration graph $\mathcal{R}_{G}(k)$ connected?
3. The shortest path problem: given a graph $G$, two dominating sets $D_{s}$ and $D_{\mathrm{t}}$ and an integer $\ell$, is the distance in $\mathcal{R}_{G}(k)$ between $D_{s}$ and $D_{\mathrm{t}}$ at most $\ell$ ? In other words, does there exist a reconfiguration sequence between $D_{s}$ and $D_{\mathrm{t}}$ of length at most $\ell$ ?

In this article, we focus on the reachability version and we will use the notation $\operatorname{DSR}_{\mathrm{TS}}$ for this problem in order to lighten notations. We adopt the same notation for Vertex Cover Reconfiguration and INDEPENDENT SET RECONFIGURATION (the reachability question under the token sliding rule) and denote these two problems by $\mathrm{VCR}_{\mathrm{TS}}$ and $\mathrm{ISR}_{\mathrm{TS}}$, respectively.

Related results. The reconfiguration of dominating sets has been mainly studied under the Token Addition and Removal model. Haas and Seyffarth gave sufficient conditions to guarantee the connectivity of the reconfiguration graph according to $k$, the cardinality threshold of dominating sets [12]. More precisely, they proved that $\mathcal{R}_{G}(n-1)$ is connected if $G$ has at least two independent edges. This value can be lowered to $\Gamma+1$ (where $\Gamma$ is the maximum size of a minimal dominating set) if the input graph $G$ is chordal or bipartite. Suzuki et al. showed that this result cannot be generalized to any graph since they constructed an infinite family of graphs for which $\mathcal{R}_{G}(\Gamma+1)$ is not connected [25]. On the positive side, they proved that $\mathcal{R}_{G}(n-\mu)$ is connected if $G$ has a matching of size $\mu+1$.

Haddadan et al. studied the complexity of the reconfiguration of dominating sets under the token addition and removal rule from a graph classes perspective [13]. They proved that the reachability problem is PSPACE-complete, even if the input graph is a split graph or a bipartite graph. On the other hand, they gave linear-time algorithms for trees or interval graphs for instance.

Mouawad et al. studied the parameterized complexity of Dominating Set Reconfiguration under token addition and removal. They proved that this problem is W[2]-hard when parameterized by $k+\ell$, where $k$ is the threshold and $\ell$ the length of the reconfiguration sequence. As a positive result, Lokshtanov et al. gave a fixed-parameter algorithm with respect to $k$ for graphs excluding $K_{d, d}$ as a subgraph, for any constant $d$ [20].

The third author also considered this problem (still in the TAR model) through the lense of an optimization variant (see Blanché et al. [1]) as recently introduced by Ito et al. for independent sets [16].

Our contribution. In this article, we are interested in the reachability question of dominating sets reconfiguration under token sliding. This reconfiguration rule has already been studied for various reconfiguration problems but not for dominating sets, to the best of the authors' knowledge.

We tackle this problem with a complexity perspective according to several graph classes: in Section 3, we prove for instance that $\mathrm{DSR}_{\mathrm{TS}}$ is PSPACE-complete for split graphs or bipartite graphs. In Section 4, we show that this problem can be solved in polynomial time on other graph classes such as cographs or dually chordal graphs. Figure 1 gives an overview of our results where $A \rightarrow B$ means that the class $B$ is properly included in the class $A$.

## 2 Preliminaries

This section is devoted to some basic definitions of graph theory used in this article. We then introduce more formally the problem we are interested in.

Each graph $G=(V, E)$ considered is simple (i.e. $G$ is undirected and has no multiple edges or loops) where $V$ represents the vertex set of $G$ and $E$ its edge set. We denote by $n=|V|$ and $m=|E|$ the number of vertices and edges of $G$. The eccentricity of a vertex $u$ denoted by $\epsilon(u)$ is the maximum distance between $u$ and any other vertex. For a subset of vertices $S \subseteq V$, we denote by $G[S]$ the subgraph induced by $S$. For a vertex $u \in V$, we denote by $N_{G}(u)$ its open neighborhood i.e. the set $\{v \mid u v \in E\}$ and by $N_{G}[u]$ its closed neighborhood i.e. the set $N_{G}(u) \cup\{u\}$. For a subset of vertices $S \subseteq V$, we define the closed neighborhood of $S$ as the union of the closed neighborhood of the vertices in $S$ i.e. $N_{G}[S]=\bigcup_{u \in S} N_{G}[u]$.

Let $G_{1}$ and $G_{2}$ be two graphs. We recall two basic binary operations on graphs: the disjoint union and the join operation. The disjoint union $G_{1} \cup G_{2}$ of two graphs on disjoint vertex sets is the graph with vertex set $V\left(G_{1} \cup G_{2}\right)=V\left(G_{1}\right) \cup V\left(G_{2}\right)$ and edge set $E\left(G_{1} \cup G_{2}\right)=E\left(G_{1}\right) \cup E\left(G_{2}\right)$. The join operation can be obtained from the disjoint union by adding all possible edges between $G_{1}$ and $G_{2}$. More formally, the join of $G_{1}$ and $G_{2}$ denoted by $G_{1}+G_{2}$ is the following graph:

- $V\left(G_{1}+G_{2}\right)=V\left(G_{1}\right) \cup V\left(G_{2}\right)$;
- $E\left(G_{1}+G_{2}\right)=E\left(G_{1}\right) \cup E\left(G_{2}\right) \cup\left\{u v \mid u \in V\left(G_{1}\right), v \in V\left(G_{2}\right)\right\}$.

A dominating set for a graph $G=(V, E)$ is a subset of vertices $D \subseteq V$ such that $N[D]=V$ i.e. each vertex either belongs to $D$ or has a neighbor in $D$. For a graph $G$, we denote by $\gamma(G)$ the domination number of $G$ defined as the minimum size of a dominating set. Let $G$ be a graph and $D$ a dominating set of $G$. We say that $u$ is a private neighbor of $v$ (with respect to $D$ ) if $u \notin D$ and $v$ is the only neighbor of $u$ in $D$. Therefore, a dominating set is minimal for inclusion if and only if each of its vertices has a private neighbor.

Our problem. In the token sliding model, a natural question is whether we should allow to place more than one token on a vertex during the reconfiguration sequence. Here is an example where it makes a difference: consider the star graph $S_{n}$ on $n+1$ vertices and two dominating sets $D_{1}$ and $D_{2}$ of $S_{n}$ of size $k$, with $k \in[2, n-1]$. Any dominating set of that size necessarily contains the central vertex. To reconfigure $D_{1}$ into $D_{2}$, we are forced to move a token from one leaf to another, which can only be done by going through the central vertex which already contains a token. Based on such examples, it seems reasonable to allow the superposition of tokens on a vertex, which is the choice we make in the following.

