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# **“He’s so fast at drawing” – Children’s use of drawings as a tool to solve word problems in multiplication and division**

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*This paper examines how young pupils (age 8–9) use drawings as a tool in exploring multiplicative situations. Analysis shows that drawings play an important role in their problem-solving process. Drawings are mainly used as information holders and as a tool to organise calculations for problems with a relatively easy structure; in problems with a more complex structure, drawings are more frequently used as a tool for reasoning. The results show no connection between the degree of abstraction in the drawing and the sophistication of the calculation strategy.*

*Keywords: Mathematical representations, Drawings, Word problems, Elementary school students.*

## **Introduction**

Representations are central to all mathematical activity and are needed both to construct understanding and to communicate mathematical ideas. Throughout the history of mathematics, conventional notation, symbols, figures and diagrams have been developed and agreed upon by the mathematical community in order to serve these purposes in an efficient manner. However, Greeno and Hall (1997) claim that, when a child is about to explore a mathematical concept for the first time, nonstandard representations can be more useful than formal ones. Self-invented representations can help pupils to keep track of ideas and connections they have already discovered, and can assist them in organising their work. Moreover, encouraging pupils to create their own representations will provide opportunities to consider the advantages and disadvantages of different forms of representation, and to use those representations as tools to build conceptual understanding.

When entering a new mathematical domain, such as multiplication and division, a simple word problem will be a mathematical problem for a pupil. Carpenter, Ansell, Franke, Fennema and Weisbeck (1993) claim that modelling, either through the use of counters or by drawing, is a natural problem-solving strategy for most kindergarten and primary-grade children. However, older pupils seem to abandon these meaningful approaches in favour of more mechanical and algorithmic ones, and may even consider arguments based on drawings as a form of cheating (Crespo & Kyriakides, 2007). To help pupils build upon and extend their intuitive modelling skills, it is worthwhile to look more closely at how primary-grade pupils use self-invented representations in problem solving in different content areas. The research question for this paper is thus: *What kind of drawings do third-graders produce when they explore a multiplicative context, and what are the function(s) of these drawings in the problem-solving process?*

## **Background**

### **Young children’s knowledge of multiplication and division**

Multiplicative thinking is fundamental for understanding more complex concepts such as ratios, fractions and linear functions. According to Steffe (1994), a multiplicative situation is characterised as one where “it is necessary to at least coordinate two composite units in such a way that one of the

composite units is distributed over the elements of the other composite unit” (p. 19). Depending on the situation, at least four different multiplicative structures can be distinguished: equal groups, multiplicative comparison, rectangular area and Cartesian product (Greer, 1992). Each structure gives rise to problems in both multiplication and division. Multiplication is often introduced prior to division, but research shows that young children spontaneously relate them and do not necessarily find division more difficult than multiplication (Bakker, van den Heuvel-Panhuizen & Robitzsch, 2014; Carpenter et al., 1993; Mulligan & Mitchelmore, 1997). However, multiplicative reasoning differs significantly from additive reasoning in terms of complexity. It is therefore not surprising that multiplicative thinking takes time to develop (Clark & Kamii, 1996). Despite this, it is widely documented that children are able to solve word problems in multiplication and division long before they receive any formal instruction in this domain. Mulligan (1992) found that most third-grade pupils are able to solve multiplicative word problems using a wide variety of strategies, and Carpenter et al. (1993) established that this is the case even for kindergarten children who have been exposed to such problems over some time. In a more recent study, these findings were sustained and extended by Bakker et al. (2014), who found that first-grade pupils are not only able to solve word problems, but to some extent are also successful in solving bare-number problems.

Young children’s strategies for solving multiplicative word problems can be classified as calculation strategies and modelling strategies (Mulligan & Watson, 1998). Calculation strategies involve increasingly sophisticated counting methods such as direct counting, rhythmic counting and skip counting, additive strategies based on repeated addition, and multiplicative strategies. Modelling strategies involve the use of physical objects or drawings. Kindergarten children almost always use direct modelling (Carpenter et al., 1993), while primary-grade children tend to use calculation strategies for small-number problems, but revert to modelling for problems involving larger numbers (Mulligan, 1992). Teachers need to assist pupils in widening their repertoire of calculation strategies. Mulligan and Mitchelmore (1997) suggest that a first step to achieve this is to help pupils to model different semantic structures so that they successfully can apply direct counting. Thereafter, pupils can be encouraged to use the equal-group structure to develop more efficient addition strategies, before these strategies are transferred to other structures. When pupils are able to use repeated addition across different semantic structures, the idea of a multiplicative operation can evolve.

