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Experimental nonlinear localisation in a system of two coupled beams

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Abstract: This study presents results showing experimental non-linear localisation in a (macro) system of two coupled beams. First a reduced order model of the system is introduced, using the so called STEP method, leading to a two dof model with cubic non-linearity. This model allows to show that non-linear localisation is possible through a 1:1 internal resonance mechanism. Moreover, one can show, using Harmonic Balance Method, that the forced localised solution stems from the principal resonance curve through pitchfork bifurcation, and the numerical model allows to compute the amplitude of bifurcation as well as the bifurcated branch. The experimental results are presented and compared to the numerical ones showing very good agreements.

1. Introduction

This study deals with the non-linear vibration of structural systems. The objective is to present both numerical and experimental results related to non-linear localisation. Localisation corresponds to vibration states where the energy is localised to a spatial subset of the structure, and is related to symmetry breaking bifurcation. Many studies have demonstrated numerically that localisation can occur in system of coupled non-linear oscillators (e.g. [7]). Some experimental results related to localisation are available for nano/micro system of beams array (e.g. [8, 1]), but a small amount of experimental results are available for macro structure. In this study, we design a macroscopic plate structure that can be considered as two coupled non-linear beams. Considering numeric and experimental results, we show that the structure allows for a non-linear modal interaction leading to symmetry breaking bifurcation and localisation.

2. Two beams system and reduced order model

2.1. Description of the system

The system considered here consists of a circular plate that has been machined in order to create two parallel beams as indicated on Fig. 1. In this study, we will consider only transverse vibrations. The body of the circular plate provides a coupling between the beams, and it also restrains the axial displacement of the beams ends, so that the behaviour of the beams is similar to clamped-clamped beams, i.e. non-linearity occurs due to a coupling between axial and transverse motions.

To observe non-linear localisation, we will consider two particular eigen-modes of the structure depicted on Fig. 1. For those modes, the beams vibrate over a first bending mode shape (in phase or out of phase) and the rest of the plate remains at quite low amplitude. The idea is to design the structure such that both mode interacts non-linearly to give a localized mode (i.e. with broken symmetry). Indeed, if one looks only at the mid-beam amplitude, both modes can be described with the shape $\phi_1 = (1, 1)$ and $\phi_2 = (1, -1)$. The idea is to generate a modal interaction of the form $q_1(t)\phi_1 + q_2(t)\phi_2$, where q_1 and q_2 are the time varying modal amplitudes. If q_1 and q_2 are in phase, then the interaction will lead to a shape of the form $(a_1 + a_2, a_1 - a_2)$ which tends to the localised shape $(1, \epsilon)$ as the amplitude of the first mode a_1 tend to the amplitude of the second mode a_2 .

In order to prevent interaction with other modes, we set the structure dimension such that the two previous modes are consecutive (i.e. there is no other modes in between). This can be achieve to a certain extend by increasing the distance between the beams. The structure was also designed such that the vibration amplitude for symmetry breaking bifurcation is quite small (about 0.23 mm at the center of the beams) in order to be observed with reasonably low forcing amplitude. The design was realized in an iterative way using a reduced order model presented briefly hereafter.

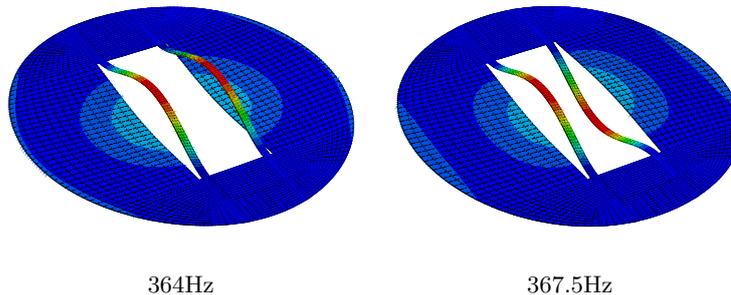


Figure 1. Linear mode shapes of interest (magnitude of the displacement) and their FEM natural frequencies

2.2. Reduced order model for plate structure

In theory, the system could be modelled by finite element method, but in practice the resolution time for non-linear computations would be prohibitive. An approximation of the system behaviour can be obtained by a so called Reduced Order Model (ROM). Here, only the form of the ROM equation for plate structure will be given, more details about the ROM procedure can be found in [3].

