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INFINITE TIME BLOW-UP IN THE PATLAK-KELLER-SEGEL SYSTEM: EXISTENCE AND STABILITY

JUAN DAVILA, MANUEL DEL PINO, JEAN DOLBEAULT, MONICA MUSSO,
AND JUNCHENG WEI

ABSTRACT. Perhaps the most classical diffusion model for chemotaxis is the Patlak-Keller-Segel system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) & \text{in } \mathbb{R}^2 \times (0, \infty), \\ v = (-\Delta_{\mathbb{R}^2})^{-1} u := \frac{1}{2\pi} \int_{\mathbb{R}^2} \log \frac{1}{|x-z|} u(z, t) dz, \\ u(\cdot, 0) = u_0 \geq 0 & \text{in } \mathbb{R}^2. \end{cases} \quad (0.1)$$

We consider the *critical mass case* $\int_{\mathbb{R}^2} u_0(x) dx = 8\pi$ which corresponds to the exact threshold between finite-time blow-up and self-similar diffusion towards zero. We find a radial function u_0^* with mass 8π such that for any initial condition u_0 sufficiently close to u_0^* the solution $u(x, t)$ of (0.1) is globally defined and blows-up in infinite time. As $t \rightarrow +\infty$ it has the approximate profile

$$u(x, t) \approx \frac{1}{\lambda^2} U_0 \left(\frac{x - \xi(t)}{\lambda(t)} \right), \quad U_0(y) = \frac{8}{(1 + |y|^2)^2},$$

where $\lambda(t) \approx \frac{c}{\sqrt{\log t}}$, $\xi(t) \rightarrow q$ for some $c > 0$ and $q \in \mathbb{R}^2$.

1. INTRODUCTION

This paper deals with the classical Keller-Segel problem in \mathbb{R}^2 ,

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) & \text{in } \mathbb{R}^2 \times (0, \infty), \\ v = (-\Delta_{\mathbb{R}^2})^{-1} u := \frac{1}{2\pi} \int_{\mathbb{R}^2} \log \frac{1}{|x-z|} u(z, t) dz \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{R}^2 \end{cases} \quad (1.1)$$

which is a well-known model for the dynamics of a population density $u(x, t)$ evolving by diffusion with a drift representing a chemotaxis effect [14, 15, 18]. We consider positive solutions which are well defined and unique and smooth up to a maximal time $0 < T \leq +\infty$. This problem formally preserves mass, in the sense that

$$\int_{\mathbb{R}^2} u(x, t) dx = \int_{\mathbb{R}^2} u_0(x) dx =: M \quad \text{for all } t \in (0, T).$$

An interesting feature of (1.1) is the connection between the second moment of the solution and its mass which is precisely given by

$$\frac{d}{dt} \int_{\mathbb{R}^2} u(x, t) |x|^2 dx = 4M - \frac{M^2}{2\pi}$$

provided that the second moments are finite. If $M > 8\pi$ the negative rate of production of the second moment and the positivity of the solution implies finite

blow-up time. If $M < 8\pi$ the solution lives at all times and eventually diffuses to zero with a self similar profile [3]. When $M = 8\pi$ the solution is globally defined in time [4]. If the initial second moment is finite, it is preserved in time, and there is *infinite time blow-up* for the solution [4].

Globally defined in time solutions of (1.1) are of course its positive finite mass steady states, which consist of the family

$$U_{\lambda,\xi}(x) = \frac{1}{\lambda^2} U_0 \left(\frac{x - \xi}{\lambda} \right), \quad U_0(y) = \frac{8}{(1 + |y|^2)^2}, \quad \lambda > 0, \quad \xi \in \mathbb{R}^2. \quad (1.2)$$

We observe that all these steady states have the exact mass 8π and infinite second moment

$$\int_{\mathbb{R}^2} U_{\lambda,\xi}(x) dx = 8\pi, \quad \int_{\mathbb{R}^2} |x|^2 U_{\lambda,\xi}(x) dx = +\infty.$$

In [1, 16, 23] blow-up in a bounded domain is studied, see also [17] and [24] for the higher-dimensional radial case. Asymptotics were derived in [25, 26] and a rigorous construction of a radial blow-up solution was found in [22]. Asymptotic stability of family of steady states (1.2) under this mass constraint has been determined in [12].

In the critical mass $M = 8\pi$ case, the infinite-time blow-up in (1.1) when the second moment is finite, takes place in the form of a bubble in the form (1.2) with $\lambda = \lambda(t) \rightarrow 0$ [4, 2]. Formal rates and precise profiles were derived in [6, 5] to be $\lambda(t) \sim \frac{c}{\sqrt{\log t}}$ as $t \rightarrow +\infty$. A radial solution with this rates was built in [13] and its stability within the radial class was established. The stability assertion for general small perturbations was conjectured and left open in [13]. The method of construction in [13] (see also [19, 20, 21, 22]) seems difficult to adapt to the general, nonradial scenario.

In this paper we construct an infinite-time blow-up solution with an entirely different method to that in [13] which in particular leads to a proof of the stability assertion. The following is our main result.

Theorem 1. *There exists a nonnegative function $u_0^*(x)$ (radially symmetric) with critical mass $\int_{\mathbb{R}^2} u_0^*(x) dx = 8\pi$ and finite second moment $\int_{\mathbb{R}^2} |x|^2 u_0^*(x) dx < +\infty$ such that for every $u_0(x)$ sufficiently close (in suitable sense) to u_0^* with $\int_{\mathbb{R}^2} u_0^* = 8\pi$ we have that the associated solution $u(x, t)$ of system (1.1) has the form*

$$u(x, t) = \frac{1}{\lambda(t)^2} U_0 \left(\frac{x - \xi(t)}{\lambda(t)} \right) (1 + o(1)), \quad U_0(y) = \frac{8}{(1 + |y|^2)^2}$$

uniformly on bounded sets of \mathbb{R}^n , and

$$\lambda(t) = \frac{c}{\sqrt{\log t}} (1 + o(1)), \quad \xi(t) \rightarrow q \quad \text{as } t \rightarrow +\infty,$$

for some numbers $c > 0$ and $q \in \mathbb{R}^2$.

Sufficiently close for the perturbation $u_0(x) := u_0^*(x) + \varphi(x)$ in this result is measured in the C^1 -weighted norm for some $\sigma > 0$

$$\|\varphi\|_{*\sigma} := \|(1 + |\cdot|^{4+\sigma})\varphi\|_{L^\infty(\mathbb{R}^2)} + \|(1 + |\cdot|^{5+\sigma})\nabla\varphi(x)\|_{L^\infty(\mathbb{R}^2)} < +\infty. \quad (1.3)$$

We observe that this decay condition implies for all $\sigma > 0$ the second moment of φ is finite, which is not the case for $\sigma \leq 0$.

We devote the rest of this paper to the proof of Theorem 1. Our approach borrows elements of constructions in the works [7, 9, 10, 11] based in the so-called *inner-outer gluing scheme*, where a system is derived for an inner equation defined near the blow-up point and expressed in the variable of the blowing-up bubble, and an outer problem that sees the whole picture in the original scale. The result of Theorem 1 has already been announced in [8] in connection with [7, 9, 10].

The scaling parameter is rather simple to find at main order from the approximate conservation of second moment. The center $\xi(t)$ actually obeys a relatively simple system of nonlocal ODEs.

