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# EXISTENCE AND STABILITY OF INFINITE TIME BLOW-UP IN THE KELLER-SEGEL SYSTEM

JUAN DÁVILA, MANUEL DEL PINO, JEAN DOLBEAULT, MONICA MUSSO,  
AND JUNCHENG WEI

ABSTRACT. Perhaps the most classical diffusion model for chemotaxis is the Keller-Segel system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) & \text{in } \mathbb{R}^2 \times (0, \infty), \\ v = (-\Delta_{\mathbb{R}^2})^{-1} u := \frac{1}{2\pi} \int_{\mathbb{R}^2} \log \frac{1}{|x-z|} u(z, t) dz, \\ u(\cdot, 0) = u_0 \geq 0 & \text{in } \mathbb{R}^2. \end{cases} \quad (*)$$

We consider the *critical mass case*  $\int_{\mathbb{R}^2} u_0(x) dx = 8\pi$  which corresponds to the exact threshold between finite-time blow-up and self-similar diffusion towards zero. We find a radial function  $u_0^*$  with mass  $8\pi$  such that for any initial condition  $u_0$  sufficiently close to  $u_0^*$  the solution  $u(x, t)$  of (\*) is globally defined and blows-up in infinite time. As  $t \rightarrow +\infty$  it has the approximate profile

$$u(x, t) \approx \frac{1}{\lambda^2} U_0 \left( \frac{x - \xi(t)}{\lambda(t)} \right), \quad U_0(y) = \frac{8}{(1 + |y|^2)^2},$$

where  $\lambda(t) \approx \frac{c}{\sqrt{\log t}}$ ,  $\xi(t) \rightarrow q$  for some  $c > 0$  and  $q \in \mathbb{R}^2$ . This result answers affirmatively the nonradial stability conjecture raised in [26].

## 1. INTRODUCTION

This paper deals with the classical Keller-Segel problem in  $\mathbb{R}^2$ ,

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) & \text{in } \mathbb{R}^2 \times (0, \infty), \\ v = (-\Delta_{\mathbb{R}^2})^{-1} u := \frac{1}{2\pi} \int_{\mathbb{R}^2} \log \frac{1}{|x-z|} u(z, t) dz, \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{R}^2, \end{cases} \quad (1.1)$$

which is a well-known model for the dynamics of a population density  $u(x, t)$  evolving by diffusion with a chemotactic drift. We consider positive solutions which are well defined, unique and smooth up to a maximal time  $0 < T \leq +\infty$ . This problem formally preserves mass, in the sense that

$$\int_{\mathbb{R}^2} u(x, t) dx = \int_{\mathbb{R}^2} u_0(x) dx =: M \quad \text{for all } t \in (0, T).$$

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An interesting feature of (1.1) is the connection between the second moment of the solution and its mass which is precisely given by

$$\frac{d}{dt} \int_{\mathbb{R}^2} |x|^2 u(x, t) dx = 4M - \frac{M^2}{2\pi},$$

provided that the second moments are finite. If  $M > 8\pi$ , the negative rate of production of the second moment and the positivity of the solution implies finite blow-up time. If  $M < 8\pi$  the solution lives at all times and diffuses to zero with a self similar profile according to [5]. When  $M = 8\pi$  the solution is globally defined in time. If the initial second moment is finite, it is preserved in time, and there is *infinite time blow-up* for the solution, as was shown in [4].

Globally defined in time solutions of (1.1) are of course its positive finite mass steady states, which consist of the family

$$U_{\lambda, \xi}(x) = \frac{1}{\lambda^2} U_0 \left( \frac{x - \xi}{\lambda} \right), \quad U_0(y) = \frac{8}{(1 + |y|^2)^2}, \quad \lambda > 0, \quad \xi \in \mathbb{R}^2. \quad (1.2)$$

We observe that all these steady states have the exact mass  $8\pi$  and infinite second moment

$$\int_{\mathbb{R}^2} U_{\lambda, \xi}(x) dx = 8\pi, \quad \int_{\mathbb{R}^2} |x|^2 U_{\lambda, \xi}(x) dx = +\infty.$$

As a consequence, if a solution of (1.1) is attracted by the family  $(U_{\lambda, \xi})$ , its mass must be larger than  $8\pi$  and if the initial second moment is finite, then blow-up occurs in a singular limit corresponding to  $\lambda \rightarrow 0_+$ .

In the critical mass  $M = 8\pi$  case, the infinite-time blow-up in (1.1) when the second moment is finite, takes place in the form of a bubble in the form (1.2) with  $\lambda = \lambda(t) \rightarrow 0$  according to [2, 4]. Formal rates and precise profiles were derived in [12, 8] to be

$$\lambda(t) \sim \frac{c}{\sqrt{\log t}} \quad \text{as } t \rightarrow +\infty.$$

A radial solution with this rate was built in [26] and its stability within the radial class was established. However, the nonradial stability was left as an outstanding open question. A positive answer was conjectured in [26] based on a crucial spectral gap estimate for a linearized operator, for which numerical evidence was obtained in [7].

In this paper we construct an infinite-time blow-up solution with an entirely different method to that in [26], which in particular leads to a proof of the stability assertion among non-radial functions. The following is our main result.

**Theorem 1.** *There exists a nonnegative, radially symmetric function  $u_0^*(x)$  with critical mass  $\int_{\mathbb{R}^2} u_0^*(x) dx = 8\pi$  and finite second moment  $\int_{\mathbb{R}^2} |x|^2 u_0^*(x) dx < +\infty$  such that for every  $u_0$  sufficiently close (in suitable sense) to  $u_0^*$  with  $\int_{\mathbb{R}^2} u_0 dx = 8\pi$ , we have that the associated solution  $u(x, t)$  of system (1.1) has the form*

$$u(x, t) = \frac{1}{\lambda(t)^2} U_0 \left( \frac{x - \xi(t)}{\lambda(t)} \right) (1 + o(1)), \quad U_0(y) = \frac{8}{(1 + |y|^2)^2}$$

*uniformly on bounded sets of  $\mathbb{R}^n$ , and*

$$\lambda(t) = \frac{c}{\sqrt{\log t}} (1 + o(1)), \quad \xi(t) \rightarrow q \quad \text{as } t \rightarrow +\infty,$$

*for some number  $c > 0$  and some  $q \in \mathbb{R}^2$ .*

*Sufficiently close* for the perturbation  $u_0(x) := u_0^*(x) + \varphi(x)$  in this result is measured in the  $C^1$ -weighted norm for some  $\sigma > 0$

$$\|\varphi\|_{*\sigma} := \|(1 + |\cdot|^{4+\sigma})\varphi\|_{L^\infty(\mathbb{R}^2)} + \|(1 + |\cdot|^{5+\sigma})\nabla\varphi(x)\|_{L^\infty(\mathbb{R}^2)} < +\infty. \quad (1.3)$$

We observe that for any  $\sigma > 0$  this decay condition implies that the second moment of  $\varphi$  is finite, which is not the case for  $\sigma \leq 0$ .

The scaling parameter is rather simple to find at main order from the approximate conservation of second moment. The center  $\xi(t)$  actually obeys a relatively simple system of nonlocal ODEs.

We devote the rest of this paper to the proof of Theorem 1. Our approach borrows elements of constructions in the works [16, 21, 18, 17] based on the so-called *inner-outer gluing scheme*, where a system is derived for an inner equation defined near the blow-up point and expressed in the variable of the blowing-up bubble, and an outer problem that sees the whole picture in the original scale. The result of Theorem 1 has already been announced in [20] in connection with [16, 21, 18].

There is a huge literature on chemotaxis in biology and in mathematics. The Patlak-Keller-Segel model [43, 35] is used in mathematical biology to describe the motion of mono-cellular organisms, like Dictyostelium Discoideum, which move randomly but experience a drift in presence of a chemo-attractant. Under certain circumstances, these cells are able to emit the chemo-attractant themselves. Through the chemical signal, they coordinate their motion and eventually aggregate. Such a self-organization scenario is at the basis of many models of chemotaxis and is considered as a fundamental mechanism in biology. Of course, the aggregation induced by the drift competes with the noise associated with the random motion so that aggregation occurs only if the chemical signal is strong enough. A classical survey of the mathematical problems in chemotaxis models can be found in [31, 32]. After a proper adimensionalization, it turns out that all coefficients in the Patlak-Keller-Segel model studied in this paper can be taken equal to 1 and that the only free parameter left is the total mass. For further considerations on chemotaxis, we shall refer to [30] for biological models and to [11] for physics backgrounds.

In many situations of interest, cells are moving on a substrate. The two-dimensional case is therefore of special interest in biology, but also turns out to be particularly interesting from the mathematical point of view as well, because of scaling properties, at least in the simplest versions of the Keller-Segel model. Boundary conditions induce various additional difficulties. In the idealized situation of the Euclidean plane  $\mathbb{R}^2$ , it is known since the early work of W. Jäger and S. Luckhaus in [33] that solutions globally exist if the mass  $M$  is small and blow-up in finite time if  $M$  is large. The blow-up in a bounded domain is studied in [33, 1, 38, 39, 45]. The precise threshold for blow-up,  $M = 8\pi$ , has been determined in [23, 5], with sufficient conditions for global existence if  $M \leq 8\pi$  in [5] (also see [22] in the radial case). The key estimate is the boundedness of the free energy, which relies on the logarithmic Hardy-Littlewood-Sobolev inequality established in optimal form in [9]. We refer to [3] for a review of related results. If  $M < 8\pi$ , diffusion dominates: intermediate asymptotic profiles and exact rates of convergence have been determined in [7]. Also see [40, 25]. In the supercritical case  $M > 8\pi$ , various formal expansions are known for many years, starting with [27, 28, 48] which were later justified in [44], in the radial case, and in [14], in the non-radially symmetric regime. This latter result is based on the analysis of the spectrum of a linearized

operator done in [15], based on the earlier work [19], and relies on a scalar product already considered in [44] and similar to the one used in [6, 7] in the subcritical mass regime. An interesting subproduct of the blow-up mechanism in [44, 29] is that the blow-up takes the form of a concentration in the form of a Dirac distribution with mass exactly  $8\pi$  at blow-up time, as was expected from [29, 24], but it is still an open question to decide whether this is, locally in space, the only mechanism of blow-up.

The critical mass case  $M = 8\pi$  is more delicate. If the second moment is infinite, there is a variety of behaviors as observed for instance in [36, 37, 42]. For solutions with finite second moment, blow-up is expected to occur as  $t \rightarrow +\infty$ : see [34] for grow-up rates in  $\mathbb{R}^2$ , and [47] for the higher-dimensional radial case. The existence in  $\mathbb{R}^2$  of a global radial solution and first results of large time asymptotics were established in [2] using cumulated mass functions. In [4], the infinite time blow-up was proved without symmetry assumptions using the free energy and an assumption of boundedness of the second moment. Also see [41, 42] for an existence result under weaker assumptions, and further estimates on the solutions. Asymptotic stability of the family of steady states determined by (1.2) under the mass constraint  $M = 8\pi$  has been determined in [10]. The blow-up rate  $\lambda(t)$  and the shape of the limiting profile  $U$  were identified in formal asymptotic expansions in [49, 50, 46, 12, 13] and also in [8, Chapter 8]. As already mentioned, a radial solution with rate  $\lambda(t) \sim (\log t)^{-1/2}$  was built and its stability within the radial class was established in [26].

## 2. AN APPROXIMATE SOLUTION AND THE INNER-OUTER GLUING SYSTEM

We consider the Keller-Segel system in entire  $\mathbb{R}^2$

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) & \text{in } \mathbb{R}^2 \times (0, \infty), \\ v = (-\Delta_{\mathbb{R}^2})^{-1} u := \frac{1}{2\pi} \int_{\mathbb{R}^2} \log \frac{1}{|x-z|} u(z, t) dz & \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{R}^2. \end{cases} \quad (2.1)$$

We will build a first approximation to a solution  $u(x, t)$  globally defined in time such that on bounded sets in  $x$  we have

$$u(x, t) = \frac{1}{\lambda(t)^2} U_0 \left( \frac{x - \xi(t)}{\lambda(t)} \right) (1 + o(1)) \quad \text{as } t \rightarrow +\infty \quad (2.2)$$

for certain functions  $0 < \lambda(t) \rightarrow 0$  and  $\xi(t) \rightarrow q \in \mathbb{R}^2$ . Here we recall

$$U_0(y) = \frac{8}{(1 + |y|^2)^2}.$$

In §2.1 we formally derive asymptotic expressions for the parameter functions. This allows us to define an adequate range for their values we consider. In §2.2 we build an additive correction of the resulting “bubble” which improves the error of approximation in the remote regime. In §2.3 we introduce the *inner-outer gluing system* for additive corrections of the approximation, that respectively distinguish local and remote regimes relative to the concentration regions.

**2.1. Formal derivation of  $\lambda(t)$ .** We know that (2.2) can only happen in the critical mass, finite second moment case:

$$\int_{\mathbb{R}^2} u(x, t) dx = 8\pi, \quad \int_{\mathbb{R}^2} |x|^2 u(x, t) dx < +\infty,$$

which according to the results in [4, 26, 12] is consistent with a behavior of the form (2.2). Since the second moment of  $U_0$  is infinite, we do not expect the approximation (2.2) be uniform in  $\mathbb{R}^2$  but sufficiently far, a faster decay in  $x$  should take place. We will find an approximate asymptotic expression for the scaling parameter  $\lambda(t)$  that matches with this behavior.

Let us introduce the function  $V_0 := (-\Delta)^{-1}U_0$ . We directly compute

$$V_0(y) = \log \frac{8}{(1 + |y|^2)^2}$$

and hence  $V_0$  solves Liouville equation

$$-\Delta V_0 = e^{V_0} = U_0 \quad \text{in } \mathbb{R}^2.$$

Then  $\nabla V_0(y) \approx -\frac{4y}{|y|^2}$  for all large  $y$ , and hence we get, away from  $x = \xi$ ,

$$-\nabla \cdot (u \nabla (-\Delta)^{-1}u) \approx 4 \nabla u \cdot \frac{x - \xi}{|x - \xi|^2}.$$

Hence defining

$$\mathcal{E}(u) := \Delta u - \nabla \cdot (u \nabla (-\Delta)^{-1}u) \quad (2.3)$$

and writing in polar coordinates

$$u(r, \theta, t) = u(x, t), \quad x = \xi(t) + r e^{i\theta},$$

we find  $\mathcal{E}(u) \approx \partial_r^2 u + \frac{5}{r} \partial_r u$  and hence, assuming  $\dot{\xi}(t) \rightarrow 0$  sufficiently fast, equation (2.1) approximately reads

$$\partial_t u = \partial_r^2 u + \frac{5}{r} \partial_r u, \quad (2.4)$$

which can be idealized as a homogeneous heat equation in  $\mathbb{R}^6$  for radially symmetric functions. It is therefore reasonable to believe that beyond the self-similar region  $r \gg \sqrt{t}$  the behavior changes into a function of  $r/\sqrt{t}$  with fast decay at  $+\infty$  that yields finiteness of the second moment. To obtain a first global approximation, we simply cut-off the bubble (2.2) beyond the self-similar zone. We introduce a further parameter  $\alpha(t)$  and set

$$u_1(x, t) = \frac{\alpha(t)}{\lambda^2} U_0 \left( \frac{|x - \xi|}{\lambda} \right) \chi(x, t), \quad (2.5)$$

where we denote

$$\chi(x, t) = \chi_0 \left( \frac{|x - \xi|}{\sqrt{t}} \right)$$

with  $\chi_0(s)$  a smooth cut-off function such that

$$\chi_0(s) = \begin{cases} 1 & \text{if } s \leq 1 \\ 0 & \text{if } s \geq 2. \end{cases}$$

The reason why we introduce the parameter  $\alpha(t)$  is because the total mass of the actual solution should equal  $8\pi$  for all  $t$ . For the moment let us just impose

$$\int_{\mathbb{R}^2} u_1(x, t) dx = 8\pi.$$

From a direct computation we arrive to the relation  $\alpha = \alpha_0$  where

$$\alpha_0(t) = 1 + a \frac{\lambda^2}{t} (1 + o(1)), \quad a = 2 \int_0^\infty \frac{1 - \chi_0(s)}{s^3} ds.$$

