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# From concrete to abstract and back: Metaphor and Representation

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*We are interested in exploring from an enactivist perspective the role of metaphor and representation in the emergence of the abstract from the concrete and also in sense making of abstract mathematical notions. After introducing our theoretical framework, we present and comment on some illustrative examples, explored with several cohorts of learners, which include both prospective mathematics secondary teachers and first year university students studying science and the humanities.*

*Keywords: Metaphor, Representation, Enaction, Concrete, Abstract.*

## Introduction

We claim that representation and metaphor (between which we distinguish, see below) play a key role in the teaching and learning of mathematics. More precisely, in this paper we focus on the ways in which *metaphorising* and *representing* can foster the emergence of abstract concepts from concrete a-didactic situations (Brousseau, 1998) and backwards, making (concrete) sense of abstract concepts. We argue, moreover, that taking advantage of such concrete situations can afford insights and motivations for classical abstract mathematical notions, which are friendlier to the learners than the traditional approach where such notions are quite often “parachuted from Olympus”. Indeed, many mathematical concepts emerge as procedural computations, which are not supposed to take a meaning by themselves. Every time a new representation, or metaphor, is found for such a “procedural” concept, this results in a deeper understanding, and quite often in new questions and results. A very famous example is that of complex numbers, invented as “impossible” numbers to solve cubic equations, whose now classical geometric representation via the complex plane was only found two centuries later. Other examples, like the metaphor of “arithmetic geometry,” are now at the base of important mathematical research programs. We argue that it is important to help students develop such representations explicitly, rather than treating them as trivial side-products of teaching. It is however a big challenge to figure out under which conditions representing and metaphorising foster mathematical thinking processes as hypothesised above.

We intend here to pursue our recent research on metaphorising, representing and enacting (Arnoux & Finkel, 2010; Diaz-Rojas & Soto-Andrade, 2015; Soto-Andrade, 2007, 2014, 2018), by presenting and commenting some non-obvious examples of enactive metaphors and representations of mathematical concepts and their use in a-didactic situations (Brousseau, 1998). The contextual background of our examples involves several cohorts of learners in Chile and France, mainly prospective secondary mathematics teachers.

## Research questions

We single out a couple of research questions, among those involved in our research:

1. Which sorts of metaphorising and representing can foster the emergence of abstract notions from concrete situations and convey a better grasp of them for the students?
2. How can teachers support, in a friendly manner, learners in developing their representational and metaphorical competences?

## **Theoretical background**

### **Metaphorising and Representing in cognitive science and mathematics education**

Metaphorising and Representing nowadays play undoubtedly a key role in cognitive science and mathematics education. See Goldin (2014) and Soto-Andrade (2014) for related comprehensive surveys. Notice that we prefer to focus in on *metaphorising*, a verb denoting a process, rather than on *metaphor*, a noun denoting an object. Indeed, what is a metaphor for someone may not be a metaphor for someone else, and for yet another person it could be a representation instead (see below). What we observe is actually the *process* of metaphorising carried out by someone, a cognitive subject.

Regarding metaphorising (“looking at something and seeing something else”, in metaphorical terms), widespread agreement has arisen in cognitive science that our ordinary conceptual system, in terms of which we both think and act, is fundamentally metaphorical in nature (Gibbs, 2008; Johnson & Lakoff, 2003). In mathematics education metaphor typically appears not just as a rhetorical device, but as a powerful cognitive tool, that helps us in grasping or constructing new concepts, as well as in solving problems in an efficient way and user-friendly way (English, 1997; Lakoff & Núñez, 2000; Sfard, 2009; Soto-Andrade, 2007, 2014).

Although in the literature the same object is sometimes described either as a representation or as a metaphor, here we draw a distinction: we *re-present* something given beforehand, usually to explain concepts already constructed, but we *metaphorise* to try to fathom something unknown or to construct a concept. Recall that Lakoff and Núñez (2000) highlight the intensive use we make of conceptual metaphors that appear – metaphorically – such as inference-preserving mappings (arrows) “going upwards” from a rather concrete ‘source domain’ into a more abstract ‘target domain’, enabling us to fathom the latter in terms of the former. Then representations naturally appear as arrows going the other way around, downwards from the more abstract domain to the more concrete one (Soto-Andrade, 2014).