2. THE ANSATZ

We consider the Keller-Segel system in the whole \mathbb{R}^2

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) & \text{in } \mathbb{R}^2 \times (0, \infty), \\ v = (-\Delta_{\mathbb{R}^2})^{-1} u := \frac{1}{2\pi} \int_{\mathbb{R}^2} \log \frac{1}{|x-z|} u(z, t) dz \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{R}^2 \end{cases}$$

and build a first approximation to a solution globally defined in time. Let us consider a smooth cut-off function χ_0 with

$$\chi_0(s) = 1 \text{ for } s < 1 \text{ and } = 0 \text{ for } s > 2. \quad (2.1)$$

We consider a positive function $\lambda(t)$ and a function $\xi(t)$ with values in \mathbb{R}^2 such that

$$\lambda(t) \rightarrow 0, \quad \xi(t) \rightarrow 0 \quad \text{as } t \rightarrow +\infty$$

and consider as a first approximation

$$u_1(x, t) = \frac{\alpha(t)}{\lambda^2} U_0(y) \chi_0\left(\frac{\lambda|y|}{\sqrt{t}}\right), \quad y = \frac{x - \xi(t)}{\lambda(t)}.$$

Here $U_0(y) = \frac{8}{(1+|y|^2)^2}$ and we write α as

$$\alpha = \alpha_1 + \alpha_2.$$

We choose $\alpha_1(t)$ in such a way that we exactly have

$$\alpha_1(t) \int_{\mathbb{R}^2} U_0(y) \chi(y, t) dy = \int_{\mathbb{R}^2} U_0(y) dy = 8\pi,$$

where

$$\chi(y, t) = \chi_0\left(\frac{\lambda|y|}{\sqrt{t}}\right). \quad (2.2)$$

We directly compute

$$\alpha_1(t) = 1 + a \frac{\lambda^2}{t} (1 + o(1)) \quad \text{as } t \rightarrow +\infty, \quad (2.3)$$

with

$$a = 2 \int_0^\infty \frac{1 - \chi_0(s)}{s^3} ds. \quad (2.4)$$

The parameters $\lambda(t) > 0$ and $\alpha_2(t) \in \mathbb{R}$ will be chosen later on, but for the moment we will assume the following bounds on them:

$$\lambda(t) \leq \lambda_*(t), \quad |\dot{\lambda}(t)| \leq |\dot{\lambda}_*(t)|, \quad (2.5)$$

where

$$\lambda_*(t) = \frac{C}{\sqrt{\log t}},$$

and $C > 0$ is a fixed constant, and

$$|\alpha_2(t)| \leq \frac{C}{t \log(t)}. \quad (2.6)$$

Also on ξ we assume the bound

$$|\dot{\xi}(t)| \leq \frac{C}{t^{1+\sigma}} \quad (2.7)$$

for some $\sigma > 0$.

Let us consider the “error operator”

$$S(u) = -u_t + \Delta u - \nabla(u \nabla v).$$

Changing variables

$$y = \frac{x - \xi}{\lambda}$$

and letting

$$\begin{aligned} \tilde{u}(y, t) &= u(x, t) \\ \tilde{v}(y, t) &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \log \frac{1}{|z - y|} \tilde{u}(z, t) dz \end{aligned}$$

we find that

$$\lambda^4 S(u) = -\lambda^2 \tilde{u}_t + \lambda \dot{\lambda} (2\tilde{u} + y \cdot \nabla_y \tilde{u}) + \lambda \dot{\xi} \cdot \nabla_y \tilde{u} + \Delta_y \tilde{u} - \nabla_y \tilde{u} \nabla_y \tilde{v} + \tilde{u}^2.$$

Let us write

$$V_0(y) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log \frac{1}{|z - y|} U_0(z) dz = \log U_0(y)$$

and also

$$\begin{aligned} U(y, t) &= U_0(y) \chi(y, t) \\ V(y) &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \log \frac{1}{|z - y|} U(z, t) dz \end{aligned}$$

where χ is defined in (2.2).

Then we compute

$$\lambda^4 S(u_1) = S_1 + S_2 + S_3$$

where

$$\begin{aligned} S_1 &= -\lambda^2 \dot{\alpha} U_0 \chi + \alpha \lambda \dot{\lambda} (2U_0 + y \cdot \nabla U_0) \chi + \frac{1}{2} \frac{\alpha \lambda^3}{t^{\frac{3}{2}}} |y| \chi'_0 U_0 \\ S_2 &= 2\alpha \nabla U_0 \cdot \nabla \chi + \alpha U_0 \Delta \chi + U_0^2 (\alpha^2 \chi^2 - \alpha \chi) \\ &\quad - \alpha^2 U_0 \nabla \chi \nabla V - \alpha(\alpha - 1) \nabla U_0 \nabla V \chi + \alpha \nabla U_0 \nabla (V - V_0) \chi \\ S_3 &= \alpha \lambda \dot{\xi} \cdot \nabla U_0 \chi + \alpha \lambda \dot{\xi} \chi'_0 \frac{\lambda}{\sqrt{t}} \cdot \frac{y}{|y|} U_0 \end{aligned}$$

We decompose

$$S_2 = \mathcal{E}_2^o + \mathcal{E}_2^i$$

where

$$\begin{aligned} \mathcal{E}_2^i &= \alpha(\alpha - 1)U_0^2\chi - \alpha(\alpha - 1)\nabla U_0\nabla V_0\chi \\ \mathcal{E}_2^o &= 2\alpha\nabla U_0 \cdot \nabla\chi + \alpha U_0\Delta\chi + U_0^2\alpha^2\chi(\chi - 1) \\ &\quad - \alpha^2U_0\nabla\chi\nabla V + \alpha\nabla U_0\nabla(V - V_0)\chi \\ &\quad - \alpha(\alpha - 1)\nabla U_0\nabla(V - V_0)\chi \end{aligned}$$

We also decompose

$$S_1 = \mathcal{E}_1^i + \mathcal{E}_1^o$$

where

$$\begin{aligned} \mathcal{E}_1^i &= -\lambda^2\dot{\alpha}U_0\chi + \alpha\lambda\dot{\lambda}(2U_0 + y \cdot \nabla U_0)\chi \\ \mathcal{E}_1^o &= \frac{1}{2} \frac{\alpha\lambda^3}{t^{\frac{3}{2}}} |y|\chi'_0 U_0 \end{aligned}$$

The main term in \mathcal{E}_2^o is

$$8 \frac{\lambda^6}{t^3 \xi^4} \left[\chi''_0 - \frac{3}{\xi} \chi'_0(\xi) \right],$$

where $\xi = \frac{r}{\sqrt{t}}$.

On the other hand, from \mathcal{E}_1^o we get a main term given by

$$\frac{1}{2} \frac{\lambda^6}{t^3} \xi \chi'_0(\xi).$$

To improve the size of the error, we introduce a correction ϕ^{o1} of the form

$$\phi^{o1} = \lambda^2 \tilde{\phi}^{o1}$$

where $\tilde{\phi}^{o1}$ is a radial function and solves

$$\begin{cases} \partial_t \tilde{\phi}^{o1} = \Delta_{\mathbb{R}^6} \tilde{\phi}^{o1} + \frac{1}{t^3} h\left(\frac{r}{\sqrt{t}}\right) \\ \tilde{\phi}^{o1}(r, 0) = 0, \end{cases} \quad (2.8)$$

with

$$h(\xi) = \frac{8}{\xi^4} \left[\chi''_0 - \frac{3}{\xi} \chi'_0(\xi) + \frac{\xi}{4} \chi'_0(\xi) \right].$$

In (2.8), $\Delta_{\mathbb{R}^6}$ means $\partial_{rr} + \frac{5}{r}\partial_r$. The solution ϕ^{o1} to (2.8) can be expressed in self-similar form as

$$\tilde{\phi}^{o1}(r, t) = \frac{1}{t^2} g(\xi), \quad \xi = \frac{r}{\sqrt{t}}.$$

We find for g the equation

$$g'' + \frac{5}{\xi}g' + \frac{\xi}{2}g' + 2g + h = 0.$$

Since $\frac{1}{\xi^4}$ is in the kernel of the homogenous equation we find an explicit solution

$$g_0(\xi) = \frac{1}{\xi^4} \int_0^\xi x^3 e^{-\frac{1}{4}x^2} \int_0^x h(y) e^{\frac{1}{4}y^2} y \, dy \, dx.$$

To find the solution ϕ^{o1} with initial condition zero we define

$$g(\xi) = g_0(\xi) - \bar{z}(\xi)I$$

where

$$\bar{z}(\xi) = \frac{1}{\xi^4} \int_0^\xi x^3 e^{-\frac{1}{4}x^2} dx$$

is a second solution of the homogeneous equation, independent of $\frac{1}{\xi^4}$ and

$$I = \int_0^\infty x^3 e^{-\frac{1}{4}x^2} \int_0^x h(y) e^{\frac{1}{4}y^2} y dy dx. \quad (2.9)$$

We note that

$$\tilde{\phi}^{o1}(0, t) = -\frac{I}{4t^2}.$$

We take then as ansatz the function

$$u_2(x, t) = \frac{\alpha(t)}{\lambda^2} U_0(y) \chi_0\left(\frac{\lambda|y|}{\sqrt{t}}\right) + \phi^{o1}(|x - \xi(t)|, t), \quad y = \frac{x - \xi(t)}{\lambda(t)}, \quad (2.10)$$

and define accordingly

$$\psi^{o1} = (-\Delta)^{-1} \phi^{o1}.$$

3. INNER OUTER SYSTEM

We look for a solution to

$$\begin{aligned} u_t &= \nabla_x \cdot (\nabla_x u - u \nabla v) \quad \text{in } \mathbb{R}^2 \times [0, \infty) \\ -\Delta_x v &= u \quad \text{in } \mathbb{R}^2 \times [0, \infty) \end{aligned}$$

of the form

$$u(x, t) = u_2(x, t) + \lambda^{-2} \phi^i\left(\frac{x - \xi}{\lambda}, t\right) \eta(x, t) + \phi^o(x, t),$$

where u_2 is the ansatz (2.10). The cut-off η is of the form

$$\eta(x, t) = \chi_0\left(\frac{|x - \xi|}{\sqrt{t}}\right)$$

and χ_0 is described in (2.1).