Next we will obtain an approximate value of the scaling parameter  $\lambda(t)$  that is consistent with the presence of a solution  $u(x, t) \approx u_1^0(x, t)$  where  $u_1^0$  is the function  $u_1$  in (2.5) with  $\alpha = \alpha_0$ . Let us consider the “error operator”

$$S(u) = -u_t + \mathcal{E}(u) \tag{2.6}$$

where  $\mathcal{E}(u)$  is defined in (2.3). We have the following well-known identities, valid for an arbitrary function  $\omega(x)$  of class  $C^2(\mathbb{R}^2)$  with finite mass and  $D^2\omega(x) = O(|x|^{-4-\sigma})$  for large  $|x|$ . We have

$$\int_{\mathbb{R}^2} |x|^2 \mathcal{E}(\omega) dx = 4M - \frac{M^2}{2\pi}, \quad M = \int_{\mathbb{R}^2} \omega(x) dx \tag{2.7}$$

and

$$\int_{\mathbb{R}^2} x \mathcal{E}(\omega) dx = 0, \quad \int_{\mathbb{R}^2} \mathcal{E}(\omega) dx = 0. \tag{2.8}$$

Let us recall the simple proof of (2.7). Integrating by parts on finite balls with large radii and using the behavior of the boundary terms we get the identities

$$\begin{aligned} \int_{\mathbb{R}^2} |x|^2 \Delta \omega dx &= 4M, \\ \int_{\mathbb{R}^2} |x|^2 \nabla \cdot (\nabla(-\Delta)^{-1}) \omega dx &= -2 \int_{\mathbb{R}^2} x \cdot \nabla(-\Delta)^{-1} \omega dx \\ &= \frac{1}{\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \omega(x) \omega(y) \frac{x \cdot (x-y)}{|x-y|^2} dx dy \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \omega(x) \omega(y) \frac{(x-y) \cdot (x-y)}{|x-y|^2} dx dy \\ &= \frac{M^2}{2\pi} \end{aligned} \tag{2.9}$$

and then (2.7) follows. The proof of (2.8) is even simpler. For a solution  $u(x, t)$  of (2.1) we then get

$$\frac{d}{dt} \int_{\mathbb{R}^2} u(x, t) |x|^2 dx = 4M - \frac{M^2}{2\pi}, \quad M = \int_{\mathbb{R}^2} u(x, t) dx. \tag{2.10}$$

At this point we also point out that the first moments, namely the center of mass of  $u$  in space is preserved since

$$\frac{d}{dt} \int_{\mathbb{R}^2} u(x, t) x_i dx = 0.$$

In particular, if  $u(x, t)$  is sufficiently close to  $u_1^0(x, t)$  and since  $\int_{\mathbb{R}^2} u_1(x, t) dx = 8\pi$  we get the approximate validity of the identity

$$\frac{d}{dt} \int_{\mathbb{R}^2} u_1(x, t) |x|^2 dx = 0.$$

This means

$$aI(t) := \int_{\mathbb{R}^2} \frac{\alpha_0}{\lambda^2} U_0 \left( \frac{x - \xi}{\lambda} \right) \chi_0 \left( \frac{|x - \xi|}{\sqrt{t}} \right) |x|^2 dx = \text{constant}.$$

We readily check that for some constant  $\kappa$

$$I(t) = 16\pi\lambda^2 \int_0^{\frac{\sqrt{t}}{\lambda}} \frac{\rho^3 d\rho}{(1 + \rho^2)^2} + \kappa + o(1) = 16\pi\lambda^2 \log \frac{\sqrt{t}}{\lambda} + \kappa + o(1).$$

Then we conclude that  $\lambda(t)$  approximately satisfies

$$\lambda^2 \log t = c^2 = \text{constant}$$

and hence we get at main order

$$\lambda(t) = \frac{c}{\sqrt{\log t}}.$$

We also notice that the center of mass is preserved for a true solution, thanks to (2.10):

$$\frac{d}{dt} \int_{\mathbb{R}^2} xu(x, t) dx = 0,$$

since the center of mass of  $u_1(x, t)$  is exactly  $\xi(t)$  we then get that approximately

$$\xi(t) = \text{constant} = q$$

**2.2. First error and improvement of approximation.** We consider as a first approximation to a solution to (2.1) the function  $u_1(x, t)$  defined by (2.5).

Motivated by the previous considerations we introduce the hypotheses that we make on the parameters  $\lambda(t) > 0$ ,  $\xi(t) \in \mathbb{R}^2$  and  $\alpha(t)$  in (2.5) that satisfy

$$\lambda(+\infty) = 0, \quad \alpha(+\infty) = 1.$$

We let

$$\lambda_*(t) = \frac{1}{\sqrt{\log t}}.$$

For fixed numbers  $M \geq 1$ ,  $t_0 > 0$  which we will later take sufficiently large, and a small  $\sigma > 0$  we assume the following bounds for the derivatives of parameters hold for all  $t \in (t_0, +\infty)$ .

$$|\dot{\lambda}(t)| \leq M|\dot{\lambda}_*(t)|, \quad |\dot{\alpha}(t)| \leq M\frac{\lambda_*^2}{t^2}, \quad |\dot{\xi}(t)| \leq M\frac{1}{t^{1+\sigma}}. \quad (2.11)$$

We will find an expression for the error of approximation  $S(u_1)$  where  $S(u)$  is the error operator (2.6), and then will build a modification

$$u_2(x, t) = u_1(x, t) + \varphi_1(x, t) \quad (2.12)$$

of  $u_1$  such that the associated error gets reduced beyond the self-similar region.

For the sake of computation, it is convenient to rewrite  $u_1(x, t)$  in (2.5) in the form

$$u_1(x, t) = \frac{\alpha}{\lambda^2} U_0(y) \chi(y, t), \quad \chi(y, t) = \chi_0 \left( \frac{\lambda|y|}{\sqrt{t}} \right), \quad y = \frac{x - \xi}{\lambda}.$$

We compute

$$S(u_1) = -\partial_t u_1 + \mathcal{E}(u_1) = S^i + S^o + \mathcal{E}(u_1)$$



where, writing  $y = \frac{x-\xi(t)}{\lambda(t)}$ ,

$$\begin{aligned} S^i &= -\frac{\dot{\alpha}}{\lambda^2} U_0(y) \chi + \alpha \frac{\dot{\lambda}}{\lambda^3} (2U_0(y) + y \cdot \nabla_y U_0(y)) \chi + \frac{\alpha}{\lambda^3} \dot{\xi} \cdot \nabla_y U_0(y) \chi \\ S^o &= \frac{1}{2} \alpha \frac{|x-\xi|}{\lambda^2 t^{\frac{3}{2}}} U_0 \chi_0' \left( \frac{|x-\xi|}{\sqrt{t}} \right) + \frac{\alpha}{\lambda^3 \sqrt{t}} \dot{\xi} \cdot \frac{y}{|y|} U_0(y) \chi_0' \left( \frac{|x-\xi|}{\sqrt{t}} \right) \end{aligned}$$

and, we recall,  $\mathcal{E}(u) = \Delta u - \nabla \cdot (u \nabla (-\Delta)^{-1} u)$ . The superscripts  $i$  and  $o$  respectively refer to “inner” and “outer” parts of the error that will later be dealt with separately.

Using the notation

$$V(z) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log \frac{1}{|z-y|} U_0(z, t) \chi(y, t) dy,$$

we decompose

$$\mathcal{E}(u_1) = \mathcal{E}^o + \mathcal{E}^i$$

where

$$\begin{aligned} \mathcal{E}^i &= \lambda^{-4} [\alpha(\alpha-1)U_0^2 \chi - \alpha(\alpha-1)\nabla_y U_0 \nabla_y V_0 \chi] \\ \mathcal{E}^o &= \lambda^{-4} [2\alpha \nabla_y U_0 \cdot \nabla_y \chi + \alpha U_0 \Delta_y \chi + U_0^2 \alpha^2 \chi(\chi-1) \\ &\quad - \alpha^2 U_0 \nabla_y \chi \nabla_y V - \alpha^2 \chi \nabla_y U_0 \nabla_y (V - V_0)]. \end{aligned}$$

Next we introduce a correction  $\varphi_1(x, t)$  as in (2.12) that eliminates the main terms of the error  $\mathcal{E}^o + S^o$ , which can be written as

$$\frac{\lambda^2}{t^3} h(\zeta), \quad \zeta = \frac{|x-\xi|}{\sqrt{t}},$$

where

$$h(\zeta) = \frac{8}{\zeta^4} \left[ \chi_0'' - \frac{3}{\zeta} \chi_0'(\zeta) + \frac{\zeta}{2} \chi_0'(\zeta) \right].$$

In agreement with the approximate expression (2.4) for the remote regime of (2.1), we look for the correction  $\varphi_1$  in the form

$$\varphi_1(x, t) = \lambda^2 \tilde{\varphi}_1(|x-\xi|, t), \quad (2.13)$$

where  $\tilde{\varphi}_1(r, t)$  solves the radial heat equation in dimension 6:

$$\begin{cases} \partial_t \tilde{\varphi}_1 = \partial_r^2 \tilde{\varphi}_1 + \frac{5}{r} \partial_r \tilde{\varphi}_1 + \frac{1}{t^3} h\left(\frac{r}{\sqrt{t}}\right), \\ \tilde{\varphi}_1(r, 0) = 0. \end{cases} \quad (2.14)$$

The solution  $\tilde{\varphi}_1(r, t)$  to problem (2.14) can be expressed in self-similar form as

$$\tilde{\varphi}_1(r, t) = \frac{1}{t^2} g(\zeta), \quad \zeta = \frac{r}{\sqrt{t}}.$$

We find for  $g$  the equation

$$g'' + \frac{5}{\zeta} g' + \frac{\zeta}{2} g' + 2g + h(\zeta) = 0, \quad \zeta \in (0, \infty). \quad (2.15)$$

Using that the function  $\frac{1}{\zeta^4}$  is in the kernel of the homogenous equation, we find the explicit solution of (2.15),

$$g_0(\zeta) = -\frac{1}{\zeta^4} \int_0^\zeta x^3 e^{-\frac{1}{4}x^2} \int_0^x h(y) e^{\frac{1}{4}y^2} y dy dx.$$

To find the solution  $\varphi_1$  with suitable decay at infinity we let

$$g(\zeta) = g_0(\zeta) + \frac{1}{8} \bar{z}(\zeta) I, \quad (2.16)$$

where

$$\bar{z}(\zeta) = \frac{1}{\zeta^4} \int_0^\zeta x^3 e^{-\frac{1}{4}x^2} dx$$

is a second solution of the homogeneous equation, linearly independent of  $\frac{1}{\zeta^4}$  and

$$I = \int_0^\infty x^3 e^{-\frac{1}{4}x^2} \int_0^x h(y) e^{\frac{1}{4}y^2} y dy dx.$$

We observe that

$$g(\zeta) = O(e^{-\frac{1}{4}\zeta^2}) \quad \text{as } \zeta \rightarrow +\infty,$$

which makes the solution (2.16) the only one with decay faster than  $O(\zeta^{-4})$  as  $\zeta \rightarrow +\infty$ . An explicit calculation gives that  $I = -8$ , and therefore

$$\varphi_1(\xi(t), t) = -\frac{\lambda(t)^2}{4t^2}, \quad (2.17)$$

an identity that will play a crucial role in later computations.

We take then as the basic approximation the function  $u_2(x, t)$  in the form (2.12) defined as

$$u_2(x, t) = \frac{\alpha(t)}{\lambda^2} U_0 \left( \frac{x - \xi(t)}{\lambda(t)} \right) \chi_0 \left( \frac{|x - \xi(t)|}{\sqrt{t}} \right) + \varphi_1(x, t), \quad (2.18)$$

where  $\varphi_1$  is defined by (2.13). Accordingly, we write

$$\psi_1 = (-\Delta_x)^{-1} \varphi_1, \quad v_2 = (-\Delta_x)^{-1} u_2. \quad (2.19)$$

A direct computation of the new error yields the validity of the following expansion.

**Lemma 2.1.** *Let  $u_2$  be given by (2.18) with  $\varphi_1(x, t)$  defined as in (2.13). The error of approximation  $S(u_2)$  can be expressed as*

$$S(u_2) = \lambda^{-4} \left[ \alpha \lambda \dot{\lambda} Z_0(y) + \alpha \lambda \dot{\xi}_1 Z_1(y) + \alpha \lambda \dot{\xi}_2 Z_2(y) + \lambda^2 \dot{\alpha} Z_3(y) \right] + \lambda^{-4} \mathcal{E} \chi \\ + \lambda^{-4} \mathcal{R}_1 \chi + \mathcal{R}_2 (1 - \chi)$$

where

$$\mathcal{E} = \alpha(\alpha - 1) [U_0^2(y) - \nabla_y U_0(y) \nabla_y V_0(y)] \\ + \lambda^4 [-\nabla_x u_1 \nabla_x (-\Delta_x)^{-1} \varphi_1 + 2u_1 \varphi_1], \quad y = \frac{x - \xi}{\lambda}, \quad (2.20)$$

the functions  $Z_j(y)$  are defined as

$$Z_0(y) = 2U_0(y) + y \cdot \nabla U_0(y), \\ Z_j(y) = \frac{\partial U_0}{\partial y_j}(y), \quad j = 1, 2, \\ Z_3(y) = -U_0(y) \quad (2.21)$$

and the remainders  $\mathcal{R}_1, \mathcal{R}_2$  respectively satisfy

$$\begin{aligned} |\mathcal{R}_1(y, t)| &\leq CM^2 \frac{\lambda^6}{t^3(1+|y|^2)} \\ |\mathcal{R}_2(x, t)| &\leq CM^2 \frac{\lambda^4}{t^3} e^{-c\frac{|x-\xi|^2}{t}} \end{aligned}$$

for universal constants  $c, C > 0$ , and where  $M$  is the number in constraints (2.11).

In addition, we check that the function  $v_2 = (-\Delta)^{-1}u_2$  can be expanded as

$$v_2(x, t) = V_0(y) + \psi_1(x, t) + \tilde{v}_2(x, t), \quad y = \frac{x - \xi}{\lambda}, \quad (2.22)$$

where the functions  $\psi_1(x, t), \tilde{v}_2(x, t)$  satisfy the gradient estimates

$$\begin{aligned} |\nabla_x \psi_1(x, t)| &\leq C \begin{cases} \frac{\lambda^2}{t^2} |x - \xi|, & |x - \xi| \leq \sqrt{t}, \\ \frac{\lambda^2}{t} \frac{1}{|x - \xi|}, & |x - \xi| \geq \sqrt{t}, \end{cases} \\ |\nabla_x \tilde{v}_2(x, t)| &\leq C \frac{\lambda^2}{t^2} \frac{1}{|x - \xi| + \lambda}. \end{aligned}$$

**2.3. The inner-outer gluing system.** We look for a solution of Equation (2.1), which we write in the form

$$\begin{cases} S(u) := -u_t + \Delta_x u - \nabla_x \cdot (u \nabla_x v) = 0 & \text{in } \mathbb{R}^2 \times (0, \infty), \\ v = (-\Delta_x)^{-1} u := \frac{1}{2\pi} \int_{\mathbb{R}^2} \log \frac{1}{|x - z|} u(z, t) dz \end{cases} \quad (2.23)$$

We look for a solution  $(u, v)$  of (2.23) as a perturbation  $(\varphi, \psi)$  of  $(u_2, v_2)$

$$\begin{aligned} u &= u_2 + \varphi, \\ v &= v_2 + \psi, \end{aligned} \quad (2.24)$$

where  $\psi = (-\Delta_x)^{-1}\varphi$ . Then (2.23) can be written as

$$-\varphi_t + \mathcal{L}_{u_2}[\varphi] - \nabla \varphi \nabla \psi + \varphi^2 + S(u_2) = 0, \quad \psi = (-\Delta_x)^{-1}\varphi, \quad (2.25)$$

where  $\mathcal{L}_{u_*}[\varphi]$  designates the linearized operator for the elliptic part of (2.23) around a function  $u_*$ , which setting  $v_* = (-\Delta_x)^{-1}u_*$  is given by

$$\mathcal{L}_{u_*}[\varphi] := \Delta_x \varphi - \nabla_x v_* \cdot \nabla_x \varphi - \nabla_x u_* \cdot \nabla_x \psi + 2u_* \varphi, \quad \psi = (-\Delta_x)^{-1}\varphi$$

For  $|x - \xi| \ll \sqrt{t}$  we have the approximate expressions

$$u_2(x, t) \approx \lambda^{-2}U_0(y), \quad v_2(x, t) \approx V_0(y) = \log U_0(y), \quad y = \frac{x - \xi}{\lambda}.$$

Writing in this region

$$\varphi(x, t) = \lambda^{-2}\phi(y, t), \quad \psi(x, t) = (-\Delta_y)^{-1}\phi(y, t), \quad y = \frac{x - \xi}{\lambda},$$

the operator  $\mathcal{L}_{u_2}[\varphi]$  gets approximated by the *inner linearized operator*

$$\mathcal{L}_{u_2}[\varphi](x, t) \approx \lambda^{-4}L^i[\phi](y, t),$$

with

$$\begin{aligned} L^i[\phi](y, t) &:= \Delta_y \phi - \nabla_y V_0(y) \cdot \nabla_y \phi - \nabla_y U_0(y) \cdot \nabla_y \psi + 2U_0(y)\phi, \\ \psi(y, t) &= (-\Delta_y)^{-1}\phi(y, t). \end{aligned} \quad (2.26)$$

The operator  $L^i$  is the linearization of the static Keller-Segel equation

$$\Delta_y U - \nabla_y \cdot [U \nabla_y (-\Delta_y)^{-1} U] = 0 \quad \text{in } \mathbb{R}^2$$

around its solution  $U = U_0(y)$ . Similarly, the *outer linearized operator* associated to (2.25) is obtained noticing that  $\nabla_x u_* \cdot \nabla_x \psi$  is comparatively small, when confronted with the other terms. The actual operator is then remotely approximated by

$$\mathcal{L}_{u_2}[\varphi](x, t) \approx L^o[\varphi](x, t),$$

where

$$L^o[\varphi] := \Delta_x \varphi - \nabla_x v_2 \cdot \nabla_x \varphi, \quad (2.27)$$

and  $v_2$  is defined in (2.19). In polar coordinates referred to  $\xi(t)$  we approximately find

$$L^o[\varphi] \approx \varphi_{rr} + \frac{5}{r} \varphi_r + \frac{1}{r^2} \varphi_{\theta\theta}.$$

which for radial functions can be idealized as a  $6d$ -Laplacian.

