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Dominating sets reconfiguration under token sliding

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Abstract

Let G be a graph and D_s and D_t be two dominating sets of G of size k . Does there exist a sequence $\langle D_0 = D_s, D_1, \dots, D_{\ell-1}, D_\ell = D_t \rangle$ of dominating sets of G such that D_{i+1} can be obtained from D_i by replacing one vertex with one of its neighbors? In this paper, we investigate the complexity of this decision problem. We first prove that this problem is PSPACE-complete, even when restricted to split, bipartite or bounded tree-width graphs. On the other hand, we prove that it can be solved in polynomial time on dually chordal graphs (a superclass of both trees and interval graphs) or cographs.

1 Introduction

General introduction. Reconfiguration problems arise when, given an instance of a problem, we want to find a step-by-step transformation (called *reconfiguration sequence*) between two feasible solutions such that all intermediate solutions are also feasible. Unfortunately, such a transformation does not always exist and some solutions may even be *frozen* ie. they can not be modified at all. In this context, two natural questions arise: (i) when can we ensure that there exists such a transformation? (ii) What is the complexity of finding such a transformation?

Interest in combinatorial reconfiguration steadily increased during the last decade. Reconfiguration of several problems, including COLORING [2, 7, 9], INDEPENDENT SET [3, 4, 19], DOMINATING SET [13, 20, 22, 25] or SATISFIABILITY [11, 21] have been studied. For an overview of recent results on reconfiguration problems, the reader is referred to the surveys of van den Heuvel [26] and Nishimura [23]. In this article, we focus on the reconfiguration of dominating sets.

A dominating set is a set of vertices such that every vertex not in the set has a neighbour in it. Visualize a dominating set as materialized by tokens placed on the vertices that are part of it. Then, modifying a dominating set corresponds to shifting the tokens according to some rule, called reconfiguration rule. In the literature, three kinds of operations have been mainly studied:

1. Token Addition and Removal (TAR): one can add or remove a token as long as the total number of tokens does not go beyond a given threshold;
2. Token Jumping (TJ): one can move a token to any vertex of the graph;
3. Token Sliding (TS): one can slide a token along an edge i.e. one moves a token to a neighbour of its current location.

One can observe that in the last two models, the size of each solution remains constant at any time, as opposed to what happens in the TAR model. In this article, we are mostly interested in the Token Sliding model.

We define the reconfiguration graph for domination denoted $\mathcal{R}_G(k)$ as follows: the vertices of $\mathcal{R}_G(k)$ are the dominating sets of size k and there is an edge between two vertices if and only if one can go from the first to the second thanks to the considered reconfiguration rule (token sliding in our case). Three natural problems can be identified:

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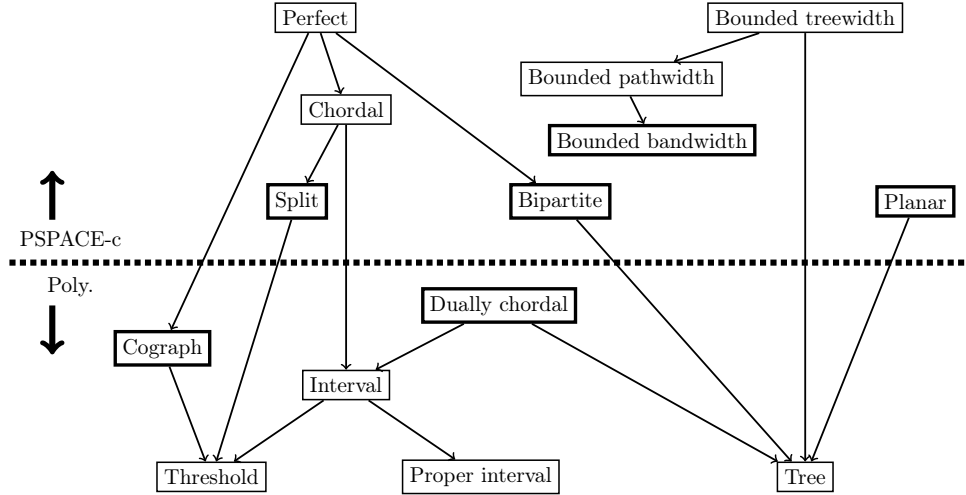


Figure 1: Our results: the frontier between PSPACE-completeness and tractability.

1. The *reachability* problem: given a graph G and two dominating sets D_s and D_t is there a path between D_s and D_t in $\mathcal{R}_G(k)$? In other words, does there exist a reconfiguration sequence between D_s and D_t ?
2. The *connectivity* problem: given a graph G , is the reconfiguration graph $\mathcal{R}_G(k)$ connected?
3. The *shortest path* problem: given a graph G , two dominating sets D_s and D_t and an integer ℓ , is the distance in $\mathcal{R}_G(k)$ between D_s and D_t at most ℓ ? In other words, does there exist a reconfiguration sequence between D_s and D_t of length at most ℓ ?

In this article, we focus on the reachability version and we will use the notation DSR_{TS} for this problem in order to lighten notations. We adopt the same notation for $\text{VERTEX COVER RECONFIGURATION}$ and $\text{INDEPENDENT SET RECONFIGURATION}$ (the reachability question under the token sliding rule) and denote these two problems by VCR_{TS} and ISR_{TS} , respectively.

Related results. The reconfiguration of dominating sets has been mainly studied under the *Token Addition and Removal* model. Haas and Seyffarth gave sufficient conditions to guarantee the connectivity of the reconfiguration graph according to k , the cardinality threshold of dominating sets [12]. More precisely, they proved that $\mathcal{R}_G(n-1)$ is connected if G has at least two independent edges. This value can be lowered to $\Gamma+1$ (where Γ is the maximum size of a minimal dominating set) if the input graph G is chordal or bipartite. Suzuki *et al.* showed that this result cannot be generalized to any graph since they constructed an infinite family of graphs for which $\mathcal{R}_G(\Gamma+1)$ is not connected [25]. On the positive side, they proved that $\mathcal{R}_G(n-\mu)$ is connected if G has a matching of size $\mu+1$.

Haddadan *et al.* studied the complexity of the reconfiguration of dominating sets under the token addition and removal rule from a graph classes perspective [13]. They proved that the reachability problem is PSPACE-complete, even if the input graph is a split graph or a bipartite graph. On the other hand, they gave linear-time algorithms for trees or interval graphs for instance.

Mouawad *et al.* studied the parameterized complexity of $\text{DOMINATING SET RECONFIGURATION}$ under token addition and removal. They proved that this problem is $\text{W}[2]$ -hard when parameterized by $k+\ell$, where k is the threshold and ℓ the length of the reconfiguration sequence. As a positive result, Lokshtanov *et al.* gave a fixed-parameter algorithm with respect to k for graphs excluding $K_{d,d}$ as a subgraph, for any constant d [20].

The third author also considered this problem (still in the TAR model) through the lense of an *optimization variant* (see Blanché *et al.* [1]) as recently introduced by Ito *et al.* for independent sets [16].

Our contribution. In this article, we are interested in the reachability question of dominating sets reconfiguration under token sliding. This reconfiguration rule has already been studied for various reconfiguration problems but not for dominating sets, to the best of the authors' knowledge.

We tackle this problem with a complexity perspective according to several graph classes: in Section 3, we prove for instance that DSR_{TS} is PSPACE-complete for split graphs or bipartite graphs. In Section 4, we show that this problem can be solved in polynomial time on other graph classes such as cographs or dually chordal graphs. Figure 1 gives an overview of our results where $A \rightarrow B$ means that the class B is properly included in the class A .

2 Preliminaries

This section is devoted to some basic definitions of graph theory used in this article. We then introduce more formally the problem we are interested in.