Let $G$ be a graph, $D_{s}$ and $D_{\mathrm{t}}$ be two dominating sets of $G$ of same size $k$. We say that $D_{s}$ is reconfigurable into $D_{\mathrm{t}}$ by token sliding if there exists a sequence $S=\left\langle D_{0}=D_{s}, D_{1}, \ldots, D_{\ell-1}, D_{\ell}=D_{\mathrm{t}}\right\rangle$ that respects the two following properties:

- each $D_{i}$ is a multiset of size $k$ that is a dominating set of $G$;
- there exists an edge $u v$ such that $D_{i+1}=D_{i} \backslash\{u\} \cup\{v\}$ i.e. we slide the token placed on the vertex $u$ along the edge $u v$.

We call such a sequence a TS-sequence and we denote this property by $D_{s} \xrightarrow{\text { TS }} D_{\mathrm{t}}$. We also say that $\left(G, D_{s}, D_{\mathrm{t}}\right)$ is a yes-instance for the $\mathrm{DSR}_{\mathrm{TS}}$ problem.

We also introduce the two following notations, recall that $D_{s}$ and $D_{\mathrm{t}}$ are two dominating sets of $G$ of size $k$.


Figure 2: Example of TS-sequence from $D_{s}$ to $D_{\mathrm{t}}$.

- $D_{s} \stackrel{\text { TAR }}{\nVdash} D_{\mathrm{t}}$ : one can reconfigure $D_{s}$ into $D_{\mathrm{t}}$ under the TAR model; each intermediate solution is of size at most $k+1$;
- $D_{s} \stackrel{\mathrm{TJ}}{\leadsto} D_{\mathrm{t}}$ : one can reconfigure $D_{s}$ into $D_{\mathrm{t}}$ under the TJ model; each intermediate solution is of size exactly $k$.

A useful observation is that each reconfiguration sequence (and thus in particular a TS-sequence) is reversible: if $D_{s} \rightsquigarrow D_{\mathrm{t}}$ holds, $D_{\mathrm{t}} \rightsquigarrow D_{s}$ holds too. We thus denote this relation by $D_{s} \leadsto \nrightarrow D_{\mathrm{t}}$. Figure 2 gives an example of a TS-sequence.

We are now ready to define properly the Dominating Set Reconfiguration problem under token sliding.

## $\mathrm{DSR}_{\mathrm{T}}$

Instance: a graph $G=(V, E)$ and two dominating sets $D_{s}$ and $D_{t}$ of cardinality $k$ of $G$.
Question: Is there a TS-sequence between $D_{s}$ and $D_{t}$, i.e. does $D_{s} \stackrel{\text { TS }}{\leadsto} D_{t}$ ?

We end this section by the following observation, showing that being reconfigurable is not a monotonous property.

Theorem 1. For every $k$ and $\ell$ with $2 \leq k<\ell$, there exist graphs where it is possible to reconfigure dominating sets of size $k$ but not of size $\ell$.

Proof. We first prove the statement for $k=2$. For every integer $\ell>2$, we define the graph $G_{\ell}$ such that $G$ contains exactly one dominating set of size $\gamma(G)=2$ but for which the dominating sets of size $\ell$ are not reconfigurable. To construct $G_{\ell}$, we first create $\ell$ pairs of triangles $\left\{\left(G_{1}^{i}, G_{2}^{i}\right), \ldots,\left(G_{1}^{\ell}, G_{2}^{\ell}\right)\right\}$ such that $G_{1}^{i}$ and $G_{2}^{i}$ share exactly one vertex $w_{i}$. Moreover, let all the $G_{1}^{i}$ share a vertex $u$ and all the $G_{2}^{i}$ share a vertex $v$ (see Figure 3 for $G_{3}$ as an example). We have $\gamma\left(G_{\ell}\right)=2$ since $\{u, v\}$ is a dominating set and $G_{\ell}$ does not contain a universal vertex (i.e. a vertex adjacent to all the other vertices). Besides, any dominating set of $G_{\ell}$ on less than $\ell$ vertices contains both $u$ and $v$. Indeed, if for instance $u$ is not in the dominating set, then $\ell$ extra vertices are necessary to dominate the triangles $G_{1}^{i}$.

Now, consider the dominating set $D_{s}=\left\{w_{1}, \ldots, w_{\ell}\right\}$ of size $\ell$ of $G_{\ell}$. By token sliding, $D_{s}$ can not be reconfigured into any other dominating set of size $\ell$. Indeed, in $D_{s}$ we can not move any $w_{i}$ in a triangle because it would leave the other triangle of the pair $\left(G_{1}^{i}, G_{2}^{i}\right)$ undominated. This settles the case $k=2$.

Now, for larger $k$, we attach to the vertex $u$ of the graph $G_{\ell-k+2}$ a path on $3(k-2)$ vertices. Dominating sets of size $k$ of this new graph are not unique anymore, but still they all contain both vertices $u$ and $v$. Now, $D_{s}$ together with all vertices of the path at distance $2 \bmod 3$ from $u$ is also a dominating set of size $\ell$ that is not reconfigurable.


Figure 3: Graph $G_{3}$.

## 3 PSPACE-completeness

In this section, we study the complexity of $\mathrm{DSR}_{\mathrm{TS}}$ in the general case. We show that this problem is PSPACE-complete, even when restricted to split graphs, bipartite graphs or bounded treewidth graphs. Let us first recall the following result from Haddadan et al., stating the complexity of the reconfiguration problem for the TAR model.

Theorem 2 ([13]). Let $G$ be a graph and $D_{s}, D_{t}$ be two dominating sets of $G$ of size $k$. Deciding whether $D_{s} \stackrel{T A R}{\sharp} D_{t}$ is PSPACE-complete.

Lemma 3. Let $G$ be a graph and $D_{s}$ and $D_{t}$ be two dominating sets of $G$ of size $k$. We have $D_{s} \stackrel{T A R}{\leadsto} D_{t}$ if and only if $D_{s} \stackrel{T J}{\longrightarrow} D_{t}$.

Proof. The proof is an adaptation of the Theorem 1 of Kamiński et al. [17]. Suppose first $D_{s} \stackrel{\text { TJ }}{\leadsto} D_{\mathrm{t}}$, let $S$ be a TJ-sequence that reconfigures $D_{s}$ into $D_{\mathrm{t}}$. This sequence corresponds to a sequence of moves $u \rightsquigarrow v$. We construct a TAR-sequence by replacing each atomic move $u \rightsquigarrow v$ by two moves of the TAR model: we first add $v$ and then delete $u$. By first adding $v$, we preserve the domination property. Besides, since we immediately delete $u$ after the addition of $v$, each intermediate solution is of size at most $k+1$, as desired.