Next we set up an ansatz for the perturbation  $\varphi$  in (2.24) which goes along with the decomposition (2.18),

$$u_2(x, t) = \frac{\alpha}{\lambda^2} U_0(y) \chi(x, t) + \varphi_1(x, t)$$

in which we make a distinction between *inner* and *outer* parts of the remainder. For certain functions  $\phi^i(y, t)$ ,  $\psi^i(y, t)$ ,  $\phi^o(x, t)$ ,  $\psi^i(x, t)$  we write

$$\begin{aligned} \varphi(x, t) &= \frac{1}{\lambda^2} \phi^i(y, t) \chi(x, t) + \phi^o(x, t), & y &= \frac{x - \xi}{\lambda} \\ \psi(x, t) &= \psi^i(y, t) \chi(x, t) + \psi^o(x, t) \end{aligned}$$

where we recall  $\chi(x, t) = \chi_0\left(\frac{|x - \xi|}{\sqrt{t}}\right)$ , with  $\chi_0(s)$  the smooth cut-off function in (2.6), and we impose the relations

$$\begin{aligned} \psi^i(y, t) &= (-\Delta_y)^{-1} \phi^i(y, t), \\ \psi^o(x, t) &= (-\Delta_x)^{-1} [\phi^o + 2\lambda^{-1} \nabla_y \psi^i \nabla_x \chi + \psi^i \Delta_x \chi], \end{aligned} \quad (2.28)$$

which readily yield  $\varphi = (-\Delta_x)^{-1} \psi$  and hence  $v = (-\Delta_x)^{-1} u$  in (2.24).

The idea is to define a coupled system of equations for the functions  $\phi^i, \phi^o, \psi^i, \psi^o$ , additional to relations (2.28), the *inner-outer gluing system*, which, if satisfied, gives a solution to (2.23) of the form (2.24). This system involves at main order parabolic evolutions of the linear elliptic operators  $L^i[\phi^i]$  and  $L^o[\phi^o]$ .

We impose on  $\phi^i$  and  $\phi^o$  a system of equations of the form

$$\lambda^2 \frac{\partial \phi^i}{\partial t} = L^i[\phi^i] + H(\phi^o, \psi^o, \phi^i, \psi^i, \lambda, \alpha, \xi) \quad (2.29)$$

$$\frac{\partial \phi^o}{\partial t} = L^o[\phi^o] + G(\phi^o, \psi^o, \phi^i, \psi^i, \lambda, \alpha, \xi). \quad (2.30)$$

The operator  $H$  involves the terms in equation (2.25) that are supported in the self-similar region  $|x - \xi(t)| \lesssim \sqrt{t}$ , and  $G$  all the remaining ones, the main of them a linear coupling with  $\phi^i$  around  $|x - \xi| \sim \sqrt{t}$  due to derivatives of  $\chi$ . This coupling will be weaker if the parameters  $\alpha(t)$ ,  $\lambda(t)$ ,  $\xi(t)$  satisfy certain solvability conditions which amount to 4 differential equations which are coupled with (2.28), (2.29), (2.30).

To define precisely the operators  $H$  and  $G$  it is convenient to introduce, for functions  $a(x)$ ,  $b(x)$ ,  $A(y)$ ,  $B(y)$ , the notation,

$$[a, b]_x = \nabla_x \cdot (a \nabla_x b), \quad [A, B]_y = \nabla_y \cdot (A \nabla_y B).$$

Precisely, we define

$$\begin{aligned} H(\phi^o, \psi^o, \phi^i, \psi^i, \lambda, \alpha, \xi)(y, t) &= \tilde{H}(\phi^o, \psi^o, \phi^i, \psi^i, \lambda, \alpha, \xi) \tilde{\chi}(y, t) \\ &\quad + [\alpha \lambda \dot{\lambda} Z_0(y) + \alpha \lambda \dot{\xi}_1 Z_1(y) + \alpha \lambda \dot{\xi}_2 Z_2(y) + \lambda^2 \dot{\alpha} Z_3(y)] \tilde{\chi}, \end{aligned} \quad (2.31)$$

where the functions  $Z_j(y)$  were defined in (2.21), the operator  $\tilde{H}$  is given by

$$\begin{aligned} \tilde{H}(\phi^o, \psi^o, \phi^i, \psi^i, \lambda, \alpha, \xi) &= \mathcal{E} + \mathcal{R}_1 + \lambda^2 U_0 \phi^o - [(\alpha \chi U_0 + \lambda^2 \varphi_1), \psi^o]_y \\ &\quad + \lambda \dot{\lambda} (2\phi^i + y \cdot \nabla_y \phi^i) + \lambda \dot{\xi} \cdot \nabla_y \phi^i \\ &\quad - (\alpha - 1)[U_0, \psi^i \chi]_y - \alpha[U_0(\chi - 1), \psi^i \chi]_y \\ &\quad - [U_0, \psi^i(1 - \chi)]_y - \lambda^2[\varphi_1, \psi^i \chi]_y - [\phi^i \chi, v_2 - V_0]_y \\ &\quad - [\phi^i \chi + \lambda^2 \phi^o, \psi^i \chi + \psi^o]_y, \end{aligned} \quad (2.32)$$

and

$$\tilde{\chi}(y, t) = \chi_0 \left( \frac{1}{2} \frac{\lambda}{\sqrt{t}} |y| \right), \quad (2.33)$$

where  $\mathcal{E}$  is defined in (2.20).

The operator  $G$  is given by

$$\begin{aligned} G(\phi^o, \psi^o, \phi^i, \psi^i, \lambda, \alpha, \xi) &= \frac{\phi^i}{\lambda^2} [\Delta_x \chi - \partial_t \chi] + \frac{2}{\lambda^2} \nabla_x \phi^i \cdot \nabla_x \chi \\ &\quad + (1 - \chi) \left[ S(u_2) + u_2 \phi^o - [u_2, \psi^o]_x - [\lambda^{-2} \phi^i \chi, v_2]_x \right. \\ &\quad \left. - [u_2, \psi^i \chi]_x - \lambda^{-2} [\phi^i(1 - \chi), V_0]_x + u_2 \phi^o - [u_2, \psi^o]_x \right. \\ &\quad \left. - [\lambda^{-2} \phi^i \chi + \phi^o, \psi^i \chi + \phi^o]_x \right]. \end{aligned} \quad (2.34)$$

It is straightforward to check that if  $\phi^i$ ,  $\psi^i$ ,  $\phi^o$ ,  $\psi^o$  satisfy the system (2.28), (2.29), (2.30), then

$$u = u_2 + \lambda^{-2} \phi^i \chi + \phi^o, \quad v = v_2 + \psi^i \chi + \psi^o$$

is a solution of the original problem (2.23), with  $u_2$ ,  $v_2$  defined in (2.18), (2.19).

The idea that motivates the solvability conditions mentioned above is to solve (2.29) for  $\phi^i(y, t)$  by temporarily ignoring the term  $\lambda^2 \phi^i$  and thus considering the elliptic equation

$$L^i[\phi^i] + H(\phi^o, \psi^o, \phi^i, \psi^i, \lambda, \alpha, \xi) = 0. \quad (2.35)$$

We would like that  $\phi^i(y, t)$  for  $y$  large becomes small so that at main order it does not influence the outer regime, mainly represented by the main coupling term  $\lambda^2 \phi^i [\Delta_x \chi - \partial_t \chi] + 2\lambda^{-2} \nabla_x \phi^i \cdot \nabla_x \chi$  in the definition of  $G$  in equation (2.30). To achieve this decay in  $y$  we need that  $H$  in (2.35) satisfies certain conditions. To state them, let us consider, more generally, the elliptic equation for  $\phi = \phi(y)$ ,  $h = h(y)$

$$L^i[\phi] + h(y) = 0 \quad \text{in } \mathbb{R}^2. \quad (2.36)$$

Let us assume that for some  $m \in (4, 6)$ ,

$$h(y) = O(|y|^{-m}) \quad \text{as } |y| \rightarrow +\infty \quad (2.37)$$

$$\int_{\mathbb{R}^2} h(y) dy = 0. \quad (2.38)$$

We recall that equation (2.36) is approximated for  $|y|$  large by the elliptic equation

$$\Delta_{\mathbb{R}^2} \phi + \frac{4y}{|y|^2} \cdot \nabla_y \phi + h(y) = 0. \quad (2.39)$$

The operator can be regarded as a 6-dimensional Laplacian when acting on radial functions. Using this, a suitable barrier argument gives that a decaying solution  $\phi(y)$  to (2.38) has the estimate

$$|\phi(y)| = O(|y|^{-m+2}) \quad \text{as } |y| \rightarrow +\infty. \quad (2.40)$$

Let us check necessary conditions for the existence of a solution to (2.36) with decay (2.40). Problem (2.36) can be rewritten in divergence form as

$$\nabla \cdot (U_0 \nabla g) + h(y) = 0 \quad \text{in } \mathbb{R}^2, \quad g = \frac{\phi}{U_0} - (-\Delta_y)^{-1} \phi. \quad (2.41)$$

Written in this form we readily see that condition (2.38) is indeed necessary. Testing the equation against the second moment factor  $|y|^2$ , integrating in large balls and taking into account the decay of boundary terms, we find

$$\int_{\mathbb{R}^2} |y|^2 \nabla \cdot (U_0 \nabla g) dy = \int_{\mathbb{R}^2} g \nabla_y \cdot (2y U_0) = 2 \int_{\mathbb{R}^2} g Z_0 dy$$

where

$$Z_0(y) = 2U_0(y) + y \cdot \nabla U_0(y) =: U_0 z_0(y).$$

Setting  $\psi = (-\Delta)^{-1} \phi$  we get

$$- \int_{\mathbb{R}^2} g Z_0 dy = \int_{\mathbb{R}^2} (\Delta \psi + U_0 \psi) z_0 = \int_{\mathbb{R}^2} (\Delta z_0 + U_0 z_0) \psi.$$

Similarly, testing equation (2.41) against the coordinate function  $y_j$  we obtain

$$\int_{\mathbb{R}^2} y_j \nabla \cdot (U_0 \nabla g) dy = \int_{\mathbb{R}^2} g \frac{\partial U_j}{\partial y_j} = 2 \int_{\mathbb{R}^2} g Z_j dy$$

and

$$\int_{\mathbb{R}^2} g Z_j dy = - \int_{\mathbb{R}^2} (\Delta z_j + U_0 z_j) \psi.$$

where  $Z_j(y) = U_0(y) z_j(y)$ . The operator  $\mathcal{L}[\psi] := \Delta \psi + U_0(y) \psi$  is classical. It corresponds to linearizing the Liouville equation

$$\Delta v + e^v = 0 \quad \text{in } \mathbb{R}^2.$$

around the solution  $V_0 = \log U_0$ . It is well known that the bounded kernel of this linearization is spanned by the generators of rigid motions, namely translations and dilation invariance of the equation, which are precisely the functions  $z_0, z_1, z_2$  introduced above, which can also be written as

$$\begin{cases} z_0(y) = \nabla V_0(y) \cdot y + 2 \\ z_j(y) = \partial_{y_j} V_0(y), \quad j = 1, 2 \end{cases} \quad (2.42)$$

Thus  $\mathcal{L}[z_j] = 0$ ,  $j = 0, 1, 2$  and then we obtain the necessary conditions

$$\int_{\mathbb{R}^2} h(y) dy = 0, \quad \int_{\mathbb{R}^2} h(y) |y|^2 dy = 0, \quad \int_{\mathbb{R}^2} h(y) y_j dy = 0, \quad j = 1, 2 \quad (2.43)$$

for existence of a solution  $\phi$  to (2.36) with decay (2.40). Conditions (2.43) will be satisfied for  $h = H(\phi^o, \psi^o, \lambda, \alpha, \xi)$  in (2.35) only if the parameters  $\lambda, \alpha, \xi$  are conveniently adjusted. We will precisely impose these constraints as additional equations to the system (2.29)-(2.30).

The form of  $H$  in (2.31) motivates the introduction of the following modification of (2.29)

$$\lambda^2 \partial_t \phi^i = L^i[\phi^i] + H(\phi^o, \psi^o, \phi^i, \psi^i, \lambda, \alpha, \xi) + \sum_{j=1}^4 c_j(t) Z_j \tilde{\chi} \quad (2.44)$$

where  $H$  is given by (2.31) and the numbers  $c_j(t)$  are precisely those such that the functions

$$h(y, t) = H(\phi^o, \psi^o, \phi^i, \psi^i, \lambda, \alpha, \xi) + \sum_{j=0}^3 c_j(t) Z_j \tilde{\chi}$$

satisfies the four integral conditions (2.43) for all  $t$ . Explicitly, we have

$$\begin{cases} c_0(t) = -\frac{\alpha}{\int_{\mathbb{R}^2} Z_0 \tilde{\chi} |y|^2 dy} \int_{\mathbb{R}^2} H(\phi^o, \psi^o, \lambda, \alpha, \xi) \tilde{\chi} |y|^2 dy \\ c_1(t) = -\frac{\alpha}{\int_{\mathbb{R}^2} Z_1 \tilde{\chi} y_1 dy} \int_{\mathbb{R}^2} H(\phi^o, \psi^o, \lambda, \alpha, \xi) \tilde{\chi} y_1 dy \\ c_2(t) = -\frac{\alpha}{\int_{\mathbb{R}^2} Z_2 \tilde{\chi} y_2 dy} \int_{\mathbb{R}^2} H(\phi^o, \psi^o, \lambda, \alpha, \xi) \tilde{\chi} y_2 dy \\ c_3(t) = -\frac{\alpha}{\int_{\mathbb{R}^2} Z_3 \tilde{\chi} dy} \int_{\mathbb{R}^2} H(\phi^o, \psi^o, \lambda, \alpha, \xi) \tilde{\chi} dy. \end{cases} \quad (2.45)$$

The scalars  $c_j$  define functionals  $c_j[\phi^o, \psi^o, \phi^i, \psi^i, \lambda, \alpha, \xi](t)$ . Equation (2.29) will then be satisfied if in addition to (2.44) we impose the relations

$$c_j[\phi^o, \psi^o, \phi^i, \psi^i, \lambda, \alpha, \xi](t) = 0, \quad j = 0, 1, 2, 3. \quad (2.46)$$

To find a solution of the original problem (2.1) it is sufficient to solve the system of equations (2.28), (2.30), (2.44), (2.46) for all  $t > t_0$ , where  $t_0$  is a large fixed number, and we impose initial conditions at  $t_0$  of the form

$$\phi^i(\cdot, t_0) = 0, \quad \phi^o(\cdot, t_0) = \phi_*^o$$

where  $\phi_*^o$  is a generic small function whose properties we will state later on.