### **Problem solving and the use of drawings**

Polya (1985) describes a four-step approach to problem solving: understand the problem, devise a plan, carry out the plan and look back. Pupils might, of course, move back and forth between these phases before they reach a solution to the problem. The use of representations in general, and drawings in particular, has been highlighted as a tool in the problem-solving process (e.g. van Essen & Hamaker, 1990; Polya, 1985). Van Essen and Hamaker (1990) claim that by translating a word problem into a picture, pupils are forced to pay attention to the given relationships in the problem. Furthermore, some problem characteristics may be more easily inferred from a drawing because they become more explicit, while drawings also relieve working memory. Yet, research on children’s use of drawings in early-years mathematics is still limited (Bakar, Way & Bobis, 2016). Some studies focus on drawings as a product, and look for (possible) relationships between the

abstractness of pupils' drawings and their success in mathematical problem solving. The findings of these studies are mixed, but there seems to be a tendency towards stating that pupils using iconic (or schematic) drawings are more successful than pupils using pictographic drawings (e.g. Crespo & Kyriakides, 2007; Veles & da Ponte, 2013). Bakar et al. (2016), however, found no such trend in their study. An *iconic* drawing contains only simple lines and shapes to embody the intended object, while a *pictographic* drawing has realistic depictions of the objects involved in the problem (Bakar et al., 2016). It is also important to determine the *mathematical matching*; that is, to what extent are the word problem and the drawing informationally equivalent? (Ott, 2017)

When it comes to viewing drawing as a process, Stylianou and Silver (2004) examined the use of representations by expert mathematicians in problem solving. They found that mathematicians consider drawings as a legitimate tool for reasoning and argumentation, and they use visual representations actively and for different purposes during the problem-solving process. Drawings can be means to understand information, recoding tools, tools that facilitate exploration, and monitoring and evaluation devices (Stylianou, 2011, p. 271). In a follow-up investigation involving middle-school pupils, Stylianou (2011) detected that pupils' use of representations resemble the experts' use in many ways. However, the use of representations as a monitoring tool is fairly limited and not very sophisticated. She recommends that teachers make pupils explicitly aware of the purposes of the representations they use, and that pupils are given opportunities to discuss and negotiate the meaning of various representations. In addition, pupils need to “develop the habit of exploring [a representation], generalising and abstracting from it, and using it as a springboard for connections among tasks and content in mathematics” (p. 277).

## **Methodology**

### **Context and data collection**

This study is part of a larger project entitled “Language Use and Development in the Mathematics Classroom” (LaUDiM), a video-based intervention project where two teachers from different primary schools and university researchers work together on planning and discussing teaching sequences. The empirical data for this particular study is drawn from a teaching sequence in one of the schools where the aim was to provide pupils with experience of different multiplicative situations. The teaching sequence consisted of two consecutive sessions, two days apart, and took place when the pupils had just entered third grade (age 8–9). In the Norwegian curriculum, multiplication and division are introduced and formalised during grades three and four, meaning the pupils in this study had not received any formal teaching on these subjects prior to this particular teaching sequence.

The pupils worked in pairs to solve word problems in multiplication and division. The context for all tasks was preparation work for a fictional school party. The teacher encouraged the pupils to write arithmetic problems or to produce drawings, but she gave no examples of what to write or draw. On the first day, two randomly chosen pairs of pupils were videotaped. In addition, a hand-held camera was used to capture glimpses of the pupils' work. On the second day, three pairs of pupils were videotaped. There was a partial overlap in terms of the pupils videotaped on the first and second day. The pupils' written work was also collected.

## Data analysis

As the research question for this paper focuses on how children use drawings in their problem-solving process, only video recordings showing the entire process of solving a given problem were considered data material. When looking through the recordings, 15 episodes were identified, where an episode is defined as “one pair of children working on a particular task, from the time they read the problem until they move on to the next task”. These episodes, together with the corresponding written work, constitute the data material for this study.