Each graph $G = (V, E)$ considered is simple (i.e. G is undirected and has no multiple edges or loops) where V represents the vertex set of G and E its edge set. We denote by $n = |V|$ and $m = |E|$ the number of vertices and edges of G . The *eccentricity* of a vertex u denoted by $\epsilon(u)$ is the maximum distance between u and any other vertex. For a subset of vertices $S \subseteq V$, we denote by $G[S]$ the subgraph induced by S . For a vertex $u \in V$, we denote by $N_G(u)$ its open neighborhood i.e. the set $\{v \mid uv \in E\}$ and by $N_G[u]$ its *closed neighborhood* i.e. the set $N_G(u) \cup \{u\}$. For a subset of vertices $S \subseteq V$, we define the closed neighborhood of S as the union of the closed neighborhood of the vertices in S i.e. $N_G[S] = \bigcup_{u \in S} N_G[u]$.

Let G_1 and G_2 be two graphs. We recall two basic binary operations on graphs: the disjoint union and the join operation. The disjoint union $G_1 \cup G_2$ of two graphs on disjoint vertex sets is the graph with vertex set $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. The join operation can be obtained from the disjoint union by adding all possible edges between G_1 and G_2 . More formally, the join of G_1 and G_2 denoted by $G_1 + G_2$ is the following graph:

- $V(G_1 + G_2) = V(G_1) \cup V(G_2)$;
- $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1), v \in V(G_2)\}$.

A dominating set for a graph $G = (V, E)$ is a subset of vertices $D \subseteq V$ such that $N[D] = V$ i.e. each vertex either belongs to D or has a neighbor in D . For a graph G , we denote by $\gamma(G)$ the domination number of G defined as the minimum size of a dominating set. Let G be a graph and D a dominating set of G . We say that u is a private neighbor of v (with respect to D) if $u \notin D$ and v is the only neighbor of u in D . Therefore, a dominating set is minimal for inclusion if and only if each of its vertices has a private neighbor.

Our problem. In the token sliding model, a natural question is whether we should allow to place more than one token on a vertex during the reconfiguration sequence. Here is an example where it makes a difference: consider the star graph S_n on $n + 1$ vertices and two dominating sets D_1 and D_2 of S_n of size k , with $k \in [2, n - 1]$. Any dominating set of that size necessarily contains the central vertex. To reconfigure D_1 into D_2 , we are forced to move a token from one leaf to another, which can only be done by going through the central vertex which already contains a token. Based on such examples, it seems reasonable to allow the superposition of tokens on a vertex, which is the choice we make in the following.

Let G be a graph, D_s and D_t be two dominating sets of G of same size k . We say that D_s is reconfigurable into D_t by token sliding if there exists a sequence $S = \langle D_0 = D_s, D_1, \dots, D_{\ell-1}, D_\ell = D_t \rangle$ that respects the two following properties:

- each D_i is a multiset of size k that is a dominating set of G ;
- there exists an edge uv such that $D_{i+1} = D_i \setminus \{u\} \cup \{v\}$ i.e. we slide the token placed on the vertex u along the edge uv .

We call such a sequence a TS-sequence and we denote this property by $D_s \stackrel{\text{TS}}{\rightsquigarrow} D_t$. We also say that (G, D_s, D_t) is a **yes**-instance for the DSR_{TS} problem.

We also introduce the two following notations, recall that D_s and D_t are two dominating sets of G of size k .

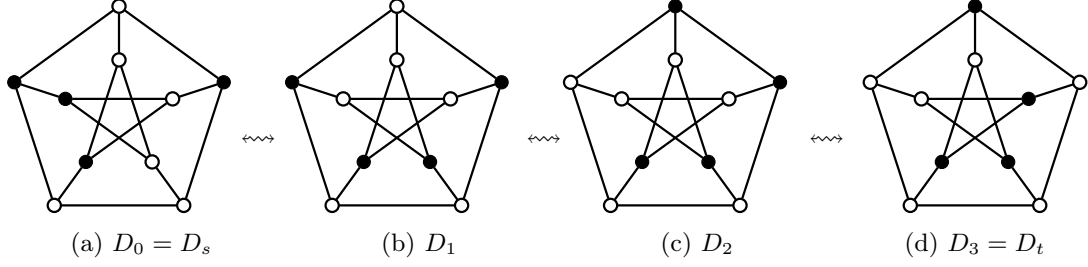


Figure 2: Example of TS-sequence from D_s to D_t .

- $D_s \overset{\text{TAR}}{\rightsquigarrow} D_t$: one can reconfigure D_s into D_t under the TAR model; each intermediate solution is of size at most $k + 1$;
- $D_s \overset{\text{TJ}}{\rightsquigarrow} D_t$: one can reconfigure D_s into D_t under the TJ model; each intermediate solution is of size exactly k .

A useful observation is that each reconfiguration sequence (and thus in particular a TS-sequence) is reversible: if $D_s \rightsquigarrow D_t$ holds, $D_t \rightsquigarrow D_s$ holds too. We thus denote this relation by $D_s \overset{\leftrightarrow}{\rightsquigarrow} D_t$. Figure 2 gives an example of a TS-sequence.

We are now ready to define properly the DOMINATING SET RECONFIGURATION problem under token sliding.

DSR_{TS}

Instance: a graph $G = (V, E)$ and two dominating sets D_s and D_t of cardinality k of G .

Question: Is there a TS-sequence between D_s and D_t , i.e. does $D_s \overset{\text{TS}}{\rightsquigarrow} D_t$?

We end this section by the following observation, showing that being reconfigurable is not a monotonous property.

Theorem 1. *For every k and ℓ with $2 \leq k < \ell$, there exist graphs where it is possible to reconfigure dominating sets of size k but not of size ℓ .*

Proof. We first prove the statement for $k = 2$. For every integer $\ell > 2$, we define the graph G_ℓ such that G contains exactly one dominating set of size $\gamma(G) = 2$ but for which the dominating sets of size ℓ are not reconfigurable. To construct G_ℓ , we first create ℓ pairs of triangles $\{(G_1^i, G_2^i), \dots, (G_1^\ell, G_2^\ell)\}$ such that G_1^i and G_2^i share exactly one vertex w_i . Moreover, let all the G_1^i share a vertex u and all the G_2^i share a vertex v (see Figure 3 for G_3 as an example). We have $\gamma(G_\ell) = 2$ since $\{u, v\}$ is a dominating set and G_ℓ does not contain a universal vertex (i.e. a vertex adjacent to all the other vertices). Besides, any dominating set of G_ℓ on less than ℓ vertices contains both u and v . Indeed, if for instance u is not in the dominating set, then ℓ extra vertices are necessary to dominate the triangles G_1^i .

Now, consider the dominating set $D_s = \{w_1, \dots, w_\ell\}$ of size ℓ of G_ℓ . By token sliding, D_s can not be reconfigured into any other dominating set of size ℓ . Indeed, in D_s we can not move any w_i in a triangle because it would leave the other triangle of the pair (G_1^i, G_2^i) undominated. This settles the case $k = 2$.

Now, for larger k , we attach to the vertex u of the graph $G_{\ell-k+2}$ a path on $3(k-2)$ vertices. Dominating sets of size k of this new graph are not unique anymore, but still they all contain both vertices u and v . Now, D_s together with all vertices of the path at distance $2 \pmod 3$ from u is also a dominating set of size ℓ that is not reconfigurable. \square

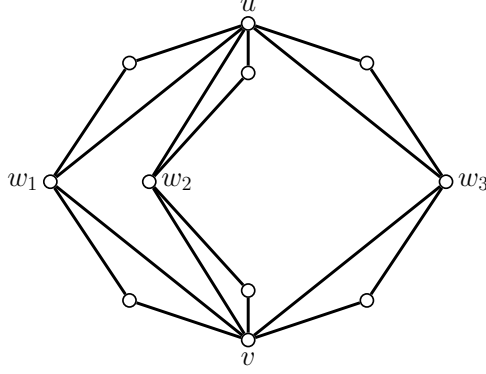


Figure 3: Graph G_3 .

