



HAL
open science

Exact controllability of the linear Zakharov-Kuznetsov equation

Mo Chen, Lionel Rosier

► **To cite this version:**

Mo Chen, Lionel Rosier. Exact controllability of the linear Zakharov-Kuznetsov equation. 2019. hal-02393893

HAL Id: hal-02393893

<https://hal.science/hal-02393893>

Preprint submitted on 4 Dec 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Exact controllability of the linear Zakharov-Kuznetsov equation

Mo Chen*

School of Mathematics and Statistics,
Center for Mathematics and Interdisciplinary Sciences,
Northeast Normal University, Changchun, 130024, P. R. China
chenmochenmo.good@163.com

Lionel Rosier†

^a Université du Littoral Côte d'Opale
Laboratoire de Mathématiques Pures et Appliquées J. Liouville,
BP 699, F-62228 Calais, France.
^b CNRS FR 2956, France
Lionel.Rosier@univ-littoral.fr

Abstract

We consider the linear Zakharov-Kuznetsov equation on a rectangle with a left Dirichlet boundary control. Using the flatness approach, we prove the null controllability of this equation and provide a space of analytic reachable states.

Keywords: Zakharov-Kuznetsov equation; null controllability; reachable states; exact controllability; flatness approach; Gevrey functions.

2010 Mathematics Subject Classification: 37L50, 93B05

1 Introduction

The Zakharov-Kuznetsov (ZK) equation

$$u_t + au_x + \Delta u_x + uu_x = 0, \quad (1.1)$$

provides a model for the propagation of nonlinear ionic-sonic waves in a plasma. In (1.1), $x, t \in \mathbb{R}$ and $y \in \mathbb{R}^d$ (with $d \in \{1, 2\}$) are the independent variables, $u = u(x, y, t)$ is the unknown, $u_t = \partial u / \partial t$, $u_x = \partial u / \partial x$, $\Delta u = \partial^2 u / \partial x^2 + \sum_{i=1}^d \partial^2 u / \partial y_i^2$, and the constant $a > 0$ stands for the sound velocity. The ZK equation is, from the mathematical point of view, a natural extension to \mathbb{R}^{d+1} of the famous Korteweg-de Vries equation

$$z_t + az_x + z_{xxx} + zz_x = 0, \quad (1.2)$$

*This author is supported by NSFC Grant (11701078) and China Scholarship Council.

†This author is supported by the ANR project Finite4SoS (ANR-15-CE23-0007)

which has been extensively studied from the control point of view (see e.g. the surveys [2, 17]). If we focus on the situation where (1.2) is supplemented with the following boundary conditions

$$z(0, t) = h(t), \quad z(L, t) = z_x(L, t) = 0, \quad (1.3)$$

where $L > 0$ is a given number and h is the control input, then it was proved in [8, 16] that (1.2)-(1.3) was null controllable on the domain $(0, L)$. Due to the smoothing effect, with such a control at the left endpoint the exact controllability can only hold in a space of analytic functions.

More recently, a space of analytic reachable states was provided in [13] for the linearized KdV equation

$$z_t + z_x + z_{xxx} = 0$$

with the same boundary conditions as in (1.3). The method of proof was based on the flatness approach, as introduced in [12] to study the reachable states of the heat equation. The aim of the paper is to extend the results given in [13] to the ZK equation.

The wellposedness of various initial boundary value problems for ZK were studied in [6, 7, 10, 11, 18, 19]. Some unique continuation property for ZK derived with a Carleman estimate was done in [3]. Exact controllability results for ZK in the same spirit as those for KdV in [15] are given in [7, 14].

Here, we limit ourselves to the case $d = 1$, so that $y \in \mathbb{R}$. By a translation, we can assume without loss of generality that $x \in (-1, 0)$ (this will be more convenient when using series to represent the solutions). We set $\Omega := (-1, 0) \times (0, 1)$. The paper is concerned with the control properties of the system:

$$u_t + u_{xxx} + u_{xyy} + au_x = 0, \quad (x, y) \in \Omega, \quad t \in (0, T), \quad (1.4)$$

$$u(0, y, t) = u_x(0, y, t) = 0, \quad y \in (0, 1), \quad t \in (0, T), \quad (1.5)$$

$$u(-1, y, t) = h(y, t), \quad y \in (0, 1), \quad t \in (0, T), \quad (1.6)$$

$$u(x, 0, t) = u(x, 1, t) = 0, \quad x \in (-1, 0), \quad t \in (0, T), \quad (1.7)$$

$$u(x, y, 0) = u_0(x, y), \quad (x, y) \in \Omega, \quad (1.8)$$

where $u_0 = u_0(x, y)$ is the initial data and $h = h(y, t)$ is the control input.

We shall address the following issues:

1. (Null controllability) Given any $u_0 \in L^2(\Omega)$, can we find a control h such that the solution u of (1.4)-(1.8) satisfies $u(\cdot, T) = 0$?
2. (Reachable states) Given any $u_1 \in \mathcal{R}$ (a subspace of $L^2(\Omega)$ defined thereafter), can we find a control h such that the solution u of (1.4)-(1.8) with $u_0 = 0$ satisfies $u(\cdot, T) = u_1$?

We shall investigate both issues by the flatness approach and derive an exact controllability in \mathcal{R} by combining our results.

To state our result, we need introduce notations. A function $u \in C^\infty([t_1, t_2])$ is said to be *Gevrey of order $s \geq 0$ on $[t_1, t_2]$* if there exist some constant $C, R \geq 0$ such that

$$|\partial_t^n u(t)| \leq C \frac{(n!)^s}{R^n} \quad \forall n \in \mathbb{N}, \quad \forall t \in [t_1, t_2].$$

The set of functions Gevrey of order s on $[t_1, t_2]$ is denoted by $G^s([t_1, t_2])$. A function $u \in C^\infty([x_1, x_2] \times [y_1, y_2] \times [t_1, t_2])$ is said to be Gevrey of order s_1 in x , s_2 in y and s_3 in t on $[x_1, x_2] \times [y_1, y_2] \times [t_1, t_2]$ if there exist some constants $C, R_1, R_2, R_3 > 0$ such that

$$|\partial_x^{n_1} \partial_y^{n_2} \partial_t^{n_3} u(x, y, t)| \leq C \frac{(n_1!)^{s_1} (n_2!)^{s_2} (n_3!)^{s_3}}{R_1^{n_1} R_2^{n_2} R_3^{n_3}} \quad \forall n_1, n_2, n_3 \in \mathbb{N}, \forall (x, y, t) \in [x_1, x_2] \times [y_1, y_2] \times [t_1, t_2].$$

The set of functions Gevrey of order s_1 in x , s_2 in y and s_3 in t on $[x_1, x_2] \times [y_1, y_2] \times [t_1, t_2]$ is denoted by $G^{s_1, s_2, s_3}([x_1, x_2] \times [y_1, y_2] \times [t_1, t_2])$.

The first main result in this paper is a null controllability result with a control input in a Gevrey class.

Theorem 1.1. *Let $u_0 \in L^2(\Omega)$ and $s \in [\frac{3}{2}, 2)$. Then there exists a control input $h \in G^{\frac{s}{2}, s}([0, 1] \times [0, T])$ such that the solution u of (1.4)-(1.8) satisfies $u(\cdot, \cdot, T) = 0$. Furthermore, it holds that*

$$u \in C([0, T]; L^2(\Omega)) \cap G^{\frac{s}{2}, \frac{s}{2}, s}([-1, 0] \times [0, 1] \times [\varepsilon, T]), \quad \forall \varepsilon \in (0, T).$$

Introduce the differential operator

$$Pu := \Delta u_x + au_x$$

and the following space

$$\mathcal{R}_{R_1, R_2} := \{u \in C^\infty([-1, 0] \times [0, 1]); \exists C > 0, |\partial_x^p \partial_y^q u(x, y)| \leq C \frac{(p!)^{\frac{2}{3}} (q!)^{\frac{2}{3}}}{R_1^p R_2^q} \quad \forall p, q \in \mathbb{N}, \forall (x, y) \in \bar{\Omega},$$

$$\text{and } P^n u(0, y) = \partial_x P^n u(0, y) = P^n u(x, 0) = P^n u(x, 1) = 0, \quad \forall n \in \mathbb{N}, \forall x \in [-1, 0], \forall y \in [0, 1]\}.$$

Our second main result provides a set of reachable states for system (1.4)-(1.8).

Theorem 1.2. *Let $R_0 := \sqrt[3]{9(a+2)}e^{(3e)^{-1}}$, and let $R_1, R_2 \in (R_0, +\infty)$. Then for any $u_1 \in \mathcal{R}_{R_1, R_2}$, there exists a control input $h \in G^{1,2}([0, 1] \times [0, T])$ such that the solution u of (1.4)-(1.8) with $u_0 = 0$ satisfies $u(\cdot, \cdot, T) = u_1$. Furthermore, $u \in G^{1,1,2}([-1, 0] \times [0, 1] \times [0, T])$, and the trajectory $u = u(x, y, t)$ and the control $h = h(y, t)$ can be expanded as series:*

$$u(x, y, t) = \sum_{j=1}^{\infty} \sum_{i=0}^{\infty} g_{i,j}(x) z_j^{(i)}(t) e_j(y), \quad (1.9)$$

$$h(y, t) = \sum_{j=1}^{\infty} \sum_{i=0}^{\infty} g_{i,j}(-1) z_j^{(i)}(t) e_j(y). \quad (1.10)$$

We refer the reader to Section 2 for the definitions of the functions $g_{i,j}$ ($i \geq 0, j \geq 1$) and of the functions e_j ($j \geq 1$).

Combining Theorem 1.1 and Theorem 1.2, we obtain the following result which implies the exact controllability of (1.4)-(1.8) in \mathcal{R}_{R_1, R_2} for $R_1 > R_0$ and $R_2 > R_0$.

Corollary 1.1. *Let $R_0 := \sqrt[3]{9(a+2)}e^{(3e)^{-1}}$, and let $R_1, R_2 \in (R_0, +\infty)$. Let $u_0 \in L^2(\Omega)$ and $u_1 \in \mathcal{R}_{R_1, R_2}$. Then there exists $h \in G^{1,2}([0, 1] \times [0, T])$ such that the solution of (1.4)-(1.8) satisfies $u(\cdot, T) = u_1$.*

The paper is outlined as follows. Section 2 introduces the eigenfunctions e_j , the generating functions $g_{i,j}$, and provide some estimates needed in the sequel. The null controllability of ZK is established in Section 3, while the reachable states of ZK are investigated in Section 4.

2 Preliminaries

First we introduce the operator

$$Au := -Pu = -\Delta u_x - au_x$$

with domain

$$\mathcal{D}(A) = \{u \in L^2(\Omega); Pu \in L^2(\Omega), u(-1, y) = u(0, y) = u_x(0, y) = 0 \text{ for a.e. } y \in (0, 1) \text{ and } u(x, 0) = u(x, 1) = 0 \text{ for a.e. } x \in (-1, 0)\}.$$

It is well-known (see e.g. [19]) that the operator A generates a semigroup of contractions in $L^2(\Omega)$. In what follows, we denote $\|f\|_{\mathcal{D}(A)} = \|f\|_{L^2(\Omega)} + \|Af\|_{L^2(\Omega)}$ for all $f \in \mathcal{D}(A)$.

It would be natural to expect, as for KdV, that the domain $\mathcal{D}(A)$ coincide with the set

$$\{u \in H^3(\Omega) \cap H_0^1(\Omega); u_x(0, y) = 0 \text{ for a.e. } y \in (0, 1)\},$$

but this is not the case. The best description (up to date) of $\mathcal{D}(A)$ is given in the following lemma.

Lemma 2.1. *We have the following inclusions:*

$$\{u \in H^2(\Omega) \cap H_0^1(\Omega); u_x \in H^2(\Omega) \text{ and } u_x(0, y) = 0 \text{ for a.e. } y \in (0, 1)\} \subset \mathcal{D}(A), \quad (2.1)$$

$$\mathcal{D}(A) \subset \{u \in H^2(\Omega) \cap H_0^1(\Omega); (x+1)u_x \in H^2(\Omega) \cap H_0^1(\Omega)\}. \quad (2.2)$$

Proof. The inclusion (2.1) is obvious. For (2.2), it follows from [19, Proposition 2] that $\mathcal{D}(A) \subset H^2(\Omega) \cap H_0^1(\Omega)$. If $u \in \mathcal{D}(A)$, then $f := \Delta u_x + au_x \in L^2(\Omega)$ and hence

$$\Delta((x+1)u_x) = (x+1)\Delta u_x + 2u_{xx} = (x+1)(f - au_x) + 2u_{xx} \in L^2(\Omega).$$

On the other hand, we claim that $(x+1)u_x \in H_0^1(\Omega)$. Indeed, $u_x \in H^1(\Omega)$ and hence $(x+1)u_x \in H^1(\Omega)$. Moreover, $u(\cdot, 0) = u(\cdot, 1) = 0$ in $H^{\frac{3}{2}}(-1, 0)$ gives $u_x(\cdot, 0) = u_x(\cdot, 1) = 0$ in $H^{\frac{1}{2}}(-1, 0)$, and finally $((x+1)u_x)(-1, \cdot) = u_x(0, \cdot) = 0$ in $H^{\frac{1}{2}}(0, 1)$. By the classical boundary H^2 regularity result for the Dirichlet problem on a Lipschitz domain, we infer that $(x+1)u_x \in H^2(\Omega) \cap H_0^1(\Omega)$. \square

Remark 2.1. *It can be shown that the inclusion (2.1) is strict.*

The following lemmas will be used several times thereafter.