### 3. THE PROOF OF THEOREM 1

In this section we provide the proof of our main result by solving the inner-outer gluing system in a suitable region for its unknowns. For the sake of exposition we postpone the proofs of suitable invertibility theories for the linear operators involved in the inner and outer equations. With the notation introduced in the previous section, we recall that the inner-outer gluing system is

$$\begin{cases} \lambda^2 \frac{\partial \phi^i}{\partial t} = L^i[\phi^i] + H(\phi^o, \psi^o, \phi^i, \psi^i \lambda, \alpha, \xi) + \sum_{j=0}^3 c_j(t) Z_j \tilde{\chi}, \\ \phi^i(y, t_0) = 0 \\ \psi^i(y, t) = (-\Delta_y)^{-1} \phi^i \end{cases} \quad (3.1)$$

$$\begin{cases} \frac{\partial \phi^o}{\partial t} = L^o[\phi^o] + G(\phi^o, \psi^o, \phi^i, \psi^i \lambda, \alpha, \xi) \\ \phi^o(x, t_0) = \phi_0^o(x) \\ \psi^o(x, t) = (-\Delta_x)^{-1} [\phi^o + 2\lambda^{-1} \nabla_y \psi^i \nabla_x \chi + \psi^i \Delta_x \chi] \end{cases} \quad (3.2)$$

$$c_j[\phi^o, \psi^o, \phi^i, \psi^i, \lambda, \alpha, \xi](t) = 0, \quad j = 0, 1, 2, 3, \quad (3.3)$$

where the  $c_j$  are defined in (2.45). We recall that on the parameters  $\alpha(t), \lambda(t), \xi(t)$  we assume that  $\lambda(+\infty) = 0, \alpha(+\infty) = 0$  and

$$|\dot{\lambda}(t)| \leq M |\dot{\lambda}_*(t)|, \quad |\dot{\alpha}(t)| \leq M \frac{\lambda_*^2}{t^2}, \quad |\dot{\xi}(t)| \leq M \frac{1}{t^{1+\sigma}}. \quad (3.4)$$

where  $\lambda_*(t) = \frac{1}{\sqrt{\log t}}$  and  $M$  is a positive constant to be later taken sufficiently large.

Next we describe the linear estimates to be used to solve the inner and outer equations (3.1) and (3.2). We consider first the inner linear equation

$$\begin{cases} \lambda^2 \partial_t \phi^i = L^i[\phi^i] + h + \sum_{j=0}^3 d_j(t) Z_j \tilde{\chi}, & \text{in } \mathbb{R}^2 \times (t_0, \infty) \\ \phi^i(\cdot, t_0) = 0, & \text{in } \mathbb{R}^2, \end{cases} \quad (3.5)$$

where  $d_j(t)$  are given by

$$\begin{aligned} d_0(t) &= -\frac{1}{\int_{\mathbb{R}^2} Z_0 \tilde{\chi} |y|^2 dy} \int_{\mathbb{R}^2} h(y, t) |y|^2 dy \\ d_1(t) &= -\frac{1}{\int_{\mathbb{R}^2} Z_1 \tilde{\chi} y_1 dy} \int_{\mathbb{R}^2} h(y, t) y_1 dy \\ d_2(t) &= -\frac{1}{\int_{\mathbb{R}^2} Z_2 \tilde{\chi} y_2 dy} \int_{\mathbb{R}^2} h(y, t) y_2 dy \\ d_3(t) &= -\frac{1}{\int_{\mathbb{R}^2} Z_3 \tilde{\chi} dy} \int_{\mathbb{R}^2} h(y, t) dy. \end{aligned}$$

We define next norms, which are suitably adapted to the terms in the inner linear problem (3.5).

We observe that in  $H$  defined in (2.31) we have the term  $\lambda^4 S(u_2) \tilde{\chi}$ , which can be estimated as

$$\begin{aligned} |\lambda^4 S(u_2) \tilde{\chi}| &\leq C \left[ \frac{|\lambda \dot{\lambda}|}{(1+|y|)^4} + \frac{|\alpha-1|}{(1+|y|)^6} + \frac{\lambda |\dot{\xi}|}{(1+|y|)^5} + \frac{\lambda |\dot{\alpha}|}{(1+|y|)^4} + \frac{M^2 \lambda^4}{t^2 (1+|y|)^4} \right] \tilde{\chi} \\ &\leq C \frac{M^2}{t^2 (\log t)^2 (1+|y|)^4} \chi_0 \left( \frac{\lambda |y|}{2\sqrt{t}} \right) \end{aligned} \quad (3.6)$$



using the hypotheses (3.4). From (3.6) we get that for  $\sigma \in (0, 1)$ .

$$|\lambda^4 S(u_2) \tilde{\chi}| \leq CM^2 \frac{1}{t^{\frac{1-\sigma}{2}} (\log t)^{\frac{3-\sigma}{2}} (1+|y|)^{5+\sigma}}. \quad (3.7)$$

Given numbers  $\nu > 0$ ,  $\mu \in \mathbb{R}$ , and  $\sigma \in (0, 1)$ , for a function  $h(y, t)$  we define the norm and  $\|h\|_{i, **}$  as the least numbers  $K \geq 0$  such that for all  $(y, t) \in \mathbb{R}^2 \times (t_0, \infty)$ ,

$$|h(y, t)| \leq K \frac{1}{t^\nu |\log t|^\mu} \frac{1}{(1+|y|)^{5+\sigma}}.$$

This norm separates space and time variables and involves a rather fast decay rate in space and product of logarithmic and algebraic decay in time. The norm in which we would like to measure a solution  $\phi(y, t)$  to (3.5) is of weighted  $C^1$  type in space.

Inside the self-similar region  $|y| \lesssim \sqrt{t}/\lambda$ , the ‘‘elliptic part’’ is dominant, namely the elliptic equation (2.39) for an  $h$  satisfying (2.37) with  $m = 5 + \sigma$ . That should lead to a control for  $\phi$  like (2.40), namely

$$\phi(y, t) = O(t^{-\nu} |\log t|^{-\mu} |y|^{2-m}).$$

For  $|y| \gtrsim \sqrt{t}/\lambda$  the time-derivative part dominates and the norm loses one factor of  $t/\lambda^2 \sim t |\log t|$  because of time integration.

Thus, for a function  $\phi(y, t)$  we let  $\|\phi\|_{i, *}$  be the least number  $K$  such that

$$|\phi(y, t)| + (1+|y|) |\nabla_y \phi(y, t)| \leq K \left[ \frac{1}{t^\nu |\log t|^\mu} \frac{\tilde{\chi}}{(1+|y|)^{3+\sigma}} + \frac{1}{t^{\nu-1} |\log t|^{\mu-1}} \frac{1-\tilde{\chi}}{(1+|y|)^{5+\sigma}} \right].$$

for  $\tilde{\chi}(y, t)$  is the cut off function in (2.33) which is supported in the self-similar region.

We have the validity of the following result which we prove in §5

**Proposition 3.1.** *Assume that  $\lambda$  satisfies (3.4). Let  $\nu > 0$ ,  $\mu \in \mathbb{R}$ ,  $\sigma \in (0, 1)$ . There exists  $C > 0$  such that for  $t_0$  sufficiently large, if  $\|h\|_{i, **} < \infty$  there is a solution  $\phi^i = \mathcal{T}_\lambda^i[h]$  to (3.5), which defines a linear operator of  $h$ , and satisfies*

$$\|\phi^i\|_{i, *} \leq C \|h\|_{i, **}.$$

Next we consider the linear outer problem:

$$\begin{cases} \partial_t \phi^o = L^o[\phi^o] + g(x, t), & \text{in } \mathbb{R}^2 \times (t_0, \infty) \\ \phi^o(\cdot, t_0) = \phi_0^o, & \text{in } \mathbb{R}^2. \end{cases} \quad (3.8)$$

For the outer problem we will only consider right hand sides that have power-like behavior in the self-similar variable  $\zeta = \frac{x-\xi}{\sqrt{t}}$ .

To define the norms for the outer problem, we take into account that  $S(u_2)(1-\chi)$  can be estimated, thanks to Lemma 2.1, by

$$|S(u_2)(1-\chi)| \leq CM^2 \frac{1}{t^3 \log^2 t} e^{-c \frac{|x-\xi|^2}{t}}. \quad (3.9)$$

To include the effect of rather general initial conditions, we consider norms that allow polynomial decay in the self-similar variable  $\frac{x-\xi}{\sqrt{t}}$ .

For a given function  $g(x, t)$  we consider the norm  $\|g\|_{**,o}$  defined as the least  $K \geq 0$  such that for all  $(x, t) \in \mathbb{R}^2 \times (t_0, \infty)$

$$|g(x, t)| \leq K \frac{1}{t^a |\log t|^\beta} \frac{1}{1 + |\zeta|^b}, \quad \zeta = \frac{x - \xi(t)}{\sqrt{t}}.$$

Accordingly, we consider for a function  $\phi^o(x, t)$  the norm  $\|\phi\|_{**,o}$  defined as the least  $K \geq 0$  such that

$$|\phi^o(x, t)| + (\lambda + |x - \xi|) |\nabla_x \phi^o(x, t)| \leq K \frac{1}{t^{a-1} |\log t|^\beta} \frac{1}{1 + |\zeta|^b}, \quad \zeta = \frac{x - \xi}{\sqrt{t}}$$

for all  $(x, t) \in \mathbb{R}^2 \times (t_0, \infty)$ . For the initial condition  $\phi_0^o$  we consider the norm  $\|\phi_0^o\|_b$  given by the least  $K$  such that

$$|\phi_0^o(x)| \leq K \frac{1}{(1 + |z|)^b}, \quad z = \frac{x - \xi(t_0)}{\sqrt{t_0}}.$$

We assume that the parameters  $a, b, \beta$  satisfy

$$a, b > 0, \quad a < 4, \quad b < 6, \quad a < 1 + \frac{b}{2}, \quad \beta \in \mathbb{R}. \quad (3.10)$$

We have the following result whose proof we postpone to §6.

**Proposition 3.2.** *Assume that  $\lambda, \alpha, \xi$  satisfy the conditions in (3.4), and that  $a, b, \mu$  satisfy (3.10). Then there is a constant  $C$  so that for  $t_0$  sufficiently large, for  $\|g\|_{**,o} < \infty$  and  $\|\phi_0^o\|_b < \infty$  there is solution  $\phi^o = \mathcal{T}_{\lambda,\alpha,\xi}^o[g]$  of (3.8), which defines a linear operator of  $g$  and  $\phi_0^o$  provided and satisfies*

$$\|\phi^o\|_{**,o} \leq C [\|g\|_{**,o} + t_0^{a-1} |\log t_0|^\beta \|\phi_0^o\|_b].$$

Next we carry out the proof of Theorem 1 assuming the validity of Propositions 3.1 and 3.2.

**Proof of Theorem 1.** Let us fix functions  $\lambda(t), \alpha(t)$  and  $\xi(t)$  that satisfy conditions (3.4) for a large  $M > 0$ . We consider the linear operator  $\phi^i = \mathcal{T}_\lambda^i[h]$  in Proposition 3.1 where we choose  $t_0 \gg 1$  sufficiently large,  $\sigma \in (0, 1)$  and, motivated by (3.7), fix values for the parameters  $\nu$  and  $\mu$  as

$$\nu = \frac{1 - \sigma}{2}, \quad \mu = \frac{3 - \sigma}{2}.$$

Proposition 3.1 tells us that  $\mathcal{T}_\lambda^i : Y^i \rightarrow X^i$  is a bounded operator where  $Y^i$  is the space of all functions  $h(y, t)$  with  $\|h\|_{**,i} < +\infty$  and  $X^i$  that of the functions  $\phi(y, t)$  such that  $\|\phi\|_{**,i} < +\infty$  and

$$\int_{\mathbb{R}^2} \phi(y, t) dy = 0 \quad \text{for all } t \in (t_0, \infty].$$

$X^i$  and  $Y^i$  are Banach spaces when endowed with their natural norms. We also fix in Proposition 3.2 the values

$$a = 3, \quad \beta = 2, \quad b = 4 + \sigma.$$

The choices of  $a$  and  $\beta$  are motivated by (3.9), while  $b = 4 + \sigma$  comes from the fact that if  $\phi_0^o(x) = O(|x|^{-4-\sigma})$  as  $|x| \rightarrow \infty$ , then from Duhamel's formula we find that the solution of the heat equation in dimension 6 with initial condition  $\phi_0^o$  has the decay  $O(t^{-2-\frac{\sigma}{2}})$  as  $t \rightarrow +\infty$ .

The linear operator  $\mathcal{T}_{\lambda,\alpha,\xi}^o : Y^o \times X_0^o \rightarrow X^o$  is bounded where  $Y^o$  is the space of functions  $g(x, t)$  with  $\|g\|_{**,o} < +\infty$ ,  $X^o$  that of functions  $\phi(x, t)$  with  $\|\phi\|_{*,o} < +\infty$  and  $X_0^o$  that of the functions  $\varphi(x)$  with  $\|\varphi\|_b < +\infty$ . Let us fix a function  $\phi_0^o \in X_0^o$  and write equations (3.1)-(3.2) as the fixed point problem in  $X^o \times X^i$ ,

$$\begin{cases} \phi^o = F^o(\phi^o, \phi^i) \\ \phi^i = F^i(\phi^o, \phi^i) \end{cases} \quad (3.11)$$

where

$$\begin{aligned} F^i(\phi^o, \phi^i) &= \mathcal{T}_\lambda^i[H(\phi^o, \psi^o, \phi^i, \psi^i, \lambda, \alpha, \xi)] \\ F^o(\phi^o, \phi^i) &= \mathcal{T}_{\lambda,\alpha,\xi}^o[G(\phi^o, \psi^o, \phi^i, \psi^i, \lambda, \alpha, \xi), \phi_0^o], \end{aligned} \quad (3.12)$$

with

$$\begin{aligned} \psi^i &= (-\Delta_y)^{-1} \phi^i \\ \psi^o &= (-\Delta_x)^{-1} [\phi^o + 2\lambda^{-1} \nabla_y \psi^i \nabla_x \chi + \psi^i \Delta_x \chi]. \end{aligned} \quad (3.13)$$

We claim that for some fixed number  $C$  independent of  $M$  and any  $t_0 \gg 1$ ,

$$\begin{aligned} \|F^i(\phi^o, \phi^i)\|_{*,i} &\leq \frac{C}{|\log t_0|} (\|\phi^i\|_{*,i} + \|\phi^o\|_{*,o} + \|\phi^i\|_{*,i}^2 + \|\phi^o\|_{*,o}^2) \\ &\quad + CM^2. \end{aligned} \quad (3.14)$$

Using Proposition 3.1, to establish (3.14) it suffices to prove

$$\begin{aligned} \|H(\phi^o, \psi^o, \phi^i, \psi^i, \lambda, \alpha, \xi)\|_{**,i} &\leq \frac{C}{|\log t_0|} (\|\phi^i\|_{*,i} + \|\phi^o\|_{*,o} + \|\phi^i\|_{*,i}^2 + \|\phi^o\|_{*,o}^2) \\ &\quad + CM^2. \end{aligned} \quad (3.15)$$

where  $\psi^i$  and  $\psi^o$  are defined by relations (3.13). We recall the expansion of the operator  $H$  in (2.31)-(2.32) and separately estimate its main terms. For the inner error term  $H_0 = \lambda^4 S(u_2) \tilde{\chi}$ , estimate (3.7) gives

$$\|H_0\|_{**,i} \leq CM^2.$$

The main terms in the operator  $H$  involving  $\phi^o, \psi^o$  are given by the linear operator

$$\begin{aligned} H_1[\phi^o, \phi^i] &= (2\lambda^2 U_0 \phi^o - [(\alpha \chi U_0 + \lambda^2 \varphi_1), \psi^o]_y) \tilde{\chi} \\ &= (\lambda^2 U_0 \phi^o - \alpha \nabla_y U_0 \nabla_y \psi^o \chi - \alpha U_0 \nabla_y \chi \nabla_y \psi^o) \tilde{\chi}, \end{aligned}$$

where we recall,  $\psi^o[\phi^o, \phi^i]$  is the linear operator in (3.13). Let us estimate  $\psi^o$ . Since  $\int_{\mathbb{R}^2} \phi^i(y, t) dy = 0$  we have from the Newtonian potential representation of  $\psi^i$ ,

$$|\psi^i(y, t)| + (1 + |y|) |\nabla_y \psi^i(y, t)| \leq C \frac{\lambda^{3-\sigma}}{t^{\frac{1-\sigma}{2}} (1 + |y|)} \|\phi^i\|_{*,i}.$$