3 PSPACE-completeness

In this section, we study the complexity of DSR_{TS} in the general case. We show that this problem is PSPACE-complete, even when restricted to split graphs, bipartite graphs or bounded treewidth graphs. Let us first recall the following result from Haddadan *et al.*, stating the complexity of the reconfiguration problem for the TAR model.

Theorem 2 ([13]). *Let G be a graph and D_s, D_t be two dominating sets of G of size k . Deciding whether $D_s \overset{TAR}{\rightsquigarrow} D_t$ is PSPACE-complete.*

Lemma 3. *Let G be a graph and D_s and D_t be two dominating sets of G of size k . We have $D_s \overset{TAR}{\rightsquigarrow} D_t$ if and only if $D_s \overset{TJ}{\rightsquigarrow} D_t$.*

Proof. The proof is an adaptation of the Theorem 1 of Kamiński *et al.* [17]. Suppose first $D_s \overset{TJ}{\rightsquigarrow} D_t$, let S be a TJ-sequence that reconfigures D_s into D_t . This sequence corresponds to a sequence of moves $u \rightsquigarrow v$. We construct a TAR-sequence by replacing each atomic move $u \rightsquigarrow v$ by two moves of the TAR model: we first add v and then delete u . By first adding v , we preserve the domination property. Besides, since we immediately delete u after the addition of v , each intermediate solution is of size at most $k + 1$, as desired.

For the other direction, let S' be a TAR-sequence that reconfigures D_s into D_t . Note that since $|D_s| = |D_t| = k$, S' is of even length. Moreover, by hypothesis, S' does not contain a configuration of size more than $k + 1$. If all the configurations of S' are of size k or $k + 1$, this means that S' corresponds to an alternation of an addition of a token on some vertex v immediately followed by the deletion of a token on a vertex u . Therefore, to get a TJ-sequence, we simply replace each such subsequence by a move $u \rightsquigarrow v$. Suppose now that S' contains some configuration of size less than k and consider a configuration, let us say D_i , of smallest size. Since D_i is a configuration of smallest size, this means that it has been obtained from D_{i-1} by the deletion of some vertex x . We also get that the configuration D_{i+1} is obtained from D_i by the addition of some vertex y . If $x = y$, then these two steps are redundant and can simply be ignored. Otherwise, observe that if we first add y and then delete x , the new sequence is still valid. If all the configurations are of size k or $k + 1$, we immediately get at TJ-sequence. Otherwise, we can repeat this process until this is the case. \square

As a corollary of Theorem 2 and Lemma 3, we get that deciding whether two dominating sets of size k of a graph G can be reconfigured under the token jumping model is a PSPACE-complete problem. We are now ready to prove Theorem 4.

Theorem 4. *DSR_{TS} is PSPACE-complete on split graphs.*

Proof. First, note that the problem is in PSPACE [15]. Let $G = (V, E)$ be a graph with $V(G) = \{v_1, \dots, v_n\}$. We construct the corresponding split graph G' as follows:

- $V(G') = V_1 \cup V_2$ where $V_1 = \{v_1, \dots, v_n\}$ and $V_2 = \{w_1, \dots, w_n\}$;

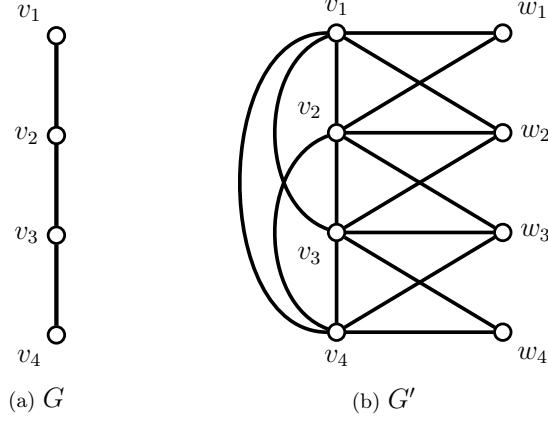


Figure 4: Example for the reduction of Theorem 4.

- $E(G') = \{uv \mid u, v \in V_1\} \cup \{v_i w_j \mid v_j \in N_G[v_i]\}$ i.e. we add all possible edges in V_1 so that V_1 forms a clique. We also add an edge between a vertex $v_i \in V_1$ and a vertex $w_j \in V_2$ if and only if the corresponding vertex v_j in the original graph G belongs to the closed neighborhood of v_i in G .

Observe that G' is a split graph since V_1 forms a clique and V_2 an independent set (see Figure 4 for an example). To a set of vertices of G , we associate the corresponding vertices of V_1 in G' . By definition of G' , any dominating set D of G is also a dominating set for G' : indeed, a vertex $v_i \in V_1$ dominates all the vertices in V_1 (since it is a clique) and all the vertices in V_2 that correspond to vertices in its closed neighborhood in G . That D dominates G allows to conclude that the corresponding set also dominates V_2 .

Let (G, D_s, D_t) be an instance of DSR_{TJ} , we reduce this instance to the instance of DSR_{TS} (G', D_s, D_t) .

This reduction can be done in linear time. Now, we need to prove that $D_s \overset{\text{TS}}{\rightsquigarrow} D_t$ if and only if there is a reconfiguration sequence between D_s and D_t in G' using the token jumping model. Consider a TJ-sequence in G , and transpose it to G' . All intermediate sets still are dominating sets, and since all pairs of vertices are joined by an edge in V_1 , this sequence is a valid TS-sequence in G' .

We now prove the other direction. Let $\langle D_0 = D_s, \dots, D_p = D_t \rangle$ be a TS-sequence in G' . If the sequence does not use vertices in V_2 , we immediately get a TJ-sequence in G from D_s to D_t , as the token jumping model does not require adjacency. Suppose on the other hand that the sequence goes through some vertices in V_2 . Since all vertices are initially in V_1 , there is a subsequence that consists in a move $v_i \overset{\text{TS}}{\rightsquigarrow} w_j$. Since $w_j \notin V_1$, there exists a later step where the token on w_j is moved to an adjacent vertex v_k in V_1 (since V_2 is independent). However, w_j does not dominate any vertex in V_2 (since V_2 is a stable set) and thus $N[w_j] \subseteq N[v_k]$. Therefore, we simply replace these two moves by a single move $v_i \rightsquigarrow v_k$. We can thus assume that the reconfiguration sequence only uses vertices in V_1 and conclude. \square

Next, we prove that DSR_{TS} is PSPACE-complete on bipartite graph. We use a reduction from the VERTEX COVER RECONFIGURATION BY TOKEN SLIDING problem (or VCR_{TS} for short). Recall that a vertex cover is a set of vertices such that every edge has an extremity in the set.

Theorem 5. DSR_{TS} is PSPACE-complete on bipartite graphs.

Proof. We show that DSR_{TS} is PSPACE-hard on bipartite graphs by a polynomial-time reduction from VCR_{TS} . This is an adaptation of the well-known reduction from VERTEX COVER to DOMINATING SET [10]. Let $G = (V, E)$ be a graph. We construct the corresponding bipartite graph $G' = (V_1 \uplus V_2, E')$ as follows: for each edge $uv \in E$, add u and v to V_1 and a new vertex v_{uv} of degree two to V_2 that is adjacent to exactly u and v . Note that E' does not contain the edge uv so that V_1 is independent. Finally, add to V_2 a vertex x adjacent to all the vertices in V_1 and attach to x a pendant vertex y which is added to V_1 (see Figure 5 for an example). Formally, the graph G' is the following:

- $V(G') = V_1 \cup V_2$ where $V_1 = V(G) \cup \{y\}$ and $V_2 = \{v_{uv} \mid uv \in E\} \cup \{x\}$;

- $E' = \{uv_{uv} \text{ and } v_{uv}v \mid u, v \in V_1 \text{ and } v_{uv} \in V_2\} \cup \{xv \mid v \in V_1\} \cup \{xy\}$.