Lemma 2.2. *For any $n \in \mathbb{N}^*$ and any $f \in \mathcal{D}(A^n)$ with $A^i f \in H^{2(n-i)}(\Omega)$ for $i = 0, 1, \dots, n$, we have*

$$\partial_y^{2p} f(x, 0) = \partial_y^{2p} f(x, 1) = 0, \quad \forall x \in [-1, 0], \forall p \in \{0, \dots, n-1\}. \quad (2.3)$$

Proof. We proceed by induction on n . For $n = 1$, the property (2.3) is obvious since $f \in \mathcal{D}(A)$. Assume now that (2.3) is true for $n-1 \geq 1$. If $f \in \mathcal{D}(A^n)$ with $A^i f \in H^{2(n-i)}(\Omega)$ for $i = 0, 1, \dots, n$, then $Pf = -Af \in \mathcal{D}(A^{n-1})$ with $A^i Pf = -A^{i+1}f \in H^{2(n-i-1)}(\Omega)$ for $i = 0, 1, \dots, n-1$, so that by (2.3) applied to Pf and $p = n-2$

$$\partial_y^{2n-4} Pf(x, 0) = \partial_y^{2n-4} Pf(x, 1) = 0.$$

This implies

$$\partial_x^3 \partial_y^{2n-4} f(x, 0) + \partial_x \partial_y^{2n-2} f(x, 0) + a \partial_x \partial_y^{2n-4} f(x, 0) = 0, \quad (2.4)$$

$$\partial_x^3 \partial_y^{2n-4} f(x, 1) + \partial_x \partial_y^{2n-2} f(x, 1) + a \partial_x \partial_y^{2n-4} f(x, 1) = 0. \quad (2.5)$$

Since (2.3) is true for $n - 1$, we obtain that $\partial_y^{2p} f(x, 0) = \partial_y^{2p} f(x, 1) = 0$ for $p = 0, 1, \dots, n - 2$, and hence (taking $p = n - 2$ and using (2.4)-(2.5))

$$\partial_x \partial_y^{2n-2} f(x, 0) = \partial_x \partial_y^{2n-2} f(x, 1) = 0.$$

This means that we have for some constants C_1 and C_2

$$\partial_y^{2n-2} f(x, 0) = C_1, \quad \partial_y^{2n-2} f(x, 1) = C_2 \quad \forall x \in [-1, 0].$$

Note that $\partial_y^{2n-2} f \in H^2(\Omega) \subset C(\bar{\Omega})$. On the other hand, it follows from the assumption $f \in \mathcal{D}(A)$ that

$$\partial_y^{2n-2} f(0, y) = 0 \quad \forall y \in [0, 1].$$

Taking $y = 0$ and next $y = 1$, we see that $C_1 = C_2 = 0$. The proof of Lemma 2.2 is complete. \square

Remark 2.2. *It will be proved in Proposition 2.1 (see below) that $\mathcal{D}(A^n) \subset H^{2n}(\Omega)$ for all $n \in \mathbb{N}$, so that the conclusion of Lemma 2.2 will be still valid when assuming solely that $f \in \mathcal{D}(A^n)$.*

The following lemma is classical. Its proof is omitted.

Lemma 2.3. *Let $A' = \partial_y^2$ with domain $\mathcal{D}(A') = H^2(0, 1) \cap H_0^1(0, 1)$. Then for any $m \in \mathbb{N}^*$, it holds*

$$\mathcal{D}(A'^{\frac{m}{2}}) = \{g \in H^m(0, 1); g^{(2p)}(0) = g^{(2p)}(1) = 0 \text{ for } 0 \leq p \leq \frac{m-1}{2}\}.$$

Let $h \in L^2(0, 1)$ be decomposed as $h(y) = \sum_{j=1}^{\infty} c_j e_j(y)$, and let $m \in \mathbb{N}^*$. Then

$$h \in \mathcal{D}(A'^{\frac{m}{2}}) \iff \sum_{j=1}^{\infty} |\lambda_j^{\frac{m}{2}} c_j|^2 < \infty.$$

Furthermore, for any $h \in \mathcal{D}(A'^{\frac{m}{2}})$, we have

$$\|h^{(q)}\|_{L^2(0,1)}^2 = \sum_{j=1}^{\infty} \lambda_j^q |c_j|^2 \quad \forall q \in \{0, \dots, m\}.$$

We are in a position to state the main result in this section.

Proposition 2.1. *For any $n \in \mathbb{N}$, it holds $\mathcal{D}(A^n) \subset H^{2n}(\Omega)$. Furthermore, there exists a constant $B \geq 1$ such that*

$$\|u\|_{H^{2n}(\Omega)} \leq B^n \sum_{i=0}^n \|P^i u\|_{L^2(\Omega)}, \quad \forall n \in \mathbb{N}, \quad \forall u \in \mathcal{D}(A^n). \quad (2.6)$$

Proof. Let $\{e_j\}_{j \geq 1}$ be an orthonormal basis in $L^2(0, 1)$ such that e_j is an eigenfunction for the Dirichlet Laplacian on $(0, 1)$, λ_j being the corresponding eigenvalue; that is

$$\begin{aligned} -e_j''(y) &= \lambda_j e_j(y), \\ e_j(0) &= e_j(1) = 0. \end{aligned}$$

A classical choice is $e_j(y) = \sqrt{2} \sin(j\pi y)$ and $\lambda_j = (j\pi)^2$ for $j \geq 1$. Following [19], we decompose any function $u \in L^2(\Omega)$ as

$$u(x, y) = \sum_{j=1}^{\infty} \hat{u}_j(x) e_j(y).$$

Note that $\|u\|_{L^2(\Omega)}^2 = \sum_{j=1}^{\infty} \|\hat{u}_j\|^2$, where we denote $\|h\| = \|h\|_{L^2(-1, 0)}$ for all $h \in L^2(-1, 0)$ for the sake of simplicity. If $u \in \mathcal{D}(A)$ and $g := \Delta u_x + a u_x$, then for any $j \geq 1$

$$\hat{u}_j''' + (a - \lambda_j) \hat{u}_j' = \hat{g}_j \quad \text{in } L^2(-1, 0) \quad (2.7)$$

where $' = d/dx$. For $n = 0$, (2.6) is obvious if we pick $C_0 \geq 1$. Let us assume first that $n = 1$. Note that $\hat{u}_j \in H^3(-1, 0)$ by (2.7). Multiplying (2.7) by $\lambda_j(x+1)\hat{u}_j$, we obtain

$$\frac{3}{2} \lambda_j \int_{-1}^0 |\hat{u}_j'|^2 dx - (a - \lambda_j) \frac{\lambda_j}{2} \int_{-1}^0 |\hat{u}_j|^2 dx = \lambda_j \int_{-1}^0 (x+1) \hat{u}_j \hat{g}_j dx.$$

Let $j_0 := \lceil \frac{\sqrt{2a}}{\pi} \rceil$. Then for $j > j_0$, we have $a \leq \lambda_j/2$ and hence $|a - \lambda_j| \lambda_j/2 \geq \lambda_j^2/4$. Using

$$\left| \lambda_j \int_{-1}^0 (x+1) \hat{u}_j \hat{g}_j dx \right| \leq \frac{\lambda_j^2}{8} \int_{-1}^0 |\hat{u}_j|^2 dx + 2 \int_{-1}^0 |\hat{g}_j|^2 dx,$$

we infer that for $j > j_0$

$$\frac{3}{2} \lambda_j \int_{-1}^0 |\hat{u}_j'|^2 dx + \frac{\lambda_j^2}{8} \int_{-1}^0 |\hat{u}_j|^2 dx \leq 2 \int_{-1}^0 |\hat{g}_j|^2 dx,$$

and that for $1 \leq j \leq j_0$

$$\frac{3}{2} \lambda_j \int_{-1}^0 |\hat{u}_j'|^2 dx + \frac{\lambda_j^2}{8} \int_{-1}^0 |\hat{u}_j|^2 dx \leq \Lambda \int_{-1}^0 |\hat{u}_j|^2 dx + 2 \int_{-1}^0 |\hat{g}_j|^2 dx, \quad (2.8)$$

where $\Lambda := \max_{1 \leq j \leq j_0} \left| \frac{\lambda_j^2}{8} + (a - \lambda_j) \frac{\lambda_j}{2} \right|$. Obviously, (2.8) is valid for any $j \geq 1$. Summing in j , we obtain

$$\frac{3}{2} \int_{\Omega} |u_{xy}|^2 dx dy + \frac{1}{8} \int_{\Omega} |u_{yy}|^2 dx dy \leq \Lambda \|u\|^2 + 2 \|g\|^2. \quad (2.9)$$

Dividing in (2.8) by $\lambda_j \geq \pi^2$ and summing in j , we obtain

$$\frac{3}{2} \int_{\Omega} |u_x|^2 dx dy + \frac{1}{8} \int_{\Omega} |u_y|^2 dx dy \leq \frac{\Lambda}{\pi^2} \|u\|^2 + \frac{2}{\pi^2} \|g\|^2. \quad (2.10)$$

It remains to estimate $\int_{\Omega} |u_{xx}|^2 dx dy$. Multiplying in (2.7) by \hat{u}'_j , we obtain

$$- \int_{-1}^0 |\hat{u}''_j|^2 dx + \hat{u}'_j \hat{u}''_j|_{-1}^0 + (a - \lambda_j) \int_{-1}^0 |\hat{u}'_j|^2 dx = \int_{-1}^0 \hat{g}_j \hat{u}'_j dx,$$

and hence

$$\int_{-1}^0 |\hat{u}''_j|^2 dx \leq |\lambda_j - a| \int_{-1}^0 |\hat{u}'_j|^2 dx + \|\hat{g}_j\| \|\hat{u}'_j\| + |\hat{u}'_j(-1) \hat{u}''_j(-1)|.$$

We are let to estimate $\hat{u}'_j(-1)$ and $\hat{u}''_j(-1)$. Multiplying in (2.7) by $\lambda_j \hat{u}_j$ results in

$$\lambda_j \frac{\hat{u}'_j(-1)^2}{2} = \lambda_j \int_{-1}^0 \hat{u}_j \hat{g}_j dx.$$

Combined with (2.8), this yields

$$\frac{\lambda_j}{2} \hat{u}'_j(-1)^2 \leq \|\lambda_j \hat{u}_j\| \cdot \|\hat{g}_j\| \leq \frac{1}{4} \|\lambda_j \hat{u}_j\|^2 + \|\hat{g}_j\|^2 \leq 5 \|\hat{g}_j\|^2 + 2\Lambda \|\hat{u}_j\|^2. \quad (2.11)$$

Multiplying in (2.7) by x , we obtain

$$- \int_{-1}^0 \hat{u}''_j dx + x \hat{u}''_j|_{-1}^0 + (a - \lambda_j) \left(- \int_{-1}^0 \hat{u}_j dx + x \hat{u}_j|_{-1}^0 \right) = \int_{-1}^0 x \hat{g}_j dx$$

which yields

$$\hat{u}''_j(-1) = -\hat{u}'_j(-1) + (a - \lambda_j) \int_{-1}^0 \hat{u}_j dx + \int_{-1}^0 x \hat{g}_j dx,$$

so that

$$|\hat{u}''_j(-1)|^2 \leq 3 (|\hat{u}'_j(-1)|^2 + 2(a^2 + |\lambda_j|^2) \|\hat{u}_j\|^2 + \|\hat{g}_j\|^2).$$

Using (2.8) and (2.11), we conclude that $|\hat{u}''_j(-1)|^2 = O(\|\hat{u}_j\|^2 + \|\hat{g}_j\|^2)$. The same is true for $\|\hat{u}''_j\|^2$. Gathering together the above estimates, we arrive at

$$\|u\|_{H^2(\Omega)}^2 \leq C_1 \left(\|u\|_{L^2(\Omega)}^2 + \|Pu\|_{L^2(\Omega)}^2 \right)$$

for some constant $C_1 = C_1(a) > 0$.

Let us check that $\mathcal{D}(A^n) \subset H^{2n}(\Omega)$ for $n \geq 2$. We proceed by induction on n . Assume that $\mathcal{D}(A^p) \subset H^{2p}(\Omega)$ for $p = 0, 1, \dots, n-1$ (with $n-1 \geq 1$), and pick any $u \in \mathcal{D}(A^n)$. Then $g = Au \in \mathcal{D}(A^{n-1}) \subset H^{2(n-1)}(\Omega)$. Let $h := (-1)^{n-1} \partial_y^{2(n-1)} g \in L^2(\Omega)$. Then, using Lemmas 2.2 and 2.3, we have that for all $j \geq 1$

$$\lambda_j^{n-1} (\hat{u}'''_j + (a - \lambda_j) \hat{u}'_j) = \hat{h}_j. \quad (2.12)$$

Multiplying in (2.12) by $\lambda_j^n (x+1) \hat{u}_j$, we obtain

$$\frac{3}{2} \lambda_j^{2n-1} \int_{-1}^0 |\hat{u}'_j|^2 dx - (a - \lambda_j) \frac{\lambda_j^{2n-1}}{2} \int_{-1}^0 |\hat{u}_j|^2 dx = \lambda_j^n \int_{-1}^0 (x+1) \hat{u}_j \hat{h}_j dx.$$

This yields

$$\lambda_j^{2n-1} \|\hat{u}'_j\|^2 + \lambda_j^{2n} \|\hat{u}_j\|^2 = O(\|\hat{u}_j\|^2 + \|\hat{h}_j\|^2), \quad (2.13)$$

Multiplying in (2.12) by $\lambda_j^n \hat{u}_j$ gives

$$\lambda_j^{2n-1} \frac{\hat{u}'_j(-1)^2}{2} = \lambda_j^n \int_{-1}^0 \hat{u}_j \hat{h}_j dx$$

and

$$\lambda_j^{2n-1} |\hat{u}'_j(-1)|^2 = O(\|\hat{u}_j\|^2 + \|\hat{h}_j\|^2).$$

From

$$\lambda_j^{n-1} \hat{u}''_j(-1) = -\lambda_j^{n-1} \hat{u}'_j(-1) + (a - \lambda_j) \lambda_j^{n-1} \int_{-1}^0 \hat{u}_j dx + \int_{-1}^0 x \hat{h}_j dx,$$

we infer that

$$\lambda_j^{2n-2} |\hat{u}''_j(-1)|^2 = O(\|\hat{u}_j\|^2 + \|\hat{h}_j\|^2).$$

It follows from

$$-\lambda_j^{2n-2} \int_{-1}^0 |\hat{u}''_j|^2 dx + \lambda_j^{2n-2} \hat{u}'_j \hat{u}''_j|_{-1}^0 + (a - \lambda_j) \lambda_j^{2n-2} \int_{-1}^0 |\hat{u}'_j|^2 dx = \lambda_j^{n-1} \int_{-1}^0 \hat{h}_j \hat{u}'_j dx,$$

that

$$\lambda_j^{2n-2} \|\hat{u}''_j\|^2 = O(\|\hat{u}_j\|^2 + \|\hat{h}_j\|^2). \quad (2.14)$$