Hence

$$\begin{aligned} |\phi^o + 2\lambda^{-1} \nabla_y \psi^i \nabla_x \chi + \psi^i \Delta_x \chi| &\leq \frac{1}{t^2 |\log t|^2 (1 + |\zeta|^b)} \|\phi^o\|_{*,o} \\ &\quad + \frac{1}{t^{2-\sigma/2} |\log t|^{2-\sigma/2} (1 + |\zeta|^b)} \|\phi^i\|_{*,i} \end{aligned}$$

where  $\zeta = \frac{x-\xi}{\sqrt{t}}$ . From (3.13) and using the Newtonian potential representation we get

$$\begin{aligned} |\nabla_x \psi^\circ(x, t)| &\leq \frac{C}{t^{3/2} |\log t|^2 (1 + |\zeta|)} \|\phi^\circ\|_{*,o} \\ &\quad + \frac{C}{t^{3/2-\sigma/2} |\log t|^{2-\sigma/2} (1 + |\zeta|)} \|\phi^i\|_{*,i}. \end{aligned}$$

Using this estimate we obtain

$$\|H_1[\phi^\circ, \phi^i]\|_{*,i} \leq \frac{C}{t_0 |\log t_0|^{\sigma/2}} \|\phi^\circ\|_{*,o} + \frac{C}{t_0^{1-\sigma/2}} \|\phi^i\|_{*,i}.$$

Next we consider the linear operator

$$H_2[\phi^i] = \lambda \dot{\lambda} (2\phi^i + y \cdot \nabla_y \phi^i) \tilde{\chi} + \lambda \dot{\xi} \cdot \nabla_y \phi^i.$$

We directly check that

$$\begin{aligned} |H_2[\phi^i]| &\leq \frac{1}{t |\log t|^2} \frac{\lambda^{3-\sigma}}{t^{\frac{1-\sigma}{2}} (1 + |y|)^{3+\sigma}} \|\phi^i\|_{*,i} \\ &\leq \frac{1}{|\log t|} \frac{\lambda^{3-\sigma}}{t^{\frac{1-\sigma}{2}} (1 + |y|)^{5+\sigma}} \|\phi^i\|_{*,i} \end{aligned}$$

and hence we obtain

$$\|H_2[\phi^i]\|_{**,i} \leq \frac{1}{|\log t_0|} \|\phi^i\|_{*,i}.$$

The remaining linear terms are

$$\begin{aligned} H_3[\phi^i] &= -(\alpha - 1)[U_0, \psi^i \chi]_y - \alpha[U_0(\chi - 1), \psi^i \chi]_y - [U_0, \psi^i(1 - \chi)]_y \\ &\quad - \lambda^2[\varphi_1, \psi^i \chi]_y - [\phi^i \chi, v_2 - V_0]_y. \end{aligned}$$

Similar computations to those above yield

$$\|H_3[\phi^i]\|_{**,i} \leq \frac{C}{t_0} \|\phi^i\|_{*,i}$$

Finally, the remaining terms in  $H$  are quadratic, and given by

$$H_4[\phi^\circ, \phi^i] = [\phi^i \chi + \lambda^2 \phi^\circ, \psi^i \chi + \lambda^2 \psi^\circ]_y \tilde{\chi}.$$

We find

$$\|H_4[\phi^\circ, \phi^i]\|_{**,i} \leq \frac{C}{t_0^{\frac{1}{2}-\sigma}} [\|\phi^i\|_{*,i}^2 + \|\phi^\circ\|_{*,o}^2].$$

Adding up these estimates we obtain (3.15) and hence (3.14).

The same type of bounds, using now Proposition 3.2 and estimating each term in the expansion of  $G$  given by (2.34), yield

$$\begin{aligned} \|F^\circ(\phi^\circ, \phi^i)\|_{*,o} &\leq C \|\phi^i\|_{*,i} + \frac{C}{t_0^{1/2}} (\|\phi^i\|_{*,i}^2 + \|\phi^\circ\|_{*,o} + \|\phi^\circ\|_{*,o}^2) \\ &\quad + C t_0^{\alpha-1} |\log t_0|^\beta \|\phi_0^\circ\|_b + C (t_0^{\alpha-1} |\log t_0|^\beta \|\phi_0^\circ\|_b)^2 + C M^2. \end{aligned} \quad (3.16)$$

Let us fix  $C$  such that (3.14) and (3.16) hold. At this point we impose that

$$C t_0^{\alpha-1} |\log t_0|^\beta \|\phi_0^\circ\|_b + C (t_0^{\alpha-1} |\log t_0|^\beta \|\phi_0^\circ\|_b)^2 \leq M^2.$$

We set up the region for the fixed point problem (3.11) as follows:

$$\mathcal{B} = \{(\phi^o, \phi^i) \in X^o \times X^i / \|\phi^i\|_{*,i} \leq 4CM^2, \quad \|\phi^o\|_{*,o} \leq (C^2 + 2C)M^2\},$$

and then the operator  $F = (F^o, F^i)$  maps  $\mathcal{B}$  into itself, for any sufficiently large  $t_0$ .

Similar computations as those leading to (3.14) and (3.16) give the validity of the following Lipschitz properties, enlarging  $t_0$  if necessary:

$$\begin{aligned} \|F^i(\phi_1^o, \phi_1^i) - F^i(\phi_2^o, \phi_2^i)\|_{*,i} &\leq \frac{C}{|\log t_0|} (\|\phi_1^i - \phi_2^i\|_{*,i} + \|\phi_1^o - \phi_2^o\|_{*,o}) \\ \|F^o(\phi_1^o, \phi_1^i) - F^o(\phi_2^o, \phi_2^i)\|_{*,o} &\leq C\|\phi_1^i - \phi_2^i\|_{*,i} + \frac{C}{t_0^{1/2}}\|\phi_1^o - \phi_2^o\|_{*,o}. \end{aligned}$$

for  $(\phi_1^o, \phi_1^i), (\phi_2^o, \phi_2^i) \in \mathcal{B}$ . Endowing the Banach space  $X^o \times X^i$  with the norm

$$\|(\phi^o, \phi^i)\| = \|\phi^o\|_{*,o} + \delta^{-1}\|\phi^i\|_{*,i}$$

we obtain that, fixing  $\delta > 0$  sufficiently small, and then taking  $t_0$  larger if necessary we then get

$$\|F\|(\phi_1^o, \phi_1^i) - F(\phi_2^o, \phi_2^i)\| \leq \frac{1}{2}\|(\phi_1^o - \phi_2^o, \phi_1^i - \phi_2^i)\|$$

so that the operator  $F : \mathcal{B} \rightarrow \mathcal{B}$  is a contraction mapping. It follows that problem (3.11) has a unique solution in  $(\phi^o, \phi^i) \in \mathcal{B}$  which define operators

$$\phi^i = \Phi^i(\lambda, \alpha, \xi), \quad \phi^o = \Phi^o(\lambda, \alpha, \xi).$$

These functions and their associates

$$\psi^i = \Psi^i(\lambda, \alpha, \xi), \quad \psi^o = \Psi^o(\lambda, \alpha, \xi)$$

satisfy equations (3.1)-(3.2). To find a solution of the full system we just need to find parameter functions  $\lambda, \alpha, \xi$  that satisfy constraints (3.4) and corresponding relations (3.3) which we write in the form

$$\bar{c}_j(\lambda, \alpha, \xi)(t) = 0, \quad \text{for all } t \in (t_0, \infty), \quad j = 0, 1, 2, 3, \quad (3.17)$$

where

$$\bar{c}_j(\lambda, \alpha, \xi) := c_j(\Phi^o(\lambda, \alpha, \xi), \Psi^o(\lambda, \alpha, \xi), \Phi^i(\lambda, \alpha, \xi), \Psi^i(\lambda, \alpha, \xi), \lambda, \alpha, \xi).$$

More, explicitly, the equations (3.17) can be written as

$$\lambda^2 \dot{\alpha}(t) = \frac{1}{\int_{\mathbb{R}^2} Z_3 \tilde{\chi} dy} \int_{\mathbb{R}^2} \bar{H}(\lambda, \alpha, \xi) dy \quad (3.18)$$

$$\alpha \lambda \dot{\lambda}(t) = \frac{1}{\int_{\mathbb{R}^2} Z_0 \tilde{\chi} |y|^2 dy} \int_{\mathbb{R}^2} \bar{H}(\lambda, \alpha, \xi) |y|^2 dy \quad (3.19)$$

$$\alpha \lambda \dot{\xi}_1(t) = \frac{1}{\int_{\mathbb{R}^2} Z_1 \tilde{\chi} y_1 dy} \int_{\mathbb{R}^2} \bar{H}(\lambda, \alpha, \xi) y_1 dy \quad (3.20)$$

$$\alpha \lambda \dot{\xi}_2(t) = \frac{1}{\int_{\mathbb{R}^2} Z_1 \tilde{\chi} y_2 dy} \int_{\mathbb{R}^2} \bar{H}(\lambda, \alpha, \xi) y_2 dy,$$

where

$$\bar{H}(\lambda, \alpha, \xi) = \tilde{H}(\Phi^o(\lambda, \alpha, \xi), \Psi^o(\lambda, \alpha, \xi), \Phi^i(\lambda, \alpha, \xi), \Psi^i(\lambda, \alpha, \xi), \lambda, \alpha, \xi) \tilde{\chi}$$

and  $\tilde{H}$  is given by (2.32). Using the expression for  $\tilde{H}$  in (2.32) and the divergence form of most of its terms, we can rewrite equation (3.18) in the form

$$8\pi\lambda^2\dot{\alpha} = 2\lambda^2 \int_{\mathbb{R}^2} U_0\varphi_1 - \lambda^2 \int_{\mathbb{R}^2} \nabla U_0 \cdot \nabla \psi_1 + f_3(\lambda, \alpha, \xi)$$

where by  $f_3$  we denote a generic function with

$$|f_3(\lambda, \alpha, \xi)(t)| \leq \frac{C(M)}{t^2 |\log t|^3} \quad \text{for all } t \in (t_0, \infty), \quad (3.21)$$

for all  $\lambda, \alpha, \xi$  satisfying (3.4). Using identity (2.17) and Taylor expanding  $\varphi_1(x, t)$  defined in (2.13) we get

$$2 \int_{\mathbb{R}^2} U_0\varphi_1 - \int_{\mathbb{R}^2} \nabla U_0 \cdot \nabla \psi_1 = \int_{\mathbb{R}^2} U_0\varphi_1 dy = -\frac{\lambda^2}{4t^2} + O(t^{-\frac{5}{2}}).$$

and hence we can rewrite (2.13) as

$$\dot{\alpha} = -\frac{1}{4} \frac{\lambda^2}{t^2} + f_3(\lambda, \alpha, \xi) \quad (3.22)$$

for a function  $f_3$  as in (3.21). Similarly, equation (3.19) can be rewritten as

$$\alpha\lambda\dot{\lambda} = \frac{\alpha(\alpha-1)}{8\pi \log t} \int_{\mathbb{R}^2} (U_0^2 - \nabla_y U_0 \nabla_y V_0) |y|^2 dy + f_0(\lambda, \alpha, \xi).$$

where  $f_0$  satisfies

$$|f_0(\lambda, \alpha, \xi)(t)| \leq \frac{C(M)}{t |\log t|^3} \quad \text{for all } t \in (t_0, \infty). \quad (3.23)$$

After an explicit computation using formula (2.9) we get

$$\int_{\mathbb{R}^2} (U_0^2 - \nabla_y U_0 \nabla_y V_0) |y|^2 dy = -32\pi$$

and we find that (3.19) can be rewritten as

$$\lambda\dot{\lambda}(t) = -2 \frac{\alpha(t)-1}{\log t} + f_0(\lambda, \alpha, \xi), \quad (3.24)$$

for a term  $f_0$  as in (3.23). Examining the possibly non-radial terms in the error due to the initial condition  $\phi_0^o(x)$ , which is not necessarily radial, gives that equation (3.20) takes the form

$$\dot{\xi}_j = f_j(\lambda, \alpha, \xi), \quad j = 1, 2, \quad |f_j(\lambda, \alpha, \xi)(t)| \leq \frac{C(M)}{t^{\frac{3}{2}-\sigma}}. \quad (3.25)$$

We rewrite the equations (3.22), (3.24), (3.25), fixing  $\xi(0) = 0$ ,  $\alpha(+\infty) = 1$  as

$$\begin{aligned} \alpha(t) - 1 &= \frac{1}{4} \int_t^\infty \frac{\lambda^2(s)}{s^2} ds + \int_t^\infty f_3(\lambda, \alpha, \xi)(s) ds \\ \xi(t) &= \int_{t_0}^t f(\lambda, \alpha, \xi)(s) ds, \quad j = 1, 2 \end{aligned} \quad (3.26)$$

Integrating by parts using (3.4) we find

$$\int_t^\infty \frac{\lambda^2(s)}{s^2} ds = \frac{\lambda^2(t)}{t} + \int_t^\infty \frac{\lambda\dot{\lambda}(s)}{s} ds,$$

and hence we can rewrite equation (3.26) as

$$\alpha(t) = 1 + \frac{\lambda^2(t)}{4t} + \int_t^\infty f_3(\lambda, \alpha, \xi)(s) ds \quad (3.27)$$

for  $f_3$  satisfying (3.21). We can also write (3.24) as

$$\lambda \dot{\lambda}(t) + \frac{1}{2} \frac{\lambda^2(t)}{t |\log t|} = f_0(\alpha, \lambda, \xi)(t) \quad \text{for all } t > t_0, \quad (3.28)$$

where  $f_0$  satisfies the bound (3.23). It is convenient to relabel

$$\eta(t) = \lambda(t)^2,$$

make the choice  $\eta(t_0) = \lambda_*(t_0)^2 = \frac{1}{\log t_0}$  and then write (3.28) as

$$\eta(t) = \frac{1}{\log t} + \frac{1}{\log t} \int_{t_0}^t f_0(\alpha, \lambda, \xi)(s) |\log s| ds. \quad (3.29)$$

Replacing (3.29) into (3.27) we obtain the equivalent equation

$$\alpha(t) = 1 + \frac{1}{4t |\log t|} + \frac{1}{4t |\log t|} \int_{t_0}^t f_0(\alpha, \lambda, \xi)(s) |\log s| ds + \int_t^\infty f_3(\alpha, \lambda, \xi)(s) ds \quad (3.30)$$

Next, for the functions in (3.29), (3.30), (3.25), we write

$$\begin{aligned} \mathcal{N}_0(\alpha, \eta, \xi) &= \frac{1}{\log t} \int_{t_0}^t f_0(\alpha, \lambda, \xi)(s) |\log s| ds. \\ \mathcal{N}_3(\alpha, \eta, \xi) &= \frac{1}{4t |\log t|} \int_{t_0}^t f_0(\alpha, \lambda, \xi)(s) |\log s| ds + \int_t^\infty f_3(\alpha, \lambda, \xi)(s) ds \\ \mathcal{N}_j(\alpha, \eta, \xi) &= \int_{t_0}^t f_j(\alpha, \lambda, \xi)(s) ds, \quad j = 1, 2 \\ \alpha_0(t) &= 1 + \frac{1}{4t |\log t|}, \quad \eta_0(t) = \frac{1}{\log t}, \quad \xi_0(t) = 0. \end{aligned}$$

We formulate the system as

$$(\alpha, \eta, \xi) = \mathcal{N}(\alpha, \eta, \xi) \in X \quad (3.31)$$

where  $\mathcal{N} = (\mathcal{N}_3, \mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2)$  and  $X$  is the Banach space of all functions  $(\alpha, \eta, \xi)$  in  $C^1[t_0, \infty)$  with  $\|(\alpha, \eta, \xi)\| < +\infty$  where

$$\|(\alpha, \eta, \xi)\| = \|(\alpha, \eta, \xi)\|_\infty + \|t^{1+\sigma} \dot{\xi}\|_\infty + \|t^2 |\log t| \dot{\alpha}\|_\infty + \|t |\log t|^2 \dot{\eta}\|_\infty.$$

Let  $\mathcal{B}$  be a closed ball centered at  $(\alpha_0, \eta_0, \xi_0)$  with a fixed small radius. Then enlarging  $t_0$  if necessary we see that  $\mathcal{N}(\mathcal{B}) \subset \mathcal{B}$ . This operator is compact on  $\mathcal{B}$  as it follows from analyzing the terms involved in the terms  $\mathcal{N}_3, \mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2$ . In fact  $F^i(\phi^\sigma, \phi^i)$ ,  $F^\sigma(\phi^\sigma, \phi^i)$  defined in (3.12) are locally uniformly Hölder continuous in time by parabolic regularity of the operators  $\mathcal{T}_\lambda^i$  and  $\mathcal{T}_{\lambda, \alpha, \xi}^\sigma$ . Using Ascoli's theorem the faster decay for derivatives in powers of  $\log t$  compared with those involved in  $\|\cdot\|$ , compactness of  $\mathcal{N}$  follows, and Schauder's theorem yields the existence of a solution in  $\mathcal{B}$  for the fixed point problem (3.31). Finally we see then that fixing  $M$ , independently of  $t_0$  corresponding to just a number slightly bigger than the one corresponding to  $(\alpha_0, \eta_0, \xi_0)$ , we find that constraints (2.11) are a posteriori satisfied.