For the other direction, let $S^{\prime}$ be a TAR-sequence that reconfigures $D_{s}$ into $D_{\mathrm{t}}$. Note that since $\left|D_{s}\right|=\left|D_{\mathrm{t}}\right|=k, S^{\prime}$ is of even length. Moreover, by hypothesis, $S^{\prime}$ does not contain a configuration of size more than $k+1$. If all the configurations of $S^{\prime}$ are of size $k$ or $k+1$, this means that $S^{\prime}$ corresponds to an alternation of an addition of a token on some vertex $v$ immediately followed by the deletion of a token on a vertex $u$. Therefore, to get a TJ-sequence, we simply replace each such subsequence by a move $u \rightsquigarrow v$. Suppose now that $S^{\prime}$ contains some configuration of size less than $k$ and consider a configuration, let us say $D_{i}$, of smallest size. Since $D_{i}$ is a configuration of smallest size, this means that it has been obtained from $D_{i-1}$ by the deletion of some vertex $x$. We also get that the configuration $D_{i+1}$ is obtained from $D_{i}$ by the addition of some vertex $y$. If $x=y$, then these two steps are redundant and can simply be ignored. Otherwise, observe that if we first add $y$ and then delete $x$, the new sequence is still valid. If all the configurations are of size $k$ or $k+1$, we immediately get at TJ-sequence. Otherwise, we can repeat this process until this is the case.

As a corollary of Theorem 2 and Lemma 3, we get that deciding whether two dominating sets of size $k$ of a graph $G$ can be reconfigured under the token jumping model is a PSPACE-complete problem. We are now ready to prove Theorem 4.

Theorem 4. $D S R_{T S}$ is PSPACE-complete on split graphs.
Proof. First, note that the problem is in PSPACE [15]. Let $G=(V, E)$ be a graph with $V(G)=\left\{v_{1}, \ldots, v_{n}\right\}$. We construct the corresponding split graph $G^{\prime}$ as follows:

- $V\left(G^{\prime}\right)=V_{1} \cup V_{2}$ where $V_{1}=\left\{v_{1}, \ldots, v_{n}\right\}$ and $V_{2}=\left\{w_{1}, \ldots, w_{n}\right\} ;$


Figure 4: Example for the reduction of Theorem 4.

- $E\left(G^{\prime}\right)=\left\{u v \mid u, v \in V_{1}\right\} \cup\left\{v_{i} w_{j} \mid v_{j} \in N_{G}\left[v_{i}\right]\right\}$ i.e. we add all possible edges in $V_{1}$ so that $V_{1}$ forms a clique. We also add an edge between a vertex $v_{i} \in V_{1}$ and a vertex $w_{j} \in V_{2}$ if and only if the corresponding vertex $v_{j}$ in the original graph $G$ belongs to the closed neighborhood of $v_{i}$ in $G$.

Observe that $G^{\prime}$ is a split graph since $V_{1}$ forms a clique and $V_{2}$ an independent set (see Figure 4 for an example). To a set of vertices of $G$, we associate the corresponding vertices of $V_{1}$ in $G^{\prime}$. By definition of $G^{\prime}$, any dominating set $D$ of $G$ is also a dominating set for $G^{\prime}$ : indeed, a vertex $v_{i} \in V_{1}$ dominates all the vertices in $V_{1}$ (since it is a clique) and all the vertices in $V_{2}$ that correspond to vertices in its closed neighborhood in $G$. That $D$ dominates $G$ allows to conclude that the corresponding set also dominates $V_{2}$.

Let $\left(G, D_{s}, D_{\mathrm{t}}\right)$ be an instance of $\mathrm{DSR}_{\mathrm{TJ}}$, we reduce this instance to the instance of $\mathrm{DSR}_{\mathrm{TS}}\left(G^{\prime}, D_{s}, D_{\mathrm{t}}\right)$. This reduction can be done in linear time. Now, we need to prove that $D_{s} \stackrel{\text { TS }}{\rightarrow} D_{\mathrm{t}}$ if and only if there is a reconfiguration sequence between $D_{s}$ and $D_{\mathrm{t}}$ in $G^{\prime}$ using the token jumping model. Consider a TJ-sequence in $G$, and transpose it to $G^{\prime}$. All intermediate sets still are dominating sets, and since all pairs of vertices are joined by an edge in $V_{1}$, this sequence is a valid TS-sequence in $G^{\prime}$.

We now prove the other direction. Let $\left\langle D_{0}=D_{s}, \ldots, D_{p}=D_{\mathrm{t}}\right\rangle$ be a TS-sequence in $G^{\prime}$. If the sequence does not use vertices in $V_{2}$, we immediately get a TJ-sequence in $G$ from $D_{s}$ to $D_{\mathrm{t}}$, as the token jumping model does not require adjacency. Suppose on the other hand that the sequence goes through some vertices in $V_{2}$. Since all vertices are initially in $V_{1}$, there is a subsequence that consists in a move $v_{i} \stackrel{\text { TS }}{\rightsquigarrow} w_{j}$. Since $w_{j} \notin V_{1}$, there exists a later step where the token on $w_{j}$ is moved to an adjacent vertex $v_{k}$ in $V_{1}$ (since $V_{2}$ is independant). However, $w_{j}$ does not dominate any vertex in $V_{2}$ (since $V_{2}$ is a stable set) and thus $N\left[w_{j}\right] \subseteq N\left[v_{k}\right]$. Therefore, we simply replace these two moves by a single move $v_{i} \rightsquigarrow v_{k}$. We can thus assume that the reconfiguration sequence only uses vertices in $V_{1}$ and conclude.

Next, we prove that $\mathrm{DSR}_{\mathrm{TS}}$ is PSPACE-complete on bipartite graph. We use a reduction from the Vertex Cover Reconfiguration by token sliding problem (or $\mathrm{VCR}_{\mathrm{TS}}$ for short). Recall that a vertex cover is a set of vertices such that every edge as an extremity in the set.

Theorem 5. $D S R_{T S}$ is PSPACE-complete on bipartite graphs.

Proof. We show that $\mathrm{DSR}_{\mathrm{TS}}$ is PSPACE-hard on bipartite graphs by a polynomial-time reduction from $\mathrm{VCR}_{\mathrm{TS}}$. This is an adaptation of the well-known reduction from Vertex Cover to Dominating Set [10]. Let $G=(V, E)$ be a graph. We construct the corresponding bipartite graph $G^{\prime}=\left(V_{1} \uplus V_{2}, E^{\prime}\right)$ as follows: for each edge $u v \in E$, add $u$ and $v$ to $V_{1}$ and a new vertex $v_{u v}$ of degree two to $V_{2}$ that is adjacent to exactly $u$ and $v$. Note that $E^{\prime}$ does not contain the edge $u v$ so that $V_{1}$ is independent. Finally, add to $V_{2}$ a vertex $x$ adjacent to all the vertices in $V_{1}$ and attach to $x$ a pendant vertex $y$ which is added to $V_{1}$ (see Figure 5 for an example). Formally, the graph $G^{\prime}$ is the following:

- $V\left(G^{\prime}\right)=V_{1} \cup V_{2}$ where $V_{1}=V(G) \cup\{y\}$ and $V_{2}=\left\{v_{u v} \mid u v \in E\right\} \cup\{x\}$;
- $E^{\prime}=\left\{u v_{u v}\right.$ and $v_{u v} v \mid u, v \in V_{1}$ and $\left.v_{u v} \in V_{2}\right\} \cup\left\{x v \mid v \in V_{1}\right\} \cup\{x y\}$.