So far, we have proved that

$$\sum_{j=1}^{\infty} \left(\lambda_j^{2n} \|\hat{u}_j\|^2 + \lambda_j^{2n-1} \|\hat{u}'_j\|^2 + \lambda_j^{2n-2} \|\hat{u}''_j\|^2 \right) < +\infty.$$

Using Lemma 2.3, this gives that $\partial_y^{2n} u$, $\partial_y^{2n-1} \partial_x u$, and $\partial_y^{2n-2} \partial_x^2 u$ belong to $L^2(\Omega)$. For the other derivatives of order $2n$, we apply the operator ∂_x^{2k} (for $k \in \mathbb{N}$ with $2k + 3 \leq 2n$) to each term in (2.7) to obtain

$$\hat{u}_j^{(2k+3)} + (a - \lambda_j) \hat{u}_j^{(2k+1)} = \hat{g}_j^{(2k)}.$$

This yields

$$\lambda_j^{(2n-3-2k)} \|\hat{u}_j^{(2k+3)}\|^2 = O(\|\hat{u}_j\|^2 + \|\hat{h}_j\|^2 + \lambda_j^{2n-3-2k} \|\hat{g}_j^{(2k)}\|^2).$$

On the other hand, (2.7) gives by differentiation with respect to x that

$$\hat{u}_j^{(4)} + (a - \lambda_j) \hat{u}_j'' = \hat{g}_j',$$

and we obtain in a similar way that

$$\lambda_j^{(2n-4-2k)} \|\hat{u}_j^{(2k+4)}\|^2 = O(\|\hat{u}_j\|^2 + \|\hat{h}_j\|^2 + \lambda_j^{2n-4-2k} \|\hat{g}_j^{(2k+1)}\|^2).$$

for $k \in \mathbb{N}$ with $2k + 4 \leq 2n$. Thus we conclude that

$$\sum_{q=0}^{2n} \sum_{j=1}^{\infty} \lambda_j^{(2n-q)} \|\hat{u}_j^{(q)}\|^2 < +\infty.$$

Using Lemma 2.3, we infer that for $q \in \{0, \dots, 2n\}$, $\partial_x^q u \in L^2(-1, 0, H^{2n-q}(0, 1))$, and hence that $\partial_y^{2n-q} \partial_x^q u \in L^2(\Omega)$. We also have that $\partial_y^{2n-1-q} \partial_x^q u \in L^2(\Omega)$ for $q \in \{0, \dots, 2n-1\}$. Taking into account the fact that $u \in \mathcal{D}(A^{n-1}) \subset H^{2(n-1)}(\Omega)$, we conclude that $u \in H^{2n}(\Omega)$. The proof of the inclusion $\mathcal{D}(A^n) \subset H^{2n}(\Omega)$ is complete.

It remains to prove that the constant in the r.h.s. of (2.6) is indeed of the form B^n . This will require a series of lemmas.

Lemma 2.4. *For any $\varepsilon_0 > 0$, there exists a constant $K = K(\varepsilon_0) > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and all $f \in H^2(-1, 0)$,*

$$\int_{-1}^0 |f'(t)|^2 dt \leq K\varepsilon \int_{-1}^0 |f''(t)|^2 dt + K\varepsilon^{-1} \int_{-1}^0 |f(t)|^2 dt. \quad (2.15)$$

Lemma 2.4 is a direct consequence of [1, Lemma 4.10] (which is concerned with twice continuously functions) by density of $C^2([-1, 0])$ in $H^2(-1, 0)$.

For any $j \in \mathbb{N}^*$, we define the operator P_j by

$$P_j f := f''' - (\lambda_j - a)f', \quad \forall f \in H^3(-1, 0).$$

Lemma 2.5. *There exists a constant $C_1 \geq 1$ such that*

$$\|f\|_{H^{2n}(-1, 0)}^2 \leq C_1^n \sum_{i=0}^n \lambda_j^{2n-2i} \|P_j^i f\|^2, \quad \forall n \in \mathbb{N}, \forall j \in \mathbb{N}^*, \forall f \in H^{3n}(-1, 0). \quad (2.16)$$

Proof. For $n = 0$, (2.16) is obvious. For $n = 1$, it follows from the definition of P_j and Lemma 2.4 that

$$\begin{aligned} \|f\|_{H^2(-1, 0)}^2 &= \|f\|^2 + \|f'\|^2 + \|f''\|^2 \\ &\leq C(\|f\|^2 + \|f''\|^2) \\ &\leq C(\|f\|^2 + \frac{1}{\lambda_j} \|f'''\|^2 + \lambda_j \|f'\|^2) \\ &\leq C(\|f\|^2 + \frac{1}{\lambda_j} \|P_j f\|^2 + \lambda_j \|f'\|^2) \\ &\leq C(\|f\|^2 + \|P_j f\|^2) + \frac{1}{2} \|f''\|^2 + C\lambda_j^2 \|f\|^2 \\ &\leq C(\lambda_j^2 \|f\|^2 + \|P_j f\|^2) + \frac{1}{2} \|f''\|^2. \end{aligned}$$

This shows that we can find a constant $C_2 \geq 1$ such that

$$\|f\|_{H^2(-1, 0)}^2 \leq C_2(\lambda_j^2 \|f\|^2 + \|P_j f\|^2).$$

Let us prove (2.16) for $n \geq 2$ by induction on n . Assume (2.16) to be true for $n-1 \geq 0$. It

follows that

$$\begin{aligned}
\|f\|_{H^{2n}(-1,0)}^2 &= \|f\|_{H^{2n-2}(-1,0)}^2 + \|f^{(2n-1)}\|^2 + \|f^{(2n)}\|^2 \\
&\leq \|f\|_{H^{2n-2}(-1,0)}^2 + \|f^{(2n-2)}\|_{H^2(-1,0)}^2 \\
&\leq \|f\|_{H^{2n-2}(-1,0)}^2 + C_2(\lambda_j^2 \|f^{(2n-2)}\|^2 + \|P_j f^{(2n-2)}\|^2) \\
&\leq 2C_2 \lambda_j^2 \|f\|_{H^{2n-2}(-1,0)}^2 + C_2 \|P_j f\|_{H^{2n-2}(-1,0)}^2 \\
&\leq 2C_2 \lambda_j^2 C_1^{n-1} \sum_{i=0}^{n-1} \lambda_j^{2n-2-2i} \|P_j^i f\|^2 + C_2 C_1^{n-1} \sum_{i=0}^{n-1} \lambda_j^{2n-2-2i} \|P_j^{i+1} f\|^2 \\
&\leq 2C_2 C_1^{n-1} \sum_{i=0}^{n-1} \lambda_j^{2n-2i} \|P_j^i f\|^2 + C_2 C_1^{n-1} \sum_{i=1}^n \lambda_j^{2n-2i} \|P_j^i f\|^2 \\
&\leq 3C_2 C_1^{n-1} \sum_{i=0}^n \lambda_j^{2n-2i} \|P_j^i f\|^2.
\end{aligned}$$

If we pick $C_1 = 3C_2$, (2.16) is true for n . □

Lemma 2.6. *There exists a positive constant C_3 such that*

$$\|u\|_{H^{2n}(\Omega)}^2 \leq C_3 \sum_{m=0}^n \sum_{k=0}^m \|\partial_x^{2k} \partial_y^{2m-2k} u\|_{L^2(\Omega)}^2, \quad \forall n \in \mathbb{N}, \quad \forall u \in \mathcal{D}(A^n).$$

Proof. For any $p \in \mathbb{N}$, we set

$$I_p := \sum_{a,b \in \mathbb{N}, a+b=p} \|\partial_x^a \partial_y^b u\|_{L^2(\Omega)}^2.$$

Decompose u as

$$u(x, y) = \sum_{j=1}^{\infty} \hat{u}_j(x) e_j(y). \tag{2.17}$$

Let us go back to the proof of Lemma 2.6. Pick any $u \in \mathcal{D}(A^n)$, for some $n \in \mathbb{N}$. Using Lemma 2.2 and applying Lemma 2.3 to the functions $\partial_x^{2m+1-k} u(x, \cdot)$ for $0 \leq m \leq n-1$,

$0 \leq k \leq 2m + 1$, and $x \in (-1, 0)$, we obtain that

$$\begin{aligned}
I_{2m+1} &= \sum_{k=0}^{2m+1} \|\partial_x^{2m+1-k} \partial_y^k u\|_{L^2(\Omega)}^2 \\
&= \sum_{k=0}^{2m+1} \sum_{j=1}^{\infty} \lambda_j^k \|\hat{u}_j^{(2m+1-k)}\|^2 \\
&= \sum_{j=1}^{\infty} \|\hat{u}_j^{(2m+1)}\|^2 + \sum_{k=1}^{2m+1} \sum_{j=1}^{\infty} \lambda_j^k \|\hat{u}_j^{(2m+1-k)}\|^2 \\
&\leq \sum_{j=1}^{\infty} \lambda_j \|\hat{u}_j^{(2m+1)}\|^2 + \frac{1}{2} \sum_{k=1}^{2m+1} \sum_{j=1}^{\infty} \lambda_j^{k-1} \|\hat{u}_j^{(2m+1-k)}\|^2 + \frac{1}{2} \sum_{k=1}^{2m+1} \sum_{j=1}^{\infty} \lambda_j^{k+1} \|\hat{u}_j^{(2m+1-k)}\|^2 \\
&= \|\partial_x^{2m+1} \partial_y u\|_{L^2(\Omega)}^2 + \frac{1}{2} \sum_{k=1}^{2m+1} \|\partial_x^{2m+1-k} \partial_y^{k-1} u\|_{L^2(\Omega)}^2 + \frac{1}{2} \sum_{k=1}^{2m+1} \|\partial_x^{2m+1-k} \partial_y^{k+1} u\|_{L^2(\Omega)}^2 \\
&\leq \frac{3}{2} I_{2m+2} + \frac{1}{2} I_{2m},
\end{aligned}$$

where we used Young's estimate. Thus, we have

$$\begin{aligned}
\|u\|_{H^{2n}(\Omega)}^2 &= \sum_{m=0}^n I_{2m} + \sum_{m=0}^{n-1} I_{2m+1} \\
&\leq \sum_{m=0}^n I_{2m} + \sum_{m=0}^{n-1} \left(\frac{3}{2} I_{2m+2} + \frac{1}{2} I_{2m} \right) \\
&\leq 3 \sum_{m=0}^n I_{2m}.
\end{aligned} \tag{2.18}$$

Next, we consider I_{2m} . For $m = 0$, $I_0 = \|u\|_{L^2(\Omega)}^2$. For $m \geq 1$, we have

$$I_{2m} = \sum_{k=0}^m \|\partial_x^{2k} \partial_y^{2m-2k} u\|_{L^2(\Omega)}^2 + \sum_{k=0}^{m-1} \|\partial_x^{2k+1} \partial_y^{2m-2k-1} u\|_{L^2(\Omega)}^2, \tag{2.19}$$

and it remains to estimate the second term in the r.h.s. of (2.19). Applying Lemma 2.4, we

obtain

$$\begin{aligned}
\sum_{k=0}^{m-1} \|\partial_x^{2k+1} \partial_y^{2m-2k-1} u\|_{L^2(\Omega)}^2 &= \sum_{k=0}^{m-1} \sum_{j=1}^{\infty} \lambda_j^{2m-2k-1} \|\hat{u}_j^{(2k+1)}\|^2 \\
&\leq C \left(\sum_{k=0}^{m-1} \sum_{j=1}^{\infty} \lambda_j^{2m-2k-2} \|\hat{u}_j^{(2k+2)}\|^2 + \sum_{k=0}^{m-1} \sum_{j=1}^{\infty} \lambda_j^{2m-2k} \|\hat{u}_j^{(2k)}\|^2 \right) \\
&= C \left(\sum_{k=1}^m \sum_{j=1}^{\infty} \lambda_j^{2m-2k} \|\hat{u}_j^{(2k)}\|^2 + \sum_{k=0}^{m-1} \sum_{j=1}^{\infty} \lambda_j^{2m-2k} \|\hat{u}_j^{(2k)}\|^2 \right) \\
&\leq C \sum_{k=0}^m \sum_{j=1}^{\infty} \lambda_j^{2m-2k} \|\hat{u}_j^{(2k)}\|^2 \\
&= C \sum_{k=0}^m \|\partial_x^{2k} \partial_y^{2m-2k} u\|_{L^2(\Omega)}^2.
\end{aligned} \tag{2.20}$$

Combining (2.18)-(2.20), the conclusion of Lemma 2.6 follows. \square

Lemma 2.7. *There exists a constant $C_4 \geq 1$ such that*

$$\lambda_j^{2m} \|P_j^i \hat{u}_j\|^2 \leq C_4^m \sum_{l=0}^m \binom{m}{l} \|P_j^{i+l} \hat{u}_j\|^2, \quad \forall m, i \in \mathbb{N}, \forall j \in \mathbb{N}^*, \forall u \in \mathcal{D}(A^{m+i}), \tag{2.21}$$

where \hat{u}_j is the Fourier coefficients of u as in (2.17).

Proof. The proof is by induction on m . For $m = 0$, (2.21) is obvious for any $C_4 \geq 1$.