The equations for  $\lambda$  and  $\xi$  is have to be solved by fixing their initial conditions independently of the initial condition  $\phi_0^o$ . The fact that perturbative initial condition  $\phi_0^o$  was arbitrary gives the stability of the blow-up, since if we assume  $\int_{\mathbb{R}^2} \phi_0^o = 0$  then necessarily  $\alpha_1(t_0) = 0$  which precisely amounts to the mass of the full initial condition to be exactly  $8\pi$ . All initial conditions in the statement of the theorem correspond to small perturbations in this form in norm (1.3). This concludes the proof.  $\square$

#### 4. PRELIMINARIES FOR THE LINEAR THEORY

**4.1. Stereographic projection.** Let  $\Pi : S^2 \setminus \{(0, 0, 1)\} \rightarrow \mathbb{R}^2$  denote the stereographic projection

$$\Pi(y_1, y_2, y_3) = \left( \frac{y_1}{1 - y_3}, \frac{y_2}{1 - y_3} \right).$$

For  $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$  we write

$$\tilde{\varphi} = \varphi \circ \Pi, \quad \tilde{\varphi} : S^2 \setminus \{(0, 0, 1)\} \rightarrow \mathbb{R}.$$

Let  $U_0$  be given by (1.2). Then we have the following formulas

$$\begin{aligned} \int_{S^2} \tilde{\varphi} &= \frac{1}{2} \int_{\mathbb{R}^2} \varphi U_0 \\ \int_{S^2} \tilde{U}_0 |\nabla_{S^2} \tilde{\varphi}|^2 &= \int_{\mathbb{R}^2} U_0 |\nabla_{\mathbb{R}^2} \varphi|^2 \\ \frac{1}{2} \tilde{U}_0 \Delta_{S^2} \tilde{\varphi} &= (\Delta_{\mathbb{R}^2} \varphi) \circ \Pi. \end{aligned}$$

The linearized Liouville equation for  $\phi, f : \mathbb{R}^2 \rightarrow \mathbb{R}$

$$\Delta \phi + U_0 \phi + U_0 f = 0 \quad \text{in } \mathbb{R}^2$$

is transformed into

$$\Delta_{S^2} \tilde{\phi} + 2\tilde{\phi} + 2\tilde{f} = 0 \quad \text{in } S^2 \setminus \{(0, 0, 1)\}.$$

#### 4.2. A quadratic form.

**Lemma 4.1.** *Let  $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}$  satisfy*

$$|\phi(y)| \leq \frac{1}{(1 + |y|)^{2+\sigma}},$$

with  $0 < \sigma < 1$ , and

$$\int_{\mathbb{R}^2} \phi dy = 0.$$

There are constants  $c_1 > 0$ ,  $c_2 > 0$  such that

$$c_1 \int_{\mathbb{R}^2} U_0 g^2 \leq \int_{\mathbb{R}^2} \phi g \leq c_2 \int_{\mathbb{R}^2} U_0 g^2$$

where

$$g = \frac{\phi}{U_0} - (-\Delta^{-1})\phi + c$$

and  $c \in \mathbb{R}$  is chosen so that

$$\int_{\mathbb{R}^2} g U_0 = 0.$$



*Proof.* We set

$$\psi_0 = (-\Delta)^{-1}\phi$$

and then note that since  $\int_{\mathbb{R}^2} \phi = 0$  we have

$$|\psi_0(y)| + (1 + |y|)|\nabla\psi_0(y)| \lesssim \frac{1}{(1 + |y|)^\sigma}. \quad (4.1)$$

From  $g = \frac{\phi}{U_0} - \psi_0 + c$  we find the estimate

$$|g(y)| \lesssim (1 + |y|)^{2-\sigma}.$$

Let  $\psi = \psi_0 - c$  and note that

$$-\Delta\psi - U_0\psi = U_0g \quad \text{in } \mathbb{R}^2.$$

We transform  $\tilde{g} = g \circ \Pi$ ,  $\tilde{\psi} = \psi \circ \Pi$  and write this equation in  $S^2$  as

$$-\Delta_{S^2}\tilde{\psi} - 2\tilde{\psi} = 2\tilde{g}, \quad \text{in } S^2. \quad (4.2)$$

Since  $\phi = U_0(g + \psi)$  we get

$$\int_{\mathbb{R}^2} \phi g = \int_{\mathbb{R}^2} U_0(g + \psi)g = \frac{1}{2} \int_{S^2} \tilde{g}^2 + \tilde{\psi}\tilde{g},$$

Multiplying (4.2) by  $\tilde{\psi}$  we find that

$$\int_{S^2} \tilde{g}\tilde{\psi} = \frac{1}{2} \int_{S^2} |\nabla_{S^2}\tilde{\psi}|^2 - \int_{S^2} \tilde{\psi}^2$$

and hence

$$\int_{\mathbb{R}^2} \phi g = \frac{1}{2} \int_{S^2} \tilde{g}^2 + \frac{1}{4} \int_{S^2} |\nabla_{S^2}\tilde{\psi}|^2 - \frac{1}{2} \int_{S^2} \tilde{\psi}^2.$$

We recall that the eigenvalues of  $-\Delta$  on  $S^2$  are given by  $\{k(k+1) \mid k \geq 0\}$ . The eigenvalue 0 has a constant eigenfunction and the eigenvalue 2 has eigenspace spanned by the coordinate functions  $\pi_i(x_1, x_2, x_3) = x_i$ , for  $(x_1, x_2, x_3) \in S^2$  and  $i = 1, 2, 3$ . Let  $(\lambda_j)_{j \geq 0}$  denote all eigenvalues, repeated according to multiplicity, with  $\lambda_0 = 0$ ,  $\lambda_1 = \lambda_2 = \lambda_3 = 2$ , and let  $(e_j)_{j \geq 0}$  denote the corresponding eigenfunctions so that they form an orthonormal system in  $L^2(S^2)$ , and  $e_1, e_2, e_3$  are multiples of the coordinate functions  $\pi_1, \pi_2, \pi_3$ . We decompose  $\tilde{\psi}$  and  $\tilde{g}$ :

$$\tilde{\psi} = \sum_{j=0}^{\infty} \tilde{\psi}_j e_j, \quad \tilde{g} = \sum_{j=0}^{\infty} \tilde{g}_j e_j, \quad (4.3)$$

where

$$\tilde{\psi}_j = \langle \tilde{\psi}, e_j \rangle_{L^2(S^2)}, \quad \tilde{g}_j = \langle \tilde{g}, e_j \rangle_{L^2(S^2)}.$$

Then

$$\begin{aligned} \int_{\mathbb{R}^2} \phi g &= \frac{1}{2} \int_{S^2} \tilde{g}^2 + \frac{1}{4} \sum_{j=0}^{\infty} (\lambda_j - 2) \tilde{\psi}_j^2 \\ &= \frac{1}{2} \int_{S^2} \tilde{g}^2 - \frac{1}{2} \tilde{\psi}_0^2 + \frac{1}{4} \sum_{j=4}^{\infty} (\lambda_j - 2) \tilde{\psi}_j^2. \end{aligned}$$

Equation (4.2) gives us that

$$\tilde{\psi}_j = \frac{2}{\lambda_j - 2} \tilde{g}_j, \quad j \notin \{1, 2, 3\},$$

and therefore

$$\begin{aligned} \int_{\mathbb{R}^2} \phi g &= \frac{1}{2} \int_{S^2} \tilde{g}^2 - \frac{1}{2} \tilde{g}_0^2 + \sum_{j=4}^{\infty} \frac{1}{\lambda_j - 2} \tilde{g}_j^2 \\ &= \frac{1}{2} \sum_{j=1}^{\infty} \tilde{g}_j^2 + \sum_{j=4}^{\infty} \frac{1}{\lambda_j - 2} \tilde{g}_j^2. \end{aligned}$$

But  $\int_{\mathbb{R}^2} g U_0 = 0$  which means that  $\tilde{g}_0 = 0$  and hence we obtain the conclusion.

We note for further reference, that by Lemma 4.2 we have also  $\tilde{g}_1 = \tilde{g}_2 = \tilde{g}_3 = 0$ . Therefore we also have the formula

$$\int_{\mathbb{R}^2} \phi g = \frac{1}{2} \sum_{j=4}^{\infty} \frac{\lambda_j}{\lambda_j - 2} \tilde{g}_j^2. \quad (4.4)$$

□

**Lemma 4.2.** *Under the same hypotheses of Lemma 4.1 we have*

$$\int_{\mathbb{R}^2} g U_0 z_j = 0, \quad j = 0, 1, 2,$$

where  $z_j$  are the functions defined in (2.42).

*Proof.* We use the notation  $\psi_0 = (-\Delta)^{-1} \phi$ ,  $\psi = \psi_0 - c$ , where  $c$  is such that  $g = \frac{\phi}{U_0} - \psi_0 + c$  and  $\int_{\mathbb{R}^2} g U_0 = 0$ . We multiply

$$-\Delta \psi - U_0 \psi = U_0 g \quad \text{in } \mathbb{R}^2.$$

by  $z_j$  in a ball  $B_R(0)$  and then let  $R \rightarrow \infty$ . Since  $z_j$  is in the kernel of  $\Delta + U_0$  we just have to check that the boundary terms

$$\int_{\partial B_R} \frac{\partial \psi}{\partial \nu} z_j - \psi \frac{\partial z_j}{\partial \nu}$$

tend to 0 as  $R \rightarrow \infty$ , where  $\nu$  is the exterior normal vector to  $\partial B_R$ . This follows from the estimates

$$|\psi(y)| \leq C, \quad |\nabla \psi| \leq \frac{C}{(1 + |y|)^{1+\sigma}}$$

due to (4.1), and the explicit bounds

$$\begin{aligned} |z_0(y)| &\leq C, \quad |z_j(y)| \leq \frac{C}{(1 + |y|)}, \quad j = 1, 2, \\ |\nabla z_j(y)| &\leq \frac{C}{(1 + |y|)^2}. \end{aligned}$$

□

**Lemma 4.3.** *Suppose that  $\phi = \phi(y, t)$ ,  $y \in \mathbb{R}^2$ ,  $t > 0$  is a function satisfying*

$$|\phi(y, t)| \leq \frac{1}{(1 + |y|)^{2+\sigma}},$$

with  $0 < \sigma < 1$ ,

$$\int_{\mathbb{R}^2} \phi(y, t) dy = 0, \quad \forall t > 0,$$

and that  $\phi$  is differentiable with respect to  $t$  and  $\phi_t$  satisfies also

$$|\phi_t(y, t)| \leq \frac{1}{(1 + |y|)^{2+\sigma}}.$$

Then

$$\int_{\mathbb{R}^2} \phi_t g = \frac{1}{2} \partial_t \int_{\mathbb{R}^2} \phi g$$

where for each  $t$ ,  $g(y, t)$  is defined as

$$g = \frac{\phi}{U_0} - (-\Delta^{-1})\phi + c(t)$$

and  $c(t) \in \mathbb{R}$  is chosen so that

$$\int_{\mathbb{R}^2} g(y, t) U_0(y) dy = 0.$$

*Proof.* Using the notation of the previous lemma, we have

$$\int_{\mathbb{R}^2} \phi_t g = \int_{\mathbb{R}^2} U_0(g_t + \psi_t) g = \frac{1}{2} \int_{S^2} (\tilde{g}_t \tilde{g} + \tilde{\psi}_t \tilde{g}).$$

We have

$$-\Delta_{S^2} \tilde{\psi} - 2\tilde{\psi} = 2\tilde{g}, \quad \text{in } S^2.$$

And differentiating in  $t$  we get

$$-\Delta_{S^2} \tilde{\psi}_t - 2\tilde{\psi}_t = 2\tilde{g}_t, \quad \text{in } S^2. \tag{4.5}$$

Multiplying by  $\tilde{g}$  and integrating we find that

$$\int_{S^2} \tilde{\psi}_t \tilde{g} = -\frac{1}{2} \int_{S^2} \Delta \tilde{\psi}_t \tilde{g} - \int_{S^2} \tilde{g}_t \tilde{g}.$$

Thus

$$\int_{\mathbb{R}^2} \phi_t g = -\frac{1}{4} \int_{S^2} \Delta \tilde{\psi}_t \tilde{g}$$

Decompose as in (4.3) and find that

$$\int_{\mathbb{R}^2} \phi_t g = \frac{1}{4} \sum_{j=0}^{\infty} \lambda_j (\tilde{\psi}_j)_t \tilde{g}_j$$

But from (4.5)

$$(\lambda_j - 2)(\tilde{\psi}_j)_t = 2(\tilde{g}_j)_t.$$

We note that  $\tilde{g}_j = 0$  for  $j = 0, 1, 2, 3$ . Indeed, this is true for  $j = 0$  by the assumption  $\int_{\mathbb{R}^2} g U_0 = 0$ . By Lemma 4.2 this is true also for  $j = 1, 2, 3$ . Then

$$\int_{\mathbb{R}^2} \phi_t g = \frac{1}{2} \sum_{j=4}^{\infty} \frac{\lambda_j}{\lambda_j - 2} (\tilde{g}_j)_t \tilde{g}_j$$

and the desired conclusion follows from (4.4).  $\square$

### 4.3. A Hardy inequality.

**Lemma 4.1.** *Let  $B_R(0) \subset \mathbb{R}^2$  be the open ball centered at 0 of radius  $R$ . There exists  $C > 0$  such that, for any  $R > 0$  large and any  $\int_{B_R} g U_0 dx = 0$*

$$\frac{C}{R^2} \int_{B_R} g^2 U_0 \leq \int_{B_R} |\nabla g|^2 U_0.$$

*Proof.* After a stereographic projection and letting  $\varepsilon = \frac{1}{R}$ ,  $A_\varepsilon = B_1(0) \setminus B_\varepsilon(0) \subset \mathbb{R}^2$ , we need to prove that for  $g \in C^1(\bar{A}_\varepsilon)$  with

$$\int_{A_\varepsilon} g dy = 0$$

we have

$$\int_{A_\varepsilon} g^2 dy \leq \frac{C}{\varepsilon^2} \int_{A_\varepsilon} |\nabla g|^2 |y|^4 dy.$$

By using polar coordinates it is sufficient to show this for radial functions, which amounts to the statement: for  $g \in C^1([\varepsilon, 1])$ , if

$$\int_\varepsilon^1 g^2 x dx = 0 \tag{4.6}$$

then

$$\int_\varepsilon^1 g^2 x dx \leq \frac{C}{\varepsilon^2} \int_\varepsilon^1 g'(x)^2 x^5 dx.$$

We write

$$\int_\varepsilon^1 g^2 x dx = \frac{1}{2} \int_\varepsilon^1 g^2 \frac{d}{dx}(x^2) dx = \frac{g^2(1)}{2} - \frac{g^2(\varepsilon)}{2} \varepsilon^2 - \frac{1}{2} \int_\varepsilon^1 gg' x^2 dx.$$