Indeed, our approach to the learning of mathematics emphasises the *poietic* (from the Greek *poiesis* = creation, production) role of metaphorising, which brings concepts into existence. For instance, we bring the concept of probability into existence when, while studying a symmetric random walk on the integers, we look at the walker (a frog jumping on a row of stones in a pond, say) and we see it *splitting* into two equal halves that go right and left instead of being equally likely to jump right or left (Diaz-Rojas & Soto-Andrade, 2015). This ‘metaphoric sleight of hand’ which turns a random process into a deterministic one, allows us to reduce probabilistic calculations to deterministic ones, where we just need to keep track of the walker’s splitting into pieces: The probability of finding the walker at a given location after  $n$  jumps is just the portion of the walker landing there after  $n$  splittings.

In the same vein, imagine that one is trying to figure a struggle between two producers A and B for a consumer market, who each month – as a consequence of intensive marketing strategies – entice consumers of the other brand to change their choice, say 20% of consumers of A going to B but only 10% conversely. If one is familiar with jumping frogs, one could metaphorise the evolution of the market as the random walk of a frog between two rocks, tagged A and B, with corresponding transition probabilities. However, someone who is more familiar with market struggles would rather metaphorise the frog's random walk as a market evolution, to benefit of his/her economic intuition.

It can be argued that we often introduce new concepts via metaphor, but giving a new meaning (and often, just meaning) to a concept that has already been taught involves a representation. We may have *internal representations* (Goldin & Janvier, 1998; Goldin, 2014) which are operationally equivalent to metaphors, as the ways a cognitive subject has of figuring out concepts unfamiliar or still opaque.

### **Methodology and experimental background**

Our methodology relies mainly on qualitative approaches like participant observation techniques and ethnographic methods (Brewer & Firmin, 2006; Spradley, 1980).

Regarding our experimental background, several cohorts of students have participated in preliminary tests at the University of Aix-Marseille, France: 60 first year university students majoring in science and humanities in a mathematics course in 2015-2018 (usually organized in 10 groups of 6) and 40 fifth year teacher students in 2 mathematics courses, of 20 students each, in 2016-2018 (usually organized in 5 groups of 4 each).

At the University of Chile, in Santiago, three cohorts of prospective secondary school mathematics and physics teachers (45 students each, on the average), in a one-semester yearly course in elementary number theory, have been involved in our teaching and learning following a metaphoric and enactivist approach, from 2016 to 2018. Working most of the time in random groups of 3 to 4, (defined by blind picking of coloured Lego cubes from a bag) they were observed and monitored by the teacher and an assistant as participant observer or ethnographer (Spradley, 1980; Brewer & Firmin, 2006).

Since we were especially interested in evaluating the impact of our approach on the student's engagement and problem solving and problem posing abilities, we observed mainly: their level of participation and horizontal (peer) interaction, the emergence of "research questions", i.e. questions they ask themselves, to be tackled by themselves (not questions addressed to the teacher to ask for a clarification or explanation) and (idiosyncratic) metaphors, arising spontaneously or under prompting, accompanying gestural language of learners and teacher, expression and explicit acknowledgement of affective reactions. Snapshots of their written products in problem solving activities were taken and processed in a worksheet, also for evaluation purposes, and some videos of their enacting moments were recorded.

### **Illustrative examples and case studies**

We present and discuss here a couple of paradigmatic examples, regarding concrete ways to introduce and motivate in a friendly way Pythagorean triples as well as arithmetical congruences. In fact, we could describe them under the same roof as arithmetic in a discrete "modular universe": a

2D pixelated, grid-like one (the lattice  $\mathbb{Z}^2$ ) in the first case, and a 1D cyclic one (the polygon  $\mathbb{Z}/m\mathbb{Z}$  formed by the integers modulo  $m$ ) in the second case. For other examples, related to computer science (finite automata) and probability, see Arnoux and Finkel (2010), Diaz-Rojas and Soto-Andrade (2015) and Soto-Andrade (2018).