We consider that the solution to the FEM plate model is expanded over the linear mode

shapes basis computed with free boundary conditions. We denote q_j (reps. p_k) the modal amplitude of the j -th transverse mode (resp. the k -th in-plane mode). Neglecting in-plane inertia and using a condensation procedure (so that the p_k 's can be expressed as a function of the q_j 's, [3]), it can be shown that the reduced set of equation for the FEM plate model can be put under the following form (i.e. only considering modal amplitudes of transverse modes) [3]:

$$\ddot{q}_j + \omega_j^2 q_j + \sum_{k,l,m=1}^{N_t} \Gamma_{jklm} q_k q_l q_m = f_j(t), \quad \forall j \in [1, N_t] \quad (1)$$

where ω_j is the natural angular frequency of the j -th transverse mode, N_t is the number of transverse modes kept in the reduction basis and f_j is the j -th transverse modal force. The Γ_{jklm} are coefficients that can be evaluated using the so called STiffness Evaluation Procedure (STEP) which is based on a series of static non-linear FEM computations [6, 3].

In this study we are interested in the interaction between the two modes described on Fig.1, so that only those two modes are kept in the reduced order model. Based on Eq.(1), the reduced set of equation have the following form:

$$\begin{aligned} \ddot{q}_1 + \omega_1^2 q_1 + G_1 q_1^3 + E_1 q_1^2 q_2 + C_1 q_1 q_2^2 + D_1 q_2^3 &= f_1(t) \\ \ddot{q}_2 + \omega_2^2 q_2 + G_2 q_2^3 + E_2 q_1 q_2^2 + C_2 q_1^2 q_2 + D_2 q_1^3 &= f_2(t) \end{aligned} \quad (2)$$

When computing the values of the coefficient, one finds that some of them are order of magnitudes smaller than the others, and are therefore neglected (see table 1).

coeff.	G_1	C_i	E_i	D_i
$i = 1$	$9.26 \cdot 10^{10}$	$3.46 \cdot 10^{11}$	≈ 0	≈ 0
$i = 2$	$1.09 \cdot 10^{11}$	$3.46 \cdot 10^{11}$	≈ 0	≈ 0

Table 1. Non-linear coefficients for the reduced order model of the final design

3. Analytic and numeric results

3.1. Equation of motion and solution methods

Based on the previously presented ROM, we consider the following equation of motion (with periodic forcing):

$$\begin{aligned} \ddot{q}_1 + 2\xi_1 \omega_1 \dot{q}_1 + \omega_1^2 q_1 + G_1 q_1^3 + C_1 q_1 q_2^2 &= f_1 \sin(\Omega t) \\ \ddot{q}_2 + 2\xi_2 \omega_2 \dot{q}_2 + \omega_2^2 q_2 + C_2 q_1^2 q_2 + G_2 q_2^3 &= f_2 \sin(\omega t) \end{aligned} \quad (3)$$

where q_i is the modal amplitude of the i -th transverse mode ($i = 1$: in phase mode, $i = 2$ out of phase mode, see Fig.1), ξ_i 's are the modal damping ratio, ω_i 's are the natural frequencies,

f_i 's are the modal forces amplitude and Ω is the excitation frequency. G_1, G_2 and C are the non-linear coefficients obtained through the reduced order model procedure.

In this study, approximated solutions to the system in Eq.(3) are obtained using the Harmonic Balance Method (HBM), coupled with a numerical continuation procedure based on the Asymptotic Numeric Method (ANM) [2]. The solution is searched for as a truncated Fourier series, and the coefficients of the Fourier series are obtained by solving an algebraic equation depending on a parameter (here the frequency). The use of the MANLAB package [4] allows to compute the solution as the parameter is varied, along with its stability.

In addition, in order to design the structure, a single harmonic approximation has been used to derive analytical conditions resulting in the appearance of a modal interaction, which will be presented hereafter.