One has

$$\begin{aligned} \int_\varepsilon^1 gg' x^2 dx &\leq \left( \int_\varepsilon^1 g'^2 x^5 dx \right)^{\frac{1}{2}} \left( \int_\varepsilon^1 g^2 x^{-1} dx \right)^{\frac{1}{2}} \\ &\leq \left( \varepsilon^{-2} \int_\varepsilon^1 g'^2 x^5 dx \right)^{\frac{1}{2}} \left( \int_\varepsilon^1 g^2 x dx \right)^{\frac{1}{2}} \\ &\leq C \varepsilon^{-2} \int_\varepsilon^1 g'^2 x^5 dx + \frac{1}{2} \int_\varepsilon^1 g^2 x dx, \end{aligned}$$

for some constant  $C$ . Inserting this inequality in the previous computation gives

$$\int_\varepsilon^1 g^2 x dx \leq g(1)^2 - g(\varepsilon)^2 \varepsilon^2 + C \varepsilon^{-2} \int_\varepsilon^1 g'^2 x^5 dx. \tag{4.7}$$

We now use (4.6) in the form

$$0 = \int_\varepsilon^1 g(x)x dx = \frac{g(1)}{2} - \frac{g(\varepsilon)}{2} \varepsilon^2 - \frac{1}{2} \int_\varepsilon^1 g' x^2 dx,$$

and so

$$g(1)^2 \leq 2g(\varepsilon)^2 \varepsilon^4 + 2 \left( \int_\varepsilon^1 g' x^2 dx \right)^2.$$

But

$$\begin{aligned} \int_{\varepsilon}^1 g' x^2 dx &\leq \left( \int_{\varepsilon}^1 g'^2 x^5 dx \right)^{\frac{1}{2}} \left( \int_{\varepsilon}^1 x^{-1} dx \right)^{\frac{1}{2}} \\ &\leq \left( |\log \varepsilon| \int_{\varepsilon}^1 g'^2 x^5 dx \right)^{\frac{1}{2}}. \end{aligned}$$

We thus get that

$$g(1)^2 \leq 2g(\varepsilon)^2 \varepsilon^4 + 2|\log \varepsilon| \int_{\varepsilon}^1 g'^2 x^5 dx$$

and this combined with (4.7) gives

$$\int_{\varepsilon}^1 g^2 x dx \leq g(\varepsilon)^2 (2\varepsilon^4 - \varepsilon^2) + (C\varepsilon^{-2} + 2|\log \varepsilon|) \int_{\varepsilon}^1 g'^2 x^5 dx.$$

For  $\varepsilon > 0$  small this gives the desired estimate.  $\square$

## 5. INNER PROBLEM

We consider equation (3.5) rewritten as

$$\begin{cases} \lambda^2 \partial_t \phi = L^i[\phi] + h & \text{in } \mathbb{R}^2 \times (t_0, \infty) \\ \phi(\cdot, t_0) = 0, & \text{in } \mathbb{R}^2, \end{cases}$$

where  $L^i$  is the operator defined in (2.26) and where we assume that

$$\begin{aligned} \int_{\mathbb{R}^2} h(y, t) dy &= 0, \quad \int_{\mathbb{R}^2} h(y, t) |y|^2 dy = 0, \\ \int_{\mathbb{R}^2} h(y, t) y_j dy &= 0, \quad j = 1, 2 \end{aligned} \tag{5.1}$$

for all  $t > t_0$ .

We change the time variable

$$\tau = \int_{t_0}^t \frac{1}{\lambda^2(s)} ds$$

and note that  $\tau \sim t \log t$ . Then this equation can be written as

$$\partial_{\tau} \phi = \nabla \cdot \left[ U \nabla \left( \frac{\phi}{U_0} - (-\Delta)^{-1} \phi \right) \right] + h.$$

We consider this equation in  $\mathbb{R}^2 \times (\tau_0, \infty)$  where  $\tau_0$  is fixed large, and with initial condition

$$\phi(y, \tau_0) \equiv 0 \quad \text{in } \mathbb{R}^2.$$

We define

$$\|h\|_{i^{*}, \nu, \mu, \sigma} = \sup \{ \tau^{\nu} \log^{\mu} \tau (1 + |y|)^{5+\sigma} |h(y, \tau)| \},$$

where

$$0 < \nu < 3, \quad \mu \in \mathbb{R}, \quad 0 < \sigma < 1.$$

*Proof of Proposition 3.1.* Let

$$g = \frac{\phi}{U_0} - (-\Delta)^{-1}\phi + c(\tau), \quad (5.2)$$

where  $c(\tau)$  is chosen so that

$$\int_{\mathbb{R}^2} g(y, \tau) U_0(y) dy = 0, \quad \forall \tau > \tau_0. \quad (5.3)$$

Note that

$$\partial_\tau \phi = \nabla \cdot (U_0 \nabla g) + h, \quad \text{in } \mathbb{R}^2 \times (\tau_0, \infty). \quad (5.4)$$

We multiply this equation by  $g$  and integrate in  $\mathbb{R}^2$ , using Lemma 4.3:

$$\frac{1}{2} \partial_\tau \int_{\mathbb{R}^2} \phi g + \int_{\mathbb{R}^2} U_0 |\nabla g|^2 = \int_{\mathbb{R}^2} h g.$$

We use the inequality in Lemma 4.1 to get

$$\frac{1}{R_1^2} \int_{B_{R_1}} (g - \bar{g}_{R_1})^2 U_0 \leq \int_{\mathbb{R}^2} U_0 |\nabla g|^2$$

where

$$\bar{g}_{R_1} = \frac{1}{\int_{B_{R_1}} U_0} \int_{B_{R_1}} g U_0.$$

Here  $R_1$  is a large positive constant to be made precise below.

Then

$$\frac{1}{2} \partial_\tau \int_{\mathbb{R}^2} \phi g + \frac{1}{R_1^2} \int_{B_{R_1}} g^2 U_0 \leq C R_1^2 \int_{\mathbb{R}^2} h^2 U_0^{-1} + \frac{1}{2 R_1^2} \left( \int_{\mathbb{R}^2} g^2 U_0 + C \bar{g}_{R_1}^2 \right).$$

But by (5.3)

$$\bar{g}_{R_1} = - \frac{1}{\int_{B_{R_1}} U_0} \int_{\mathbb{R}^2 \setminus B_{R_1}} g U_0$$

so

$$\bar{g}_{R_1}^2 \leq C \int_{\mathbb{R}^2 \setminus B_{R_1}} g^2 U_0.$$

Therefore

$$\frac{1}{2} \partial_\tau \int_{\mathbb{R}^2} \phi g + \frac{1}{2 R_1^2} \int_{B_{R_1}} g^2 U_0 \lesssim R_1^2 \int_{\mathbb{R}^2} h^2 U_0^{-1} + \frac{1}{R_1^2} \int_{\mathbb{R}^2 \setminus B_{R_1}} g^2 U_0.$$

We now use Lemma 4.1 to get

$$\partial_\tau \int_{\mathbb{R}^2} \phi g + \frac{1}{C} \int_{\mathbb{R}^2} \phi g \lesssim R_1^2 \int_{\mathbb{R}^2} h^2 U_0^{-1} + \frac{1}{R_1^2} \int_{\mathbb{R}^2 \setminus B_{R_1}} g^2 U_0. \quad (5.5)$$

Define

$$A^2 = \sup_{\tau \geq \tau_0} \left\{ \tau^{2\nu} \log^{2\mu} \tau \int_{\mathbb{R}^2 \setminus B_R} g^2(t) U_0 \right\}.$$

Integrating (5.5) and using Lemma 4.1 we find

$$\int_{\mathbb{R}^2} g^2 U_0 \lesssim \frac{R_1^4 \|h\|_{i^{**}, \nu, \mu, \sigma}^2}{\tau^{2\nu} \log^{2\mu} \tau} + \frac{A^2}{\tau^{2\nu} \log^{2\mu} \tau}. \quad (5.6)$$

Let us use the notation

$$\|g\|_{L^2(U_0^{1/2})}^2 = \int_{\mathbb{R}^2} g^2 U_0$$

and we record the estimate (5.6) as

$$\|g(\tau)\|_{L^2(U_0^{1/2})} \lesssim \frac{M}{\tau^\nu \log^\mu \tau}, \quad (5.7)$$

where

$$M = R_1^2 \|h\|_{i^{**}, \nu, \mu, \sigma} + A$$

The idea now is to obtain decay of  $g$ , and use this decay to show that  $A$  can be eliminated from the estimate (5.6).

We define

$$g_0 = U_0 g$$

and obtain from (5.4) the equation

$$\begin{aligned} \partial_\tau g_0 &= U_0 \partial_\tau g = \partial_\tau \phi + U_0 \Delta^{-1} \partial_\tau \phi \\ &= \nabla \cdot (U_0 \nabla g) + h - U_0 (-\Delta)^{-1} [\nabla \cdot (U_0 \nabla g) + h] \\ &= \nabla \cdot \left[ U_0 \nabla \left( \frac{g_0}{U_0} \right) \right] + h - U_0 v - U_0 (-\Delta)^{-1} h, \end{aligned} \quad (5.8)$$

where

$$v := (-\Delta)^{-1} [\nabla \cdot (U_0 \nabla g)]. \quad (5.9)$$

We claim that

$$|v(y, \tau)| \lesssim \frac{M}{\tau^\nu \log^\mu \tau (1 + |y|^2)^{-\varepsilon}}, \quad (5.10)$$

for any  $\varepsilon > 0$ .

To prove this, we first compute

$$\nabla \cdot (U_0 \nabla g) = \Delta g U_0 + \nabla U_0 \cdot \nabla g = \Delta(g U_0) - \nabla U_0 \cdot \nabla g - g \Delta U_0,$$

and hence

$$v = -g U_0 - (-\Delta)^{-1} [\nabla U_0 \cdot \nabla g + g \Delta U_0].$$

Let

$$v_2 = (-\Delta)^{-1} [\nabla U_0 \cdot \nabla g + g \Delta U_0], \quad (5.11)$$

so that

$$-\Delta v_2 = \nabla U_0 \cdot \nabla g + g \Delta U_0 = \nabla(g \nabla U_0) \quad \text{in } \mathbb{R}^2. \quad (5.12)$$

We write equation (5.12) on the sphere  $S^2$

$$-\Delta_{S^2} \tilde{v}_2 = \operatorname{div}_{S^2}(\tilde{g} \nabla_{S^2} \tilde{U}_0) \quad \text{in } S^2, \quad (5.13)$$

where  $\tilde{v}_2 = v_2 \circ \Pi$ ,  $\tilde{g} = g \circ \Pi$ ,  $\tilde{U}_0 = U_0 \circ \Pi$ , and  $\Pi$  is the stereographic projection defined in section 4.1. We note that the solution of (5.13) is defined up to an additive constant. In  $\tilde{v}_2$  this constant is fixed by the condition  $\tilde{v}_2(P) = 0$ , which corresponds to the solution selected by the formula (5.11). Observe that

$$\int_{S^2} \tilde{g}^2 |\nabla_{S^2} \tilde{U}_0|^2 = \int_{\mathbb{R}^2} g^2 |\nabla_{\mathbb{R}^2} U_0|^2 \lesssim \|g\|_{L^2(U_0^{1/2})}^2.$$

Using standard elliptic theory we find that  $\tilde{v}_2 \in H^1(S^2)$  and  $\|\tilde{v}_2\|_{H^1} \lesssim \|g\|_{L^2(U_0^{1/2})}$ . Hence for any  $p > 1$ ,  $\|\tilde{v}_2\|_{L^p} \lesssim \|g\|_{L^2(U_0^{1/2})}$  and this implies that

$$\left( \int_{\mathbb{R}^2} |v_2|^p U_0 \right)^{1/p} \lesssim \|g\|_{L^2(U_0^{1/2})}. \quad (5.14)$$

We write (5.8) as

$$\partial_\tau g_0 = \Delta g_0 - \nabla g_0 \nabla V_0 + h + 2U_0 g_0 + U_0 v_2 - U_0 (-\Delta)^{-1} h. \quad (5.15)$$

Consider a point  $y \in \mathbb{R}^2$ . From (5.7) we see that

$$\|g_0\|_{L^2(B_1(y))} \lesssim \frac{M}{\tau^\nu \log^\mu \tau (1 + |y|)^2},$$

and from (5.14) we have

$$\|U_0 v_2\|_{L^p(B_1(y))} \lesssim U_0^{1-\frac{1}{p}} \frac{M}{\tau^\nu \log^\mu \tau},$$

With a similar argument we get that

$$|(-\Delta)^{-1} h(y, \tau)| \lesssim \frac{\|h\|_{i^{**}, \nu, \mu, \sigma}}{\tau^\nu \log^\mu \tau (1 + |y|)^{1-\varepsilon}}.$$

Applying standard parabolic  $L^p$  to (5.15) restricted to  $B_1(y) \times (\tau, \tau + 1)$  and embedding into Hölder spaces we deduce that

$$|g_0(y, \tau)| \lesssim \frac{M}{\tau^\nu \log^\mu \tau (1 + |y|)^2}, \quad (5.16)$$

Then for  $g$  we obtain the estimate

$$|g(y, \tau)| \lesssim \frac{M(1 + |y|)^2}{\tau^\nu \log^\mu \tau},$$

This implies that in (5.13),  $\|\tilde{g} \nabla_{S^2} \tilde{U}_0\|_{L^\infty(S^2)} \lesssim \frac{M}{\tau^\nu \log^\mu \tau}$ , and by elliptic regularity we get  $\|\tilde{v}_2\|^{C^\alpha} \lesssim \frac{M}{\tau^\nu \log^\mu \tau}$  for any  $\alpha \in (0, 1)$ . Since  $\tilde{v}_2(P) = 0$  we get

$$|v_2(y, \tau)| \lesssim \frac{M}{\tau^\nu \log^\mu \tau (1 + |y|)^\alpha},$$

for any  $\alpha \in (0, 1)$ .