### Example 1: Pythagorean triples, with Lego bricks

This activity has been carried out at the University of Aix-Marseille, in several contexts, with the cohorts described above: first year science and humanities students and fifth year teacher students (last year of teacher initial formation in mathematics).

It has been tested informally in the last two years, using qualitative monitoring by an assistant, and posterior evaluation by the participants; we are setting up now an interdisciplinary team (mathematicians, didacticians and cognitive scientists) to study it more in depth in the framework of a methodology unit next year.

We give to groups of 4 to 6 science and humanities students or teacher students one  $16 \times 16$  horizontal plaque, two elementary  $1 \times 1$  bricks, and one  $1 \times 16$  brick. The question is: *for which positions on the plaque can the small bricks be connected by the long one?* See Figure 1, showing a  $8 \times 8$  plaque, two  $1 \times 1$  bricks in a good position, and a  $1 \times 8$  brick superposed on them; it is of course cumbersome to explain this in writing, but the real model leaves no room for hesitation.

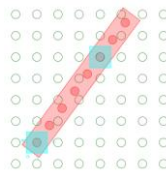


Figure 1: Linking brick for Pythagorean triples

By construction, the plaque is endowed with a discrete grid (a metaphor for  $\mathbb{Z}^2$  with the canonical metric, points being the centres of the small circles and  $1 \times 1$  bricks playing the role of dots); no explanation of this is needed. But some time is needed to understand that the long brick can only be set on the pair of small bricks if the distance between these two bricks is an integer multiple of the unit distance: after half an hour, most groups discover the concept of Pythagorean triples as implied by this situation (they all know the Pythagorean Theorem, but not Pythagorean triples). This exercise is very rich. It works well, because all the distances are quadratic numbers, hence, if they are not integers, they are quite far from an integer: *at small distance*, there are only exact solutions, no “near-solutions”. It is also very engaging for students, because it is concrete. It evolves naturally into finding the classical Babylonian solutions (it may be pertinent at that point to present the famous Plimpton 322 tablet). One can then ask to find non-trivial rational points on the unit circle; students usually start by trial and error, and come first to the conjecture that there are no such points. It takes some time to make the relation with the previous question; this can be continued with a geometric parametrisation by the rational lines through the point  $(-1,0)$ , which makes the link with the quadratic equation, and, via elementary geometry, with the formula for sine and cosine as a function of the tangent of the half-angle: these formulas appear as a reformulation of the Babylonian formula. Most students seem surprised to see that there can be links between very different domains of mathematics. This can be taken to much more elaborate questions, like the number of triples within a bounded distance, and in fact to difficult research questions. One remark

is that, in our experience, it is not important whether or not the students have already studied these notions: in the case of prospective teachers, they are always surprised by the exercises, and do not immediately link them to notions they know quite well, but in an abstract way.

### **Example 2: Congruence mod $m$ and dynamical systems**

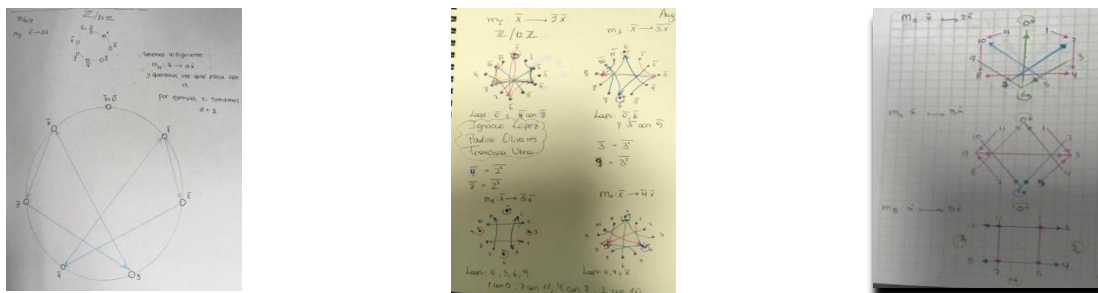
We report on some developments of our metaphoric enactive approach to arithmetic congruences, with third year prospective maths and physics secondary teachers taking a one semester course in elementary number theory, in 2016, 2017 and 2018, at the University of Chile. Since congruences mod  $m$  are unavoidable in this course, we wanted to motivate them or make their study friendlier. To this end, following a radical enactivist approach (Proulx & Maheux, 2017; Soto-Andrade, 2018), we just proposed a *situational seed* first and let the action emerge freely.