The first stage of the data analysis involved watching and transcribing all episodes. The drawings were categorised as mainly *iconic* or mainly *pictorial* (Bakar, 2016), the *mathematical matching* (Ott, 2017) was examined and the *solution strategy* (Mulligan & Watson, 1998) was identified. In the next step, the episodes were re-watched several times with the purpose of identifying the functions of the drawings. A more detailed description of the production and use of drawings, in the form of pointing, making additional markings and so on, was added to the transcripts, and an inductive analysis was conducted. Examples of questions asked about the material are “in what phases of the problem solving is the drawing produced and/or used?” and “how is the making and the use of the drawing linked to the solution strategy?”. Polya’s (1985) description of the four phases of a problem-solving process served as a tool for structuring this work. The analysis yielded several parameters, such as problem structure, abstractness, mathematical matching, solution strategy and different uses of drawings. The analyses of the 15 episodes were compared and contrasted with regard to these parameters to look for possible interrelationships.

## Findings

### Pupils’ drawings

One or more drawings are produced in all 15 episodes. Two of the drawings can be seen as mainly decorative as they do not resemble the mathematical structure. Figure 1 provides an example of a decorative drawing related to the problem of how many eggs one needs to make twelve portions of muffins, given that one needs four eggs for one portion. The pupils begin by drawing three rows of four eggs, which seems to resemble the equal-group structure of the problem, but then they add three more eggs to each row. After counting a total of 21 eggs, the pupils erase the additional eggs and decide to draw the entire baking process, from the beating of the eggs to ready baked muffins.

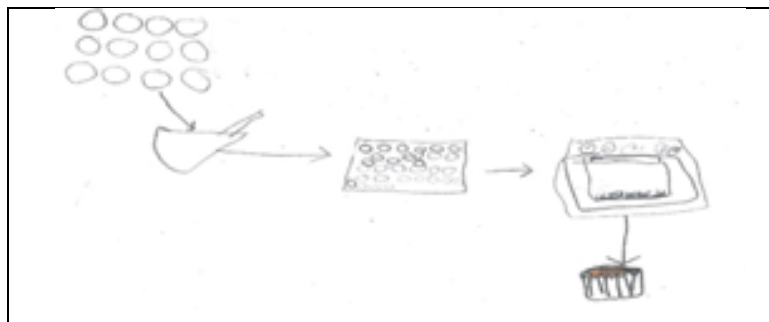
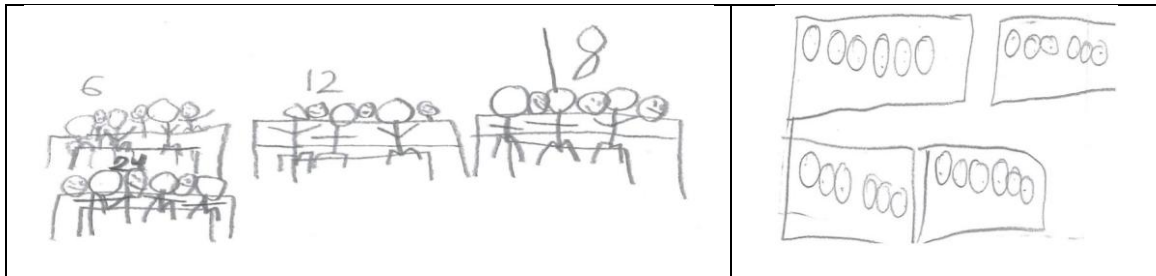


Figure 1: A decorative drawing

Of the thirteen remaining drawings, three are categorised as pictographic, and the rest as iconic. Figure 2 provides examples of both a pictographic and an iconic drawing of the problem of how many tables are needed for twenty-four people if there can be six at each table.