Observe that G' is bipartite and the reduction can be done in polynomial time. We now prove that the vertex covers of G of size k are reconfigurable if and only if the dominating sets of G' of size $k + 1$ are. Let (G, C_s, C_t) be an instance for the VCR_{TS} problem. We define the corresponding instance for the DSR_{TS} problem as $(G', C_s \cup \{x\}, C_t \cup \{x\})$. Since C_s is a vertex cover of G , for every edge $uv \in E(G)$ we have $\{u, v\} \cap C_s \neq \emptyset$ and thus the vertices u, v, v_{uv} are dominated by C_s in G' . Now x dominates both x and y , so $D_s = C_s \cup \{x\}$ is a dominating set of G' , and by the same argument, so is $D_t = C_t \cup \{x\}$.

We start with the only if direction. First, it immediately follows from the definition of D_s and D_t that $D_s \setminus \{x\} = C_s$ and $D_t \setminus \{x\} = C_t$. Let us assume that (G, C_s, C_t) is a **yes**-instance for the VCR_{TS} problem. Then, there exists a reconfiguration sequence S using the token sliding model between C_s and C_t . One can construct a sequence S' for G' by replacing a move $u \rightsquigarrow v$ (where $uv \in E(G)$) of S into two moves: $u \rightsquigarrow v_{uv}$ followed by $v_{uv} \rightsquigarrow v$. We need to prove that the domination property is preserved at any step. First, observe that each intermediate solution contains x and thus that each move of the form $v_{uv} \rightsquigarrow v$ is safe because u is still dominated by x and v_{uv} by v . Therefore, the only risk is to leave some vertex v_{wu} undominated after a move $u \rightsquigarrow v_{uv}$. If so, this implies, that w does not belong to the solution and thus the edges wu and wv are covered only by u . Therefore, the move $u \rightsquigarrow v$ of the sequence S is not valid (because the edge wu is no more covered), a contradiction. Therefore, (G', D_s, D_t) is a **yes**-instance for the DSR_{TS} problem.

It remains to prove the if direction. Suppose that (G', D_s, D_t) is a **yes**-instance for the DSR_{TS} problem. Then, there exists a reconfiguration sequence $S' = \langle D_s, \dots, D_t \rangle$ in G' . First, observe that at each step, y needs to be dominated and thus either x or y belongs to each solution. Moreover, initially, D_s does not contain y . If a move is of the form $x \rightsquigarrow y$ (it will be followed in a further step by a move $y \rightsquigarrow x$), we can simply ignore it and assume that x contains at least one token in each solution. Therefore, the only vertices that we need to take care (in the sense that they are the only one not immediately dominated by x) are the vertices of the form v_{uv} i.e. the vertices that correspond to the edges of G . We consider sequentially the two other possible moves $u \rightsquigarrow v$ and focus on the next operation (which may not be consecutive) that touches the vertex v :

- $u \rightsquigarrow v_{uv}$: if the next move that touches v_{uv} is $v_{uv} \rightsquigarrow u$, these two operations are useless and can be simply ignored. Otherwise, this means that the next operation that touches v_{uv} is $v_{uv} \rightsquigarrow v$: it can be replaced by a single one $u \rightsquigarrow v$ in G .
- $u \rightsquigarrow x$: in the case, note that x contains at least two tokens. As before, if the next move is of the form $x \rightsquigarrow u$, we just ignore it. If it is a move $x \rightsquigarrow v$, note that the new solution must contain u (if u initially contained at least two vertices) or v since v_{uv} needs to be dominated. Therefore, we can safely replace these two moves $u \rightsquigarrow x$ and $x \rightsquigarrow v$ by d moves (where d represents the length of a shortest path from u to v in G).

Therefore, one can obtain from S' a TS-sequence that reconfigures C_s into C_t and thus (G, C_s, C_t) is a **yes**-instance for VCR_{TS} , as desired. This concludes the proof. \square

Next, we prove that DSR_{TS} is PSPACE -complete on planar graphs and bounded bandwidth graphs. Recall that a graph has *bandwidth* at most k if there exists a numbering ℓ of the vertices with distinct

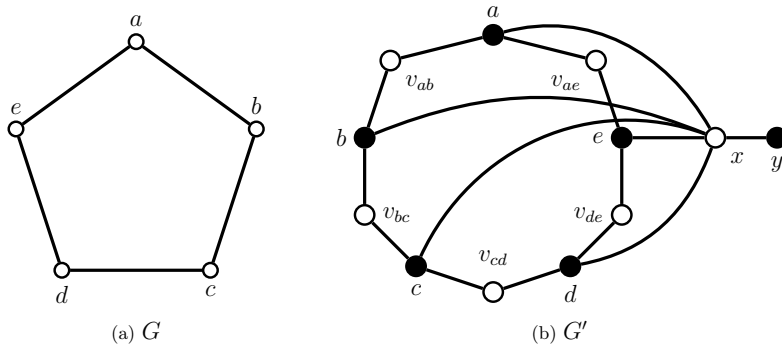


Figure 5: Example for the reduction of Theorem 5.

integers between 1 and n (where n is the number of vertices of the graph) such that adjacent vertices get label at distance less than k (i.e. for every edge $uv \in E$, $|\ell(u) - \ell(v)| \leq k$).

The complement of a vertex cover is thus an independent set, whose reconfiguration is known to be PSPACE-complete on planar graphs [14] and bounded bandwidth graphs [27].

Theorem 6. *DSR_{TS} is PSPACE-complete on planar graphs and bounded bandwidth graphs.*

Proof. The proof for dominating sets reconfiguration under TAR on planar graphs from [13] works also here since VCR_{TS} is PSPACE-complete on planar graphs. We use the well-known reduction mentioned in Theorem 5 which is the following: start with a copy of the original graph G and for each edge uv , add a vertex v_{uv} of degree two adjacent to u and v (note that the planarity property is preserved).

Recall that VCR_{TS} is PSPACE-complete on bounded bandwidth graphs. Let G be a graph whose bandwidth is bounded by some constant, let us say k . This implies that the maximum degree of G is bounded by $2k$ since a vertex can have at most k neighbors of lower label and k neighbors of higher label. Therefore, the graph G' obtained from the reduction has its bandwidth bounded by $k \cdot (k + 1)$. We explain how to label G' . The idea is to leave k free values between two vertices labelled consecutively in the original labelling i.e. two vertices u and v such that $\ell(v) = \ell(u) + 1$: this means that for all $i > 1$, we relabel the (unique) vertex labelled i by $1 + (i - 1) \cdot (k + 1)$. It is clearly sufficient as discussed above. Besides, one can check that the difference between the new labels of two adjacent vertices is at most $k \cdot (k + 1)$. Finally, to get a proper labelling where each vertex is labelled by a value between 1 and $|V(G')|$, we can simply remove all the unused values by some successive shifting operations. \square

Bottcher *et al.* observed that the *pathwidth* and thus the *treewidth* of a graph are bounded by its bandwidth [6]. Therefore, we immediately get from Theorem 6 that DSR_{TS} is PSPACE-complete for bounded pathwidth and bounded treewidth graphs.