For $m = 1$ and $u \in \mathcal{D}(A^{1+i})$, we have that $P^i u \in \mathcal{D}(A)$ and, by [19, Lemma 4.1],

$$(P^i u)(x, y) = \sum_{j=1}^{\infty} (P_j^i \hat{u}_j)(x) e_j(y),$$

where the function $P_j^i \hat{u}_j$ satisfies for each $j \in \mathbb{N}^*$

$$\begin{cases} (P_j^i \hat{u}_j)''' - (\lambda_j - a)(P_j^i \hat{u}_j)' = P_j^{i+1} \hat{u}_j, & x \in (-1, 0), \\ (P_j^i \hat{u}_j)(-1) = (P_j^i \hat{u}_j)(0) = (P_j^i \hat{u}_j)'(0) = 0. \end{cases} \tag{2.22}$$

Multiplying the first equation in (2.22) by $\lambda_j(x+1)P_j^i \hat{u}_j$ and integrating over $(-1, 0)$ results in

$$\frac{3}{2} \lambda_j \int_{-1}^0 |(P_j^i \hat{u}_j)'|^2 dx + (\lambda_j - a) \frac{\lambda_j}{2} \int_{-1}^0 |P_j^i \hat{u}_j|^2 dx = \lambda_j \int_{-1}^0 (x+1)(P_j^i \hat{u}_j)(P_j^{i+1} \hat{u}_j) dx.$$

After some elementary calculations, we can find a constant $C_4 = C_4(a) \geq 1$ such that

$$\lambda_j \|(P_j^i \hat{u}_j)'\|^2 + \lambda_j^2 \|P_j^i \hat{u}_j\|^2 \leq C_4 (\|P_j^i \hat{u}_j\|^2 + \|P_j^{i+1} \hat{u}_j\|^2).$$

Therefore, (2.21) holds for $m = 1$. Pick now any $m \geq 2$, and assume that (2.21) is true for $m - 1 \geq 0$. For any $u \in \mathcal{D}(A^{m+i})$, we have

$$\lambda_j^{2m} \|P_j^i \hat{u}_j\|^2 = \lambda_j^2 \lambda_j^{2m-2} \|P_j^i \hat{u}_j\|^2 \leq \lambda_j^2 C_4^{m-1} \sum_{l=0}^{m-1} \binom{m-1}{l} \|P_j^{i+l} \hat{u}_j\|^2.$$

Since $u \in \mathcal{D}(A^{m+i})$, for any $l = 0, 1, \dots, m-1$, system (2.22) is satisfied with $P_j^{i+l} \hat{u}_j$ substituted to $P_j^i \hat{u}_j$, and it follows as above that

$$\lambda_j^2 \|P_j^{i+l} \hat{u}_j\|^2 \leq C_4 (\|P_j^{i+l} \hat{u}_j\|^2 + \|P_j^{i+l+1} \hat{u}_j\|^2).$$

We infer that

$$\begin{aligned} \lambda_j^{2m} \|P_j^i \hat{u}_j\|^2 &\leq C_4^m \sum_{l=0}^{m-1} \binom{m-1}{l} (\|P_j^{i+l} \hat{u}_j\|^2 + \|P_j^{i+l+1} \hat{u}_j\|^2) \\ &= C_4^m (\|P_j^i \hat{u}_j\|^2 + \sum_{l=1}^{m-1} \binom{m-1}{l} \|P_j^{i+l} \hat{u}_j\|^2 + \sum_{l=1}^{m-1} \binom{m-1}{l-1} \|P_j^{i+l} \hat{u}_j\|^2 + \|P_j^{i+m} \hat{u}_j\|^2) \\ &= C_4^m (\|P_j^i \hat{u}_j\|^2 + \sum_{l=1}^{m-1} \binom{m}{l} \|P_j^{i+l} \hat{u}_j\|^2 + \|P_j^{i+m} \hat{u}_j\|^2) \\ &= C_4^m \sum_{l=0}^m \binom{m}{l} \|P_j^{i+l} \hat{u}_j\|^2 \end{aligned}$$

where we used Pascal's Rule. The proof of Lemma 2.7 is achieved. \square

We are in a position to complete the proof of Proposition 2.1. The estimate (2.6) is obvious for $n = 0$. Let $n \geq 1$. Using Lemmas 2.5, 2.6, and 2.7, we obtain that

$$\begin{aligned} \|u\|_{H^{2n}(\Omega)}^2 &\leq C_3 \sum_{m=0}^n \sum_{k=0}^m \|\partial_x^{2k} \partial_y^{2m-2k} u\|_{L^2(\Omega)}^2 \\ &= C_3 \sum_{m=0}^n \sum_{k=0}^m \sum_{j=1}^{\infty} \lambda_j^{2m-2k} \|\hat{u}_j^{(2k)}\|^2 \\ &\leq C_3 \sum_{m=0}^n \sum_{k=0}^m \sum_{j=1}^{\infty} \lambda_j^{2m-2k} C_1^k \sum_{i=0}^k \lambda_j^{2k-2i} \|P_j^i \hat{u}_j\|^2 \\ &\leq C_3 C_1^n \sum_{m=0}^n \sum_{k=0}^m \sum_{j=1}^{\infty} \sum_{i=0}^k \lambda_j^{2m-2i} \|P_j^i \hat{u}_j\|^2. \end{aligned}$$

Using the fact that $i \leq k \leq m \leq n$ in the sum above, we obtain

$$\begin{aligned}
\|u\|_{H^{2n}(\Omega)}^2 &\leq C_3 C_1^n \sum_{m=0}^n \sum_{k=0}^n \sum_{j=1}^{\infty} \sum_{i=0}^n \lambda_j^{2n-2i} \|P_j^i \hat{u}_j\|^2 \\
&\leq C_3 C_1^n (n+1)^2 \sum_{j=1}^{\infty} \sum_{i=0}^n \lambda_j^{2n-2i} \|P_j^i \hat{u}_j\|^2 \\
&\leq C_3 C_1^n (n+1)^2 \sum_{j=1}^{\infty} \sum_{i=0}^n C_4^{n-i} \sum_{l=0}^{n-i} \binom{n-i}{l} \|P_j^{i+l} \hat{u}_j\|^2 \\
&\leq C_3 C_1^n (n+1)^2 C_4^n 2^n \sum_{j=1}^{\infty} \sum_{i=0}^n \sum_{l=i}^n \|P_j^l \hat{u}_j\|^2 \\
&\leq C_3 C_1^n (n+1)^2 C_4^n 2^n \sum_{j=1}^{\infty} \sum_{i=0}^n \sum_{l=0}^n \|P_j^l \hat{u}_j\|^2 \\
&\leq C_3 C_1^n (n+1)^3 C_4^n 2^n \sum_{l=0}^n \sum_{j=1}^{\infty} \|P_j^l \hat{u}_j\|^2 \\
&\leq B^n \sum_{l=0}^n \|P^l u\|_{L^2(\Omega)}^2
\end{aligned}$$

with $B := 16C_1C_3C_4$. Indeed, it is easy to see that $(n+1)^3 \leq 8^n$ for all $n \in \mathbb{N}$. The proof of Proposition 2.1 is achieved. \square

Recall that $\lambda_j = (j\pi)^2$ for $j \geq 1$. For any $j \geq 1$, we consider a sequence of generating functions $g_{i,j}$ ($i \geq 0$), where $g_{0,j}$ is the solution of the Cauchy problem

$$\begin{cases} g_{0,j}'''(x) - (\lambda_j - a)g_{0,j}'(x) = 0, & x \in (-1, 0), \\ g_{0,j}(0) = g_{0,j}'(0) = 0, \quad g_{0,j}''(0) = 1, \end{cases} \quad (2.23)$$

while $g_{i,j}$ for $i \geq 1$ is defined inductively as the solution of the Cauchy problem

$$\begin{cases} g_{i,j}'''(x) - (\lambda_j - a)g_{i,j}'(x) = -g_{i-1,j}(x), & x \in (-1, 0), \\ g_{i,j}(0) = g_{i,j}'(0) = g_{i,j}''(0) = 0. \end{cases} \quad (2.24)$$

Proposition 2.2. *For any $i \geq 0, j \geq 1$ and $x \in [-1, 0]$, we have*

$$|g_{i,j}(x)| \leq e^{\sqrt{\lambda_j}} \frac{3^i i!}{(3i+2)!}. \quad (2.25)$$

Proof. It follows from (2.23) and (2.24) that

$$\begin{aligned}
g_{i,j}(x) &= - \int_0^x g_{0,j}(x-\xi) g_{i-1,j}(\xi) d\xi \\
&= - \int_0^x g_{0,j}''(x-\xi) \left(\int_0^\xi \left(\int_0^\zeta g_{i-1,j}(\sigma) d\sigma \right) d\zeta \right) d\xi, \quad i, j \geq 1.
\end{aligned}$$

(1) if $\lambda_j \leq a$, it is not difficult to obtain that

$$g_{0,j}(x) = \begin{cases} \frac{1}{a - \lambda_j} (1 - \cos(\sqrt{a - \lambda_j}x)), & \lambda_j < a; \\ \frac{1}{2}x^2, & \lambda_j = a, \end{cases}$$

this implies

$$0 \leq g_{0,j}(x) \leq \frac{x^2}{2}, \quad \forall j \geq 1, x \in [-1, 0].$$

Then it follows from [13, Lemma 2.1] that

$$|g_{i,j}(x)| \leq \frac{|x|^{3i+2}}{(3i+2)!} \leq e^{\sqrt{\lambda_j}} \frac{3^i i!}{(3i+2)!}, \quad \forall i \geq 0, \forall j \geq 1, \forall x \in [-1, 0].$$

(2) if $\lambda_j > a$, we claim that

$$g_{i,j}(x) \leq \cosh(\sqrt{\lambda_j - ax}) \frac{(-x)^{3i+2} 3^i i!}{(3i+2)!}, \quad \forall i \geq 0, \forall j \geq 1, \forall x \in [-1, 0] \quad (2.26)$$

which implies (2.25).

Let us prove (2.26) by induction on i . For $i = 0$,

$$\begin{aligned} 0 \leq g_{0,j}(x) &= \frac{1}{\lambda_j - a} (\cosh(\sqrt{\lambda_j - ax}) - 1) \\ &= \sum_{q=1}^{\infty} \frac{(\lambda_j - a)^{q-1} x^{2q}}{(2q)!} \\ &\leq \sum_{q=1}^{\infty} \frac{(\lambda_j - a)^{q-1} x^{2q-2}}{(2q-2)!} \frac{x^2}{2!} \\ &= \cosh(\sqrt{\lambda_j - ax}) \frac{x^2}{2!}, \end{aligned}$$

so that (2.26) is true for $i = 0$.

Assume now that (2.26) is true for $i - 1 \geq 0$. We can deduce that for $x \in [-1, 0]$

$$\begin{aligned} |g_{i,j}(x)| &\leq - \int_0^x g''_{0,j}(x - \xi) \left(\int_0^\xi \left(\int_0^\zeta |g_{i-1,j}(\sigma)| d\sigma \right) d\zeta \right) d\xi \\ &\leq - \int_0^x \sum_{p=0}^{\infty} \frac{(\lambda_j - a)^p (x - \xi)^{2p}}{(2p)!} \left(\int_0^\xi \left(\int_0^\zeta 3^{i-1} (i-1)! \sum_{q=0}^{\infty} \frac{(\lambda_j - a)^q (-\sigma)^{3i-1+2q}}{(2q)!(3i-1)!} d\sigma \right) d\zeta \right) d\xi \\ &= - 3^{i-1} (i-1)! \int_0^x \sum_{p=0}^{\infty} \frac{(\lambda_j - a)^p (x - \xi)^{2p}}{(2p)!} \sum_{q=0}^{\infty} \frac{(\lambda_j - a)^q (-\xi)^{3i+1+2q}}{(2q)!(3i-1)!(3i+2q)(3i+2q+1)} d\xi \\ &= - 3^{i-1} (i-1)! \int_0^x \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{(\lambda_j - a)^{p+q} (x - \xi)^{2p} (-\xi)^{3i+1+2q}}{(2p)!(2q)!(3i-1)!(3i+2q)(3i+2q+1)} d\xi. \end{aligned}$$

Then, integrating by parts $2p$ times, we obtain

$$\begin{aligned} |g_{i,j}(x)| &\leq -3^{i-1}(i-1)! \int_0^x \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{(\lambda_j - a)^{p+q} (-\xi)^{3i+1+2q+2p} (3i+2q-1)!}{(2q)!(3i-1)!(3i+1+2q+2p)!} d\xi \\ &= 3^{i-1}(i-1)! \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{(\lambda_j - a)^{p+q} (-x)^{3i+2+2q+2p} (3i+2q-1)!}{(2q)!(3i-1)!(3i+2+2q+2p)!}. \end{aligned}$$

Next, we will show that

$$\frac{3^{i-1}(i-1)!(3i+2q-1)!}{(2q)!(3i-1)!(3i+2+2q+2p)!} \leq \frac{3^i i!}{p+q+1} \frac{1}{(2p+2q)!(3i+2)!} \quad \forall p, q \geq 0, i \geq 1. \quad (2.27)$$

It is easy to see that (2.27) is equivalent to

$$\begin{aligned} \frac{(3i+2q-1)!}{(2q)!(3i-1)!} &\leq \frac{3i}{p+q+1} \frac{(3i+2+2q+2p)!}{(2p+2q)!(3i+2)!} \\ &= 6i \frac{(2p+2q+1)(2p+2q+3)(2p+2q+4) \cdots (2p+2q+3i+2)}{(3i+2)!}. \end{aligned} \quad (2.28)$$

Since the left hand side of (2.28) is independent of p and the right hand side of (2.28) is increasing in p , we only need to prove (2.27) for $p = 0$, namely, we need to show that

$$\frac{(3i+2q-1)!}{(3i-1)!} \leq \frac{3i}{q+1} \frac{(3i+2+2q)!}{(3i+2)!} \quad \forall q \geq 0, \forall i \geq 1,$$

this is obvious due to the fact that

$$\begin{aligned} \frac{(3i+2)!(3i+2q-1)!}{3i(3i-1)!(3i+2+2q)!} &= \frac{(3i+1)(3i+2)}{(3i+2q)(3i+2q+1)(3i+2q+2)} \\ &\leq \frac{1}{3i+2q} \\ &\leq \frac{1}{q+1}. \end{aligned}$$

Applying (2.27), we infer that

$$\begin{aligned} |g_{i,j}(x)| &\leq \frac{(-x)^{3i+2} 3^i i!}{(3i+2)!} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{(\lambda_j - a)^{p+q} x^{2p+2q}}{(p+q+1)(2p+2q)!} \\ &= \frac{(-x)^{3i+2} 3^i i!}{(3i+2)!} \sum_{k=0}^{\infty} \frac{(\lambda_j - a)^k x^{2k}}{(2k)!} \\ &= \cosh(\sqrt{\lambda_j - ax}) \frac{(-x)^{3i+2} 3^i i!}{(3i+2)!}, \end{aligned}$$

where we have used the fact that for any function $f : \mathbb{N} \rightarrow \mathbb{R}_+$, it holds

$$\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} f(p+q) = \sum_{k=0}^{\infty} (k+1) f(k).$$

This ends the proof of Proposition 2.2. □

Remark 2.3. Compared with the result in [13, Lemma 2.1], it seems that a more natural estimate of $g_{i,j}$ is

$$|g_{i,j}(x)| \leq \cosh(\sqrt{\lambda_j - a}) \frac{R^i (-x)^{3i+2}}{(3i+2)!}$$

for some constant $R > 0$. According to the proof of Proposition 2.2, to prove this result, we need to obtain that

$$\frac{(3i+2q-1)!}{(3i-1)!} \leq \frac{R}{q+1} \frac{(3i+2+2q)!}{(3i+2)!} \quad \forall q \geq 0, \forall i \geq 1.$$

This is equivalent to

$$\frac{(q+1)(3i)(3i+1)(3i+2)}{(3i+2q)(3i+2q+1)(3i+2q+2)} \leq R \quad \forall q \geq 0, \forall i \geq 1.$$

However, this is impossible if we pick $q = 3i$.