Observe that $G^{\prime}$ is bipartite and the reduction can be done in polynomial time. We now prove that the vertex covers of $G$ of size $k$ are reconfigurable if and only if the dominating sets of $G^{\prime}$ of size $k+1$ are. Let $\left(G, C_{s}, C_{t}\right)$ be an instance for the $\mathrm{VCR}_{\mathrm{TS}}$ problem. We define the corresponding instance for the $\mathrm{DSR}_{\mathrm{TS}}$ problem as $\left(G^{\prime}, C_{s} \cup\{x\}, C_{t} \cup\{x\}\right)$. Since $C_{s}$ is a vertex cover of $G$, for every edge $u v \in E(G)$ we have $\{u, v\} \cap C_{s} \neq \emptyset$ and thus the vertices $u, v, v_{u v}$ are dominated by $C_{s}$ in $G^{\prime}$. Now $x$ dominates both $x$ and $y$, so $D_{s}=C_{s} \cup\{x\}$ is a dominating set of $G^{\prime}$, and by the same argument, so is $D_{\mathrm{t}}=C_{t} \cup\{x\}$.

We start with the only if direction. First, it immediately follows from the definition of $D_{s}$ and $D_{\mathrm{t}}$ that $D_{s} \backslash\{x\}=C_{s}$ and $D_{\mathrm{t}} \backslash\{x\}=C_{t}$. Let us assume that $\left(G, C_{s}, C_{t}\right)$ is a yes-instance for the VCR $\mathrm{VA}_{\mathrm{TS}}$ problem. Then, there exists a reconfiguration sequence $S$ using the token sliding model between $C_{s}$ and $C_{t}$. One can construct a sequence $S^{\prime}$ for $G^{\prime}$ by replacing a move $u \rightsquigarrow v$ (where $u v \in E(G)$ ) of $S$ into two moves: $u \rightsquigarrow v_{u v}$ followed by $v_{u v} \rightsquigarrow v$. We need to prove that the domination property is preserved at any step. First, observe that each intermediate solution contains $x$ and thus that each move of the form $v_{u v} \rightsquigarrow v$ is safe because $u$ is still dominated by $x$ and $v_{u v}$ by $v$. Therefore, the only risk is to leave some vertex $v_{w u}$ undominated after a move $u \rightsquigarrow v_{u v}$. If so, this implies, that $w$ does not belong to the solution and thus the edges $w u$ and $u v$ are covered only by $u$. Therefore, the move $u \rightsquigarrow v$ of the sequence $S$ is not valid (because the edge $w u$ is no more covered), a contradiction. Therefore, $\left(G^{\prime}, D_{s}, D_{\mathrm{t}}\right)$ is a yes-instance for the $\mathrm{DSR}_{\mathrm{TS}}$ problem.

It remains to prove the if direction. Suppose that $\left(G^{\prime}, D_{s}, D_{\mathrm{t}}\right)$ is a yes-instance for the $\mathrm{DSR}_{\mathrm{TS}}$ problem. Then, there exists a reconfiguration sequence $S^{\prime}=\left\langle D_{s}, \ldots, D_{\mathrm{t}}\right\rangle$ in $G^{\prime}$. First, observe that at each step, $y$ needs to be dominated and thus either $x$ or $y$ belongs to each solution. Moreover, initially, $D_{s}$ does not contain $y$. If a move is of the form $x \rightsquigarrow y$ (it will be followed in a further step by a move $y \rightsquigarrow x$ ), we can simply ignore it and assume that $x$ contains at least one token in each solution. Therefore, the only vertices that we need to take care (in the sense that they are the only one not immediately dominated by $x$ ) are the vertices of the form $v_{u v}$ i.e. the vertices that correspond to the edges of $G$. We consider sequentially the two other possible moves $u \rightsquigarrow v$ and focus on the next operation (which may not be consecutive) that touches the vertex $v$ :

- $u \rightsquigarrow v_{u v}$ : if the next move that touches $v_{u v}$ is $v_{u v} \rightsquigarrow u$, these two operations are useless and can be simply ignored. Otherwise, this means that the next operation that touches $v_{u v}$ is $v_{u v} \rightsquigarrow v$ : it can be replaced by a single one $u \rightsquigarrow v$ in $G$.
- $u \rightsquigarrow x$ : in the case, note that $x$ contains at least two tokens. As before, if the next move is of the form $x \rightsquigarrow u$, we just ignore it. If it is a move $x \rightsquigarrow v$, note that the new solution must contain $u$ (if $u$ initially contained at least two vertices) or $v$ since $v_{u v}$ needs to be dominated. Therefore, we can safely replace these two moves $u \rightsquigarrow x$ and $x \rightsquigarrow v$ by $d$ moves (where $d$ represents the length of a shortest path from $u$ to $v$ in $G$ ).

Therefore, one can obtain from $S^{\prime}$ a TS-sequence that reconfigures $C_{s}$ into $C_{t}$ and thus ( $G, C_{s}, C_{t}$ ) is a yes-instance for $\mathrm{VCR}_{\mathrm{TS}}$, as desired. This concludes the proof.

Next, we prove that $\mathrm{DSR}_{\mathrm{TS}}$ is PSPACE-complete on planar graphs and bounded bandwidth graphs. Recall that a graph has bandwidth at most $k$ if there exists a numbering $\ell$ of the vertices with distinct


Figure 5: Example for the reduction of Theorem 5.
integers between 1 and $n$ (where $n$ is the number of vertices of the graph) such that adjacent vertices get label at distance less than $k$ (i.e. for every edge $u v \in E,|\ell(u)-\ell(v)| \leq k$ ).

The complement of a vertex cover is thus an independent set, whose reconfiguration is known to be PSPACE-complete on planar graphs [14] and bounded bandwidth graphs [27].

Theorem 6. $D S R_{T S}$ is PSPACE-complete on planar graphs and bounded bandwidth graphs.
Proof. The proof for dominating sets reconfiguration under TAR on planar graphs from [13] works also here since $\mathrm{VCR}_{\text {TS }}$ is PSPACE-complete on planar graphs. We use the well-known reduction mentioned in Theorem 5 which is the following: start with a copy of the original graph $G$ and for each edge $u v$, add a vertex $v_{u v}$ of degree two adjacent to $u$ and $v$ (note that the planarity property is preserved).

Recall that $\mathrm{VCR}_{\mathrm{TS}}$ is PSPACE-complete on bounded bandwidth graphs. Let $G$ be a graph whose bandwidth is bounded by some constant, let us say $k$. This implies that the maximum degree of $G$ is bounded by $2 k$ since a vertex can have at most $k$ neighbors of lower label and $k$ neighbors of higher label. Therefore, the graph $G^{\prime}$ obtained from the reduction has its bandwidth bounded by $k \cdot(k+1)$. We explain how to label $G^{\prime}$. The idea is to leave $k$ free values between two vertices labelled consecutively in the original labelling i.e. two vertices $u$ and $v$ such that $\ell(v)=\ell(u)+1$ : this means that for all $i>1$, we relabel the (unique) vertex labelled $i$ by $1+(i-1) \cdot(k+1)$. It is clearly sufficient as discussed above. Besides, one can check that the difference between the new labels of two adjacent vertices is at most $k \cdot(k+1)$. Finally, to get a proper labelling where each vertex is labelled by a value between 1 and $\left|V\left(G^{\prime}\right)\right|$, we can simply remove all the unused values by some successive shifting operations.