Recalling that $y = \frac{x - \xi}{\lambda}$, we compute

$$\begin{aligned} -u_t &= -\partial_t u_2 + \frac{\dot{\lambda}}{\lambda^3} (2\phi^i + \nabla_y \phi^i \cdot y) \eta \\ &\quad - \nabla_y \phi^i \frac{\dot{\xi}}{\lambda^3} \eta - \frac{1}{\lambda^2} \phi_t^i \eta - \frac{1}{\lambda^2} \phi \eta_t - \phi_t^o \end{aligned}$$

Define

$$\begin{aligned} \psi^i &= (-\Delta_y)^{-1} \phi^i, \\ v_2 &= (-\Delta_x)^{-1} u_2 \end{aligned}$$

and then we take

$$v = v_2 + \psi^i \eta + \psi^o$$

We then get the following system of equations for $\phi^i, \psi^i, \phi^o, \psi^o$:

$$\begin{cases} \lambda^2 \partial_t \phi^i = \Delta_y \phi^i - \nabla_y \phi^i \nabla_y V_0 - \nabla_y \psi^i \nabla_y U_0 + 2U_0 \phi^i \\ \quad + 2\lambda^2 U_0 \chi \phi^o + 2\lambda^2 U_0 \chi \phi^{o1} - \lambda \nabla_y [U_0 \chi] \nabla_x \psi^o - \lambda \nabla_y [U_0 \chi] \nabla_x \psi^{o1} + E_{in} \\ -\Delta_y \psi^i = \phi^i \end{cases} \quad (3.1)$$

$$\begin{cases} \partial_t \phi^o = L_{out} \phi^o + G(\phi^i, \phi^o, \psi^i, \psi^o, \lambda, \alpha) \\ -\Delta_x \psi^o = \phi^o + 2\lambda^{-1} \nabla_y \psi^i \nabla_x \eta + \psi^i \Delta_x \eta \end{cases} \quad (3.2)$$

where

$$L_{out} \phi^o = \Delta_x \phi^o - \nabla_x \phi^o \nabla_x V_0 \left(\frac{x - \xi(t)}{\lambda(t)} \right)$$

$$G(\phi^i, \phi^o, \psi^i, \psi^o, \lambda, \alpha) = A_1 \phi^o + A_2 \psi^o + B_1 \phi^i + B_2 \psi^i + E_{out} + N(\phi^i, \phi^o, \psi^i, \psi^o)$$

with

$$\begin{aligned} A_1 \phi^o &= 2u_2 \phi^o (1 - \eta) - \nabla_x \phi^o \nabla_x \tilde{v}_2 + 2[(\alpha - 1)\lambda^{-2} U_0 \eta + \phi^{o1}] \phi^o \\ A_2 \psi^o &= -\nabla_x u_2 \nabla_x \psi^o (1 - \eta) - \lambda^{-3} \nabla_x \psi^o \nabla_y [(\alpha - 1)U_0 \chi + \lambda^2 \phi^{o1}] \\ B_1 \phi^i &= \frac{\dot{\lambda}}{\lambda^3} - \frac{\dot{\xi}}{\lambda^3} \nabla_y \phi^i \eta + (2\phi^i + \nabla_y \phi^i y) \eta + \lambda^{-2} (\Delta_x \eta - \eta_t) \phi^i \\ &\quad + 2\lambda^{-2} \nabla_x \phi^i \nabla_x \eta - \lambda^{-2} \phi^i \nabla_x \eta \nabla_x v_2 \\ &\quad + 2\lambda^{-4} [U_0 (1 - \eta) + (\alpha - 1)U_0 \eta] \phi^i \\ B_2 \psi^i &= -\psi^i \nabla_x u_2 \nabla_x \eta - \lambda^{-4} \nabla_y \psi^i \nabla_y [U_0 (1 - \chi) + (\alpha - 1)U_0 \chi + \lambda^2 \phi^{o1}] \eta \end{aligned}$$

In the above expressions \tilde{v}_2 is given by

$$-\Delta_x \tilde{v}_2 = \lambda^{-2} U_0(y) (\chi - 1) + (\alpha - 1) \lambda^{-2} U_0(y) \chi + \phi^{o1},$$

and the terms E_{in} is given by

$$\begin{aligned} E_{in} &= \alpha(\alpha - 1)U_0^2 - \alpha(\alpha - 1)\nabla U_0 \nabla V_0 - \lambda^2 \dot{\alpha} U_0 \chi + \alpha \lambda \dot{\lambda} (2U_0 + y \cdot \nabla U_0) \chi \\ &\quad + \alpha \lambda \dot{\xi} \cdot \nabla U_0 \chi, \end{aligned}$$

and E_{out} satisfies the estimate

$$|E_{out}| \leq C \begin{cases} \frac{1}{t^3 \log^2 t} & r \leq \sqrt{t} \\ \frac{1}{t \log^2 t r^4} & r \geq \sqrt{t}. \end{cases}$$

Finally, $N(\phi^i, \phi^o, \psi^i, \psi^o)$ are the nonlinear terms:

$$N(\phi^i, \phi^o, \psi^i, \psi^o) = -\nabla \Phi \nabla \Psi + 2\Phi^2$$

with

$$\begin{aligned} \Phi &= \lambda^{-2} \phi^i \left(\frac{x - \xi}{\lambda}, t \right) \eta(x, t) + \phi^o(x, t) \\ \Psi &= \psi^i \eta + \psi^o. \end{aligned}$$

Instead of solving the system (3.1)-(3.2) directly, we consider the following modification of (3.1)

$$\begin{cases} \lambda^2 \partial_t \phi^i = \Delta_y \phi^i - \nabla_y \phi^i \nabla_y V_0 - \nabla_y \psi^i \nabla_y U_0 + 2U_0 \phi^i \\ \quad + H(\phi^o, \psi^o, \phi^{o1}, \psi^{o1}, \lambda, \alpha_2, \xi) - \sum_{j=1}^4 c_j(t) \tilde{Z}_j \\ -\Delta_y \psi^i = \phi^i \end{cases} \quad (3.3)$$

where

$$H(\phi^o, \psi^o, \phi^{o1}, \psi^{o1}, \lambda, \alpha_2, \xi) = 2\lambda^2 U_0 \chi \phi^o + 2\lambda^2 U_0 \chi \phi^{o1} \quad (3.4)$$

$$- \lambda \nabla_y [U_0 \chi] \nabla_x \psi^o - \lambda \nabla_y [U_0 \chi] \nabla_x \psi^{o1}, \quad (3.5)$$

$$\tilde{Z}_j(y) = Z_j(y) \chi(y)$$

and

$$\begin{cases} Z_0(y) = 2U_0(y) + \nabla U_0(y)y, \\ Z_1(y) = \partial_{y_1} U_0(y), \\ Z_2(y) = \partial_{y_3} U_0(y), \\ Z_3(y) = U_0(y). \end{cases} \quad (3.6)$$

For the proof of the main theorem, we will use the following norms.

For the inner problem we define

$$\begin{aligned} \|\phi^i\|_{i^*, \nu, \mu, \sigma} &= \sup_{|y| \leq \sqrt{t}} \{t^\nu \log^\mu t [(1 + |y|)^{3+\sigma} |\phi^i(y, t)| + (1 + |y|)^{4+\sigma} |\nabla \phi^i(y, t)|]\} \\ &\quad + \sup_{|y| \geq \sqrt{t}} \{t^{\nu-1} \log^\mu t [(1 + |y|)^{5+\sigma} |\phi^i(y, t)| + (1 + |y|)^{6+\sigma} |\nabla \phi^i(y, t)|]\} \\ \|h^i\|_{i^{**}, \nu, \mu, \sigma} &= \sup \{t^\nu (1 + |y|)^{5+\sigma} |h^i(y, t)|\} \end{aligned}$$

where $\nu > 0$, $\mu \geq 0$, and $\sigma > 0$.