Using Proposition 2.2, we can obtain the following corollary which will be used in the proof of the main results.

Corollary 2.1. For any $i \geq 0, j \geq 1$ and $x \in [-1, 0]$, we have

$$|g_{i,j}(x)| \leq C e^{\sqrt{\lambda_j}} \frac{1}{(2i)!}, \quad (2.29)$$

where the constant C is independent of i and j .

Proof. By Stirling's formula $i! \sim (i/e)^i \sqrt{2\pi i}$, and it follows from (2.25) that for $i \geq 1$ and $j \geq 1$ we have

$$\begin{aligned} |g_{i,j}(x)| &\leq e^{\sqrt{\lambda_j}} \frac{3^i i!}{(3i+2)!} \\ &\leq C e^{\sqrt{\lambda_j}} \frac{3^i i!}{(3i+1)(3i+2) \frac{3^{3i}}{2^{2i}} \frac{\sqrt{6\pi i}}{\sqrt{2\pi i} \sqrt{4\pi i}} (2i)! i!} \\ &\leq C e^{\sqrt{\lambda_j}} \frac{1}{(2i)!}. \end{aligned}$$

□

3 Null controllability

Proposition 3.1. Let $s \in [0, 2)$, $0 < t_1 < t_2 \leq T$ and $z_j \in G^s([t_1, t_2])$ satisfy

$$|z_j^{(i)}(t)| \leq M_j \frac{(i!)^s}{R^i},$$

where R is a positive constant and the positive constants M_j are such that

$$\sum_{j=1}^{\infty} M_j e^{\sqrt{\lambda_j}} < \infty. \quad (3.1)$$

Then the function u defined by (1.9) solves system (1.4)-(1.8) and $u \in G^{\frac{s}{2}, \frac{s}{2}, s}([-1, 0] \times [0, 1] \times [t_1, t_2])$.

Proof. As the proof is similar to that of [13, Proposition 2.1], it is only sketched.

Let $m, p, q \in \mathbb{N}$. By applying Proposition 2.1 and (1.9), we obtain that

$$\begin{aligned}
|\partial_t^m \partial_x^p \partial_y^q u(x, y, t)| &\leq C \|\partial_t^m u(\cdot, \cdot, t)\|_{H^{p+q+2}(\Omega)} \\
&\leq CB^{[\frac{p+q+2}{2}]+1} \sum_{n=0}^{[\frac{p+q+2}{2}]+1} \|P^n \partial_t^m u(\cdot, \cdot, t)\|_{L^2(\Omega)} \\
&\leq CB^{[\frac{p+q+2}{2}]+1} \sum_{n=0}^{[\frac{p+q+2}{2}]+1} \sup_{(x,y) \in \Omega} |\partial_t^m P^n u(x, y, t)| \\
&\leq CB^{[\frac{p+q+2}{2}]+1} \sum_{n=0}^{[\frac{p+q+2}{2}]+1} \sup_{(x,y) \in \Omega} \sum_{j=1}^{\infty} \sum_{i=0}^{\infty} |\partial_t^m P^n (g_{i,j}(x) z_j^{(i)}(t) e_j(y))|.
\end{aligned}$$

By the definitions of $g_{i,j}$ and e_j , it is clear that

$$\partial_t^m P^n (g_{i,j}(x) z_j^{(i)}(t) e_j(y)) = \begin{cases} z_j^{(i+m)}(t) (-1)^n g_{i-n,j}(x) e_j(y), & i \geq n; \\ 0, & i < n. \end{cases}$$

Setting $k = i - n$ and $N = n + m$, arguing as in [13, Proposition 2.1], we infer from Corollary 2.1 that

$$\begin{aligned}
\sum_{j=1}^{\infty} \sum_{i=0}^{\infty} |\partial_t^m P^n (g_{i,j}(x) z_j^{(i)}(t) e_j(y))| &= \sum_{j=1}^{\infty} \sum_{i=n}^{\infty} |z_j^{(i+m)}(t) g_{i-n,j}(x) e_j(y)| \\
&\leq C \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} M_j \frac{(k+N)!^s}{R^{k+N}} e^{\sqrt{\lambda_j}} \frac{1}{(2k)!} \\
&\leq C \frac{(N!)^s}{(\frac{R}{2^s})^N} \\
&\leq C \frac{(n!)^s (m!)^s}{R_1^n R_2^m},
\end{aligned}$$

where $R_1 = R_2 = R/4^s$.

Gathering the above estimates together, we obtain that

$$\begin{aligned}
|\partial_t^m \partial_x^p \partial_y^q u(x, y, t)| &\leq CB^{[\frac{p+q+2}{2}]+1} \sum_{n=0}^{[\frac{p+q+2}{2}]+1} \sup_{(x,y) \in \Omega} \sum_{j=1}^{\infty} \sum_{i=0}^{\infty} |\partial_t^m P^n (g_{i,j}(x) z_j^{(i)}(t) e_j(y))| \\
&\leq CB^{[\frac{p+q+2}{2}]+1} \sum_{n=0}^{[\frac{p+q+2}{2}]+1} \frac{(n!)^s (m!)^s}{R_1^n R_2^m} \\
&\leq C \frac{(p!)^{\frac{s}{2}} (q!)^{\frac{s}{2}} (m!)^s}{\bar{R}_1^p \bar{R}_2^q \bar{R}_3^m}
\end{aligned}$$

for some positive constants $\bar{R}_1, \bar{R}_2, \bar{R}_3$. Finally, it is easily seen that u is indeed a solution of the ZK system. \square

Let \bar{u} denote the solution of the free evolution for the ZK system:

$$\begin{cases} \bar{u}_t + a\bar{u}_x + \Delta\bar{u}_x = 0, & (x, y) \in (-1, 0) \times (0, 1), t \in (0, T), \\ \bar{u}(-1, y, t) = \bar{u}(0, y, t) = \bar{u}_x(0, y, t) = 0, & y \in (0, 1), t \in (0, T), \\ \bar{u}(x, 0, t) = \bar{u}(x, 1, t) = 0, & x \in (-1, 0), t \in (0, T), \\ \bar{u}(x, y, 0) = u_0(x, y), & x \in (-1, 0), y \in (0, 1). \end{cases} \quad (3.2)$$

As for KdV, we have a Kato smoothing effect.

Proposition 3.2. *Let $u_0 \in L^2(\Omega)$.*

1. *System (3.2) admits a unique solution $\bar{u} \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$ and we have*

$$\sup_{t \in [0, T]} \|\bar{u}(\cdot, \cdot, t)\|_{L^2(\Omega)}^2 + \int_0^T \|\bar{u}(\cdot, \cdot, t)\|_{H^1(\Omega)}^2 dt \leq C \|u_0\|_{L^2(\Omega)}^2. \quad (3.3)$$

2. *If, in addition, $u_0 \in \mathcal{D}(A) \cap H^3(\Omega)$, then $\bar{u} \in C([0, T]; H^3(\Omega)) \cap L^2(0, T; H^4(\Omega))$ and we have*

$$\sup_{t \in [0, T]} \|\bar{u}(\cdot, \cdot, t)\|_{H^3(\Omega)}^2 + \int_0^T \|\bar{u}(\cdot, \cdot, t)\|_{H^4(\Omega)}^2 dt \leq C \|u_0\|_{H^3(\Omega)}^2. \quad (3.4)$$

Proof. (i) comes from [19]. Let us proceed with the proof of (ii). For any $u_0 \in \mathcal{D}(A) \cap H^3(\Omega)$, we have that $\bar{u} \in C([0, T]; \mathcal{D}(A))$ by the semigroup theory, and hence $\bar{u} \in C([0, T]; H^2(\Omega) \cap H_0^1(\Omega))$. Let $w_0 = Au_0$ and $w = A\bar{u}$. It is well known that w is the solution of (3.2) with initial value $w_0 \in L^2(\Omega)$. According to (i), we have

$$-\Delta\bar{u}_x - a\bar{u}_x = A\bar{u} = w \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega)).$$

Therefore $\Delta\bar{u}_x \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$. Assume finally that $u_0 \in \mathcal{D}(A) \cap H^3(\Omega)$, and let us prove that $u \in C([0, T], H^3(\Omega)) \cap L^2(0, T, H^4(\Omega))$. Decompose u as $u(x, y, t) = \sum_{j=1}^{\infty} \hat{u}_j(x, t)e_j(y)$. Then for $j \geq 1$, \hat{u}_j solves

$$\frac{d\hat{u}_j}{dt} + \hat{u}_j''' + (a - \lambda_j)\hat{u}_j' = 0, \quad (3.5)$$

$$\hat{u}_j(-1, t) = \hat{u}_j(0, t) = \hat{u}_j'(0, t) = 0, \quad (3.6)$$

$$\hat{u}_j(\cdot, 0) = \hat{u}_j^0, \quad (3.7)$$

where $u_0(x, y) = \sum_{j=1}^{\infty} \hat{u}_j^0(x)e_j(y)$. Multiplying in (3.5) by \hat{u}_j (resp. by $(x+1)\hat{u}_j$) and integrating over $(-1, 0)_x \times (0, T)_t$, we obtain respectively

$$\int_{-1}^0 |\hat{u}_j(x, T)|^2 dx + \int_0^T |\hat{u}_j'(-1, t)|^2 dt = \int_{-1}^0 |\hat{u}_j^0(x)|^2 dx, \quad (3.8)$$

$$\int_{-1}^0 (x+1)|\hat{u}_j(x, T)|^2 dx + 3 \int_0^T \int_{-1}^0 |\hat{u}_j'|^2 dx dt + (\lambda_j - a) \int_0^T \int_{-1}^0 |\hat{u}_j|^2 dx dt$$

$$= \int_{-1}^0 (x+1)|\hat{u}_j^0(x)|^2 dx. \quad (3.9)$$

It follows from (3.8) that for any $k \in \mathbb{N}$

$$\sum_{j=1}^{\infty} \lambda_j^k \|\hat{u}_j(\cdot, t)\|^2 \leq \sum_{j=1}^{\infty} \lambda_j^k \|\hat{u}_j^0\|^2, \quad \forall t \in \mathbb{R}_+ \quad (3.10)$$

(that is, $\|\partial_y^k u(\cdot, \cdot, t)\|_{L^2(\Omega)}^2 \leq \|\partial_y^k u_0\|_{L^2(\Omega)}^2$ for all $t \in \mathbb{R}_+$), and from (3.9) that

$$\int_0^T \sum_{j=1}^{\infty} (\lambda_j^k \|\hat{u}_j'(\cdot, t)\|^2 + \lambda_j^{k+1} \|\hat{u}_j(\cdot, t)\|^2) dt \leq (1+aT) \sum_{j=1}^{\infty} \lambda_j^k \|\hat{u}_j^0\|^2, \quad \forall T > 0 \quad (3.11)$$

(that is, $\int_0^T \|\nabla \partial_y^k u(\cdot, \cdot, t)\|_{L^2(\Omega)}^2 dt \leq (1+aT) \|\partial_y^k u_0\|_{L^2(\Omega)}^2$ for all $T > 0$). We need the following lemma.