4 Polynomial-time algorithms

In this section, we focus on graph classes for which DSR_{TS} can be solved in polynomial time. A natural way to solve this problem is to distinguish a special dominating set (that we call *canonical*) and then show that each dominating set can be reconfigured into this special one [13]. Of course, since the canonical dominating set is not part of the original instance, we must be able to compute it in polynomial time. We emphasize the fact that this canonical dominating set must be uniquely defined i.e. the set of vertices that hold a token as well as the number of tokens on each such vertices must be identical.

4.1 Joins and cographs

In this section, we prove the following theorem, that will allow us to conclude about cographs as a special case. Recall that the domination number of a join $G_1 + G_2$ is always at most two, since taking a vertex from each operand of the join dominates the whole graph.

Theorem 7. *Let G_1 and G_2 be two graphs, D_s and D_t two dominating sets of $G_1 + G_2$ of same size. The dominating set D_s can be reconfigured into D_t by token sliding if and only if one of the three following conditions holds:*

- (i) $|D_s| = |D_t| \geq 3$,
- (ii) the domination number of G_1 or of G_2 is at most two,
- (iii) both G_1 and G_2 are connected.

Proof. We first show that if none of these conditions hold, then the graph is not reconfigurable. Let G_1 and G_2 be two graphs with $\gamma(G_1) > 2$ and $\gamma(G_2) > 2$, and assume without loss of generality that G_1 is not connected, say with two components C_1 and C_2 . Note that $\gamma(G_1 + G_2) = 2$ since neither G_1 nor G_2 has a universal vertex.

Let $D_s = \{u, v\}$ and $D_t = \{w, v\}$ be two minimum dominating sets of G with $u \in C_1$, $w \in C_2$ and $v \in V(G_2)$. We prove that D_s can not be reconfigured into D_t . Since G_1 is not connected, there is no path between u and w in $G[V_1]$. Therefore, the only way to reach w from u is to go through $V(G_2)$. But

since $\gamma(G_2) > 2$ no pair of vertices in G_2 can dominate G_2 , and thus no move bringing the vertex in $V(G_1)$ to $V(G_2)$ is possible.

We now prove that each of the above conditions is sufficient for the dominating sets to be reconfigured.

Condition (i) Suppose $|D_s| = |D_t| \geq 3$. Recall that picking a vertex of G_1 and one of G_2 always forms a dominating set of $G_1 + G_2$. We infer that it is always possible to make one move from D_s to reach a configuration with tokens in both G_1 and G_2 , then from such position tokens can be slid freely in their part, until reaching $|D_t|$ with a last move.

We assume now that $|D_s| = |D_t| \leq 2$.

Condition (ii) For the case when G_1 or G_2 has domination number at most 2, we consider different cases depending of whether a graph has domination number one or not.

Case 1. If $\gamma(G_1) = 1$ or $\gamma(G_2) = 1$: then $G_1 + G_2$ contains a universal vertex. Then, from D_s , one can place a token on this vertex, reconfigure possible other tokens freely, then move that token to reach D_t .

Case 2. If $\gamma(G_1) = 2$ or $\gamma(G_2) = 2$. Without loss of generality, assume that $\gamma(G_1) = 2$. Note that in this case, $\gamma(G_1 + G_2) = 2$, let $D_s = \{v_1, v_2\}$. We define an arbitrary canonical dominating set C by taking a vertex (e.g. of smallest index) in each of G_1 and G_2 ; we denote these vertices $u_1 \in V(G_1)$ and $u_2 \in V(G_2)$.

Suppose first that v_1 and v_2 belong to the same original graph, say $v_1, v_2 \in V(G_1)$. We show how to reconfigure D_s into C in at most two steps. First, observe that since C is a dominating set of G , we have that $u_1 \in N[\{v_1, v_2\}]$, say u_1 belongs to $N[v_1]$. Our first step is to slide the token from v_2 to u_2 , along the corresponding edge of the join. Then, by our observation that $u_1 \in N[v_1]$, we can slide if necessary the token from v_1 to u_1 .

Suppose now that v_1 and v_2 belong respectively to $V(G_1)$ and $V(G_2)$. Since $\gamma(G_1) = 2$, let $\{w_1, w_2\}$ be a dominating set of G_1 and thus of $G_1 + G_2$ (it can be computed naively in quadratic time). It dominates v_1 so assume without loss of generality that $v_1 w_1$ is an edge. First moving the token from v_1 to w_1 , at most two steps permit to reconfigure D_s into $\{w_1, w_2\}$, which we can then reconfigure into C by the above argument.

Condition (iii) Suppose finally that $\gamma(G_1) \geq 3$ and $\gamma(G_2) \geq 3$ but G_1 and G_2 are connected. Then $\gamma(G) = 2$ and minimum dominating sets of $G_1 + G_2$ are exactly the sets containing a vertex in G_1 and a vertex in G_2 . Let $D_s = \{v_1, v_2\}$ and $D_t = \{w_1, w_2\}$ with $v_1, w_1 \in V(G_1)$ and $v_2, w_2 \in V(G_2)$. Since G_1 is connected, there exists a path from v_1 to w_1 in $G[V(G_1)]$. Moving the token along this path, we always keep a dominating set by the above observation. Doing similarly along a path from v_2 to w_2 , we have a reconfiguration from D_s to D_t . \square

We now consider the special case of cographs. Recall that the family of cographs can be defined as the family of graphs with no induced P_4 , or equivalently by the following recursive definition:

- K_1 is a cograph;
- for G_1 and G_2 any two cographs, the disjoint union $G_1 \cup G_2$ is a cograph;
- for G_1 and G_2 any two cographs, the join $G_1 + G_2$ is a cograph.

Recall that computing the domination number of a cograph is polynomial time solvable, using the above definition. By the previous theorem, we infer that if a cograph is constructed as a join, whether it is reconfigurable is polynomial time decidable. The case when $G = K_1$, is straightforward. If $G = G_1 \cup G_2$ is the disjoint union of two cographs, then for two dominating set D_s and D_t , deciding whether $D_s \rightsquigarrow D_t$ is equivalent to deciding whether $D_s \cap V(G_1) \rightsquigarrow D_t \cap V(G_1)$ in G_1 , and $D_s \cap V(G_2) \rightsquigarrow D_t \cap V(G_2)$ in G_2 , which can be done inductively by induction.

Theorem 8. *There is a polynomial time algorithm deciding DSR_{TS} in cographs.*

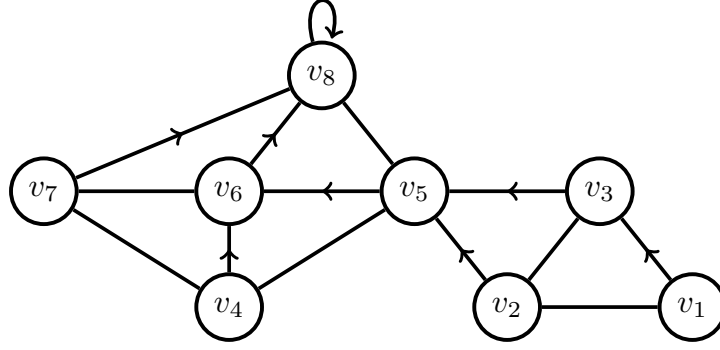


Figure 6: A dually chordal graph.

4.2 Dually chordal graphs

Let $G = (V, E)$ be a graph with $V = \{v_1, v_2, \dots, v_n\}$. We denote by G_i the graph $G[\{v_i, v_{i+1}, \dots, v_n\}]$. A *maximum neighbor* of a vertex u is a vertex $v \in N[u]$ such that we have $N[w] \subseteq N[v]$ for every vertex $w \in N[u]$. In other words, v contains in its closed neighborhood every vertex at distance at most two from u . A *maximum neighborhood ordering* (or *mno* for short) is an ordering of the vertices in such a way that v_i has a maximum neighbor in the graph G_i . A graph is *dually chordal* if it has a maximum neighborhood ordering. This ordering can be computed in linear time [5]. Moreover, the *mno* computed by this algorithm is such that for every vertex v_i (with $i < n$), v_i 's maximum neighbor is different from v_i (for connected graphs). An alternative proof of a similar statement for not necessarily connected graphs can be found in [8]. In the following, we always assume that a *mno* is associated with a function $mn : V \rightarrow V$ that associates to each vertex a maximum neighbor.