For the outer problem we use the norms

$$\begin{aligned} \|\phi^o\|_{o^*, a, \mu, b} &= \sup \left\{ \left[t^a \log^\mu t \left(\frac{|x - \xi|}{\lambda} + 1 \right)^{b-2} + \lambda^2 t^{a-1} \log^\mu t \left(\frac{|x - \xi|}{\lambda} + 1 \right)^b \right] |\phi^o(x, t)| \right\} \\ &\quad + \sup \left\{ \left[t^a \log^\mu t \left(\frac{|x - \xi|}{\lambda} + 1 \right)^{b-1} + \lambda^2 t^{a-1} \log^\mu t \left(\frac{|x - \xi|}{\lambda} + 1 \right)^{b+1} \right] |\nabla \phi^o(x, t)| \right\} \\ \|h\|_{o^{**}, a, \mu, b} &= \sup \left\{ \lambda^2 t^a \log^\mu t \left(\frac{|x - \xi|}{\lambda} + 1 \right)^b |h(x, t)| \right\}, \end{aligned}$$

where $2 < b < 6$, $1 < a < 3$, $\mu \geq 0$.

Proof of Theorem 1. We let $\sigma > 0$ be a small number. We work in both equations (3.3) and (3.2) with $t \geq t_0$ and t_0 a large constant. In (3.3) we impose zero initial condition and in (3.2) we put initial condition $\phi^o(x, t_0) = \phi_0^o(x)$ a function with small norm (1.3). We assume that λ , α_2 and ξ satisfy the conditions (2.5), (2.6), (2.7). Given a function ϕ^i with $\|\phi^i\|_{*i, 1, 2, \sigma}$ small we solve (3.2) using Theorem 3 and the Banach fixed point theorem, and we find a solution ϕ^o with the estimate

$$\|\phi^o\|_{*o, 2, 2, 0} \lesssim \|\phi^i\|_{*i, 1, 2, \sigma} + \|\phi_0^o\|_{*\sigma}.$$

Here we have used in a crucial manner the fact that

$$\phi_0^o(x) = O(|x|^{-4-\sigma}) \quad \text{as } |x| \rightarrow \infty,$$

since from here and Duhamel's formula we find that $\phi^o(x, t) = O(t^{-2-\frac{\sigma}{2}})$.

It is straightforward to check that ϕ^o, ψ^o are Lipschitz functions of ϕ^i . Inserting these functions in (3.3) we solve now for ϕ^i , having λ, α_2 and ξ as given parameters. The functions $c_j(t)$ in (3.3) are chosen such that $H(\phi^o, \psi^o, \phi^{o1}, \psi^{o1})$ satisfies the four conditions in Theorem 2. Using that theorem and the Banach fixed point theorem, we find that (3.3) has a solution ϕ^i with $\|\phi^i\|_{*i,1,2,\sigma}$ small. To find a solution to the problem we need the following conditions to be satisfied

$$\begin{aligned} \lambda^2 \dot{\alpha} \int_{\mathbb{R}^2} U_0 \chi \, dy &= \alpha(\alpha - 1) \int_{\mathbb{R}^2} [U_0^2 - \nabla U_0 \nabla V_0] \chi \, dy - \alpha \lambda \dot{\lambda} \int_{\mathbb{R}^2} [2U_0 + y \nabla U_0] \, dy \\ &+ \int_{\mathbb{R}^2} H(\phi^o, \psi^o, \phi^{o1}, \psi^{o1}, \lambda, \alpha_2, \xi) \, dy \end{aligned} \quad (3.7)$$

$$\begin{aligned} \alpha \lambda \dot{\lambda} &= \int_{\mathbb{R}^2} H(\phi^o, \psi^o, \phi^{o1}, \psi^{o1}, \lambda, \alpha_2, \xi) |y|^2 \, dy \\ &- \alpha(\alpha - 1) \int_{\mathbb{R}^2} [U_0^2 - \nabla U_0 \nabla V_0] |y|^2 \, dy \\ &- \lambda^2 \dot{\alpha} \int_{\mathbb{R}^2} U_0 \chi |y|^2 \, dy \end{aligned} \quad (3.8)$$

$$\alpha \lambda \dot{\xi} = \int_{\mathbb{R}^2} H(\phi^o, \psi^o, \phi^{o1}, \psi^{o1}, \lambda, \alpha_2, \xi) y \, dy, \quad (3.9)$$

where H is given by (3.4). At main order, after some computation we find the equation

$$-\lambda^2(\dot{\alpha}_1 + \dot{\alpha}_2)8\pi + 2\lambda^2 \int_{\mathbb{R}^2} U_0 \phi^{o1} - \lambda^2 \int_{\mathbb{R}^2} \nabla U_0 \cdot \nabla \psi^{o1} = \frac{1}{t^2 \log^3 t} \mathcal{N}_1(\dot{\lambda}, \dot{\alpha}_2, \dot{\xi})$$

where \mathcal{N}_1 is a uniformly bounded operator, Lipschitz in its arguments for their associated topologies in (2.5), (2.6), (2.7). After an integration by parts, and integrating in time from t to $+\infty$, and using the definition of α_1 in (2.3) we find that at main order

$$\alpha_2(t) + \frac{\lambda^2}{t}(I - a) + \frac{1}{t \log^2 t} \mathcal{N}_2(\dot{\lambda}, \dot{\alpha}_2, \dot{\xi}) = 0,$$

where I and a are the numbers introduced in (2.9) and (2.4).

Computing the explicitly constant $I - a$ and replacing this in (3.8) gives at main order

$$\lambda \dot{\lambda} = -c_1 \frac{\lambda^4}{t} (1 + O(\frac{1}{\log t})),$$

for some positive constant c_1 , and solving for λ gives the desired rate after an application of contraction mapping principle to system (3.7), (3.8), (3.9). The equations for λ and ξ is have to be solved by fixing their initial conditions independently of ϕ_0^o . The fact that perturbative initial condition ϕ_0^o was arbitrary gives the stability of the blow-up, since if we assume $\int_{\mathbb{R}^2} \phi_0^o = 0$ then necessarily $\alpha_2(t_0) = 0$ which precisely amounts to the mass of the full initial condition to be exactly 8π . We check that for the topology we are using in the initial conditions we can represent an arbitrary perturbation in this form. This concludes the proof. \square

4. PRELIMINARIES FOR THE LINEAR THEORY

4.1. **The Liouville equation.** Let

$$\begin{aligned} U_0(y) &= \frac{8}{(1 + |y|^2)^2} \\ V_0(y) &= \log \frac{8}{(1 + |y|^2)^2} \end{aligned} \tag{4.1}$$

with $y \in \mathbb{R}^2$ and note that

$$\Delta V_0 + e^{V_0} = 0 \quad \text{in } \mathbb{R}^2.$$

The linearization around of V_0 is given by the operator

$$\psi \mapsto \Delta \psi + U_0 \psi$$

and we note that the following explicit functions

$$\begin{cases} z_0(y) = \nabla V_0(y) \cdot y + 2 \\ z_j(y) = \partial_{y_j} V_0(y), \quad j = 1, 2 \end{cases}$$

are elements in the kernel of $\Delta + U_0$. These are the only bounded elements in the kernel.

4.2. **Stereographic projection.** Let $\Pi : S^2 \setminus \{(0, 0, 1)\} \rightarrow \mathbb{R}^2$ denote the stereographic projection

$$\Pi(y_1, y_2, y_3) = \left(\frac{y_1}{1 - y_3}, \frac{y_2}{1 - y_3} \right).$$

For $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$ we write

$$\tilde{\varphi} = \varphi \circ \Pi, \quad \tilde{\varphi} : S^2 \setminus \{(0, 0, 1)\} \rightarrow \mathbb{R}.$$

Let U_0 be given by (4.1). Then we have the following formulas

$$\begin{aligned} \int_{S^2} \tilde{\varphi} &= \frac{1}{2} \int_{\mathbb{R}^2} \varphi U_0 \\ \int_{S^2} \tilde{U}_0 |\nabla_{S^2} \tilde{\varphi}|^2 &= \int_{\mathbb{R}^2} U_0 |\nabla_{\mathbb{R}^2} \varphi|^2 \\ \frac{1}{2} \tilde{U}_0 \Delta_{S^2} \tilde{\varphi} &= (\Delta_{\mathbb{R}^2} \varphi) \circ \Pi. \end{aligned}$$

The linearized Liouville equation for $\phi, f : \mathbb{R}^2 \rightarrow \mathbb{R}$

$$\Delta \phi + U_0 \phi + U_0 f = 0 \quad \text{in } \mathbb{R}^2$$

is transformed into

$$\Delta_{S^2} \tilde{\phi} + 2\tilde{\phi} + 2\tilde{f} = 0 \quad \text{in } S^2 \setminus \{(0, 0, 1)\}.$$