Bottcher et al. observed that the pathwidth and thus the treewidth of a graph are bounded by its bandwidth [6]. Therefore, we immediately get from Theorem 6 that DSR ${ }_{T S}$ is PSPACE-complete for bounded pathwidth and bounded treewidth graphs.

## 4 Polynomial-time algorithms

In this section, we focus on graph classes for which $\mathrm{DSR}_{\mathrm{TS}}$ can be solved in polynomial time. A natural way to solve this problem is to distinguish a special dominating set (that we call canonical) and then show that each dominating set can be reconfigured into this special one [13]. Of course, since the canonical dominating set is not part of the original instance, we must be able to compute it in polynomial time. We emphasize the fact that this canonical dominating set must be uniquely defined i.e. the set of vertices that hold a token as well as the number of tokens on each such vertices must be identical.

### 4.1 Joins and cographs

In this section, we prove the following theorem, that will allow us to conclude about cographs as a special case. Recall that the domination number of a join $G_{1}+G_{2}$ is always at most two, since taking a vertex from each operand of the join dominates the whole graph.

Theorem 7. Let $G_{1}$ and $G_{2}$ be two graphs, $D_{s}$ and $D_{t}$ two dominating sets of $G_{1}+G_{2}$ of same size. The dominating set $D_{s}$ can be reconfigured into $D_{t}$ by token sliding if and only if one of the three following conditions holds:
(i) $\left|D_{s}\right|=\left|D_{t}\right| \geq 3$,
(ii) the domination number of $G_{1}$ or of $G_{2}$ is at most two,
(iii) both $G_{1}$ and $G_{2}$ are connected.

Proof. We first show that if none of these conditions hold, then the graph is not reconfigurable. Let $G_{1}$ and $G_{2}$ be two graphs with $\gamma\left(G_{1}\right)>2$ and $\gamma\left(G_{2}\right)>2$, and assume without loss of generality that $G_{1}$ is not connected, say with two components $C_{1}$ and $C_{2}$. Note that $\gamma\left(G_{1}+G_{2}\right)=2$ since neither $G_{1}$ nor $G_{2}$ has a universal vertex.

Let $D_{s}=\{u, v\}$ and $D_{\mathrm{t}}=\{w, v\}$ be two minimum dominating sets of $G$ with $u \in C_{1}, w \in C_{2}$ and $v \in V\left(G_{2}\right)$. We prove that $D_{s}$ can not be reconfigured into $D_{\mathrm{t}}$. Since $G_{1}$ is not connected, there is no path between $u$ and $w$ in $G\left[V_{1}\right]$. Therefore, the only way to reach $w$ from $u$ is to go through $V\left(G_{2}\right)$. But
since $\gamma\left(G_{2}\right)>2$ no pair of vertices in $G_{2}$ can dominate $G_{2}$, and thus no move bringing the vertex in $V\left(G_{1}\right)$ to $V\left(G_{2}\right)$ is possible.

We now prove that each of the above conditions is sufficient for the dominating sets to be reconfigured.

Condition (i) Suppose $\left|D_{s}\right|=\left|D_{\mathrm{t}}\right| \geq 3$. Recall that picking a vertex of $G_{1}$ and one of $G_{2}$ always forms a dominating set of $G_{1}+G_{2}$. We infer that it is always possible to make one move from $D_{s}$ to reach a configuration with tokens in both $G_{1}$ and $G_{2}$, then from such position tokens can be slid freely in their part, until reaching $\left|D_{\mathrm{t}}\right|$ with a last move.

We assume now that $\left|D_{s}\right|=\left|D_{\mathrm{t}}\right| \leq 2$.

Condition (ii) For the case when $G_{1}$ or $G_{2}$ has domination number at most 2, we consider different cases depending of whether a graph has domination number one or not.

Case 1. If $\gamma\left(G_{1}\right)=1$ or $\gamma\left(G_{2}\right)=1$ : then $G_{1}+G_{2}$ contains a universal vertex. Then, from $D_{s}$, one can place a token on this vertex, reconfigure possible other tokens freely, then move that token to reach $D_{\mathrm{t}}$.
Case 2. If $\gamma\left(G_{1}\right)=2$ or $\gamma\left(G_{2}\right)=2$. Without loss of generality, assume that $\gamma\left(G_{1}\right)=2$. Note that in this case, $\gamma\left(G_{1}+G_{2}\right)=2$, let $D_{s}=\left\{v_{1}, v_{2}\right\}$. We define an arbitrary canonical dominating set $C$ by taking a vertex (e.g. of smallest index) in each of $G_{1}$ and $G_{2}$; we denote these vertices $u_{1} \in V\left(G_{1}\right)$ and $u_{2} \in V\left(G_{2}\right)$.

Suppose first that $v_{1}$ and $v_{2}$ belong to the same original graph, say $v_{1}, v_{2} \in V\left(G_{1}\right)$. We show how to reconfigure $D_{s}$ into $C$ in at most two steps. First, observe that since $C$ is a dominating set of $G$, we have that $u_{1} \in N\left[\left\{v_{1}, v_{2}\right\}\right]$, say $u_{1}$ belongs to $N\left[v_{1}\right]$. Our first step is to slide the token from $v_{2}$ to $u_{2}$, along the corresponding edge of the join. Then, by our observation that $u_{1} \in N\left[v_{1}\right]$, we can slide if necessary the token from $v_{1}$ to $u_{1}$.

Suppose now that $v_{1}$ and $v_{2}$ belong respectively to $V\left(G_{1}\right)$ and $V\left(G_{2}\right)$. Since $\gamma\left(G_{1}\right)=2$, let $\left\{w_{1}, w_{2}\right\}$ be a dominating set of $G_{1}$ and thus of $G_{1}+G_{2}$ (it can be computed naively in quadratic time). It dominates $v_{1}$ so assume without loss of generality that $v_{1} w_{1}$ is an edge. First moving the token from $v_{1}$ to $w_{1}$, at most two steps permit to reconfigure $D_{s}$ into $\left\{w_{1}, w_{2}\right\}$, which we can then reconfigure into $C$ by the above argument.

Condition (iii) Suppose finally that $\gamma\left(G_{1}\right) \geq 3$ and $\gamma\left(G_{2}\right) \geq 3$ but $G_{1}$ and $G_{2}$ are connected. Then $\gamma(G)=2$ and minimum dominating sets of $G_{1}+G_{2}$ are exactly the sets containing a vertex in $G_{1}$ and a vertex in $G_{2}$. Let $D_{s}=\left\{v_{1}, v_{2}\right\}$ and $D_{\mathrm{t}}=\left\{w_{1}, w_{2}\right\}$ with $v_{1}, w_{1} \in V\left(G_{1}\right)$ and $v_{2}, w_{2} \in V\left(G_{2}\right)$. Since $G_{1}$ is connected, there exists a path from $v_{1}$ to $w_{1}$ in $G\left[V\left(G_{1}\right)\right]$. Moving the token along this path, we always keep a dominating set by the above observation. Doing similarly along a path from $v_{2}$ to $w_{2}$, we have a reconfiguration from $D_{s}$ to $D_{\mathrm{t}}$.