Note that a dually chordal graph is not necessarily chordal. Figure 6 gives an example of a graph which is dually chordal but not chordal, since it contains an induced cycle on four vertices. The label inside each vertex corresponds to its rank in the ordering, and its maximum neighbor is the vertex pointed by the outgoing edge (note that v_8 's maximum neighbor is itself). Moreover, observe that any tree T is a dually chordal graph: root the tree in some vertex and orient all edges toward the root, any numbering keeping all G_i connected is a *mno* where arcs point towards the vertex maximum neighbor.

Link with interval graphs. An interval graph is the intersection graph of a family of intervals on the real line. In other words, let $\{I_1, I_2, \dots, I_n\}$ be a set of intervals. Each interval I can be represented by its extremities $\ell(I), r(I)$ with $\ell(I) \leq r(I) \in \mathbb{R}$. We call these values respectively ℓ -value and r -value (for left and right). The corresponding interval graph $G = (V, E)$ is the following:

- $V = \{I_1, I_2, \dots, I_n\}$;
- $I_i I_j \in E \Leftrightarrow I_i \cap I_j \neq \emptyset$ i.e. $\ell(I_j) \leq r(I_i)$ and $\ell(I_i) \leq r(I_j)$.

Let $G = (V, E)$ be an interval graph. For convenience, we denote by v_i the vertex related to the interval I_i . We now order the vertices of G with respect to their r -value i.e. $v_i < v_j$ if and only if $r(I_i) < r(I_j)$ (or $r(I_i) = r(I_j)$ and $\ell(I_i) < \ell(I_j)$). Then, we get the following useful property:

Property 9. Let v_i and v_j be two vertices of G such that $v_i < v_j$. If $v_i v_j \in E$, then for any v_k such that $v_i < v_k < v_j$, we have $v_k v_j \in E$.

Proof. Since $v_i < v_k < v_j$, we have $r(I_i) \leq r(I_k) \leq r(I_j)$. Since $v_i v_j$ is an edge, $\ell(I_j) \leq r(I_i)$. Thus, we get that $\ell(I_j) \leq r(I_k)$. Adding that $\ell(I_k) \leq r(I_k) \leq r(I_j)$, the conclusion follows. \square

Observation 10. Interval graphs are dually chordal graphs.

To see this observation, we prove that the ordering described above is a *mno*. For every vertex v_i , we set its neighbor of maximum index v_j in the ordering to be its maximum neighbor. Then for any neighbor v_k of v_i , $v_i < v_k < v_j$, on the one hand v_k is adjacent to v_j but also, any other neighbor $v_\ell > v_i$ of v_k satisfies either $v_i < v_\ell < v_j$ or $v_k < v_j < v_\ell$, and in all cases property 9 concludes. See Figure 7 for an example: the maximum neighbor of I_i is the vertex pointed by its only outgoing edge.

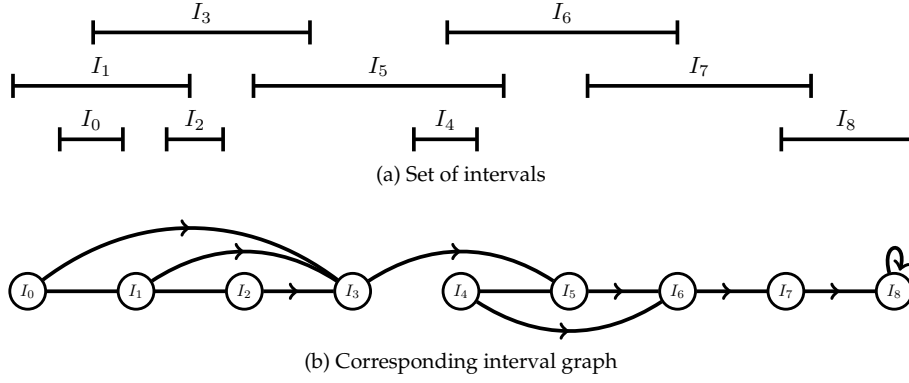


Figure 7: Interval graph and its maximum neighborhood ordering.

Algorithm 1 MDS

Require: A dually chordal graph G with a mno .

Ensure: A minimum triggered dominating set C and its set of triggering vertices T .

```

1: Mark all vertices BOUNDED
2:  $C \leftarrow \emptyset$ 
3: for all  $i$  from 1 to  $n$  do
4:   if  $v_i$  is labelled BOUNDED then
5:     label  $mn(v_i)$  with REQUIRED
6:     Add  $v_i$  to the set of triggering vertices  $T$ 
7:     for all  $u \in N[mn(v_i)]$  do
8:       if  $u$  is not labelled REQUIRED then
9:         Label  $u$  with FREE
10:      end if
11:    end for
12:  end if
13:  if  $v_i$  is labelled REQUIRED then
14:     $C \leftarrow C \cup \{v_i\}$ 
15:  end if
16: end for
17: return  $C$  and  $T$ 

```

Computing the canonical dominating set. It is known that the MINIMUM DOMINATING PROBLEM is linear-time solvable on dually chordal graphs [5]. In our case, we give another algorithm to compute a triggered dominating set that we define now, that will serve as a canonical dominating set.

Let G be a dually chordal graph, whose vertices are ordered by a mno . Let $C = \{c_1, c_2, \dots, c_k\}$ be a dominating set of G and $T = \{t_1, t_2, \dots, t_k\}$ a set of vertices, both sets in increasing order according to the mno . We say that C is a *triggered dominating set* with triggering vertices T if and only if:

- (i) $c_i = mn(t_i)$ for all $1 \leq i \leq k$,
- (ii) following the mno , t_i is the least vertex not in $N[c_1, \dots, c_{i-1}]$, for all $1 \leq i \leq k$.

Observe that with a mno is associated exactly one triggered dominating set.

The following algorithm, called MDS, is strongly inspired by the classical algorithm for computing minimum dominating sets in trees. It takes as input a dually chordal graph $G = (V, E)$ with a mno and computes a triggered dominating set C of size $\gamma(G)$ and its corresponding set of triggering vertices T in running time $O(|V| + |E|)$.

Lemma 11 is devoted to prove the correctness of the algorithm MDS.

Lemma 11. *Given a dually chordal graph $G = (V, E)$, the algorithm MDS computes a triggered dominating set of G of order $\gamma(G)$ in time $O(|V| + |E|)$.*

Algorithm 2 DUALY-CHORDAL-RECONF

Require: A dually chordal graph $G = (V, E)$, a minimum dominating set D of G

```
1: Compute a mno for  $G$ 
2:  $(C, T) \leftarrow \text{MDS}(G)$ .
3: for  $i$  from 1 to  $\gamma(G)$  do
4:   Let  $x_i$  be the least vertex of  $D \cap N[t_i]$ 
5:   if  $x_i c_i \in E$  then
6:      $x_i \overset{\text{TS}}{\rightsquigarrow} c_i$ 
7:   else
8:      $y \leftarrow mn(x_i)$ 
9:      $x_i \overset{\text{TS}}{\rightsquigarrow} y_i$ 
10:     $y_i \overset{c}{\rightsquigarrow} c_i$ 
11:   end if
12: end for
```

Proof. The fact that C is a triggered dominating set with triggering vertices T is a direct consequence of the construction of the algorithm. Still we need to prove that this dominating set is of size $\gamma(G)$.