4.3. A quadratic form.

Lemma 4.1. *Let $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfy*

$$|\phi(y)| \leq \frac{1}{(1 + |y|)^{2+\sigma}},$$

with $0 < \sigma < 1$, and

$$\int_{\mathbb{R}^2} \phi \, dy = 0.$$

There are constants $c_1 > 0$, $c_2 > 0$ such that

$$c_1 \int_{\mathbb{R}^2} U_0 g^2 \leq \int_{\mathbb{R}^2} \phi g \leq c_2 \int_{\mathbb{R}^2} U_0 g^2$$

where

$$g = \frac{\phi}{U_0} - (-\Delta^{-1})\phi + c$$

and $c \in \mathbb{R}$ is chosen so that

$$\int_{\mathbb{R}^2} g U_0 = 0.$$

Proof. Since $\int_{\mathbb{R}^2} \phi = 0$ there is a unique bounded ψ such that

$$-\Delta \psi = \phi \quad \text{in } \mathbb{R}^2,$$

and this is the function denoted by $\psi = (-\Delta)^{-1}\phi$. Moreover

$$|\psi(y)| + (1 + |y|)|\nabla \psi(y)| \lesssim \frac{1}{(1 + |y|)^\sigma}.$$

Since

$$g = \frac{\phi}{U_0} - \psi + c$$

we find the estimate

$$|g(y)| \lesssim (1 + |y|)^{2-\sigma}.$$

Note that

$$-\Delta \psi - U_0 \psi = U_0 g \quad \text{in } \mathbb{R}^2.$$

We transform $\tilde{g} = g \circ \Pi$, $\tilde{\psi} = \psi \circ \Pi$ and write this equation in S^2 as

$$-\Delta_{S^2} \tilde{\psi} - 2\tilde{\psi} = 2\tilde{g}, \quad \text{in } S^2. \tag{4.2}$$

Since $\phi = U_0(g + \psi - c)$ we get

$$\int_{\mathbb{R}^2} \phi g = \int_{\mathbb{R}^2} U_0(g + \psi)g = \frac{1}{2} \int_{S^2} \tilde{g}^2 + \tilde{\psi} \tilde{g},$$

Multiplying (4.2) by $\tilde{\psi}$ we find that

$$\int_{S^2} \tilde{g} \tilde{\psi} = \frac{1}{2} \int_{S^2} |\nabla_{S^2} \tilde{\psi}|^2 - \int_{S^2} \tilde{\psi}^2$$

and hence

$$\int_{\mathbb{R}^2} \phi g = \frac{1}{2} \int_{S^2} \tilde{g}^2 + \frac{1}{4} \int_{S^2} |\nabla_{S^2} \tilde{\psi}|^2 - \frac{1}{2} \int_{S^2} \tilde{\psi}^2.$$

We recall that the eigenvalues of $-\Delta$ on S^2 are given by $\{k(k+1) \mid k \geq 0\}$. The eigenvalue 0 has a constant eigenfunction and the eigenvalue 2 has eigenspace

spanned by the coordinate functions $\pi_i(x_1, x_2, x_3) = x_i$, for $(x_1, x_2, x_3) \in S^2$ and $i = 1, 2, 3$. Let $(\lambda_j)_{j \geq 0}$ denote all eigenvalues, repeated according to multiplicity, with $\lambda_0 = 0$, $\lambda_1 = \lambda_2 = \lambda_3 = 2$, and let $(e_j)_{j \geq 0}$ denote the corresponding eigenfunctions so that they form an orthonormal system in $L^2(S^2)$, and e_1, e_2, e_3 are multiples of the coordinate functions π_1, π_2, π_3 . We decompose $\tilde{\psi}$ and \tilde{g} :

$$\tilde{\psi} = \sum_{j=0}^{\infty} \tilde{\psi}_j e_j, \quad \tilde{g} = \sum_{j=0}^{\infty} \tilde{g}_j e_j, \quad (4.3)$$

where

$$\tilde{\psi}_j = \langle \tilde{\psi}, e_j \rangle_{L^2(S^2)}, \quad \tilde{g}_j = \langle \tilde{g}, e_j \rangle_{L^2(S^2)}.$$

Then

$$\begin{aligned} \int_{\mathbb{R}^2} \phi g &= \frac{1}{2} \int_{S^2} \tilde{g}^2 + \frac{1}{4} \sum_{j=0}^{\infty} (\lambda_j - 2) \tilde{\psi}_j^2 \\ &= \frac{1}{2} \int_{S^2} \tilde{g}^2 - \frac{1}{2} \tilde{\psi}_0^2 + \frac{1}{4} \sum_{j=4}^{\infty} (\lambda_j - 2) \tilde{\psi}_j^2. \end{aligned}$$

Equation (4.2) gives us that

$$\tilde{\psi}_j = \frac{2}{\lambda_j - 2} \tilde{g}_j, \quad j \notin \{1, 2, 3\},$$

and therefore

$$\begin{aligned} \int_{\mathbb{R}^2} \phi g &= \frac{1}{2} \int_{S^2} \tilde{g}^2 - \frac{1}{2} \tilde{g}_0^2 + \sum_{j=4}^{\infty} \frac{1}{\lambda_j - 2} \tilde{g}_j^2 \\ &= \frac{1}{2} \sum_{j=1}^{\infty} \tilde{g}_j^2 + \sum_{j=4}^{\infty} \frac{1}{\lambda_j - 2} \tilde{g}_j^2. \end{aligned}$$

But $\int_{\mathbb{R}^2} g U_0 = 0$ which means that $\tilde{g}_0 = 0$ and hence we obtain the conclusion. \square

Lemma 4.2. *Suppose that $\phi = \phi(y, t)$, $y \in \mathbb{R}^2$, $t > 0$ is a function satisfying*

$$|\phi(y, t)| \leq \frac{1}{(1 + |y|)^{2+\sigma}},$$

with $0 < \sigma < 1$,

$$\int_{\mathbb{R}^2} \phi(y, t) dy = 0, \quad \forall t > 0,$$

and that ϕ is differentiable with respect to t and ϕ_t satisfies also

$$|\phi_t(y, t)| \leq \frac{1}{(1 + |y|)^{2+\sigma}}.$$

Then

$$\int_{\mathbb{R}^2} \phi_t g = \frac{1}{2} \partial_t \int_{\mathbb{R}^2} \phi g$$

where for each t , $g(y, t)$ is defined as

$$g = \frac{\phi}{U_0} - (-\Delta^{-1})\phi + c(t)$$

and $c(t) \in \mathbb{R}$ is chosen so that

$$\int_{\mathbb{R}^2} g(y, t) U_0(y) dy = 0.$$

Proof. Using the notation of the previous lemma, we have

$$\int_{\mathbb{R}^2} \phi_t g = \int_{\mathbb{R}^2} U_0(g_t + \psi_t) g = \frac{1}{2} \int_{S^2} (\tilde{g}_t \tilde{g} + \tilde{\psi}_t \tilde{g}).$$

We have

$$-\Delta_{S^2} \tilde{\psi} - 2\tilde{\psi} = 2\tilde{g}, \quad \text{in } S^2.$$

And differentiating in t we get

$$-\Delta_{S^2} \tilde{\psi}_t - 2\tilde{\psi}_t = 2\tilde{g}_t, \quad \text{in } S^2. \quad (4.4)$$

Multiplying by \tilde{g} and integrating we find that

$$\int_{S^2} \tilde{\psi}_t \tilde{g} = -\frac{1}{2} \int_{S^2} \Delta \tilde{\psi}_t \tilde{g} - \int_{S^2} \tilde{g}_t \tilde{g}.$$

Thus

$$\int_{\mathbb{R}^2} \phi_t g = -\frac{1}{4} \int_{S^2} \Delta \tilde{\psi}_t \tilde{g}$$

Decompose as in (4.3) and find that

$$\int_{\mathbb{R}^2} \phi_t g = \frac{1}{4} \sum_{j=0}^{\infty} \lambda_j (\tilde{\psi}_j)_t \tilde{g}_j$$

But from (4.4)

$$(\lambda_j - 2)(\tilde{\psi}_j)_t = 2(\tilde{g}_j)_t.$$

If

$$\tilde{g}_j = 0, \quad j = 0, 1, 2, 3,$$

then

$$\int_{\mathbb{R}^2} \phi_t g = \frac{1}{2} \sum_{j=4}^{\infty} \frac{\lambda_j}{\lambda_j - 2} (\tilde{g}_j)_t \tilde{g}_j$$