We now consider the special case of cographs. Recall that the family of cographs can be defined as the family of graphs with no induced $P_{4}$, or equivalently by the following recursive definition:

- $K_{1}$ is a cograph;
- for $G_{1}$ and $G_{2}$ any two cographs, the disjoint union $G_{1} \cup G_{2}$ is a cograph;
- for $G_{1}$ and $G_{2}$ any two cographs, the join $G_{1}+G_{2}$ is a cograph.

Recall that computing the domination number of a cograph is polynomial time solvable, using the above definition. By the previous theorem, we infer that if a cograph is constructed as a join, whether it is reconfigurable is polynomial time decidable. The case when $G=K_{1}$, is straightforward. If $G=$ $G_{1} \cup G_{2}$ is the disjoint union of two cographs, then for two dominating set $D_{s}$ and $D_{\mathrm{t}}$, deciding whether $D_{s} \leftrightarrow \leftrightarrow D_{\mathrm{t}}$ is equivalent to deciding whether $D_{s} \cap V\left(G_{1}\right) \leadsto D_{\mathrm{t}} \cap V\left(G_{1}\right)$ in $G_{1}$, and $D_{s} \cap V\left(G_{2}\right) \leftrightarrow$ $D_{\mathrm{t}} \cap V\left(G_{2}\right)$ in $G_{2}$, which can be done inductively by induction.

Theorem 8. There is a polynomial time algorithm deciding $D S R_{T S}$ in cographs.


Figure 6: A dually chordal graph.

### 4.2 Dually chordal graphs

Let $G=(V, E)$ be a graph with $V=\left\{v_{1}, v_{2}, \ldots, v_{n}\right\}$. We denote by $G_{i}$ the graph $G\left[\left\{v_{i}, v_{i+1}, \ldots, v_{n}\right\}\right]$. A maximum neighbor of a vertex $u$ is a vertex $v \in N[u]$ such that we have $N[w] \subseteq N[v]$ for every vertex $w \in N[u]$. In other words, $v$ contains in its closed neighborhood every vertex at distance at most two from $u$. A maximum neighborhood ordering (or mno for short) is an ordering of the vertices in such a way that $v_{i}$ has a maximum neighbor in the graph $G_{i}$. A graph is dually chordal if it has a maximum neighborhood ordering. This ordering can be computed in linear time [5]. Moreover, the mno computed by this algorithm is such that for every vertex $v_{i}$ (with $i<n$ ), $v_{i}^{\prime} s$ maximum neighbor is different from $v_{i}$ (for connected graphs). An alternative proof of a similar statement for not necessarily connected graphs can be found in [8]. In the following, we always assume that a mno is associated with a function $m n: V \longrightarrow V$ that associates to each vertex a maximum neighbor.

Note that a dually chordal graph is not necessarily chordal. Figure 6 gives an example of a graph which is dually chordal but not chordal, since it contains an induced cycle on four vertices. The label inside each vertex corresponds to its rank in the ordering, and its maximum neighbor is the vertex pointed by the outgoing edge (note that $v_{8}{ }^{\prime}$ s maximum neighbor is itself). Moreover, observe that any tree $T$ is a dually chordal graph: root the tree in some vertex and orient all edges toward the root, any numbering keeping all $G_{i}$ connected is a mno where arcs point towards the vertex maximum neighbor.

Link with interval graphs. An interval graph is the intersection graph of a family of intervals on the real line. In other words, let $\left\{I_{1}, I_{2}, \ldots, I_{n}\right\}$ be a set of intervals. Each interval $I$ can be represented by its extremities $\ell(I), r(I)$ with $\ell(I) \leq r(I) \in \mathbb{R}$. We call these values respectively $\ell$-value and $r$-value (for left and right). The corresponding interval graph $G=(V, E)$ is the following:

- $V=\left\{I_{1}, I_{2}, \ldots, I_{n}\right\} ;$
- $I_{i} I_{j} \in E \Leftrightarrow I_{i} \cap I_{j} \neq \emptyset$ i.e. $\ell\left(I_{j}\right) \leq r\left(I_{i}\right)$ and $\ell\left(I_{i}\right) \leq r\left(I_{j}\right)$.

Let $G=(V, E)$ be an interval graph. For convenience, we denote by $v_{i}$ the vertex related to the interval $I_{i}$. We now order the vertices of $G$ with respect to their $r$-value i.e. $v_{i}<v_{j}$ if and only if $r\left(I_{i}\right)<r\left(I_{j}\right)$ (or $r\left(I_{i}\right)=r\left(I_{j}\right)$ and $\ell\left(I_{i}\right)<\ell\left(I_{j}\right)$ ). Then, we get the following useful property:

Property 9. Let $v_{i}$ and $v_{j}$ be two vertices of $G$ such that $v_{i}<v_{j}$. If $v_{i} v_{j} \in E$, then for any $v_{k}$ such that $v_{i}<v_{k}<v_{j}$, we have $v_{k} v_{j} \in E$.
Proof. Since $v_{i}<v_{k}<v_{j}$, we have $r\left(I_{i}\right) \leq r\left(I_{k}\right) \leq r\left(I_{j}\right)$. Since $v_{i} v_{j}$ is an edge, $\ell\left(I_{j}\right) \leq r\left(I_{i}\right)$. Thus, we get that $\ell\left(I_{j}\right) \leq r\left(I_{k}\right)$. Adding that $\ell\left(I_{k}\right) \leq r\left(I_{k}\right) \leq r\left(I_{j}\right)$, the conclusion follows.

Observation 10. Interval graphs are dually chordal graphs.
To see this observation, we prove that the ordering described above is a mno. For every vertex $v_{i}$, we set its neighbor of maximum index $v_{j}$ in the ordering to be its maximum neighbor. Then for any neighbor $v_{k}$ of $v_{i}, v_{i}<v_{k}<v_{j}$, on the one hand $v_{k}$ is adjacent to $v_{j}$ but also, any other neighbor $v_{\ell}>v_{i}$ of $v_{k}$ satisfies either $v_{i}<v_{\ell}<v_{j}$ or $v_{k}<v_{j}<v_{\ell}$, and in all cases property 9 concludes. See Figure 7 for an example: the maximum neighbor of $I_{i}$ is the vertex pointed by its only outgoing edge.


Figure 7: Interval graph and its maximum neighborhood ordering.

```
Algorithm 1 MDS
Require: A dually chordal graph G with a mno.
Ensure: A minimum triggered dominating set C and its set of triggering vertices T.
    Mark all vertices BOUNDED
    C\leftarrow\emptyset
    for all i from 1 to }n\mathrm{ do
        if }\mp@subsup{v}{i}{}\mathrm{ is labelled BOUNDED then
            label mn(vi) with REQUIRED
            Add v}\mp@subsup{v}{i}{}\mathrm{ to the set of triggering vertices T
            for all }u\inN[mn(\mp@subsup{v}{i}{})] d
                if u}\mathrm{ is not labelled REQUIRED then
                    Label u with FreE
            end if
            end for
        end if
        if }\mp@subsup{v}{i}{}\mathrm{ is labelled REQUIRED then
            C\leftarrowC\cup{\mp@subsup{v}{i}{}}
        end if
    end for
    return C and T
```

Computing the canonical dominating set. It is known that the Minimum Dominating Problem is linear-time solvable on dually chordal graphs [5]. In our case, we give another algorithm to compute a triggered dominating set that we define now, that will serve as a canonical dominating set.