3.2. Non-linear modal analysis

In this section, we consider the undamped, unforced version of the equation of motion (3). We search for a single harmonic solution under the form $q_n(t) = A_n e^{i\omega t} + c.c.$ (where *c.c.* stands for complex conjugate). The following polar representation is introduced: $A_n = a_n e^{i\gamma_n}$. The derivatives \dot{q}_n and \ddot{q}_n are computed and substituted into (the unforced and undamped version of) (3) leading to the following (after separation of real and imaginary parts):

$$\begin{aligned} a_1 (-\omega^2 + \omega_1^2 + 3G_1 a_1^2 + C_1 a_2^2 (\cos(2\gamma_2 - 2\gamma_1) + 2)) &= 0 \\ a_2 (-\omega^2 + \omega_2^2 + C_2 a_1^2 (\cos((2\gamma_1 - 2\gamma_2) + 2) + 3G_2 a_2^2) &= 0 \\ a_1 a_2^2 C_1 \sin(2\gamma_1 - 2\gamma_2) &= 0 \\ a_1^2 a_2 C_1 \sin(2\gamma_2 - 2\gamma_1) &= 0 \end{aligned} \quad (4)$$

The free HBM solutions can be separated into uncoupled and coupled solutions.

The uncoupled solutions are obtained by setting $a_2 = 0$ (mode 1) or $a_1 = 0$ (mode 2) in Eq.(4), leading to the expression of the backbone curves for the first and the second mode:

$$\begin{aligned} \omega_{nl,1}^2 &= \omega_1^2 + 3G_1 a_1^2, \quad \text{and } a_2 = 0 \\ \omega_{nl,2}^2 &= \omega_2^2 + 3G_2 a_2^2, \quad \text{and } a_1 = 0 \end{aligned} \quad (5)$$

The coupled solutions are obtained by setting $\sin(2\gamma_2 - 2\gamma_1) = 0$ in Eq.(4). There is two cases to be considered, namely $\cos(2\gamma_2 - 2\gamma_1) = \pm 1$, referred to as "linear coupling" (+1, the modes interact in phase, or in phase opposition) and "elliptic coupling" (-1, the modes interact in phase quadrature).

- For the linear coupling, the following equation can be obtained from Eq.(4):

$$3(G_1 - C_2)a_1^2 + 3(C_1 - G_2)a_2^2 = \omega_2^2 - \omega_1^2 \quad (6)$$

It shows that a linear-coupled solution can bifurcate from mode one only if $G_1 - C_2 > 0$, and from mode 2 only if $C_1 - G_2 > 0$. The amplitudes of bifurcation are respectively $a_1^2 = \frac{\omega_2^2 - \omega_1^2}{3(G_1 - C_2)}$ and $a_2^2 = \frac{\omega_2^2 - \omega_1^2}{3(C_1 - G_2)}$

- For the elliptic coupling, the following equation can be obtained from Eq.(4):

$$(3G_1 - C_2)a_1^2 + (C_1 - 3G_2)a_2^2 = \omega_2^2 - \omega_1^2 \quad (7)$$

It shows that a elliptic-coupled solution can bifurcate from mode one only if $3G_1 - C_2 > 0$, and from mode 2 only if $C_1 - 3G_2 > 0$. The amplitudes of bifurcation are respectively $a_1^2 = \frac{\omega_2^2 - \omega_1^2}{3G_1 - C_2}$ and $a_2^2 = \frac{\omega_2^2 - \omega_1^2}{C_1 - 3G_2}$

Different behaviour may happen, based on the values of the non linear coefficients (see e.g. [5]). In this study, the structure has been designed to allow both modes to interact in order to generate localisation. Indeed, using the coefficients of the reduced order model given in table 1, one has the following:

- $G_1 - C_2 < 0$ and $C_1 - G_2 > 0$, so that a linear-coupled mode can bifurcate only from the out-of-phase mode when the mid-beam amplitude is close to 0.23mm.
- $3G_1 - C_2 \approx 0$ and $C_1 - 3G_2 \approx 0$ so that the "elliptic-coupled" bifurcation points are at very high amplitude (considered here as infinity)

The summary of the non-linear analysis is given on Fig.2. The backbone curve of both modes are represented, along with the bifurcated solution stemming from the out-of-phase mode. A stability analysis, similar to the one use in the multiple scale method, allows to draw an instability zone for the out-of-phase mode, defined by the following:

$$(2\omega\xi_1\omega_1)^2 + (\omega_1^2 - \omega^2 + 2C_1a_2^2)^2 = (a_2^2C_1)^2 \quad (8)$$

As soon as the second mode enter the instability zone, it bifurcates to a coupled solution where the first mode gets more and more activated though non-linear interaction. In the physical domain, this correspond to a solution where both beams vibrate out-of-phase with (increasing) different amplitude (see Fig.2, left panel), indicating the break of symmetry and the localisation.