□

4.4. A Hardy inequality.

Lemma 4.1. *There exists $C > 0$ such that, for any $R > 0$ large and any $\int_{B_R} g U_0 = 0$*

$$\frac{C}{R^2} \int_{B_R} g^2 U_0 \leq \int_{B_R} |\nabla g|^2 U_0.$$

Proof. After a stereographic projection and letting $\varepsilon = \frac{1}{R}$, $A_\varepsilon = B_1(0) \setminus B_\varepsilon(0) \subset \mathbb{R}^2$, we need to prove that for $g \in C^1(\bar{A}_\varepsilon)$ with

$$\int_{A_\varepsilon} g dy = 0$$

we have

$$\int_{A_\varepsilon} g^2 dy \leq \frac{C}{\varepsilon^2} \int_{A_\varepsilon} |\nabla g|^2 |y|^4 dy.$$

By using polar coordinates it is sufficient to show this for radial functions, which amounts to the statement: for $g \in C^1([\varepsilon, 1])$, if

$$\int_{\varepsilon}^1 g^2 x \, dx = 0 \quad (4.5)$$

then

$$\int_{\varepsilon}^1 g^2 x \, dx \leq \frac{C}{\varepsilon^2} \int_{\varepsilon}^1 g'(x)^2 x^5 \, dx.$$

We write

$$\int_{\varepsilon}^1 g^2 x \, dx = \frac{1}{2} \int_{\varepsilon}^1 g^2 \frac{d}{dx}(x^2) \, dx = \frac{g^2(1)}{2} - \frac{g^2(\varepsilon)}{2} \varepsilon^2 - \frac{1}{2} \int_{\varepsilon}^1 gg' x^2 \, dx.$$

One has

$$\begin{aligned} \int_{\varepsilon}^1 gg' x^2 \, dx &\leq \left(\int_{\varepsilon}^1 g'^2 x^5 \, dx \right)^{\frac{1}{2}} \left(\int_{\varepsilon}^1 g^2 x^{-1} \, dx \right)^{\frac{1}{2}} \\ &\leq \left(\varepsilon^{-2} \int_{\varepsilon}^1 g'^2 x^5 \, dx \right)^{\frac{1}{2}} \left(\int_{\varepsilon}^1 g^2 x \, dx \right)^{\frac{1}{2}} \\ &\leq C \varepsilon^{-2} \int_{\varepsilon}^1 g'^2 x^5 \, dx + \frac{1}{2} \int_{\varepsilon}^1 g^2 x \, dx, \end{aligned}$$

for some constant C . Inserting this inequality in the previous computation gives

$$\int_{\varepsilon}^1 g^2 x \, dx \leq g(1)^2 - g(\varepsilon)^2 \varepsilon^2 + C \varepsilon^{-2} \int_{\varepsilon}^1 g'^2 x^5 \, dx. \quad (4.6)$$

We now use (4.5) in the form

$$0 = \int_{\varepsilon}^1 g(x) x \, dx = \frac{g(1)}{2} - \frac{g(\varepsilon)}{2} \varepsilon^2 - \frac{1}{2} \int_{\varepsilon}^1 g' x^2 \, dx,$$

and so

$$g(1)^2 \leq 2g(\varepsilon)^2 \varepsilon^4 + 2 \left(\int_{\varepsilon}^1 g' x^2 \, dx \right)^2.$$

But

$$\begin{aligned} \int_{\varepsilon}^1 g' x^2 \, dx &\leq \left(\int_{\varepsilon}^1 g'^2 x^5 \, dx \right)^{\frac{1}{2}} \left(\int_{\varepsilon}^1 x^{-1} \, dx \right)^{\frac{1}{2}} \\ &\leq \left(|\log \varepsilon| \int_{\varepsilon}^1 g'^2 x^5 \, dx \right)^{\frac{1}{2}}. \end{aligned}$$

We thus get that

$$g(1)^2 \leq 2g(\varepsilon)^2 \varepsilon^4 + 2|\log \varepsilon| \int_{\varepsilon}^1 g'^2 x^5 \, dx$$

and this combined with (4.6) gives

$$\int_{\varepsilon}^1 g^2 x \, dx \leq g(\varepsilon)^2 (2\varepsilon^4 - \varepsilon^2) + (C\varepsilon^{-2} + 2|\log \varepsilon|) \int_{\varepsilon}^1 g'^2 x^5 \, dx.$$

For $\varepsilon > 0$ small this gives the desired estimate. \square

5. INNER PROBLEM

The relevant linear equation for ϕ^i appearing in (3.3) is

$$\lambda^2 \partial_t \phi^i = \Delta_y \phi^i - \nabla_y \phi^i \nabla_y V_0 - \nabla_y \psi^i \nabla_y U_0 + 2U_0 \phi^i + h.$$

We change the time variable

$$\tau = \int_{t_0}^t \frac{1}{\lambda^2(s)} ds$$

and note that $\tau \sim t \log t$. Then this equation can be written as

$$\partial_\tau \phi = \nabla \cdot \left[U \nabla \left(\frac{\phi}{U_0} - (-\Delta)^{-1} \phi \right) \right] + h, \quad \text{in } \mathbb{R}^2 \times (\tau_0, \infty).$$

where

$$(-\Delta)^{-1} \phi(x, t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log \left(\frac{1}{|x-y|} \right) \phi(y, t) dy,$$

and τ_0 is fixed large. We consider it with initial condition

$$\phi(y, \tau_0) \equiv 0 \quad \text{in } \mathbb{R}^2.$$

We assume that

$$|h(y, \tau)| \leq \frac{1}{\tau^\nu \log^\mu \tau (1+|y|)^{5+\sigma}}$$

with

$$0 < \nu < 3, \quad \mu \in \mathbb{R}, \quad 0 < \sigma < 1.$$

We define

$$\|h\|_{i^{**}, \nu, \mu, \sigma} = \sup \{ \tau^\nu \log^\mu \tau (1+|y|)^{5+\sigma} |h(y, \tau)| \}.$$

We also assume that

$$\int_{\mathbb{R}^2} h(y, \tau) dy = 0, \tag{5.1}$$

$$\int_{\mathbb{R}^2} h(y, \tau) |y|^2 dy = 0, \tag{5.2}$$

$$\int_{\mathbb{R}^2} h(y, \tau) y_j dy = 0, \quad j = 1, 2, \tag{5.3}$$

for all $\tau > \tau_0$.

We have:

Theorem 2. *Assume that h satisfies $\|h\|_{i^{**}, \nu, \mu, \sigma} < \infty$ and the conditions (5.1), (5.2) and (5.3). Then*

$$|\phi(y, \tau)| (1+|y|) |\nabla \phi(y, \tau)| \lesssim \|h\|_{i^{**}, \nu, \mu, \sigma} \begin{cases} \frac{1}{\tau^\nu \log^\mu \tau (1+\rho)^{3+\sigma}} & \rho \leq \sqrt{\tau} \\ \frac{1}{\tau^{\nu-1} \log^\mu \tau (1+\rho)^{5+\sigma}} & \rho \geq \sqrt{\tau}. \end{cases}$$

Proof. Let

$$g = \frac{\phi}{U_0} - (-\Delta)^{-1} \phi + c(\tau),$$

where $c(\tau)$ is chosen so that

$$\int_{\mathbb{R}^2} g U_0 = 0. \tag{5.4}$$

Note that

$$\partial_\tau \phi = \nabla \cdot (U_0 \nabla g) + h, \quad \text{in } \mathbb{R}^2 \times (\tau_0, \infty).$$

We multiply this equation by g and integrate in \mathbb{R}^2 , using Lemma 4.2:

$$\frac{1}{2} \partial_\tau \int_{\mathbb{R}^2} \phi g + \int_{\mathbb{R}^2} U_0 |\nabla g|^2 = \int_{\mathbb{R}^2} h g.$$

We use the inequality in Lemma 4.1 to get

$$\frac{1}{R_1^2} \int_{B_{R_1}} (g - \bar{g}_{R_1})^2 U_0 \leq \int_{\mathbb{R}^2} U_0 |\nabla g|^2$$

where

$$\bar{g}_{R_1} = \frac{1}{\int_{B_{R_1}} U_0} \int_{B_{R_1}} g U_0.$$

Here R_1 is a large positive constant, to be made precise below.