From our theoretical perspective, taking into account the previous mathematical training of our teacher students, at secondary school and also in most courses of their initial formation, we could predict a “metaphoric deficit” in their understanding of arithmetical congruences. Eventually they will be able to recite the definition and calculate, but without having a favourite metaphor (or representation) for them, and also with no appreciation of their usefulness. Notice that if they metaphorise the integers mod  $m$  as a finite “shadow” of the integers, they might easily have the idea that a necessary condition for a property to hold for the integers is that it holds for its projection onto their arithmetical shadows, something much easier to investigate. So, if a Diophantine equation is solvable (in the integers) its shadow should be solvable in the shadow integers mod  $m$ . In this way, they get necessary conditions for a Diophantine equation to be solvable, which allows them to prove unsolvability in several cases.

Indeed, we found that although our prospective teachers know by heart the definition of congruence mod  $m$  in the integers, when we ask them how they imagine, metaphorise or visualise congruence mod  $m$  or which are their internal representations for it in the sense of Goldin (2014), they are at a loss. After a while, some of them think of kangaroos, rabbits, frogs, jumping on the integer line, or they begin to paint the integers in different colours (five colours for congruence mod 5). Slowly the metaphor emerges in several groups, of winding the integer number line on a polygon, e. g. a pentagon for congruence mod 5. They naturally carry out a spiral-like winding on the plane to begin with. If nobody suggests a different way, we ask them what they do when they want to wind up a long garden hose. They realise then that it is smarter to wind it in 3D, cylinder like. From there they come to visualise congruence mod 5, say, as a helical winding of the integer line above a regular pentagon. They metaphorise then this congruence relation as a covering space! Typically, however, this helix does not remind them of the intuitive construction of the imaginary exponential that they were exposed to in Calculus 1, where the real line was wound around or above the unit circle. Our enactivist theoretical perspective emphasises *processes and dynamics* more than (the more traditional) *objects and static structures*. So setting the integers mod  $m$  as a stage, for instance  $m = 12$  (or even  $m = 6$  or  $7$ , for simpler examples of different nature), we prompt the students to wonder what interesting phenomena may arise in these universes with just 6, 7 or 12 sites, where they can add and multiply.

Among other ideas, they can look at the transformation  $M_2$  given by multiplying by 2, for example, which when iterated launches a *dynamical system* in the integers mod 12. The students try then to

study the generated dynamics, something more natural for those who have developed a systemic perspective on phenomena, more often biology and physics students or humanistic students than mathematics students. Working for an hour, in random groups of 3 to 4 (defined by blind picking of coloured Lego cubes from a bag), a class of 30 to 40 students dutifully iterates  $M_2$  and tries to *represent* the phenomenon. They explore then  $M_3$  and so on. See Figure 2, which shows some drawings by the students, for  $m = 7$  and  $m = 12$ . More generally, they investigate the dynamics spawned by the multiplication by an integer  $k$  in the integers mod  $m$ , discovering its variegated (forward) orbit structure. They metaphorise idiosyncratically these orbits as forward trajectories and they see fixed points, sinks, attractors, 2 – cycles, 3-cycles, etc., and become able to figure out the “fate” of different integers mod  $m$ . They begin to conjecture on the number of orbits, or the number of  $n$ -cycles for a given  $n$ . Level of engagement and participation is high, as well as their horizontal interaction, where several ideas and approaches meet. Some of them want to find an arithmetical explanation of this dynamic geometric behaviour. Others do not. But we suggest everyone to look at this geometric phenomenon with arithmetical eyes and vice versa. Most of them draw pictures, but others just write down numbers and tables, and compare. We prompt them to exert some hermeneutical effort (Isoda & Katagiri, 2012), so that all try to understand other’s viewpoints. In this way, some predict the existence of an “ubiquitous orbit” for some  $k$  for prime  $m$ . The students are quite excited and motivated by this challenge when working in small random groups, where horizontal interaction is fostered. Quite often some groups want absolutely to show the teacher and assistant their progress so far. Each group summarises its findings in a report on a sheet of their notebooks, which is photographed by the teacher and assistant. Samples of their work are shown in Figure 2. These reports are evaluated, and a huge worksheet is set up with the corresponding grades.