3.3. Forced response

In order to evaluate the structure of the forced response the MANLAB package is used [4]. In order to see clearly the non-linear interaction with the first mode, only the second mode is excited (i.e. $f_1(t) = 0$ in Eq.(3)). Fig.3 shows the forced response of the second mode

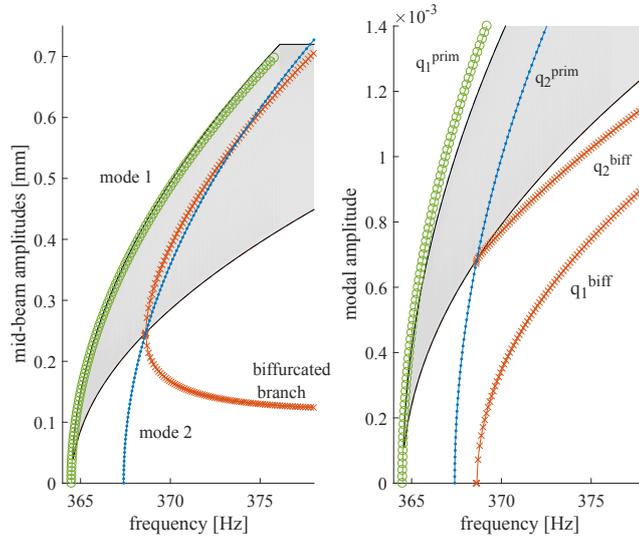


Figure 2. Backbone curves for the two uncoupled modes (o: in-phase mode, ·: out-of-phase mode) along with the bifurcated solution (×) and the instability zone of the out-of-phase mode (shaded area). left panel: physical representation, right panel: modal representation

for several amplitude of forcing (for $f_2(t)$), along with the instability zone presented in the previous section. As the forcing amplitude increase, the response becomes more and more non-linear, and turning points appear. As the amplitude increases even further the response enters the shaded area and becomes unstable.

MANLAB allows to compute the bifurcated branch as depicted on Fig.4 (for a forcing amplitude of $f_2 = 2$). It turns out that, for this forcing amplitude, the bifurcated solution is stable, then experiments a Neimark-Saker bifurcation, then gets stable again until the turning point to the right. This stability analysis indicates that the bifurcated solution could be observed in practice, as will be demonstrated by the experimental measurements.

The right panel of Fig.4 depicts the same forced response results using the physical representation, i.e. the amplitude at the middle of each beam. It clearly shows that the bifurcated branch corresponds to a non-symmetric state, which gets more and more localised as the amplitude increases.

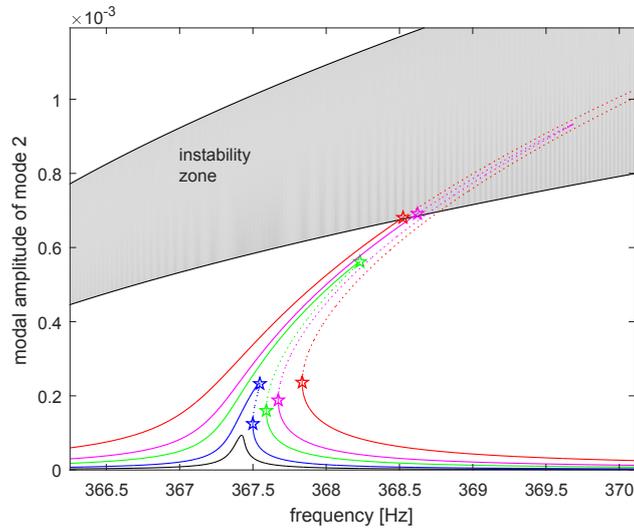


Figure 3. Numerical forced response for an excitation on the second mode only, for several amplitude of modal forcing ($f_2 = 0.1, 0.25, 0.6, 1.0, 2.0$). Solid lines: stable, dotted lines: unstable. The stars indicate the changes of stability.