Lemma 3.1. *Let $a \geq 0$ and $\lambda > 0$ be given. Let H^k ($k \in \mathbb{N}$) denote the Sobolev space $H^k(-1, 0)$, and let $\mathcal{H}^3 := \{u \in H^3(-1, 0); u(-1) = u(0) = u'(0) = 0\}$. Let $\|\cdot\|$ denote the norm $\|\cdot\|_{L^2(-1, 0)}$.*

1. *There exists a constant $C > 0$ such that*

$$\sum_{k=0}^3 \lambda^k \|\partial_x^{3-k} y\|^2 \leq C (\|y'''\|^2 + (a-\lambda)y'\|^2 + \lambda^3 \|y\|^2) \quad \forall y \in \mathcal{H}^3, \forall \lambda \geq \lambda_0. \quad (3.12)$$

2. *There exists a constant $C' > 0$ such that*

$$\sum_{k=0}^4 \lambda^k \|\partial_x^{4-k} y\|^2 \leq C' \left(\|y^{(4)}\|^2 + (a-\lambda)y''\|^2 + \|y'''\|^2 + (a-\lambda)y'\|^2 + \lambda^4 \|y\|^2 \right) \\ \forall y \in \mathcal{H}^3 \cap H^4, \forall \lambda \geq \lambda_0. \quad (3.13)$$

Proof of Lemma 3.1: 1. Pick any $y \in \mathcal{H}^3$ and any $\lambda \geq 0$. By the Interpolation Theorem and Young inequality, we have that

$$\lambda^2 \|y'\|^2 \leq C \lambda^2 \|y\|^{\frac{4}{3}} \|y'''\|^{\frac{2}{3}} \leq \varepsilon \|y'''\|^2 + C_\varepsilon \lambda^3 \|y\|^2, \\ \lambda \|y''\|^2 \leq C \lambda \|y\|^{\frac{2}{3}} \|y'''\|^{\frac{4}{3}} \leq \varepsilon \|y'''\|^2 + C'_\varepsilon \lambda^3 \|y\|^2.$$

We infer that if $\lambda \geq \lambda_0 > 0$

$$\|y'''\|^2 \leq 2\|y'''\|^2 + (a-\lambda)y'\|^2 + 2(a-\lambda)^2 \|y'\|^2 \\ \leq 2\|y'''\|^2 + (a-\lambda)y''\|^2 + 2\varepsilon \|y'''\|^2 + 2C_\varepsilon |a-\lambda|^3 \|y\|^2 \\ \leq 2\|y'''\|^2 + (a-\lambda)y'\|^2 + 2\varepsilon \|y'''\|^2 + C''_\varepsilon \lambda^3 \|y\|^2$$

and (3.12) follows by picking $\varepsilon < 1/4$.

2. Pick now any $y \in \mathcal{H}^3 \cap H^4$ and any $\lambda \geq 0$. Then we have

$$\lambda^3 \|y'\|^2 \leq C \lambda^3 \|y\|^{\frac{3}{2}} \|y\|_{H^4}^{\frac{1}{2}} \leq \varepsilon (\|y^{(4)}\|^2 + \|y'''\|^2) + C_\varepsilon \lambda^4 \|y\|^2,$$

$$\begin{aligned}\lambda^2 \|y''\|^2 &\leq C\lambda^2 \|y\| \|y\|_{H^4} \leq \varepsilon(\|y^{(4)}\|^2 + \|y'''\|^2) + C'_\varepsilon \lambda^4 \|y\|^2, \\ \lambda \|y'''\|^2 &\leq C\lambda \|y\|^{\frac{1}{2}} \|y\|_{H^4}^{\frac{3}{2}} \leq \varepsilon(\|y^{(4)}\|^2 + \|y'''\|^2) + C''_\varepsilon \lambda^4 \|y\|^2.\end{aligned}$$

On the other hand, we have that for $\lambda \geq \lambda_0 > 0$

$$\begin{aligned}\|y^{(4)}\|^2 &\leq 2\|y^{(4)} + (a - \lambda)y''\|^2 + 2(a - \lambda)^2 \|y''\|^2 \\ &\leq 2\|y^{(4)} + (a - \lambda)y''\|^2 + 2\varepsilon(\|y^{(4)}\|^2 + \|y'''\|^2) + C'''_\varepsilon \lambda^4 \|y\|^2,\end{aligned}$$

and (3.13) follows by picking $\varepsilon < 1/4$ and by using (3.12).

Assuming that $u_0 \in \mathcal{D}(A) \cap H^3(\Omega)$ and using (3.10) and (3.12), we obtain that for any $t \in [0, T]$ (with a constant C that may vary from line to line)

$$\begin{aligned}\|u(\cdot, \cdot, t)\|_{H^3(\Omega)}^2 &= \|u(\cdot, \cdot, t)\|_{H^2(\Omega)}^2 + \sum_{k=0}^3 \|\partial_y^k \partial_x^{3-k} u(\cdot, \cdot, t)\|_{L^2(\Omega)}^2 \\ &\leq C\|u_0\|_{\mathcal{D}(A)}^2 + \sum_{k=0}^3 \sum_{j=1}^{\infty} \lambda_j^k \|\partial_x^{3-k} \hat{u}_j(\cdot, t)\|^2 \\ &\leq C\|u_0\|_{\mathcal{D}(A)}^2 + C \sum_{k=0}^3 \sum_{j=1}^{\infty} (\|\hat{u}_j''(\cdot, t) + (a - \lambda_j)\hat{u}_j'(\cdot, t)\|^2 + \lambda_j^3 \|\hat{u}_j(\cdot, t)\|^2) \\ &\leq C\|u_0\|_{\mathcal{D}(A)}^2 + C\|\partial_y^3 u_0\|_{L^2(\Omega)}^2 \\ &\leq C\|u_0\|_{H^3(\Omega)}^2.\end{aligned}$$

On the other hand $\|u(\cdot, \cdot, t)\|_{H^4(\Omega)}^2 = \|u(\cdot, \cdot, t)\|_{H^3(\Omega)}^2 + \sum_{k=0}^4 \|\partial_y^k \partial_x^{4-k} u(\cdot, \cdot, t)\|_{L^2(\Omega)}^2$ and it is clear

that $\int_0^T \|u(\cdot, \cdot, t)\|_{H^3(\Omega)}^2 dt \leq C\|u_0\|_{H^3(\Omega)}^2$. Using (3.13), we obtain

$$\begin{aligned}&\int_0^T \sum_{k=0}^4 \|\partial_y^k \partial_x^{4-k} u(\cdot, \cdot, t)\|_{L^2(\Omega)}^2 dt \\ &= \int_0^T \sum_{k=0}^4 \sum_{j=1}^{\infty} \lambda_j^k \|\partial_x^{4-k} \hat{u}_j(\cdot, t)\|^2 dt \\ &\leq C \int_0^T \sum_{j=1}^{\infty} (\|\hat{u}_j^{(4)} + (a - \lambda_j)\hat{u}_j''\|^2 + \|\hat{u}_j''' + (a - \lambda_j)\hat{u}_j'\|^2 + \lambda_j^4 \|\hat{u}_j\|^2) dt \\ &\leq C \int_0^T (\|Au(\cdot, \cdot, t)\|_{H^1(\Omega)}^2 + \|\partial_y^4 u(\cdot, \cdot, t)\|_{L^2(\Omega)}^2) dt \\ &\leq C\|u_0\|_{H^3(\Omega)}^2\end{aligned}$$

where we used (3.11) with $k = 3$. This completes the proof of the proposition. \square

Interpolating between (3.3) and (3.4), we obtain

$$\begin{aligned} \sup_{t \in [0, T]} \|\bar{u}(\cdot, \cdot, t)\|_{H^1(\Omega)}^2 + \int_0^T \|\bar{u}(\cdot, \cdot, t)\|_{H^2(\Omega)}^2 dt &\leq C \|u_0\|_{H^1(\Omega)}^2, \\ \sup_{t \in [0, T]} \|\bar{u}(\cdot, \cdot, t)\|_{H^2(\Omega)}^2 + \int_0^T \|\bar{u}(\cdot, \cdot, t)\|_{H^3(\Omega)}^2 dt &\leq C \|u_0\|_{H^2(\Omega)}^2. \end{aligned}$$

This gives

$$\|\bar{u}(\cdot, \cdot, t)\|_{H^{n+1}(\Omega)} \leq \frac{C}{\sqrt{t}} \|u_0\|_{H^n(\Omega)}, \quad \text{for } n \in \{0, 1, 2, 3\}.$$

Proceeding as in [13, Proposition 2.2], we can show that if $u_0 \in L^2(\Omega)$, then $\bar{u}(t) \in \mathcal{D}(A^n)$ for any $t \in (0, T]$ and $n \in \mathbb{N}$, and it holds

$$\|A^n \bar{u}(\cdot, \cdot, t)\|_{L^2(\Omega)} \leq \frac{C^n}{t^{\frac{3n}{2}}} n^{\frac{3n}{2}} \|u_0\|_{L^2(\Omega)}. \quad (3.14)$$

Without loss of generality, we assume that $T = 1$. Then for any $p, q \in \mathbb{N}$, we infer from Proposition 2.1 that

$$\begin{aligned} |\partial_x^p \partial_y^q \bar{u}(x, y, t)| &\leq \|\bar{u}(\cdot, \cdot, t)\|_{H^{p+q+2}(\Omega)} \\ &\leq C_0 B^{[\frac{p+q+2}{2}]+1} \sum_{n=0}^{[\frac{p+q+2}{2}]+1} \|P^n \bar{u}(\cdot, \cdot, t)\|_{L^2(\Omega)} \\ &\leq C_0 B^{[\frac{p+q+2}{2}]+1} \sum_{n=0}^{[\frac{p+q+2}{2}]+1} \frac{C^n}{t^{\frac{3n}{2}}} n^{\frac{3n}{2}} \|u_0\|_{L^2(\Omega)} \\ &\leq C t^{-\frac{3}{2}[\frac{p+q}{2}]-3} \frac{(p!)^{\frac{3}{4}} (q!)^{\frac{3}{4}}}{R_1^p R_2^q} \end{aligned}$$

for some $R_1, R_2 > 0$. This means that $\bar{u}(\cdot, \cdot, t) \in G^{\frac{3}{4}, \frac{3}{4}}([-1, 0] \times [0, 1])$ for any $t \in (0, T]$.

Let

$$f_j(t) := \int_0^1 e_j(y) \partial_x^2 \bar{u}(0, y, t) dy.$$

Lemma 3.2. *For any $j \geq 1$ and $n \geq 0$, there exist positive constants R_1, R_2 and C such that*

$$|f_j^{(n)}(t)| \leq \frac{C}{(j\pi)^j} t^{-\frac{3}{2}(n+[\frac{j}{2}]+3)} \frac{(n!)^{\frac{3}{2}} (j!)^{\frac{3}{4}}}{R_1^n R_2^j}.$$

Proof. Without loss of generality, we can assume that $T = 1$. Since $\bar{u}(\cdot, \cdot, t) \in \mathcal{D}(A^n)$ for any $t \in (0, T]$ and $n \in \mathbb{N}$, it follows from Lemma 2.2 that

$$\partial_x^2 \partial_y^{2n} \bar{u}(x, 0, t) = \partial_x^2 \partial_y^{2n} \bar{u}(x, 1, t) = 0, \quad \forall x \in [-1, 0], \forall t \in (0, T], \forall n \in \mathbb{N}.$$

Then, integrating by parts j -times, we deduce that

$$\begin{aligned}
f_j(t) &= \sqrt{2} \int_0^1 \sin(j\pi y) \partial_x^2 \bar{u}(0, y, t) dy \\
&= \frac{\sqrt{2}}{j\pi} \int_0^1 \cos(j\pi y) \partial_x^2 \partial_y \bar{u}(0, y, t) dy \\
&= \frac{\sqrt{2}}{(j\pi)^2} \int_0^1 \sin(j\pi y) \partial_x^2 \partial_y^2 \bar{u}(0, y, t) dy \\
&= \begin{cases} \frac{\sqrt{2}}{(j\pi)^j} \int_0^1 \sin(j\pi y) \partial_x^2 \partial_y^j \bar{u}(0, y, t) dy, & \text{if } j \text{ is even;} \\ -\frac{\sqrt{2}}{(j\pi)^j} \int_0^1 \cos(j\pi y) \partial_x^2 \partial_y^j \bar{u}(0, y, t) dy, & \text{if } j \text{ is odd.} \end{cases}
\end{aligned} \tag{3.15}$$

To estimate $|f_j^{(n)}(t)|$ ($n \in \mathbb{N}$), it remains to estimate $|\partial_t^n \partial_x^2 \partial_y^j \bar{u}(0, y, t)|$. Let

$$l = \lceil \frac{j+4}{2} \rceil + 1.$$

Taking (2.6) (with $u = P^i \bar{u}$) and (3.14) into account, we obtain that

$$\begin{aligned}
|\partial_t^n \partial_x^2 \partial_y^j \bar{u}(x, y, t)| &= |P^n \partial_x^2 \partial_y^j \bar{u}(x, y, t)| \\
&\leq C \|P^n \bar{u}(\cdot, \cdot, t)\|_{H^{j+4}(\Omega)} \\
&\leq CB^l \sum_{k=0}^l \|P^{n+k} \bar{u}(\cdot, \cdot, t)\|_{L^2(\Omega)} \\
&\leq CB^l \sum_{k=0}^{n+l} \|P^k \bar{u}(\cdot, \cdot, t)\|_{L^2(\Omega)} \\
&\leq CB^l \sum_{k=0}^{n+l} \frac{C^k k^{\frac{3}{2}k}}{t^{\frac{3}{2}k}} \|u_0\|_{L^2(\Omega)} \\
&\leq CB^l \frac{C^{n+l} (n+l+1) (n+l)^{\frac{3}{2}(n+l)}}{t^{\frac{3}{2}(n+l)}} \|u_0\|_{L^2(\Omega)} \\
&\leq Ct^{-\frac{3}{2}(n+\lceil \frac{j}{2} \rceil + 3)} \frac{(n!)^{\frac{3}{2}} (j!)^{\frac{3}{4}}}{R_1^n R_2^j} \|u_0\|_{L^2(\Omega)}
\end{aligned} \tag{3.16}$$

for some $R_1, R_2 > 0$.

Combining (3.15) and (3.16), we obtain

$$\begin{aligned}
|f_j^{(n)}(t)| &\leq \frac{C}{(j\pi)^j} \sup_{y \in [0,1]} |\partial_t^n \partial_x^2 \partial_y^j \bar{u}(0, y, t)| \\
&\leq \frac{C}{(j\pi)^j} t^{-\frac{3}{2}(n+\lceil \frac{j}{2} \rceil + 3)} \frac{(n!)^{\frac{3}{2}} (j!)^{\frac{3}{4}}}{R_1^n R_2^j}.
\end{aligned}$$

□

Now, we can prove the first main result in this paper.