**Figure 2: Orbit structure for  $M_k$  ( $k = 2, 3, 4, 5$ ) in the integers mod 7 and mod 12**

At the end of the course each student gets a grade in group work which is only taken into account if it is higher than the average the grades in the usual compulsory traditional tests taken by the student during the course. Interestingly, in our courses the percentage of (the non-compulsory) student attendance is much higher than the average (80 to 85% against 50%).

Overall, we see that students previously exposed to traditional teaching show a severe metaphorical and dynamical deficit, remaining confined in the arithmetic-algebraic realm. However, progressively, most of them turn out to be able to open up to geometric and dynamical insights, even though initially they remain tied to the purely arithmetical approach that they are more familiar with. At the end of the work session consensus arises on the importance of being able to “change register” and move seamlessly between the arithmetic-algebraic realm and the geometric-visual one. Most students seem surprised to discover, in this way, links between domains of mathematics usually perceived as being far apart, like arithmetic and dynamical systems.

## Discussion and open ends

We have shown how Pythagorean triples can be concretely embodied and enactively explored and also how a dynamical systems approach can make the arithmetic modulo  $m$  more lively, visual and motivating.

Our examples show that concrete embodiment of abstract mathematical notions and properties, before or instead of their “abstract parachuting”, may bear a dramatic impact on the level of participation and engagement of the students in otherwise unappealing tasks. Such impact is also seen on their mathematical performance and their ability to put forth their own research questions, like investigating the “fate” of various numbers mod  $m$  or wondering about an arithmetical explanation of the dynamical phenomena observed in the universe of integers mod  $m$ .

Our didactic activities are not staircases to climb sequentially but a rather highly intertwined space to explore, where moving flexibly among several representations or metaphors is a must for a meaningful and fruitful mathematical experience.

Our examples also show that representations and metaphors appear sometimes deeply intertwined in a circular relationship, so that it may be somewhat artificial to try to separate them. If we focus in on *metaphorising* and *representing* it may very well happen that in a class, some students are metaphorising while others are representing.

Also, quite often the teacher will be striving to represent – in a friendly way for the students – a notion that she is familiar with (maybe in a rather abstract way though), while students will be metaphorising the same notion, which they hardly fathom. This is visible in our preliminary example of the frog’s random walk: the teacher will be eventually representing the probability of finding the frog in a given stone after  $m$  jumps by the portion of the frog landing there, while the students will be metaphorically just constructing this probability as the frog’s portion landing in this stone.

The same holds for Pythagorean triples and arithmetical congruences. Some arithmetically minded students may realise for the first time that Pythagorean triples may be embodied as an integer fitting phenomenon on a grid and that congruence modulo 5 may be represented by winding the integer number line over a regular pentagon. Other students may discover arithmetical congruences, which they ignored or felt to be an esoteric notion, with the help of this enactive geometric winding. Notice here the emerging connection between arithmetical congruences and geometric congruences.

A recurring observation in all our examples, is the surprise that arises among the learners when they realise hitherto hidden connections between realms of mathematics which seemed to lie far apart. Furthermore, it appears that usual problem solving, as found in the literature, tends to neglect the important role of metaphorisation and representation, as a learner’s first reaction when tackling a problem that looks opaque to him or her. Not only because this may allow him or her to solve an otherwise unyielding problem but also because it may allow him or her to “see” a solution, turning a hitherto blind calculation into pellucid insight.

The various metaphors we have presented here are only a sample; there are many others, in analysis, probability, computer science, etc. Invariably, when these metaphors are presented or emerge, they attract the attention of students, and make for a motivating and meaningful classroom



experience. They may be however hard to find, so a systematic “catalogue” of them would be most commendable.

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