Let $G$ be a dually chordal graph, whose vertices are ordered by a $m n o$. Let $C=\left\{c_{1}, c_{2}, \ldots, c_{k}\right\}$ be a dominating set of $G$ and $T=\left\{t_{1} t_{2}, \ldots, t_{k}\right\}$ a set of vertices, both sets in increasing order according to the mno. We say that $C$ is a triggered dominating set with triggering vertices $T$ if and only if:
(i) $c_{i}=m n\left(t_{i}\right)$ for all $1 \leq i \leq k$,
(ii) following the $m n o, t_{i}$ is the least vertex not in $N\left[c_{1}, \ldots, c_{i-1}\right]$, for all $1 \leq i \leq k$.

Observe that with a mno is associated exactly one triggered dominating set.
The following algorithm, called MDS, is strongly inspired by the classical algorithm for computing minimum dominating sets in trees. It takes as input a dually chordal graph $G=(V, E)$ with a mno and computes a triggered dominating set $C$ of size $\gamma(G)$ and its corresponding set of triggering vertices $T$ in running time $O(|V|+|E|)$.

Lemma 11 is devoted to prove the correctness of the algorithm MDS.
Lemma 11. Given a dually chordal graph $G=(V, E)$, the algorithm MDS computes a triggered dominating set of $G$ of order $\gamma(G)$ in time $O(|V|+|E|)$.

```
Algorithm 2 DUALLY-CHORDAL-RECONF
Require: A dually chordal graph \(G=(V, E)\), a minimum dominating set \(D\) of \(G\)
    Compute a mno for \(G\)
    \((C, T) \leftarrow \operatorname{MDS}(\mathrm{G})\).
    for \(i\) from 1 to \(\gamma(G)\) do
        Let \(x_{i}\) be the least vertex of \(D \cap N\left[t_{i}\right]\)
        if \(x_{i} c_{i} \in E\) then
            \(x_{i} \stackrel{\text { TS }}{\rightsquigarrow} c_{i}\)
        else
            \(y \leftarrow m n\left(x_{i}\right)\)
            \(x_{i} \xrightarrow{\text { TS }} y_{i}\)
            \(y_{i} \stackrel{c}{\rightsquigarrow}_{i}\)
        end if
    end for
```

Proof. The fact that $C$ is a triggered dominating set with triggering vertices $T$ is a direct consequence of the construction of the algorithm. Still we need to prove that this dominating set is of size $\gamma(G)$.

We call a labelled graph a graph with the vertices labeled FREE, REQUIRED or BOUNDED, such that a vertex is labelled Free if and only if it is adjacent to a vertex labelled Required and it is not labelled REQUIRED. Observe that the algorithm MDS maintains all along a labelled graph. In a labelled graph, we define a labelled dominating set a set of vertices containing all the vertices labelled REQUIRED and dominating all vertices labelled BOUNDED. We call the labelled domination number the minimum size of a labelled dominating set, and show that the algorithm MDS keeps the labelled domination number of the graph invariant. Since at the beginning, when all the vertices are labelled BOUNDED, the labelled domination number of the graph is exactly its domination number, this proves that the set of vertices marked REQUIRED at the end forms a minimum dominating set of $G$, of order $\gamma(G)$.

Let $S$ be a minimum labelled dominating set of a labelled graph $G$, and let $v_{i}$ be the minimum vertex in the mno that is labelled BoUnded. Let $w$ be the maximum neighbor of $v_{i}$ in $G_{i}$. If $w \in S$, then $S$ is also a minimum labelled dominating set of the graph $G$ where $w$ is labelled REQUIRED and all its neighbors previously labelled BOUNDED are labelled FREE, so the algorithm does not change the labelled domination number of $G$.

Otherwise, say $v_{j}$ is the vertex that dominates $v_{i}$ in $S$. Since $v_{i}$ is marked BOUNDED, $v_{j}$ is not marked required. If $j \geq i$, then by the maximum neighbor property, $w$ is adjacent to all the neighbors of $v_{j}$ that are in $G_{i}$, so $w$ dominates all neighbors of $v_{j}$ that are still marked BOUNDED. Thus we can replace $v_{j}$ by $w$ in $S$ and keep a minimum labelled dominating set of $G$, the conclusion follows. Suppose now $j<i$. Consider $v_{k}$ the maximum neighbor of $v_{j}$ in $G_{j}$. Observe that by the maximum neighbor definition and since no vertex less than $v_{i}$ is BOUNDED, again we can replace $v_{j}$ by $v_{k}$ in $S$. We can iterate until the vertex dominating $v_{i}$ in $S$ is no less than $v_{i}$, and then use the above argument. This concludes the proof that the algorithm MDS produces a minimum dominating set of $G$.

For the time complexity of the algorithm, observe that the algorithm visits every vertex at most once in the main loop, and it visits the neighborhoods of each vertex at most once when it possibly labels it REQUIRED. So the complexity is upper bounded by $\sum_{v \in V} O(1+|N(v)|)=O(|V|+|E|)$.

The reconfiguration algorithm. We show how to use the canonical triggered dominating set $C$ computed by the MDS algorithm in order to reconfigure two dominating sets of a dually chordal graph. To do that, we propose an algorithm DUALLY-CHORDAL-RECONF that modifies any dominating set $D$ of a dually chordal graph in such a way that $C \subseteq D$. The idea of this algorithm is to pick one vertex in $D$ that dominates the triggering vertex $t_{i}$ and to replace it by the corresponding vertex $c_{i}$ of $C$.
Lemma 12. Given a dually chordal graph $G=(V, E)$ and a dominating set $D$, DUALLY-ChORDAL-RECONF modifies $D$ with respect to the token sliding model in such a way that $C \subseteq D$ in $O(|V|)$ time, where $C$ is the canonical triggered dominating set computed by the MDS algorithm.
Proof. Let $T=\left(t_{1}, t_{2}, \ldots, t_{\gamma}\right)$ and $C=\left\{c_{1}, c_{2}, \ldots, c_{\gamma}\right\}$ be the output of algorithm MDS, with $c_{i}=m n\left(t_{i}\right)$. We denote by $C_{i}=\left\{c_{1}, \ldots, c_{i}\right\}$ the set of $i$ first vertices of $C$ according to the mno.

In order to prove the correctness of the algorithm, we need to prove the two following constraints are satisfied:
(i) each move is valid with respect to the token sliding model.
(ii) every intermediate set is a dominating set of $G$. (Note that this ensures the existence of the $x_{i}$ of line 4.)