Then

$$\frac{1}{2} \partial_\tau \int_{\mathbb{R}^2} \phi g + \frac{1}{R_1^2} \int_{B_{R_1}} g^2 U_0 \leq C R_1^2 \int_{\mathbb{R}^2} h^2 U_0^{-1} + \frac{1}{2 R_1^2} \left(\int_{\mathbb{R}^2} g^2 U_0 + C \bar{g}_{R_1}^2 \right).$$

But by (5.4)

$$\bar{g}_{R_1} = - \frac{1}{\int_{B_{R_1}} U_0} \int_{\mathbb{R}^2 \setminus B_{R_1}} g U_0$$

so

$$\bar{g}_{R_1}^2 \leq C \int_{\mathbb{R}^2 \setminus B_{R_1}} g^2 U_0.$$

Therefore

$$\frac{1}{2} \partial_\tau \int_{\mathbb{R}^2} \phi g + \frac{1}{2 R_1^2} \int_{B_{R_1}} g^2 U_0 \lesssim R_1^2 \int_{\mathbb{R}^2} h^2 U_0^{-1} + \frac{1}{R_1^2} \int_{\mathbb{R}^2 \setminus B_{R_1}} g^2 U_0.$$

We now use Lemma 4.1 to get

$$\partial_\tau \int_{\mathbb{R}^2} \phi g + \frac{1}{C} \int_{\mathbb{R}^2} \phi g \lesssim R_1^2 \int_{\mathbb{R}^2} h^2 U_0^{-1} + \frac{1}{R_1^2} \int_{\mathbb{R}^2 \setminus B_{R_1}} g^2 U_0. \quad (5.5)$$

Define

$$A^2 = \sup_{\tau \geq \tau_0} \left\{ \tau^{2\nu} \log^{2\mu} \tau \int_{\mathbb{R}^2 \setminus B_R} g^2(t) U_0 \right\}.$$

Integrating (5.5) and using Lemma 4.1 we find

$$\int_{\mathbb{R}^2} g^2 U_0 \lesssim \frac{R_1^4 \|h\|_{i^{**}, \nu, \mu, \sigma}^2}{\tau^{2\nu} \log^{2\mu} \tau} + \frac{A^2}{\tau^{2\nu} \log^{2\mu} \tau}. \quad (5.6)$$

Let us use the notation

$$\|g\|_{L^2(U_0^{1/2})}^2 = \int_{\mathbb{R}^2} g^2 U_0$$

and we record the estimate (5.6) as

$$\|g(\tau)\|_{L^2(U_0^{1/2})} \lesssim \frac{R_1^2 \|h\|_{i^{**}, \nu, \mu, \sigma}}{\tau^\nu \log^\mu \tau} + \frac{A}{\tau^\nu \log^\mu \tau}.$$

The idea now is to obtain decay of g , and then show that A can be eliminated from the estimate (5.6).

Let us rewrite the equation for $\tilde{g} = U_0 g$ as

$$\begin{aligned} \partial_\tau \tilde{g} &= U_0 \partial_\tau g = \partial_\tau \phi + U_0 \Delta^{-1} \partial_\tau \phi \\ &= \nabla \cdot (U_0 \nabla g) + h - U_0 (-\Delta)^{-1} [\nabla \cdot (U_0 \nabla g) + h] \\ &= \nabla \cdot \left[U_0 \nabla \left(\frac{\tilde{g}}{U_0} \right) \right] + h - U_0 v, \end{aligned} \quad (5.7)$$

where

$$v := (-\Delta)^{-1} [\nabla \cdot (U_0 \nabla g) + h].$$

We claim that

$$|v(y)| \lesssim \frac{1}{1 + |y|^{2-\varepsilon}} \left[\frac{R_1^2 \|h\|_{i^{**}, \nu, \mu, \sigma}}{\tau^\nu \log^\mu \tau} + \frac{A}{t^\nu \log^\mu \tau} \right], \quad (5.8)$$

for any $\varepsilon > 0$.

This can be proved by noticing that

$$\begin{aligned} \nabla \cdot (U_0 \nabla g) &= \Delta g U_0 + \nabla U_0 \cdot \nabla g \\ &= \Delta(g U_0) - \nabla U_0 \cdot \nabla g - g \Delta U_0, \end{aligned}$$

and hence

$$v = -g U_0 - (-\Delta)^{-1} [\nabla U_0 \cdot \nabla g + g \Delta U_0].$$

Thus we can decompose

$$v = v_1 + v_2 + v_3$$

where

$$\begin{aligned} v_1 &= -g U_0 \\ v_2 &= -(-\Delta)^{-1} [\nabla U_0 \cdot \nabla g] \\ v_3 &= -(-\Delta)^{-1} [g \Delta U_0]. \end{aligned}$$

Then we solve the equations for v_2 and v_3 in the sphere, using the stereographic projection and the basic estimate

$$\|g\|_{L^2(B_1(y))} \lesssim (1 + |y|)^2 \|g(\tau)\|_{L^2(U_0^{1/2})} \lesssim (1 + |y|)^2 \left[\frac{R^2 \|h\|_{i^{**}, \nu, \mu, \sigma}}{\tau^\nu \log^\mu \tau} + \frac{A}{\tau^\nu \log^\mu \tau} \right],$$

for any y . From this estimate and equation (5.7) we get

$$|g(y, t)| \lesssim (1 + |y|)^2 \frac{R^2 \|h\|_{i^{**}, \nu, \mu, \sigma} + A}{\tau^\nu \log^\mu \tau}.$$

Using again standard elliptic estimate we deduce (5.8).

Now we choose a large constant R_0 so that we can use the maximum principle for the parabolic operator $\partial_\tau f - \nabla \cdot [U_0 \nabla (\frac{f}{U_0})]$ is valid outside the ball $B_{R_0}(0)$. Indeed, we have

$$\nabla \cdot \left[U_0 \nabla \left(\frac{f}{U_0} \right) \right] = \Delta f - \nabla V_0 \nabla f + U_0 f = \partial_{\rho\rho} f + \frac{5}{\rho} \partial_\rho f + \frac{1}{\rho^2} \partial_{\theta\theta} f + Df$$

where $Df = O(\frac{1}{\rho^3})\partial_\rho f + O(\frac{1}{\rho^4})f$ represent lower order terms. Using the maximum principle and an appropriate barrier in $\mathbb{R}^2 \setminus B_{R_0}$, as constructed in Theorem 3 below, we get that

$$|\tilde{g}(y, \tau)| \lesssim \frac{R_1^2 \|h\|_{i^{**}, \nu, \mu, \sigma} + A}{\tau^\nu \log^\mu \tau (1 + |y|)^{3+\sigma}}, \quad |y| \leq \sqrt{\tau}$$

and

$$|\tilde{g}(y, \tau)| \lesssim \frac{R^2 \|h\|_{i^{**}, \nu, \mu, \sigma} + A}{\tau^{\nu-1} \log^\mu \tau (1 + |y|)^{5+\sigma}}, \quad |y| \geq \sqrt{\tau}.$$

We use this estimate to compute

$$\begin{aligned} \int_{\mathbb{R}^2 \setminus B_{R_1}} g^2 U &= \int_{\mathbb{R}^2 \setminus B_{R_1}} \tilde{g}^2 U^{-1} \\ &\lesssim \frac{1}{R_1^{2\sigma}} \frac{R_1^4 \|h\|_{i^{**}, \nu, \mu, \sigma}^2 + A^2}{\tau^{2\nu} \log^{2\mu} \tau}. \end{aligned}$$

This implies that

$$A^2 \leq C \frac{1}{R_1^{2\sigma}} (R_1^4 \|h\|_{i^{**}, \nu, \mu, \sigma}^2 + A^2),$$

where C is a constant from previous inequalities, which is independent of R_1 . Choosing a fixed R_1 large then implies that

$$A^2 \lesssim R_1^{4-2\sigma} \|h\|_{i^{**}, \nu, \mu, \sigma}^2.$$

We then conclude that

$$|\tilde{g}(y, t)| \lesssim \|h\|_{i^{**}, \nu, \mu, \sigma} \begin{cases} \frac{1}{\tau^\nu \log^\mu \tau (1+|y|)^{3+\sigma}}, & |y| \leq \sqrt{\tau} \\ \frac{1}{\tau^{\nu-1} \log^\mu \tau (1+|y|)^{5+\sigma}}, & |y| \geq \sqrt{\tau}. \end{cases}$$

From parabolic estimates we also find

$$|\nabla \tilde{g}(y, t)| \lesssim \|h\|_{i^{**}, \nu, \mu, \sigma} \begin{cases} \frac{1}{\tau^\nu \log^\mu \tau (1+|y|)^{4+\sigma}}, & |y| \leq \sqrt{\tau} \\ \frac{1}{\tau^{\nu-1} \log^\mu \tau (1+|y|)^{6+\sigma}}, & |y| \geq \sqrt{\tau}. \end{cases} \quad (5.9)$$