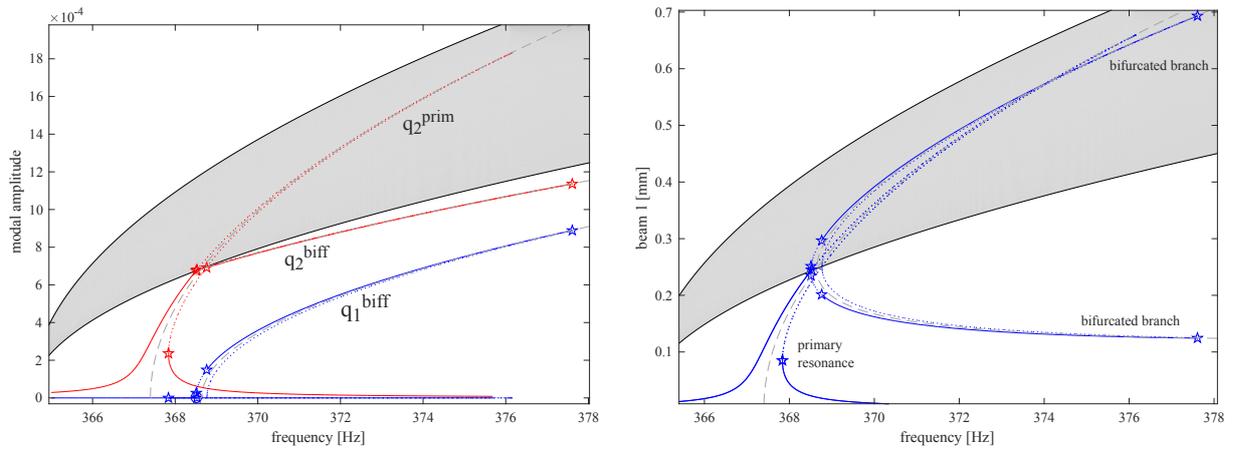


Figure 4. Numerical forced response for an excitation on the second mode with an amplitude of $f_2 = 2.0$). Solid lines: stable, dotted lines: unstable, the stars indicate the change of stability. The dashed lines represents the backbone curves obtained in the modal analysis. Right panel: modal representation, Left panel: Physical representation

4. Experimental set-up and protocols

4.1. Description of the experimental setup

The plate structure depicted on Fig.5 has been machined out of a stainless steel plate (thickness $e = 1.5$ mm, density $\rho = 7850\text{kg}\cdot\text{m}^{-3}$, Young modulus $E = 190\text{GPa}$) using wire cutting. For the experiments, the structure is hanged using four nylon wires, which allows to be close to free boundary conditions.

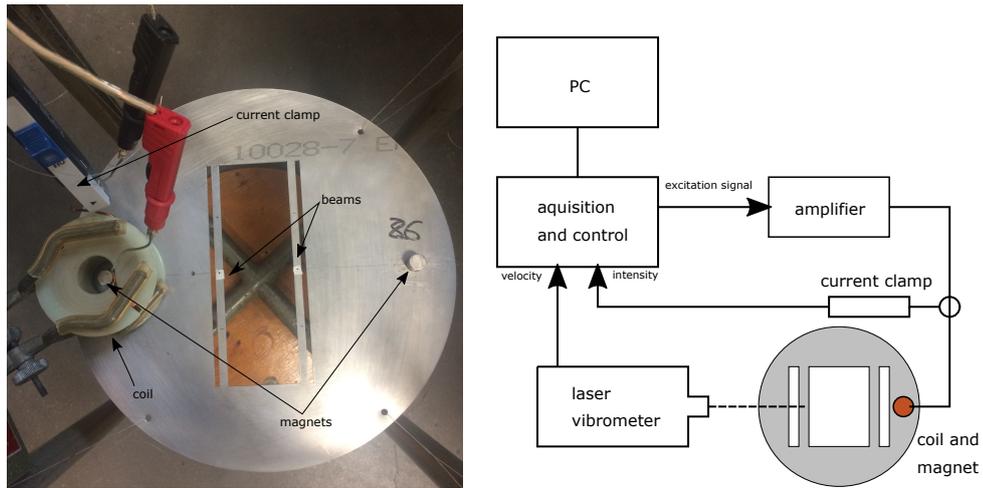


Figure 5. Picture of the experimental setup and scheme of the measurement chain

In order to provides excitation to the structure, a small magnet (about 6 grams) is stuck at a given excitation point, and a coil is then placed around it. Sending periodic current thought the coil allows to transmit a force without having an actual physical contact with the structure (Lorentz/Laplace forces). The amplitude of the force is controlled by the intensity of the current sent to the coil, which is monitored using a current clamp.