We prove these two properties by induction on the index $i(0<i \leq \gamma)$. For some $i>0$, assume that the algorithm reconfigured properly $D$ into $D_{i-1}=\left(D \backslash\left\{x_{1}, \ldots, x_{i-1}\right\}\right) \cup\left\{c_{1}, \ldots, c_{i-1}\right\}$. We explain how to extend this up to rank $i$. By definition, $t_{i}$ is the least vertex which is not dominated by $N\left[C_{i-1}\right]=$ $N\left[\left\{c_{1}, \ldots, c_{i-1}\right\}\right]$. Let $x_{i}$ be the least vertex dominating $t_{i}$ in $D$. Observe that $x_{i} \notin\left\{c_{1}, \ldots, c_{i-1}\right\}$ since $t_{i}$ is the triggering vertex of $c_{i}$. For simplifying notations, we denote by $G^{\prime}$ the subgraph of $G$ induced by vertices larger than $t_{i}$ in the mno (i.e. the subgraph $G_{j}$ where $j$ is the index of $t_{i}$ in the mno). Note that since $C_{i-1} \subset D_{i-1}$, all vertices in $G \backslash G^{\prime}$ are dominated. We consider two cases:

Case 1. If $x_{i}$ is adjacent to $c_{i}$. Observe first that this case occurs whenever $x_{i} \geq t_{i}$ in the mno (where $\left.x_{i} \in N_{G^{\prime}}\left[t_{i}\right] \subseteq N_{G^{\prime}}\left[c_{i}\right]\right)$. In that case, the algorithm executes the line 6 and the token sliding constraint i is satisfied. Now, since $c_{i}=m n\left(t_{i}\right)$ and $x_{i}$ is adjacent to $t_{i}, N_{G^{\prime}}\left[x_{i}\right] \subseteq N\left[c_{i}\right]$, and the conclusion follows from the fact that all vertices in $G \backslash G^{\prime}$ are dominated.

Case 2. If $x_{i}$ is not adjacent to $c_{i}$. This is possibly the case when $x_{i}<t_{i}$ in the mno. The algorithm then first reconfigure $x_{i}$ into its maximum neighbor $y_{i}$, which is adjacent to $x_{i}$ and dominates all vertices of $x_{i}$ that might not be dominated yet. Moreover, $x_{i}$ is adjacent to $t_{i}$ and $x_{i}<t_{i}$ in the $m n o$, so $y_{i}$, as the maximum neighbor of $x_{i}$, must be adjacent to all neighbors of $t_{i}$, which contains in particular $c_{i}$. So the next move to $c_{i}$ satisfies the token sliding constraint i .

We now need to prove the dominating constraint ii is satisfied. Let $z$ be a vertex adjacent to $x_{i}$ and larger than $t_{i}$ (so possibly undominated). We show that $z$ has a common neighbor with $t_{i}$ no smaller than $t_{i}$. From this, we infer that it is adjacent to a neighbor of $t_{i}$ in $G^{\prime}$ and thus dominated by $m n\left(t_{i}\right)=c_{i}$.

First note that since $x_{i}$ is adjacent to both $t_{i}$ and $z$, and $x_{i}<t_{i}<z, y_{i}$ as its maximum neighbor is also adjacent to both $t_{i}$ and $z$. If $y_{i}>t_{i}$, this closes the case. Otherwise, $y_{i}$ has a neighbor larger than itself, so it must have a maximum neighbor $y_{i}^{\prime}$ distinct from itself, thus larger, and adjacent to both $t_{i}$ and $z$. Repeating inductively, we eventually get a vertex $y_{i}^{\prime \prime}$ adjacent to both $t_{i}$ and $z$, and larger than $t_{i}$. This concludes the proof.

Theorem 13. $D S R_{T S}$ can be solved in quadratic time on dually chordal graphs.
Proof. Let $G=(V, E)$ be a dually chordal graph and $D_{s}, D_{\mathrm{t}}$ be two dominating sets of $G$ of size $k$ (i.e. $\left(G, D_{s}, D_{\mathrm{t}}\right)$ is an instance of the $\mathrm{DSR}_{\mathrm{TS}}$ problem). Assume that $G$ is connected (otherwise we proceed independently for each connected component, checking first that the number of tokens on each component fit). We explain how to reconfigure $D_{s}$ into $D_{\mathrm{t}}$ in at most quadratic time.

First, we compute the canonical dominating set $C$ of $G$ with the algorithm MDS. By Lemma 12, one can transform $D_{s}$ and $D_{\mathrm{t}}$ in such a way that both contain $C$. This can be done in linear time (with respect to the order of $G$ ) since we move at most $\gamma(G)$ tokens and each move requires at most two steps. If $k=\gamma$, we are done. Otherwise, choose a vertex $v$ of minimum eccentricity and move all the remaining tokens by a shortest path to $v$. Therefore, the total time complexity is $O(|V|)+O((k-\gamma) \cdot \epsilon(v))$, which is at most quadratic (when $k=\Omega(n)$ ).

Observe that when $k$ is close to $\gamma$, the algorithm is linear. However, when the number of extra tokens is large (i.e. is linear in $n$ ), the quadratic overhead may be necessary. Indeed, consider a path on $n$ vertices $P_{n}$. The minimum eccentricity of $P_{n}$ is that of the middle vertex $v$ which is $\lfloor n / 2\rfloor$. Therefore, if all the extra tokens are on an extremity of the path, the time needed to move all of them to $v$ is quadratic. Since a path is a dually chordal graph, the conclusion follows.

Open questions. In all of our polynomial results presented in Section 4, computing a minimum dominating set can be bone in polynomial or even linear time on the graph classes considered. Therefore, a challenging question is the following: does there exist a graph class for which computing a minimum dominating set is NP-complete but $\mathrm{DSR}_{\mathrm{TS}}$ can be solved in polynomial time? As a result of Theorem 13 , we get that $\mathrm{DSR}_{\mathrm{TS}}$ is polynomial-time solvable on interval graphs. Recall that a circle graph is the
intersection graph set of chords of a circle. The Dominating Set Problem has been shown to be NP-complete on this graph class [18]. Hence, we ask the following question:

Question 1. What is the complexity of $D S R_{T S}$ on circle graphs?
If the answer is positive, note that it would generalize our result on cographs. A circular-arc graph is the intersection graph of a set of arcs on the circle. Even if computing a minimum dominating set can be computed in linear time on circular-arc graphs [24], we are interested in the complexity of the reconfiguration version under token sliding. More precisely, we ask for the following:

Question 2. Is $D S R_{T S}$ polynomial-time solvable on circular-arc graphs?
Besides, we found polynomial-time algorithms for cographs and dually chordal graph but the underlying reconfiguration sequence is most likely not optimal. Indeed, it may be possible that the shortest path in $\mathcal{R}_{G}(k)$ between the two given solutions does not go through the canonical dominating set. Therefore, can we bound the diameter of the reconfiguration graph? In other words, what is the maximum length of a reconfiguration sequence? Besides, what is the complexity of finding the optimal solution i.e. the shortest reconfiguration sequence between two dominating set on cographs or dually chordal graphs?

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