Now we estimate ϕ . We decompose

$$\phi = \phi^\perp + \omega(\tau) Z_0,$$

where Z_0 is defined in (3.6). We then have

$$g = \frac{\phi^\perp}{U_0} - (\Delta^{-1})\phi^\perp + c(t).$$

We let $\psi = (-\Delta^{-1})\phi^\perp$ and see that

$$gU = \Delta\psi + U_0\psi.$$

Integrating the equation times $|y|^2$ we get

$$\int_{\mathbb{R}^2} \phi(y, \tau) |y|^2 dy = 0, \quad \forall \tau > \tau_0.$$

and this is equivalent to

$$\int_{\mathbb{R}^2} g Z_0 = 0, \quad \forall \tau > \tau_0,$$

where Z_0 is defined in (3.6). We then can solve the equation for ψ and find

$$|\psi(y, \tau)| \lesssim \|h\|_{i^{**}, \nu, \mu, \sigma} \begin{cases} \frac{1}{\tau^\nu \log^\mu \tau (1+|y|)^{1+\sigma}}, & |y| \leq \sqrt{\tau} \\ \frac{1}{\tau^{\nu-1} \log^\mu \tau (1+|y|)^{3+\sigma}}, & |y| \geq \sqrt{\tau}. \end{cases}$$

Since

$$\phi^\perp = U_0(g - \psi)$$

we find that

$$|\phi^\perp(y, \tau)| \lesssim \|h\|_{i^{**}, \nu, \mu, \sigma} \begin{cases} \frac{1}{\tau^\nu \log^\mu \tau (1+|y|)^{3+\sigma}}, & |y| \leq \sqrt{\tau} \\ \frac{1}{\tau^{\nu-1} \log^\mu \tau (1+|y|)^{5+\sigma}}, & |y| \geq \sqrt{\tau}. \end{cases} \quad (5.10)$$

Finally we estimate $\omega(\tau)$. We have

$$\partial_\tau \phi^\perp + \omega_\tau z = L\phi + h.$$

We multiply by $|y|^2$ and integrate in B_{R_2} where $R_2 \rightarrow \infty$ and in a time interval $[\tau_1, \tau_2]$. We get

$$\int_{B_{R_2}} [\phi(\tau_2)^\perp - \phi(\tau_1)^\perp] |y|^2 dy + (\omega(\tau_2) - \omega(\tau_1)) \int_{B_{R_2}} Z_0 |y|^2 dy = \int_{\tau_1}^{\tau_2} \int_{B_{R_2}} L\phi |y|^2 dy d\tau.$$

Let us observe that if $R_2 \geq \sqrt{\tau}$ then

$$\int_{B_{R_2}} |\phi(y, \tau)^\perp| |y|^2 dy \lesssim \frac{1}{\tau^{\nu-1} \log^\mu \tau R_2^{1+\sigma}}.$$

On the other hand

$$\int_{B_{R_2}} L\phi |y|^2 dy = \int_{B_{R_2}} g Z_0 dy + \int_{\partial B_{R_2}} U_0 |y|^2 \nabla g \cdot \nu - \int_{\partial B_{R_2}} g U_0 y \cdot \nu,$$

and

$$\left| \int_{B_{R_2}} g Z_0 dy \right| \leq \int_{B_{R_2}} |\tilde{g}| dy \lesssim \frac{1}{\tau^\nu \log^\mu \tau},$$

$$\left| \int_{\partial B_{R_2}} U_0 |y|^2 \nabla g \cdot \nu \right| + \left| \int_{\partial B_{R_2}} g U_0 y \cdot \nu \right| \lesssim \frac{1}{\tau^\nu \log^\mu \tau R_2^{1+\sigma}}.$$

Since

$$\int_{B_{R_2}} Z_0 |y|^2 dy \sim \log R_2,$$

letting $R_2 \rightarrow \infty$ we find that $\omega(\tau_2) = \omega(\tau_1)$. Hence $\omega \equiv \text{const}$ and since we start with $\omega(0) = 0$ we deduce $\omega \equiv 0$. Hence the estimate (5.10) gives the desired estimate for ϕ . The estimate for the gradient of ϕ comes from the corresponding estimate for the gradient of ϕ^\perp , which is obtained similarly from (5.9). \square

6. OUTER PROBLEM

Theorem 3. *Consider*

$$\partial_t \phi^o = L_{out} \phi^o + h,$$

where

$$|h(x, t)| \leq \frac{\lambda^{-2}}{t^a \log^\mu t \left(\frac{|x-\xi|}{\lambda} + 1 \right)^b},$$

If $2 < b < 6$ and $1 < a < 3$, then

$$|\phi^o(x, t)| \lesssim \min \left(\frac{1}{t^a \log^\mu t \left(\frac{|x-\xi|}{\lambda} + 1 \right)^{b-2}}, \frac{\lambda^{-2}}{t^{a-1} \log^\mu t \left(\frac{|x-\xi|}{\lambda} + 1 \right)^b} \right).$$

Proof. We solve

$$\partial_t \varphi = \partial_{\rho\rho} \varphi + \frac{5}{\rho} \partial_\rho \varphi + \bar{h}(\rho, t)$$

Define

$$\rho = \sqrt{r^2 + \lambda^2},$$

$$\tilde{\varphi}(r, t) = \varphi(\sqrt{r^2 + \lambda^2}, t).$$

We compute

$$\begin{aligned} \partial_t \tilde{\varphi} &= \partial_t \varphi + \partial_\rho \varphi \frac{\lambda \dot{\lambda}}{\sqrt{r^2 + \lambda^2}} \\ \partial_r \tilde{\varphi} &= \partial_\rho \varphi \frac{r}{\sqrt{r^2 + \lambda^2}} \\ \partial_{rr} \tilde{\varphi} &= \partial_{\rho\rho} \varphi \frac{r^2}{r^2 + \lambda^2} + \partial_\rho \varphi \left(\frac{1}{\sqrt{r^2 + \lambda^2}} - \frac{r^2}{(r^2 + \lambda^2)^{\frac{3}{2}}} \right) \\ &= \partial_{\rho\rho} \varphi \frac{r^2}{r^2 + \lambda^2} + \partial_\rho \varphi \frac{\lambda^2}{(r^2 + \lambda^2)^{\frac{3}{2}}} \end{aligned}$$

Then

$$\begin{aligned} & -\partial_t \tilde{\varphi} + \partial_{rr} \tilde{\varphi} + \left(\frac{1}{r} + \frac{4r}{r^2 + \lambda^2} \right) \partial_r \tilde{\varphi} \\ &= -\partial_t \varphi - \partial_\rho \varphi \frac{\lambda \dot{\lambda}}{\sqrt{r^2 + \lambda^2}} + \partial_{\rho\rho} \varphi \frac{r^2}{r^2 + \lambda^2} + \partial_\rho \varphi \frac{\lambda^2}{(r^2 + \lambda^2)^{\frac{3}{2}}} \\ & \quad + \partial_\rho \varphi \frac{r}{\sqrt{r^2 + \lambda^2}} \left(\frac{1}{r} + \frac{4r}{r^2 + \lambda^2} \right) \\ &= -\partial_t \varphi + \partial_{\rho\rho} \varphi + \frac{5}{\rho} \partial_\rho \varphi - \partial_{\rho\rho} \varphi \frac{\lambda^2}{\rho^2} + \partial_\rho \varphi \left(-\frac{\lambda \dot{\lambda}}{\rho} + \frac{\lambda^2}{\rho^3} + \frac{1}{\rho} + \frac{4r^2}{\rho^3} - \frac{5}{\rho} \right) \\ &= -\partial_t \varphi + \partial_{\rho\rho} \varphi + \frac{5}{\rho} \partial_\rho \varphi - \partial_{\rho\rho} \varphi \frac{\lambda^2}{\rho^2} + \partial_\rho \varphi \left(-\frac{\lambda \dot{\lambda}}{\rho} - 3 \frac{\lambda^2}{\rho^3} \right) \\ &= \bar{h}(\rho, t) - \partial_{\rho\rho} \varphi \frac{\lambda^2}{\rho^2} + \partial_\rho \varphi \left(-\frac{\lambda \dot{\lambda}}{\rho} - 3 \frac{\lambda^2}{\rho^3} \right) \end{aligned}$$

Therefore $\tilde{\varphi}$ serves as a supersolution. Then we take a cut-off function $\eta(x, t) = \chi_0(\frac{r}{\sqrt{t}})$. Using an appropriate self-similar solution ψ we find a supersolution for the problem of the form $\eta\tilde{\varphi} + \psi$.

□

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