We call a labelled graph a graph with the vertices labeled FREE, REQUIRED or BOUNDED, such that a vertex is labelled FREE if and only if it is adjacent to a vertex labelled REQUIRED and it is not labelled REQUIRED. Observe that the algorithm MDS maintains all along a labelled graph. In a labelled graph, we define a *labelled dominating set* a set of vertices containing all the vertices labelled REQUIRED and dominating all vertices labelled BOUNDED. We call the labelled domination number the minimum size of a labelled dominating set, and show that the algorithm MDS keeps the labelled domination number of the graph invariant. Since at the beginning, when all the vertices are labelled BOUNDED, the labelled domination number of the graph is exactly its domination number, this proves that the set of vertices marked REQUIRED at the end forms a minimum dominating set of G , of order $\gamma(G)$.

Let S be a minimum labelled dominating set of a labelled graph G , and let v_i be the minimum vertex in the *mno* that is labelled BOUNDED. Let w be the maximum neighbor of v_i in G_i . If $w \in S$, then S is also a minimum labelled dominating set of the graph G where w is labelled REQUIRED and all its neighbors previously labelled BOUNDED are labelled FREE, so the algorithm does not change the labelled domination number of G .

Otherwise, say v_j is the vertex that dominates v_i in S . Since v_i is marked BOUNDED, v_j is not marked required. If $j \geq i$, then by the maximum neighbor property, w is adjacent to all the neighbors of v_j that are in G_i , so w dominates all neighbors of v_j that are still marked BOUNDED. Thus we can replace v_j by w in S and keep a minimum labelled dominating set of G , the conclusion follows. Suppose now $j < i$. Consider v_k the maximum neighbor of v_j in G_j . Observe that by the maximum neighbor definition and since no vertex less than v_i is BOUNDED, again we can replace v_j by v_k in S . We can iterate until the vertex dominating v_i in S is no less than v_i , and then use the above argument. This concludes the proof that the algorithm MDS produces a minimum dominating set of G .

For the time complexity of the algorithm, observe that the algorithm visits every vertex at most once in the main loop, and it visits the neighborhoods of each vertex at most once when it possibly labels it REQUIRED. So the complexity is upper bounded by $\sum_{v \in V} O(1 + |N(v)|) = O(|V| + |E|)$. \square

The reconfiguration algorithm. We show how to use the canonical triggered dominating set C computed by the MDS algorithm in order to reconfigure two dominating sets of a dually chordal graph. To do that, we propose an algorithm DUALY-CHORDAL-RECONF that modifies any dominating set D of a dually chordal graph in such a way that $C \subseteq D$. The idea of this algorithm is to pick one vertex in D that dominates the triggering vertex t_i and to replace it by the corresponding vertex c_i of C .

Lemma 12. *Given a dually chordal graph $G = (V, E)$ and a dominating set D , DUALY-CHORDAL-RECONF modifies D with respect to the token sliding model in such a way that $C \subseteq D$ in $O(|V|)$ time, where C is the canonical triggered dominating set computed by the MDS algorithm.*

Proof. Let $T = (t_1, t_2, \dots, t_\gamma)$ and $C = \{c_1, c_2, \dots, c_\gamma\}$ be the output of algorithm MDS, with $c_i = mn(t_i)$. We denote by $C_i = \{c_1, \dots, c_i\}$ the set of i first vertices of C according to the *mno*.

In order to prove the correctness of the algorithm, we need to prove the two following constraints are satisfied:

- (i) each move is valid with respect to the token sliding model.
- (ii) every intermediate set is a dominating set of G . (Note that this ensures the existence of the x_i of line 4.)

We prove these two properties by induction on the index i ($0 < i \leq \gamma$). For some $i > 0$, assume that the algorithm reconfigured properly D into $D_{i-1} = (D \setminus \{x_1, \dots, x_{i-1}\}) \cup \{c_1, \dots, c_{i-1}\}$. We explain how to extend this up to rank i . By definition, t_i is the least vertex which is not dominated by $N[C_{i-1}] = N[\{c_1, \dots, c_{i-1}\}]$. Let x_i be the least vertex dominating t_i in D . Observe that $x_i \notin \{c_1, \dots, c_{i-1}\}$ since t_i is the triggering vertex of c_i . For simplifying notations, we denote by G' the subgraph of G induced by vertices larger than t_i in the mno (i.e. the subgraph G_j where j is the index of t_i in the mno). Note that since $C_{i-1} \subset D_{i-1}$, all vertices in $G \setminus G'$ are dominated. We consider two cases:

Case 1. If x_i is adjacent to c_i . Observe first that this case occurs whenever $x_i \geq t_i$ in the mno (where $x_i \in N_{G'}[t_i] \subseteq N_{G'}[c_i]$). In that case, the algorithm executes the line 6 and the token sliding constraint i is satisfied. Now, since $c_i = mn(t_i)$ and x_i is adjacent to t_i , $N_{G'}[x_i] \subseteq N[c_i]$, and the conclusion follows from the fact that all vertices in $G \setminus G'$ are dominated.

Case 2. If x_i is not adjacent to c_i . This is possibly the case when $x_i < t_i$ in the mno . The algorithm then first reconfigure x_i into its maximum neighbor y_i , which is adjacent to x_i and dominates all vertices of x_i that might not be dominated yet. Moreover, x_i is adjacent to t_i and $x_i < t_i$ in the mno , so y_i , as the maximum neighbor of x_i , must be adjacent to all neighbors of t_i , which contains in particular c_i . So the next move to c_i satisfies the token sliding constraint i.

We now need to prove the dominating constraint ii is satisfied. Let z be a vertex adjacent to x_i and larger than t_i (so possibly undominated). We show that z has a common neighbor with t_i no smaller than t_i . From this, we infer that it is adjacent to a neighbor of t_i in G' and thus dominated by $mn(t_i) = c_i$.

First note that since x_i is adjacent to both t_i and z , and $x_i < t_i < z$, y_i as its maximum neighbor is also adjacent to both t_i and z . If $y_i > t_i$, this closes the case. Otherwise, y_i has a neighbor larger than itself, so it must have a maximum neighbor y'_i distinct from itself, thus larger, and adjacent to both t_i and z . Repeating inductively, we eventually get a vertex y''_i adjacent to both t_i and z , and larger than t_i . This concludes the proof. \square

Theorem 13. DSR_{TS} can be solved in quadratic time on dually chordal graphs.

Proof. Let $G = (V, E)$ be a dually chordal graph and D_s, D_t be two dominating sets of G of size k (i.e. (G, D_s, D_t) is an instance of the DSR_{TS} problem). Assume that G is connected (otherwise we proceed independently for each connected component, checking first that the number of tokens on each component fit). We explain how to reconfigure D_s into D_t in at most quadratic time.

First, we compute the canonical dominating set C of G with the algorithm MDS. By Lemma 12, one can transform D_s and D_t in such a way that both contain C . This can be done in linear time (with respect to the order of G) since we move at most $\gamma(G)$ tokens and each move requires at most two steps. If $k = \gamma$, we are done. Otherwise, choose a vertex v of minimum eccentricity and move all the remaining tokens by a shortest path to v . Therefore, the total time complexity is $O(|V|) + O((k - \gamma) \cdot \epsilon(v))$, which is at most quadratic (when $k = \Omega(n)$). \square

Observe that when k is close to γ , the algorithm is linear. However, when the number of extra tokens is large (i.e. is linear in n), the quadratic overhead may be necessary. Indeed, consider a path on n vertices P_n . The minimum eccentricity of P_n is that of the middle vertex v which is $\lfloor n/2 \rfloor$. Therefore, if all the extra tokens are on an extremity of the path, the time needed to move all of them to v is quadratic. Since a path is a dually chordal graph, the conclusion follows.