Proof of Theorem 1.1. Pick any $\tau \in (0, T)$, $s \in [3/2, 2)$ and let

$$z_j(t) = \phi_s\left(\frac{t-\tau}{T-\tau}\right) f_j(t), \quad 0 \leq t \leq T,$$

where

$$\phi_s(\rho) = \begin{cases} 1 & \text{if } \rho \leq 0, \\ 0 & \text{if } \rho \geq 1, \\ \frac{e^{-\frac{M}{(1-\rho)^\sigma}}}{e^{-\frac{M}{\rho^\sigma}} + e^{-\frac{M}{(1-\rho)^\sigma}}} & \text{if } \rho \in (0, 1) \end{cases}$$

with $M > 0$ and $\sigma = (s-1)^{-1}$. As ϕ_s is Gevrey of order s , there exist $R_\phi > 0$ such that

$$|\phi_s^{(p)}(\rho)| \leq C \frac{(p!)^s}{R_\phi^p} \quad \forall p \in \mathbb{N}, \rho \in \mathbb{R}.$$

Then, applying Lemma 3.2, for any $\varepsilon \in (0, T)$ and $t \in [\varepsilon, T]$, we have

$$\begin{aligned} |z_j^{(i)}(t)| &\leq \sum_{n=0}^i \binom{i}{n} \left| \partial_t^{i-n} \left[\phi_s\left(\frac{t-\tau}{T-\tau}\right) \right] \right| |f_j^{(n)}(t)| \\ &\leq C \sum_{n=0}^i \binom{i}{n} \frac{(i-n)!^s}{R_\phi^{i-n}} \left(\frac{1}{T-\tau}\right)^{i-n} \frac{1}{(j\pi)^j} t^{-\frac{3}{2}(n+[\frac{j}{2}]+3)} \frac{(n!)^{\frac{3}{2}}(j!)^{\frac{3}{4}}}{R_1^n R_2^j} \\ &\leq C \frac{1}{(j\pi)^j} \varepsilon^{-\frac{3}{2}([\frac{j}{2}]+3)} \frac{(j!)^{\frac{3}{4}}}{R_2^j} \sum_{n=0}^i \binom{i}{n} \frac{(i-n)!^s}{R_\phi^{i-n}} \left(\frac{1}{T-\tau}\right)^{i-n} \varepsilon^{-\frac{3}{2}n} \frac{(n!)^{\frac{3}{2}}}{R_1^n} \\ &\leq C \frac{1}{(j\pi)^j} \varepsilon^{-\frac{3}{2}([\frac{j}{2}]+3)} \frac{(j!)^{\frac{3}{4}}}{R_2^j} \frac{(i!)^s}{\min\{R_\phi, R_1\}^i} \sum_{n=0}^i \binom{i}{n} \left(\frac{1}{T-\tau}\right)^{i-n} \varepsilon^{-\frac{3}{2}n} \\ &\leq M_j \frac{(i!)^s}{\widehat{R}^i}, \end{aligned}$$

where M_j satisfies (3.1). Let

$$u(x, y, t) = \begin{cases} u_0(x, y) & \text{if } x \in [-1, 0], y \in [0, 1], t = 0, \\ \sum_{j=1}^{\infty} \sum_{i=0}^{\infty} g_{i,j}(x) z_j^{(i)}(t) e_j(y) & \text{if } x \in [-1, 0], y \in [0, 1], t \in (0, T]. \end{cases}$$

Then, it is easy to see that $u(\cdot, \cdot, T) = 0$. By Proposition 3.1, $u \in G^{\frac{s}{2}, \frac{s}{2}, s}([-1, 0] \times [0, 1] \times [\varepsilon, T])$ for any $\varepsilon \in (0, T)$. Furthermore, we have

$$\begin{aligned} u_t + au_x + \Delta u_x &= 0 = \bar{u}_t + a\bar{u}_x + \Delta \bar{u}_x \quad \text{in } \Omega \times (0, T), \\ u(0, y, t) &= 0 = \bar{u}(0, y, t), \quad \forall y \in [0, 1], \forall t \in (0, \tau), \\ \partial_x u(0, y, t) &= 0 = \partial_x \bar{u}(0, y, t), \quad \forall y \in [0, 1], \forall t \in (0, \tau), \\ \partial_x^2 u(0, y, t) &= \sum_{j=1}^{\infty} z_j(t) e_j(y) = \partial_x^2 \bar{u}(0, y, t), \quad \forall y \in [0, 1], \forall t \in (0, \tau). \end{aligned}$$

It follows from Holmgren theorem that $u(x, y, t) = \bar{u}(x, y, t)$ for any $(x, y, t) \in [-1, 0] \times [0, 1] \times (0, \tau)$. In particular, $u \in C([0, T]; L^2(\Omega))$ and $h = 0$ for $t \in [0, \tau)$, so that $h \in G^{\frac{s}{2}, s}([0, 1] \times [0, T])$. The proof of Theorem 1.1 is complete. \square

4 Reachable states

Proposition 4.1. *For any $j \geq 1$, assume that $z_j \in G^2([0, T])$ is such that*

$$|z_j^{(i)}(t)| \leq M_j \frac{(2i)!}{R^{2i}}, \quad \forall i \geq 0, t \in [0, T],$$

where $R > 1$ and M_j satisfies (3.1). Then the function u defined by (1.9) solves system (1.4)-(1.8) and $u \in G^{1,1,2}([-1, 0] \times [0, 1] \times [0, T])$.

Proof. According to the proof of Proposition 3.1, for any $m, p, q \in \mathbb{N}$, we have

$$|\partial_t^m \partial_x^p \partial_y^q u(x, y, t)| \leq C B^{\lceil \frac{p+q+2}{2} \rceil + 1} \sum_{n=0}^{\lceil \frac{p+q+2}{2} \rceil + 1} \sup_{(x,y) \in \Omega} \sum_{j=1}^{\infty} \sum_{i=n}^{\infty} |z_j^{(i+m)}(t) g_{i-n,j}(x) e_j(y)|.$$

Let $k = 2i - 2n$ and $N = 2n + 2m$. We can obtain by the same arguments as in [13, Proposition 3.1] that

$$\begin{aligned} \sum_{j=1}^{\infty} \sum_{i=n}^{\infty} |z_j^{(i+m)}(t) g_{i-n,j}(x) e_j(y)| &\leq \sum_{j=1}^{\infty} \sum_{i=n}^{\infty} M_j \frac{(2i + 2m)!}{R^{2i+2m}} \frac{C e^{\sqrt{\lambda_j}}}{(2i - 2n)!} \\ &= \sum_{j=1}^{\infty} C M_j e^{\sqrt{\lambda_j}} \sum_{k=0}^{\infty} \frac{(k + N)!}{R^{k+N} k!} \\ &\leq C \sum_{k=0}^{\infty} \frac{(k + N)!}{R^{k+N} k!} \\ &= C \sum_{k=0}^{\infty} \frac{(k + 1) \cdots (k + N)}{R^{k+N}} \\ &\leq C \left(\frac{\alpha e}{R^\sigma} \right)^N N! \sqrt{N} \\ &\leq C \frac{(2n)!(2m)!}{R_1^n R_2^m}, \end{aligned}$$

where R_1, R_2 are two positive constants, $\sigma \in (0, 1)$ and

$$\alpha = \sup_{k \geq 0} \frac{k + 2}{(R^{1-\sigma})^{k+1}}.$$

It follows from the above estimates that

$$\begin{aligned} |\partial_t^m \partial_x^p \partial_y^q u(x, y, t)| &\leq C B^{\lceil \frac{p+q+2}{2} \rceil + 1} \sum_{n=0}^{\lceil \frac{p+q+2}{2} \rceil + 1} \frac{(2n)!(2m)!}{R_1^n R_2^m} \\ &\leq C \frac{p!q!(m!)^2}{\widehat{R}_1^p \widehat{R}_2^q \widehat{R}_3^m} \end{aligned}$$

for some positive constants $\widehat{R}_1, \widehat{R}_2$ and \widehat{R}_3 . This ends the proof of Proposition 4.1. \square

As a particular case of [12, Proposition 3.6] (with $a_0 = 1, a_p = [2p(2p - 1)]^{-1}$ for $p \geq 1$), we have the following result.

Proposition 4.2. *Let $\{d_q\}_{q \geq 0}$ be a sequence of real numbers such that*

$$|d_q| \leq CH^q(2q)! \quad \forall q \geq 0$$

for some $H > 0$ and $C > 0$. Then for all $\widetilde{H} > e^{e^{-1}}H$, there exists a function $f \in C^\infty(\mathbb{R})$ such that

$$\begin{aligned} f^{(q)}(0) &= d_q \quad \forall q \geq 0, \\ |f^{(q)}(x)| &\leq C\widetilde{H}^q(2q)! \quad \forall q \geq 0, x \in \mathbb{R}. \end{aligned}$$

Let

$$\begin{aligned} \mathcal{X} &:= \{u \in C^\infty([-1, 0] \times [0, 1]); \\ &P^n u(0, y) = \partial_x P^n u(0, y) = P^n u(x, 0) = P^n u(x, 1) = 0, \quad \forall n \in \mathbb{N}, \forall x \in [-1, 0], \forall y \in [0, 1]\}. \end{aligned}$$

A result similar to Lemma 2.2 can be derived.

Lemma 4.1. *For any $n \in \mathbb{N}$, we have*

$$\partial_y^{2n} f(x, 0) = \partial_y^{2n} f(x, 1) = 0, \quad \forall f \in \mathcal{X}, \forall x \in [-1, 0]. \quad (4.1)$$

Proof. We proceed by induction on n . For $n = 0$, (4.1) is obvious since $f \in \mathcal{X}$. Assume now that (4.1) is true for $n - 1 \geq 0$. If $f \in \mathcal{X}$, then $Pf \in \mathcal{X}$, so that by the induction hypothesis

$$\partial_y^{2n-2} Pf(x, 0) = \partial_y^{2n-2} Pf(x, 1) = 0.$$

This implies

$$\begin{aligned} \partial_x^3 \partial_y^{2n-2} f(x, 0) + \partial_x \partial_y^{2n} f(x, 0) + a \partial_x \partial_y^{2n-2} f(x, 0) &= 0, \\ \partial_x^3 \partial_y^{2n-2} f(x, 1) + \partial_x \partial_y^{2n} f(x, 1) + a \partial_x \partial_y^{2n-2} f(x, 1) &= 0. \end{aligned}$$

Since (4.1) is true for $n - 1$, we obtain that

$$\partial_x \partial_y^{2n} f(x, 0) = \partial_x \partial_y^{2n} f(x, 1) = 0.$$

This means that for some constants C_1 and C_2 ,

$$\partial_y^{2n} f(x, 0) = C_1, \quad \partial_y^{2n} f(x, 1) = C_2 \quad \forall x \in [-1, 0].$$

On the other hand, we infer from the assumption $f \in \mathcal{X}$ that

$$\partial_y^{2n} f(0, y) = 0 \quad \forall y \in [0, 1].$$

Taking $y = 0$ and next $y = 1$, we see that $C_1 = C_2 = 0$. The proof of Lemma 4.1 is complete. \square

Lemma 4.2. *If $f \in \mathcal{X}$ is such that*

$$\int_0^1 e_l(y) P^n f(0, y) dy = \int_0^1 e_l(y) \partial_x P^n f(0, y) dy = \int_0^1 e_l(y) \partial_x^2 P^n f(0, y) dy = 0 \quad (4.2)$$

for any $l \geq 1$ and any $n \geq 0$, then

$$\int_0^1 e_l(y) \partial_x^m f(0, y) dy = 0 \quad (4.3)$$

holds for any $l \geq 1$ and any $m \geq 0$.

Proof. To prove that (4.3) holds for any $l \geq 1$ and any $m \geq 0$, it is sufficient to show that for any $M \in \mathbb{N}$, (4.3) holds for any $l \geq 1$ and any $m \leq 3M + 2$. We proceed by induction on M .

For $M = 0$, we can take $n = 0$ in (4.2) to see that (4.3) holds for any $l \geq 1$ and $m \leq 2$.

Assume that (4.3) is true for any $l \geq 1$ and any $m \leq 3M - 1$. We claim that (4.3) holds for any $l \geq 1$ and $m = 3M, 3M + 1, 3M + 2$. Indeed, taking $n = M$ in (4.2), we have

$$\begin{aligned} 0 &= (-1)^M \int_0^1 e_l(y) P^M f(0, y) dy \\ &= \int_0^1 e_l(y) (\partial_x^2 + \partial_y^2 + a)^M \partial_x^M f(0, y) dy \\ &= \int_0^1 e_l(y) \partial_x^{3M} f(0, y) dy \\ &\quad + \int_0^1 e_l(y) \sum_{k=0}^{M-1} \binom{M}{k} \sum_{i=0}^{M-k} \binom{M-k}{i} a^{M-k-i} \partial_x^{2k+M} \partial_y^{2i} f(0, y) dy. \end{aligned} \quad (4.4)$$

Since $f \in \mathcal{X}$, it follows from Lemma 4.1 that

$$\partial_y^{2n} f(x, 0) = \partial_y^{2n} f(x, 1) = 0, \quad \forall x \in [-1, 0], \quad \forall n \in \mathbb{N}.$$

Then, we obtain by integrations by parts that for $k \in \{0, \dots, M - 1\}$ and $i \in \{0, \dots, M - k\}$

$$\int_0^1 e_l(y) \partial_x^{2k+M} \partial_y^{2i} f(0, y) dy = (-1)^i (l\pi)^{2i} \int_0^1 e_l(y) \partial_x^{2k+M} f(0, y) dy = 0.$$

In the last step, we used the fact that $2k + M \leq 3M - 1$. Thus, we infer from (4.4) that

$$\int_0^1 e_l(y) \partial_x^{3M} f(0, y) dy = 0, \quad \forall l \geq 1.$$

We can show in the same way that (4.3) is true for $m = 3M + 1, 3M + 2$ by using the fact that

$$\int_0^1 e_l(y) \partial_x P^M f(0, y) dy = \int_0^1 e_l(y) \partial_x^2 P^M f(0, y) dy = 0, \quad \forall l \geq 1.$$

The proof of Lemma 4.2 is complete. \square

Now, we are in a position to prove the second main result in this paper.