**Figure 2: A pictographic and an iconic drawing of the table problem**

All of the pictographic drawings are produced by the same male pupil. One girl initially starts to draw pictographically (using coloured pencils), but is encouraged by the teacher and her peer to change to an iconic drawing:

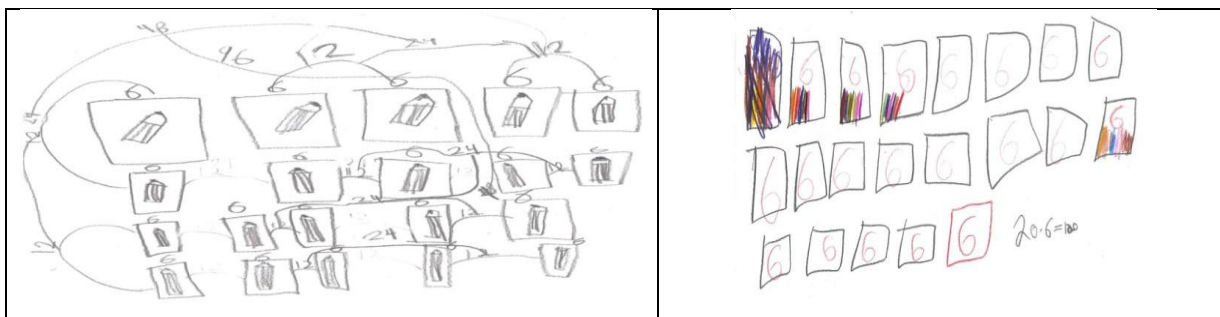
Teacher: Stop for a while. This is very nice, and it works well, but can it be done in a different way so that you don't have to draw every one? There is still a lot remaining.

Peer: Draw circles.

Girl: Can write 6.

The drawing produced in this episode is seen as the drawing on the right-hand side in Figure 3.

The iconic drawings still relate to the given context, as eggs, muffins and cookies are drawn as circles or dots, tables and trays drawn as circles or quadrilaterals, and pencils drawn as tally marks. This might suggest that classifying drawings from pictographic to iconic on a continuum, is more fruitful than seeing it as a dichotomy. All drawings, apart from the purely decorative ones, are considered to have a high degree of mathematical matching. As one can see in Figure 2, both drawings show the right number of people for each table, and twenty-four people in total. Three of the drawings involve number symbols as an important element. Figure 3 shows two of those, both produced in relation to the problem of how many pencils there are in twenty boxes, given that there are six pencils in each box. The pupils are successful in solving the word problems, barring minor counting errors, in all but two of the episodes. The two episodes where the pupils fail to solve the problem are the ones where only a decorative drawing is produced.



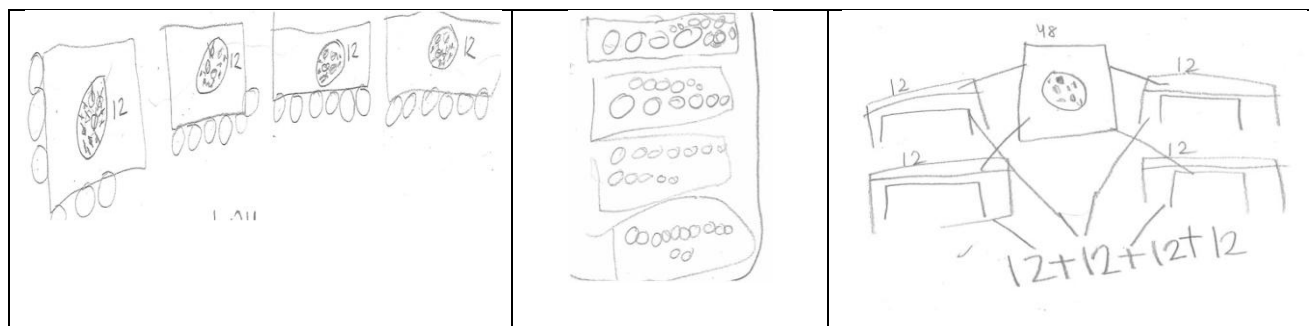
**Figure 3: Drawings with number symbols for the pencil problem**

### **Pupils' strategies and use of drawings**

The pupils' solution strategies are dominated by direct counting. For the multiplication tasks, there are only two examples of more advanced strategies. One is seen in the drawing on the left in Figure 3, where a strategy based on doubling is used; for the other drawing in Figure 3, the pupils count by ones, tapping a finger rhythmically six times for each box of pencils. The other example of a more advanced strategy is seen when a pair of pupils work on the problem of how many muffins there are on seven trays if there are ten muffins on each tray. They draw an iconic picture of all the trays and all the muffins, before they count ten muffins on each tray, simultaneously writing  $10+10+10+\dots$  on the paper. They reach the answer of 70 muffins with no further counting or visible reasoning.