Applying now parabolic estimates to (5.15) and a scaling argument we find

$$|\nabla g_0(y, \tau)| \lesssim \frac{M}{\tau^\nu \log^\mu \tau (1 + |y|)^3}. \quad (5.17)$$

We reconsider now (5.13) and observe that

$$\begin{aligned} \operatorname{div}_{S^2}(\tilde{g} \nabla_{S^2} \tilde{U}_0) &= \nabla_{S^2} \tilde{g} \nabla_{S^2} \tilde{U}_0 + \tilde{g} \Delta_{S^2} \tilde{U}_0 \\ &= \frac{(1 + |y|^2)^2}{4} [\nabla_{\mathbb{R}^2} g \nabla_{\mathbb{R}^2} U_0 + g \Delta_{\mathbb{R}^2} U_0]. \end{aligned}$$

Using (5.16), (5.17) and  $g_0 = gU_0$  we get that

$$|\operatorname{div}_{S^2}(\tilde{g} \nabla_{S^2} \tilde{U}_0)| \lesssim \frac{M}{\tau^\nu \log^\mu \tau}.$$



Using standard elliptic regularity we conclude that  $\tilde{v}_2 \in C^{1,\alpha}(S^2)$  for any  $\alpha \in (0, 1)$  and the estimate

$$\|\tilde{v}_2\|_{C^{1,\alpha}(S^2)} \lesssim \frac{M}{\tau^\nu \log^\mu \tau}.$$

Since  $\tilde{v}_2(P) = 0$ , from a Taylor expansion of  $\tilde{v}_2$  about  $P$  we obtain for the original  $v_2$  the expansions

$$\begin{cases} \left| v_2(y, \tau) - \frac{a(\tau) \cdot y}{|y|^2} \right| \lesssim \frac{M}{\tau^\nu \log^\mu \tau (1 + |y|)^{1+\alpha}}, \\ \left| \nabla v_2(y, \tau) - \frac{a(\tau)|y|^2 - 2ya(\tau) \cdot y}{|y|^4} \right| \lesssim \frac{M}{\tau^\nu \log^\mu \tau (1 + |y|)^{2+\alpha}}, \end{cases} \quad (5.18)$$

for some  $a(\tau) = (a_1(\tau), a_2(\tau))$ , for any  $\alpha \in (0, 1)$ . By the definition of  $v$  (5.9), the estimate for  $g_0$  (5.16), (5.17), and the expansion (5.18) we obtain also for  $v$ :

$$\begin{cases} \left| v(y, \tau) - \frac{a(\tau) \cdot y}{|y|^2} \right| \lesssim \frac{M}{\tau^\nu \log^\mu \tau (1 + |y|)^{1+\alpha}}, \\ \left| \nabla v(y, \tau) - \frac{a(\tau)|y|^2 - 2ya(\tau) \cdot y}{|y|^4} \right| \lesssim \frac{M}{\tau^\nu \log^\mu \tau (1 + |y|)^{2+\alpha}}, \end{cases} \quad (5.19)$$

We will show next that actually  $a_1(\tau) = a_2(\tau) = 0$ . For this use the definition of  $v$  (5.9) to write

$$-\Delta v = \nabla \cdot (U_0 \nabla g) \quad \text{in } \mathbb{R}^2. \quad (5.20)$$

and we multiply by  $y_i$  and integrate by parts. First we observe first that for  $i = 1, 2$

$$\int_{\mathbb{R}^2} \nabla(U_0 \nabla g) y_i \, dy = 0. \quad (5.21)$$

Indeed,

$$\int_{\mathbb{R}^2} \nabla(U_0 \nabla g) y_i \, dy = - \int_{\mathbb{R}^2} U_0 \nabla g e_i = \int_{\mathbb{R}^2} g \nabla U_0 e_i.$$

But from (5.2), letting  $\psi_0 = (-\Delta)^{-1} \phi$  and  $\psi = \psi_0 - c(\tau)$  we have

$$-\Delta \psi - U_0 \psi = U_0 g.$$

Multiplying this equation by  $z_i = \nabla V_0 e_i$  defined in (2.42) and integrating we get

$$\int_{\mathbb{R}^2} g U_0 \nabla V_0 e_i = 0,$$

which is the desired claim (5.21). We note that the integrations by parts are justified by the decay

$$|\psi_0(y, \tau)| + (1 + |y|) |\nabla \psi_0(y, \tau)| \lesssim C(\tau) \frac{1}{(1 + |y|)^\sigma}.$$

Now we multiply (5.20) by  $y_1$  and integrate in a ball  $B_R(0)$ , where  $R > 0$  and later we let  $R \rightarrow \infty$ . Integrating we get

$$\int_{\partial B_R} (-\partial_\nu v y_1 + v e_1) = \int_{B_R} \nabla(U_0 \nabla g) y_1 \, dy$$

Using polar coordinates  $y = \rho e^{i\theta}$  and (5.19), we see that

$$\int_{\partial B_R} (-\partial_\nu v y_1 + v e_1) = 2\pi a_1(\tau) + O(R^{-\alpha}).$$

Letting  $R \rightarrow \infty$  and using (5.21) we conclude that  $a_1(\tau) = 0$ . Similarly  $a_2(\tau) = 0$ . We deduce from this and (5.19) that

$$|v(y, \tau)| \lesssim \frac{M}{\tau^\nu \log^\mu \tau (1 + |y|)^{1+\alpha}}.$$

This is the desired conclusion (5.10).

A similar proof, using (5.1) yields

$$|(-\Delta)^{-1}h(y, \tau)| \lesssim \frac{\|h\|_{i^{**}, \nu, \mu, \sigma}}{\tau^\nu \log^\mu \tau (1 + |y|)^{2-\varepsilon}}.$$

Now we choose a large constant  $R_0$  so that we can use the maximum principle for the parabolic operator  $\partial_\tau f - \nabla \cdot [U_0 \nabla (\frac{f}{U_0})]$  is valid outside the ball  $B_{R_0}(0)$ . Indeed, we have

$$\nabla \cdot \left[ U_0 \nabla \left( \frac{f}{U_0} \right) \right] = \Delta f - \nabla V_0 \nabla f + U_0 f = \partial_{\rho\rho} f + \frac{5}{\rho} \partial_\rho f + \frac{1}{\rho^2} \partial_{\theta\theta} f + Df$$

where  $Df = O(\frac{1}{\rho^3})\partial_\rho f + O(\frac{1}{\rho^4})f$  represent lower order terms. Using the maximum principle and an appropriate barrier in  $\mathbb{R}^2 \setminus B_{R_0}$ , as constructed in Theorem 3.2 below, we get that

$$|g_0(y, \tau)| \lesssim \frac{R_1^2 \|h\|_{i^{**}, \nu, \mu, \sigma} + A}{\tau^\nu \log^\mu \tau (1 + |y|)^{3+\sigma}}, \quad |y| \leq \sqrt{\tau}$$

and

$$|g_0(y, \tau)| \lesssim \frac{R^2 \|h\|_{i^{**}, \nu, \mu, \sigma} + A}{\tau^{\nu-1} \log^\mu \tau (1 + |y|)^{5+\sigma}}, \quad |y| \geq \sqrt{\tau}.$$

We use this estimate to compute

$$\begin{aligned} \int_{\mathbb{R}^2 \setminus B_{R_1}} g^2 U &= \int_{\mathbb{R}^2 \setminus B_{R_1}} g_0^2 U^{-1} \\ &\lesssim \frac{1}{R_1^{2\sigma}} \frac{R_1^4 \|h\|_{i^{**}, \nu, \mu, \sigma}^2 + A^2}{\tau^{2\nu} \log^{2\mu} \tau}. \end{aligned}$$

This implies that

$$A^2 \leq C \frac{1}{R_1^{2\sigma}} (R_1^4 \|h\|_{i^{**}, \nu, \mu, \sigma}^2 + A^2),$$

where  $C$  is a constant from previous inequalities, which is independent of  $R_1$ . Choosing a fixed  $R_1$  large then implies that

$$A^2 \lesssim R_1^{4-2\sigma} \|h\|_{i^{**}, \nu, \mu, \sigma}^2.$$

We then conclude that

$$|g_0(y, t)| \lesssim \|h\|_{i^{**}, \nu, \mu, \sigma} \begin{cases} \frac{1}{\tau^\nu \log^\mu \tau (1+|y|)^{3+\sigma}}, & |y| \leq \sqrt{\tau} \\ \frac{1}{\tau^{\nu-1} \log^\mu \tau (1+|y|)^{5+\sigma}}, & |y| \geq \sqrt{\tau}. \end{cases}$$

From parabolic estimates we also find

$$|\nabla g_0(y, t)| \lesssim \|h\|_{i^{**}, \nu, \mu, \sigma} \begin{cases} \frac{1}{\tau^\nu \log^\mu \tau (1+|y|)^{4+\sigma}}, & |y| \leq \sqrt{\tau} \\ \frac{1}{\tau^{\nu-1} \log^\mu \tau (1+|y|)^{6+\sigma}}, & |y| \geq \sqrt{\tau}. \end{cases} \quad (5.22)$$

Now we estimate  $\phi$ . We decompose

$$\phi = \phi^\perp + \omega(\tau)Z_0,$$

where  $Z_0$  is defined in (2.21). We then have

$$g = \frac{\phi^\perp}{U_0} - (\Delta^{-1})\phi^\perp + c(t).$$

We let  $\psi = (-\Delta^{-1})\phi^\perp$  and see that

$$gU = \Delta\psi + U_0\psi.$$

Integrating the equation times  $|y|^2$  we get

$$\int_{\mathbb{R}^2} \phi(y, \tau)|y|^2 dy = 0, \quad \forall \tau > \tau_0.$$

and this is equivalent to

$$\int_{\mathbb{R}^2} gZ_0 = 0, \quad \forall \tau > \tau_0,$$

where  $Z_0$  is defined in (2.21). We then can solve the equation for  $\psi$  and find

$$|\psi(y, \tau)| \lesssim \|h\|_{i^{**}, \nu, \mu, \sigma} \begin{cases} \frac{1}{\tau^\nu \log^\mu \tau (1+|y|)^{1+\sigma}}, & |y| \leq \sqrt{\tau} \\ \frac{1}{\tau^{\nu-1} \log^\mu \tau (1+|y|)^{3+\sigma}}, & |y| \geq \sqrt{\tau}. \end{cases}$$

Since

$$\phi^\perp = U_0(g - \psi)$$

we find that

$$|\phi^\perp(y, \tau)| \lesssim \|h\|_{i^{**}, \nu, \mu, \sigma} \begin{cases} \frac{1}{\tau^\nu \log^\mu \tau (1+|y|)^{3+\sigma}}, & |y| \leq \sqrt{\tau} \\ \frac{1}{\tau^{\nu-1} \log^\mu \tau (1+|y|)^{5+\sigma}}, & |y| \geq \sqrt{\tau}. \end{cases} \quad (5.23)$$

Finally we estimate  $\omega(\tau)$ . We have

$$\partial_\tau \phi^\perp + \omega_\tau z = L[\phi] + h.$$

We multiply by  $|y|^2$  and integrate in  $B_{R_2}$  where  $R_2 \rightarrow \infty$  and in a time interval  $[\tau_1, \tau_2]$ . We get

$$\begin{aligned} & \int_{B_{R_2}} [\phi(\tau_2)^\perp - \phi(\tau_1)^\perp] |y|^2 dy + (\omega(\tau_2) - \omega(\tau_1)) \int_{B_{R_2}} Z_0 |y|^2 dy \\ &= \int_{\tau_1}^{\tau_2} \int_{B_{R_2}} L\phi |y|^2 dy d\tau. \end{aligned}$$

Let us observe that if  $R_2 \geq \sqrt{\tau}$  then

$$\int_{B_{R_2}} |\phi(y, \tau)^\perp| |y|^2 dy \lesssim \frac{1}{\tau^{\nu-1} \log^\mu \tau R_2^{1+\sigma}}.$$

On the other hand

$$\int_{B_{R_2}} L[\phi] |y|^2 dy = \int_{B_{R_2}} gZ_0 dy + \int_{\partial B_{R_2}} U_0 |y|^2 \nabla g \cdot \nu - \int_{\partial B_{R_2}} gU_0 y \cdot \nu,$$

and

$$\left| \int_{B_{R_2}} g Z_0 dy \right| \leq \int_{B_{R_2}} |g_0| dy \lesssim \frac{1}{\tau^\nu \log^\mu \tau},$$

$$\left| \int_{\partial B_{R_2}} U_0 |y|^2 \nabla g \cdot \nu \right| + \left| \int_{\partial B_{R_2}} g U_0 y \cdot \nu \right| \lesssim \frac{1}{\tau^\nu \log^\mu \tau R_2^{1+\sigma}}.$$

Since

$$\int_{B_{R_2}} Z_0 |y|^2 dy \sim \log R_2,$$

letting  $R_2 \rightarrow \infty$  we find that  $\omega(\tau_2) = \omega(\tau_1)$ . Hence  $\omega \equiv \text{const}$  and since we start with  $\omega(0) = 0$  we deduce  $\omega \equiv 0$ . Hence the estimate (5.23) gives the desired estimate for  $\phi$ . The estimate for the gradient of  $\phi$  comes from the corresponding estimate for the gradient of  $\phi^\perp$ , which is obtained similarly from (5.22).  $\square$

## 6. OUTER PROBLEM

We consider here the solution  $\phi^o$  of

$$\begin{cases} \partial_t \phi^o = L^o[\phi^o] + h(x, t), & \text{in } \mathbb{R}^2 \times (t_0, \infty) \\ \phi^o(\cdot, t_0) = \phi_0^o, & \text{in } \mathbb{R}^2 \end{cases}$$

given by Duhamel's formula, where  $L^o$  is the operator (2.27).

*Proof of Theorem 3.2.* To obtain the desired estimate we construct barriers. Using polar coordinates  $x - \xi(t) = r e^{i\theta}$  and the notation in (2.22), the outer operator  $L^o$  (c.f. (2.27)) can be written as:

$$L^o[\phi^o] = \partial_{rr} \phi^o + \left( \frac{1}{r} + \frac{4r}{\lambda^2 + r^2} - \partial_r \psi_1 - \partial_r \tilde{v}_2 \right) \partial_r \phi^o + \frac{1}{r^2} \partial_{\theta\theta} \phi^o.$$

First we construct a supersolution  $\bar{\phi}_1$  valid in the outer region, of the form

$$\bar{\phi}_1(x, t) = \frac{1}{t^{a-1} |\log t|^\beta} g_0 \left( \frac{|x - \xi|}{\sqrt{t}} \right)$$

where  $g = g(s)$  satisfies

$$g_0'' + \left( \frac{5}{s} + \frac{s}{2} \right) g_0' + (a-1)g_0 + \frac{1}{1+s^b} \leq 0, \quad s > 0,$$

and

$$\frac{c_1}{1+s^b} \leq g_0(s) \leq \frac{c_2}{1+s^b},$$

for some  $c_1, c_2 > 0$  and all  $s > 0$ . The function  $g$  can be explicitly taken as

$$g_0(s) = M_0 e^{-\frac{s^2}{4}} + \frac{1}{(1+s^2)^{b/2}},$$

for a constant  $M$  sufficiently large. Here we have used the hypothesis  $-\frac{b}{2} + a - 1 < 0$  in (3.10) and the fact  $G(s) = e^{-\frac{s^2}{4}}$  satisfies  $G'' + \left(\frac{5}{s} + \frac{s}{2}\right)G' + 3G = 0$ .

We find that

$$-\partial_t \bar{\phi}_1 + L^o[\bar{\phi}_1] \leq -\frac{c}{t^a |\log t|^\beta (1 + |\zeta|^b)} + C \frac{\lambda^2}{r(r^2 + \lambda^2)} |\partial_r \bar{\phi}_1|,$$

for some  $c > 0$ , for  $t \geq t_0$  and  $t_0$  large. But

$$|\partial_r \bar{\phi}_1| \leq \frac{C}{t^{a-1} |\log t|^\beta} \begin{cases} \frac{r}{\sqrt{t}} & r \leq \sqrt{t} \\ \left(\frac{r}{\sqrt{t}}\right)^{-b-1} & r \geq \sqrt{t}, \end{cases}$$

where  $r = |x - \xi|$  and therefore in the region  $r \leq \sqrt{t}$  we have

$$\frac{\lambda^2}{r(r^2 + \lambda^2)} |\partial_r \bar{\phi}_1| \leq C \frac{|\log t|^{-1}}{(r^2 + |\log t|^{-1})} \frac{1}{t^{a-1/2} |\log t|^\beta}.$$

To deal with this term we solve the elliptic problem for  $\varphi(x, t)$ ,  $x \in B_{\sqrt{t}}(0) \subset \mathbb{R}^2$

$$-\Delta \varphi = \frac{|\log t|^{-1}}{|x|^2 + |\log t|^{-1}} \quad \text{in } B_{\sqrt{t}}(0), \quad \varphi = 0 \quad \text{on } \partial B_{\sqrt{t}}(0),$$

and obtain

$$|\varphi(x, t)| \leq \frac{C}{\log t} \log^2 \left( \frac{\sqrt{t}}{|x| + |\log t|^{-1/2}} \right).$$

Then we define

$$\bar{\phi}_2(x, t) = \frac{1}{t^{a-1/2} |\log t|^\beta} \varphi(x - \xi, t) \chi_0 \left( 2 \frac{x - \xi}{\sqrt{t}} \right).$$

and

$$\bar{\phi} = A \bar{\phi}_1 + \bar{\phi}_2 + t_0^{b/2-a+1} |\log t_0|^{-\beta} \bar{\phi}_3$$

with a constant  $A$  sufficiently large. Then  $\bar{\phi}$  satisfies

$$-\partial_t \bar{\phi} + L^o[\bar{\phi}] \leq -c \frac{1}{t^a |\log t|^\beta} \frac{1}{1 + |\zeta|^b}, \quad \zeta = \frac{x - \xi(t)}{\sqrt{t}}$$

for some  $c > 0$  and

$$\frac{c_1}{t_0^{a-1} |\log t_0|^b} \frac{1}{1 + |z|^b} \leq \bar{\phi}(t_0, x) \leq \frac{c_2}{t_0^{a-1} |\log t_0|^b} \frac{1}{1 + |z|^b},$$

where  $z = \frac{x - \xi(t_0)}{\sqrt{t_0}}$ , for some  $c_1, c_2 > 0$ . It follows from the maximum principle that the solution  $\phi^o$  of (3.8) given by Duhamel's formula satisfies

$$|\phi^o(x, t)| \leq C \bar{\phi}(x, t) [\|g\|_{**o} + t_0^{a-1} |\log t_0|^\beta \|\phi_0^o\|_b].$$

□

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