Proof of Theorem 1.2. Assume that $R := \min\{R_1, R_2\} > R_0 = \sqrt[3]{9(a+2)e^{(3e)^{-1}}}$ and pick any $u_1 \in \mathcal{R}_{R_1, R_2}$. We intend to expand u_1 in the following form:

$$u_1(x, y) = \sum_{j=1}^{\infty} \sum_{i=0}^{\infty} b_{i,j} g_{i,j}(x) e_j(y),$$

where

$$b_{i,j} = (-1)^i \int_0^1 e_j(y) \partial_x^2 P^i u_1(0, y) dy.$$

Since $u_1 \in \mathcal{R}_{R_1, R_2} \subset \mathcal{X}$, we have that $P^i u_1 \in \mathcal{X}$ for any $i \in \mathbb{N}$. By Lemma 4.1, we infer that

$$\partial_y^{2n} P^i u_1(x, 0) = \partial_y^{2n} P^i u_1(x, 1) = 0, \quad \forall x \in [-1, 0].$$

Then, by integration by parts, we have

$$|b_{i,j}| = \left| \int_0^1 e_j(y) \partial_x^2 P^i u_1(0, y) dy \right| \leq \frac{C}{(j\pi)^j} \sup_{(x,y) \in \Omega} |\partial_x^2 \partial_y^j P^i u_1(x, y)|.$$

Next, we estimate $|\partial_x^2 \partial_y^j P^i u_1(x, y)|$.

$$\begin{aligned} |\partial_x^2 \partial_y^j P^i u_1(x, y)| &= \left| \partial_x^2 \partial_y^j \sum_{n=0}^i \binom{i}{n} (\partial_x^2 + \partial_y^2)^n \partial_x (a \partial_x)^{i-n} u_1(x, y) \right| \\ &= \left| \partial_x^{i+2} \partial_y^j \sum_{n=0}^i \binom{i}{n} a^{i-n} (\partial_x^2 + \partial_y^2)^n u_1(x, y) \right| \\ &= \left| \partial_x^{i+2} \partial_y^j \sum_{n=0}^i \binom{i}{n} a^{i-n} \sum_{m=0}^n \binom{n}{m} \partial_x^{2m} \partial_y^{2n-2m} u_1(x, y) \right| \\ &\leq \sum_{n=0}^i \sum_{m=0}^n \binom{i}{n} \binom{n}{m} a^{i-n} |\partial_x^{2m+i+2} \partial_y^{2n-2m+j} u_1(x, y)| \\ &\leq C \sum_{n=0}^i \sum_{m=0}^n \binom{i}{n} \binom{n}{m} a^{i-n} \frac{(2m+i+2)^{\frac{2}{3}} (2n-2m+j)^{\frac{2}{3}}}{R_1^{2m+i+2} R_2^{2n-2m+j}} \\ &\leq C \sum_{n=0}^i \sum_{m=0}^n \binom{i}{n} \binom{n}{m} a^{i-n} \frac{(2m+i+2)^{\frac{2}{3}} (2n-2m+j)^{\frac{2}{3}}}{R^{2n+i+j+2}}. \end{aligned}$$

We notice that

$$\begin{aligned} (2m+i+2)!(2n-2m+j)! &= \binom{2m+i+2}{2} \binom{2n-2m+j}{j} 2! j! (2m+i)! (2n-2m)! \\ &\leq \binom{2m+i+2}{2} \binom{2n-2m+j}{j} 2! j! (2n+i)!, \end{aligned}$$

where we used the fact that

$$\binom{2n+i}{2m+i} = \frac{(2n+i)!}{(2m+i)!(2n-2m)!} \geq 1.$$

According to [9, Lemma A.1], we have

$$\binom{2m+i+2}{2} \binom{2n-2m+j}{j} \leq \binom{2n+i+j+2}{j+2}.$$

This implies

$$\begin{aligned} (2m+i+2)!(2n-2m+j)! &\leq \binom{2n+i+j+2}{j+2} 2!j!(2n+i)! \\ &= \frac{(2n+i+j+2)!2!j!(2n+i)!}{(j+2)!(2n+i)!} \\ &\leq (2n+i+j+2)!. \end{aligned}$$

Combining the above estimates, we infer that

$$\begin{aligned} |b_{i,j}| &\leq \frac{C}{(j\pi)^j} \sum_{n=0}^i \sum_{m=0}^n \binom{i}{n} \binom{n}{m} a^{i-n} \frac{(2n+i+j+2)!^{\frac{2}{3}}}{R^{2n+i+j+2}} \\ &\leq \frac{C}{(j\pi)^j} \sum_{n=0}^i \binom{i}{n} 2^n a^{i-n} \frac{(3i+j+2)!^{\frac{2}{3}}}{R^{3i+j+2}} \\ &= \frac{C}{(j\pi)^j} \frac{(3i+j+2)!^{\frac{2}{3}}}{R^{3i+j+2}} (2+a)^i \\ &\leq \frac{C 2^{\frac{2}{3}(3i+j+2)} (3i)!^{\frac{2}{3}} (j+2)!^{\frac{2}{3}} (2+a)^i}{(j\pi)^j R^{3i+j+2}} \\ &\leq \frac{C 2^{\frac{2}{3}(j+2)} (j+2)!^{\frac{2}{3}} 2^{2i} (3i)!^{\frac{2}{3}} (2+a)^i}{(j\pi)^j R^{j+2} R^{3i}} \\ &\leq \frac{C 2^{\frac{2}{3}(j+2)} (j+2)!^{\frac{2}{3}} 3^{2i} (6\pi i)^{\frac{1}{3}} (4\pi i)^{-\frac{1}{2}} (2i)! (2+a)^i}{(j\pi)^j R^{j+2} R^{3i}} \\ &\leq \frac{C 2^{\frac{2}{3}(j+2)} (j+2)!^{\frac{2}{3}} [9(2+a)]^i (2i)!}{(j\pi)^j R^{j+2} R^{3i}} \\ &= M_j \frac{[9(2+a)]^i (2i)!}{R^{3i}}, \end{aligned}$$

where M_j satisfies (3.1).

By Proposition 4.2, for any $j \geq 1$, there exists a function $h_j \in G^2([0, T])$ and a number $\tilde{R} > 1$ such that

$$\begin{aligned} h_j^{(i)}(T) &= b_{i,j} \quad \forall i \geq 0, \\ |h_j^{(i)}(t)| &\leq M_j \frac{(2i)!}{\tilde{R}^{2i}} \quad \forall i \geq 0, t \in [0, T]. \end{aligned} \tag{4.5}$$

Pick any $\tau \in (0, T)$, $s \in (1, 2)$ and let

$$g(t) = 1 - \phi_s\left(\frac{t-\tau}{T-\tau}\right) \quad \text{for } t \in [0, T].$$

Setting

$$z_j(t) = h_j(t)g(t) \quad \forall t \in [0, T],$$

following the method developed in [12, Theorem 3.2], and taking into account the fact that $s < 2$, we see that z_j satisfies

$$\begin{aligned} z_j^{(i)}(T) &= b_{i,j} \quad \forall j \geq 1, i \geq 0, \\ z_j^{(i)}(0) &= 0 \quad \forall j \geq 1, i \geq 0, \\ |z_j^{(i)}(t)| &\leq CM_j \frac{(2i)!}{\tilde{R}^{2i}} \quad \forall j \geq 1, i \geq 0, t \in [0, T], \end{aligned} \tag{4.6}$$

where \tilde{R} is the same as in (4.5) and C is a positive constant independent of i and j .

Let u be as in (1.9). According to (4.6), we have $u_0 = 0$ and

$$u(x, y, T) = \sum_{j=1}^{\infty} \sum_{i=0}^{\infty} g_{i,j}(x) z_j^{(i)}(T) e_j(y) = \sum_{j=1}^{\infty} \sum_{i=0}^{\infty} b_{i,j} g_{i,j}(x) e_j(y).$$

By Proposition 4.1, u solves system (1.4)-(1.8) and $u \in G^{1,1,2}([-1, 0] \times [0, 1] \times [0, T])$. Let

$$h(y, t) = u(-1, y, t) \quad \forall y \in [0, 1], \forall t \in [0, T].$$

Then $h \in G^{1,2}([0, 1] \times [0, T])$.

Finally, for any $l \geq 1$ and $n \geq 0$, we have

$$\begin{aligned} \int_0^1 e_l(y) P^n u(0, y, T) dy &= \int_0^1 e_l(y) \sum_{j=1}^{\infty} \sum_{i=n}^{\infty} b_{i,j} (-1)^n g_{i-n,j}(0) e_j(y) dy = 0 \\ &= \int_0^1 e_l(y) P^n u_1(0, y) dy, \\ \int_0^1 e_l(y) \partial_x P^n u(0, y, T) dy &= \int_0^1 e_l(y) \sum_{j=1}^{\infty} \sum_{i=n}^{\infty} b_{i,j} (-1)^n g'_{i-n,j}(0) e_j(y) dy = 0 \\ &= \int_0^1 e_l(y) \partial_x P^n u_1(0, y) dy, \\ \int_0^1 e_l(y) \partial_x^2 P^n u(0, y, T) dy &= \int_0^1 e_l(y) \sum_{j=1}^{\infty} \sum_{i=n}^{\infty} b_{i,j} (-1)^n g''_{i-n,j}(0) e_j(y) dy = b_{nl} \\ &= \int_0^1 e_l(y) \partial_x^2 P^n u_1(0, y) dy. \end{aligned}$$

Since $u(\cdot, \cdot, T), u_1 \in \mathcal{X}$, it follows from Lemma 4.2 that

$$\int_0^1 e_l(y) [\partial_x^m u(0, y, T) - \partial_x^m u_1(0, y)] dy = 0 \quad \forall l \geq 1, \forall m \geq 0,$$

and hence

$$\partial_x^m u(0, y, T) - \partial_x^m u_1(0, y) = 0 \quad \forall m \geq 0, \forall y \in [0, 1].$$

Since the map $x \rightarrow u(x, y, T) - u_1(x, y)$ is in $G^1([-1, 0])$ (i.e. is analytic) for any $y \in [0, 1]$, we infer that

$$u(x, y, T) = u_1(x, y) \quad \forall (x, y) \in [-1, 0] \times [0, 1].$$

The proof of Theorem 1.2 is complete. \square

Acknowledgements

Lionel Rosier was partially supported by the ANR project Finite4SoS (ANR-15-CE23-0007). Mo Chen was supported by NSFC Grant (11701078) and China Scholarship Council (No. 201806625055).

References

- [1] R. A. Adams, *Sobolev Spaces*. Academic Press, New York, 1975.
- [2] E. Cerpa, *Control of a Korteweg-de Vries equation: a tutorial*, *Math. Control Relat. Fields* **4** (2014), no. 1, 45–99.
- [3] M. Chen, *Unique continuation property for the Zakharov-Kuznetsov equation*, *Comput. Math. Appl.* **77** (2019), no. 5, 1273–1281.
- [4] G. G. Doronin, N. A. Larkin, *Stabilization for the linear Zakharov-Kuznetsov equation without critical size restrictions*, *J. Math. Anal. Appl.* **428** (2015), no. 1, 337–355.
- [5] G. G. Doronin, N. A. Larkin, *Stabilization of regular solutions for the Zakharov-Kuznetsov equation posed on bounded rectangles and on a strip*, *Proc. Edinb. Math. Soc. (2)* **58** (2015), no. 3, 661–682.
- [6] A. V. Faminski. *The Cauchy problem for the Zakharov-Kuznetsov equation*, *Differ. Uravn.* **31** (1995) 1070–1081, English transl. in *Differential Equ.* **31** (1995).
- [7] A. V. Faminskii, *Initial-boundary value problems in a rectangle for two-dimensional Zakharov-Kuznetsov equation*, *J. Math. Anal. Appl.* **463** (2018), no. 2, 760–793.
- [8] O. Glass, S. Guerrero, *Some exact controllability results for the linear KdV equation and uniform controllability in the zero-dispersion limit*, *Asymptotic Analysis* **60** (2008), 61–100.
- [9] C. Laurent, L. Rosier, *Exact Controllability of Semilinear Heat Equations in Spaces of Analytic Functions*, submitted.
- [10] F. Linares and J.-C. Saut, *The Cauchy problem for the 3D Zakharov-Kuznetsov equation*, *Discrete Contin. Dynam. Syst. A* **24** (2009), 547–565.
- [11] F. Linares, A. Pastor and J.-C. Saut, *Well-posedness for the ZK equation in a cylinder and on the background of a KdV Soliton*, *Commun. PDEs* **35** (2010), 1674–1689.
- [12] P. Martin, L. Rosier, P. Rouchon. *On the reachable states for the boundary control of the heat equation*, *Appl. Math. Res. Express. AMRX* 2016, 2, 181-216.

- [13] P. Martin, I. Rivas, L. Rosier, P. Rouchon. *Exact controllability of a linear Korteweg-de Vries equation by the flatness approach*, SIAM J. Control Optim. **57** (2019), no. 4, 2467–2486.
- [14] G. Perla-Menzala, L. Rosier, J.-C. Saut, R. Temam, *Boundary control of the Zakharov-Kuznetsov equation*, in preparation.
- [15] L. Rosier, *Exact boundary controllability for the Korteweg-de Vries equation on a bounded domain*, ESAIM Control Optim. Calc. Var. **2** (1997), 33–55.
- [16] L. Rosier, *Control of the surface of a fluid by a wavemaker*, ESAIM Control Optim. Calc. Var. **10** (3) (2004), 346–380.
- [17] L. Rosier, B.-Y. Zhang, *Control and stabilization of the Korteweg-de Vries equation: recent progresses*. J. Syst. Sci. Complex. **22** (2009), no. 4, 647–682.
- [18] J.-C. Saut, R. Temam, *An initial boundary-value problem for the Zakharov-Kuznetsov equation*, Adv. Diff. Equations, **15** (11-12) (2010), 1001–1031.
- [19] J.-C. Saut, R. Temam and C. Wang. *An initial and boundary-value problem for the Zakharov-Kuznetsov equation in a bounded domain*, Journal of Mathematical Physics, 2012, 53, 115612.