For the multiplication problems, the drawing is usually produced immediately after the pupils have read the text. This indicates that they instantly visualise the problem situation and use the drawing as an information holder. In cases where the pupils struggle to understand the problem, they call for a teacher to help them re-read the text and explain particular words. When the drawing of all objects is completed, the pupils use the drawing as a calculation tool. Usually, the drawing is used to execute a counting plan, but in one case it acts as a means to develop a more advanced doubling strategy, as seen in Figure 3. There are instances where the pupils count one by one, even though the video recordings reveal that they master more advanced strategies. We see this, for example, in connection to the problem of finding the total number of muffins on a tray consisting of five rows with seven muffins in each row. The pupils first find the answer by direct counting, but later they write the addition problem  $5+5+5+5+5+5$  and skip count to 35.

For the division problems involving smaller numbers, such as the table problem (see Figure 2), all pairs use direct counting in groups of six. For the division problems involving larger numbers – 48 cookies being distributed equally between four tables – one pair use direct counting to distribute one cookie at a time until there are no cookies left. In the other two episodes involving this task, the pupils state that there will be ten cookies for each table and then divide the remaining eight cookies. One of these pairs draw all the cookies (first ten on each table, then adding two more), while the other pair write number symbols after performing calculations in their heads. Figure 4 shows the corresponding drawings for this problem.



**Figure 4: Drawings of the cookie problem**

The use of drawings in solving division problems differs somewhat from the use of drawings for multiplication. The actual calculations are more often made based on the numbers, not on the

drawing. Nevertheless, the drawing is important for reasoning and for devising and executing a calculation plan. This is exemplified by the following short extract from an episode related to the cookie problem. A pack of cookies has been drawn, with the number 48 above, as we enter the situation (drawing is seen to the far right in Figure 4):

Pupil 1: You have to draw four tables with people. No, just four tables without any people. (*P2 draws four tables*)

Pupil 2: We have to divide those by four. (*P2 points to the pack of cookies*)

Pupil 1: Yes, that is what I plan to do. Ten. (*P1 points to each of the tables*) 11, 11, 11, 11. (*P1 points to each of the tables*) 12, 12, 12, 12. (*P1 points to each of the tables*)

This extract demonstrates how the drawing offers a visual support for the mental calculations by giving meaning to the numbers and the operation.

## **Discussion and implications**

It is not surprising that the pupils successfully solve word problems, as this result has emerged in previous research (e.g. Bakker et al., 2014). Somewhat more surprising is that there is no evidence in the material showing that iconic drawings are used in a more sophisticated way than pictographic drawings; rather, the opposite is true, but the data material in this study is rather limited.

When it comes to pupils' strategies and use of drawings, the amount of drawing and counting is striking. The pupils are not discouraged by higher numbers, such as  $20:6$ ; as one girl exclaims about her partner in relation to the pencil problem: "He's so fast at drawing". This supports the impression that even though the pupils are capable of using more advanced strategies, they prefer drawing and direct counting. One hypothesis is that the pupils in this particular class are used to showing their thinking, and that they consider drawings more suitable for this purpose than number symbols. Overall, they appear to regard drawing as a legitimate way to reason and argue, a view that needs to be acknowledged and nurtured by the teacher (Crespo & Kyriakides, 2007). Another hypothesis is that drawing serves as a form of confirmation for the pupils. Because multiplication and division are a new school topic for them, they need to model the situations in order to fully grasp the meaning of the numbers and the relations between them. There is therefore no rush for the teacher to push for more advanced strategies at this point; instead, they can use the pupils' work to discuss, compare and contrast the different multiplicative situations and possible ways to represent them. As claimed by Carpenter et al. (1993), children who are taught to approach problem solving as an effort to make sense out of problem situations, may come to believe that learning and doing mathematics involves the solution of problems in ways that always make sense. (p. 440)

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