Velocity measurements are carried out using a laser vibrometer, which allows to measure the transverse velocity at each user-defined points on the top surface of the structure (most of the time at the middle of the beams).

4.2. Measurement protocols

In this study, the measurement are made using swept sine excitation. The scheme of the experimental set-up is depicted on Fig.5. For the analysis, a sinusoidal current with fixed intensity is send to the coil, and the velocity amplitude is measured at the center of a particular beam. Then the excitation frequency is slightly increased (or decreased) and the velocity measurements is repeated, until a given excitation frequency has been reached. For

a given excitation level, this allows to plots the amplitude of the displacement at the middle of each beam as as function of the excitation frequency (forced response).

Note that since only one laser vibrometer was available, the measurements are done one after another (i.e. for a given level of excitation, the measurements is done for beam 1, then the experiment is reset and the measurement is done for beam 2).

4.3. Location of the excitation

Following the numerical analysis of the previous section, the excitation point is chosen such that it excites only the out-of-phase modes. This is done by positioning the magnets at a node of the in-plane modes. In order to keep the structure symmetric, the magnet is positioned on a node of the in-plane mode belonging to the horizontal axis of symmetry, and a second magnet (without coil) is positioned in a symmetrical position with relation to the vertical axis (see Fig.5).

5. Results and discussion

5.1. Linear modal analysis

An experimental modal analysis is conducted on the structure in order to evaluates its symmetry. The experimental natural frequency are approximatively 364Hz and 367Hz, which is in good agreement as compared to the FEM frequencies given in Fig.1. The modal damping ratios are estimated approximatively using the -3dB bandwidth, leading to $\xi_1 = \xi_2 = 10^{-4}$.

The experimental mode shapes of the the out-of-phase mode is evaluated by sending a low intensity sinusoidal current to the coil (at the natural frequency) and scanning the velocity over the structure. The results is depicted on Fig.6. The mode shape is very close to the FEM mode shape presented in Fig.1. The maximum amplitudes at the middle of the beam are respectively $2.2 \cdot 10^{-3}$ and $1.9 \cdot 10^{-3}$ mm (about 10% difference), which indicates a small disturbance in the symmetry, mainly due to the magnet positioning.



Figure 6. Experimental mode shape for the out-of phase mode (top view and 3D view), only the velocity amplitude of the beams have been depicted

5.2. Forced response

The results of the sweep-sine analysis are presented on Fig.7, where the mid-beam amplitudes are plotted as a function of the excitation frequency for several current amplitudes.

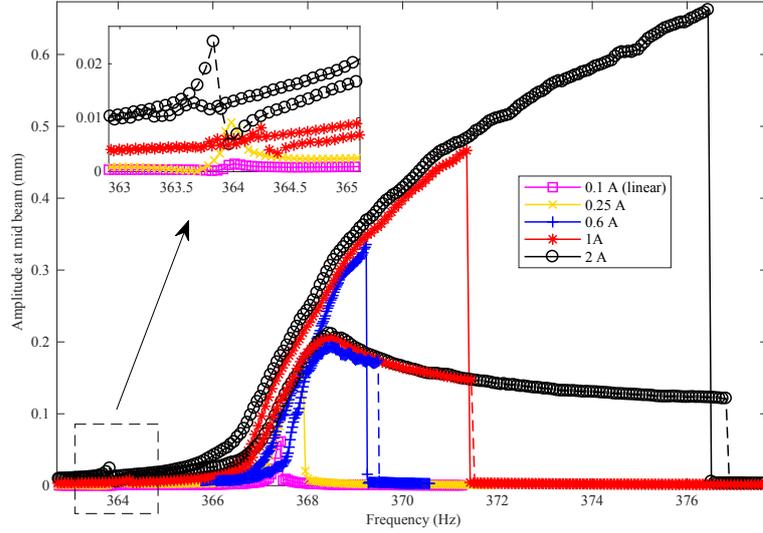


Figure 7. Mid-beam amplitude as a function of frequency for several current intensity (solid line: beam one, dashed line: beam 2)

It can be observed that the response of the first mode (around 364Hz) is very low, which indicates that the magnets are well positioned, and that the coil excites mainly the out-of-phase mode. Since each measurement is carried out one after another, over time, it might happen that the natural frequencies of the structure change a bit (about 0.5Hz) due to temperature variations or other reasons (as observed for the resonance peak of the in-phase mode in the zoom of Fig.7).