Open questions. In all of our polynomial results presented in Section 4, computing a minimum dominating set can be done in polynomial or even linear time on the graph classes considered. Therefore, a challenging question is the following: does there exist a graph class for which computing a minimum dominating set is NP-complete but DSR_{TS} can be solved in polynomial time? As a result of Theorem 13, we get that DSR_{TS} is polynomial-time solvable on interval graphs. Recall that a circle graph is the

intersection graph set of chords of a circle. The DOMINATING SET PROBLEM has been shown to be NP-complete on this graph class [18]. Hence, we ask the following question:

Question 1. *What is the complexity of DSR_{TS} on circle graphs?*

If the answer is positive, note that it would generalize our result on cographs. A circular-arc graph is the intersection graph of a set of arcs on the circle. Even if computing a minimum dominating set can be computed in linear time on circular-arc graphs [24], we are interested in the complexity of the reconfiguration version under token sliding. More precisely, we ask for the following:

Question 2. *Is DSR_{TS} polynomial-time solvable on circular-arc graphs?*

Besides, we found polynomial-time algorithms for cographs and dually chordal graph but the underlying reconfiguration sequence is most likely not optimal. Indeed, it may be possible that the shortest path in $\mathcal{R}_G(k)$ between the two given solutions does not go through the canonical dominating set. Therefore, can we bound the diameter of the reconfiguration graph? In other words, what is the maximum length of a reconfiguration sequence? Besides, what is the complexity of finding the optimal solution i.e. the shortest reconfiguration sequence between two dominating set on cographs or dually chordal graphs ?

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References

- [1] Alexandre Blanché, Haruka Mizuta, Paul Ouvrard, and Akira Suzuki. Decremental optimization of dominating sets under reachability constraints. *CoRR*, abs/1906.05163, 2019.
- [2] Marthe Bonamy, Nicolas Bousquet, Carl Feghali, and Matthew Johnson. On a conjecture of mohar concerning kempe equivalence of regular graphs. *Journal of Combinatorial Theory, Series B*, 135:179 – 199, 2019.
- [3] Paul S. Bonsma. Independent set reconfiguration in cographs and their generalizations. *Journal of Graph Theory*, 83(2):164–195, 2016.
- [4] Nicolas Bousquet, Arnaud Mary, and Aline Parreau. Token jumping in minor-closed classes. In Ralf Klasing and Marc Zeitoun, editors, *Fundamentals of Computation Theory*, pages 136–149, Berlin, Heidelberg, 2017. Springer Berlin Heidelberg.
- [5] Andreas Brandstädt, Victor D. Chepoi, and Feodor F. Dragan. The algorithmic use of hypertree structure and maximum neighbourhood orderings. *Discrete Applied Mathematics*, 82(1):43 – 77, 1998.
- [6] Julia Böttcher, Klaas P. Pruessmann, Anusch Taraz, and Andreas Würfl. Bandwidth, expansion, treewidth, separators and universality for bounded-degree graphs. *European Journal of Combinatorics*, 31(5):1217 – 1227, 2010.
- [7] Luis Cereceda, Jan van den Heuvel, and Matthew Johnson. Connectedness of the graph of vertex-colourings. *Discrete Mathematics*, 308(5):913 – 919, 2008. Selected Papers from 20th British Combinatorial Conference.
- [8] Paul Dorbec, Gašper Košmrlj, and Gabriel Renault. The domination game played on unions of graphs. *Discrete Math.*, 338(1):71–79, 2015.
- [9] Carl Feghali. Paths between colourings of sparse graphs. *European Journal of Combinatorics*, 75:169 – 171, 2019.
- [10] Michael R. Garey and David S. Johnson. *Computers and Intractability; A Guide to the Theory of NP-Completeness*. W. H. Freeman & Co., New York, NY, USA, 1990.

- [11] Parikshit Gopalan, Phokion G. Kolaitis, Elitza Maneva, and Christos H. Papadimitriou. The connectivity of boolean satisfiability: Computational and structural dichotomies. *SIAM J. Comput.*, 38(6):2330–2355, March 2009.
- [12] R. Haas and K. Seyffarth. The k-dominating graph. *Graphs and Combinatorics*, 30(3):609–617, May 2014.
- [13] Arash Haddadan, Takehiro Ito, Amer E. Mouawad, Naomi Nishimura, Hirotaka Ono, Akira Suzuki, and Youcef Tebbal. The complexity of dominating set reconfiguration. *Theor. Comput. Sci.*, 651(C):37–49, October 2016.
- [14] Robert A. Hearn and Erik D. Demaine. Pspace-completeness of sliding-block puzzles and other problems through the nondeterministic constraint logic model of computation. *Theor. Comput. Sci.*, 343(1-2):72–96, October 2005.
- [15] Takehiro Ito, Erik D. Demaine, Nicholas J.A. Harvey, Christos H. Papadimitriou, Martha Sideri, Ryuhei Uehara, and Yushi Uno. On the complexity of reconfiguration problems. *Theoretical Computer Science*, 412(12):1054 – 1065, 2011.
- [16] Takehiro Ito, Haruka Mizuta, Naomi Nishimura, and Akira Suzuki. Incremental optimization of independent sets under the reconfiguration framework. In *Computing and Combinatorics - 25th International Conference, COCOON 2019, Xi'an, China, July 29-31, 2019, Proceedings*, pages 313–324, 2019.
- [17] Marcin Kamiński, Paul Medvedev, and Martin Milanič. Complexity of independent set reconfigurability problems. *Theoretical Computer Science*, 439:9 – 15, 2012.
- [18] J.Mark Keil. The complexity of domination problems in circle graphs. *Discrete Applied Mathematics*, 42(1):51 – 63, 1993.
- [19] Daniel Lokshtanov and Amer E. Mouawad. The complexity of independent set reconfiguration on bipartite graphs. In *Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2018, New Orleans, LA, USA, January 7-10, 2018*, pages 185–195, 2018.
- [20] Daniel Lokshtanov, Amer E. Mouawad, Fahad Panolan, M.S. Ramanujan, and Saket Saurabh. Reconfiguration on sparse graphs. *Journal of Computer and System Sciences*, 95:122 – 131, 2018.
- [21] A. Mouawad, N. Nishimura, V. Pathak, and V. Raman. Shortest reconfiguration paths in the solution space of boolean formulas. *SIAM Journal on Discrete Mathematics*, 31(3):2185–2200, 2017.
- [22] Amer E. Mouawad, Naomi Nishimura, Venkatesh Raman, Narges Simjour, and Akira Suzuki. On the parameterized complexity of reconfiguration problems. *Algorithmica*, 78(1):274–297, May 2017.
- [23] Naomi Nishimura. Introduction to reconfiguration. *Algorithms*, 11(4):52, 2018.
- [24] Maw shang Chang. Efficient algorithms for the domination problems on interval and circular-arc graphs. *SIAM J. Comput.*, 27:1671–1694, 1998.
- [25] Akira Suzuki, Amer E. Mouawad, and Naomi Nishimura. Reconfiguration of dominating sets. In Zhipeng Cai, Alex Zelikovskiy, and Anu Bourgeois, editors, *Computing and Combinatorics*, pages 405–416, Cham, 2014. Springer International Publishing.
- [26] Jan van den Heuvel. The complexity of change. In Simon R. Blackburn, Stefanie Gerke, and Mark Wildon, editors, *Surveys in Combinatorics*, volume 409 of *London Mathematical Society Lecture Note Series*, pages 127–160. Cambridge University Press, 2013.
- [27] Marcin Wrochna. Reconfiguration in bounded bandwidth and tree-depth. *Journal of Computer and System Sciences*, 93:1 – 10, 2018.