For low intensity currents ($I=0.1\text{A}$ and $I=0.25\text{A}$), only the mid-beam amplitude of beam 1 is depicted on Fig.7 (solid lines). For $I=0.1\text{A}$, the behaviour of the structure is linear, and both beams vibrate with the same amplitude. For $I=0.25\text{A}$, the non-linearity starts to be activated, and a jumps occurs as the excitation frequency is increased.

For higher intensity, both the amplitude of beam one (solid lines) and two (dashed lines) are depicted. For the three cases ($I=0.6\text{A}$, $I=1\text{A}$ and $I=2\text{A}$) the behaviour is similar and can be interpreted as follows. When the mid-beam amplitude is below 0.2mm, both beams vibrate out-of-phase with roughly the same amplitude. However, when the vibration amplitude increases, there is a break of symmetry in the vibration shape of the structure,

both beams vibrates with clearly different amplitude. As the current increases, the difference in the beam amplitude increases as well leading to a localisation. For the highest excitation current, one can see that the first beam vibrates with an amplitude 7 times greater than the second beam.

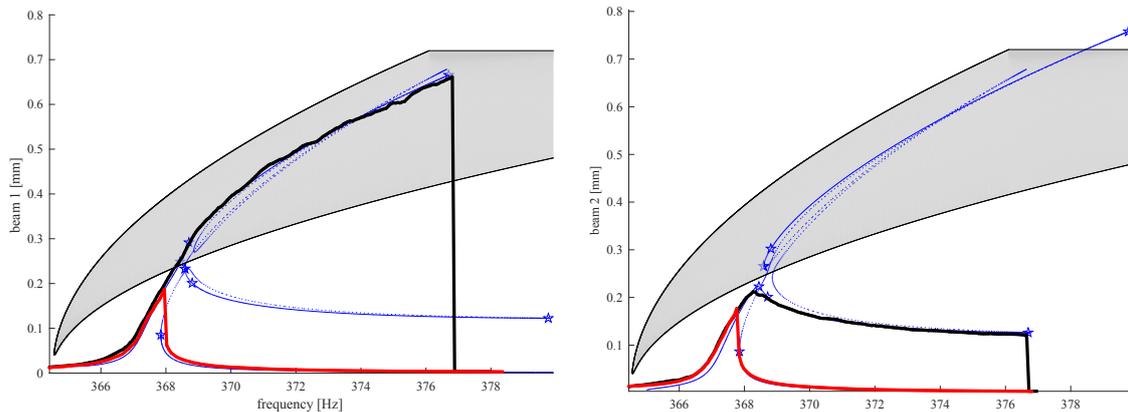


Figure 8. Comparison between experimental and numerical results for a current of 2A (left: beam 1, right beam 2). The thick lines represent experimental sweep-sine responses (upward and downward sweep). The thin lines represent numerical responses (solid: stable, dotted: unstable). The stars indicate the stability changes

A comparison to the numerical results obtained using the reduced order model is given on Fig.8. The current in the coil is fixed to the maximum tested amplitude (i.e. $I = 2A$). In the experimental case both an upward and a downward sweep-sine analysis is carried out for both beam (thick lines). In the numerical case, the solution to Eq.(2) is computed using the HBM and the ANM (thin lines). In order to take into account the fact that magnet is not exactly positioned at a node of the in-phase mode, we consider that the in-phase mode is also forced, with an amplitude ten times smaller than the out-of-phase modes. This perturbation leads to the destruction of the branching points, and the localisation appears in a continuous way, as in the experimental data. One can see that the numerical simulations agree very well with the experimental results, which validates the reduced order model procedure and the theoretical analysis.

6. Conclusion

In this paper, we considered the transverse non-linear vibration of a system of coupled beams. Numerical simulations based on a reduced model of the structure have shown that a non-linear modal interaction can occur, leading to symmetry breaking bifurcation and

localisation. Experimental measurements using swept sine excitation were carried out, and demonstrate that the modal interaction can be observed in practice (with good repeatability), if the excitation amplitude is high enough. The numerical and the experimental results agree very well, showing that the reduced order model procedure is able to generate a good representation of the physical system with only a few degree of freedom. Further study should consider systems with more than two beams, in order to generate travelling waves and/or breathers.

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