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Exact controllability of the linear Zakharov-Kuznetsov equation

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Abstract

We consider the linear Zakharov-Kuznetsov equation on a rectangle with a left Dirichlet boundary control. Using the flatness approach, we prove the null controllability of this equation and provide a space of analytic reachable states.

Keywords: Zakharov-Kuznetsov equation; null controllability; reachable states; exact controllability; flatness approach; Gevrey functions.
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1 Introduction

The Zakharov-Kuznetsov (ZK) equation

$$u_t + au_x + \Delta u_x + uu_x = 0, \tag{1.1}$$

provides a model for the propagation of nonlinear ionic-sonic waves in a plasma. In (1.1), $x, t \in \mathbb{R}$ and $y \in \mathbb{R}^d$ (with $d \in \{1, 2\}$) are the independent variables, u = u(x, y, t) is the unknown, $u_t = \partial u/\partial t$, $u_x = \partial u/\partial x$, $\Delta u = \partial^2 u/\partial x^2 + \sum_{i=1}^d \partial^2 u/\partial y_i^2$, and the constant a > 0 stands for the sound velocity. The ZK equation is, from the mathematical point of view, a natural extension to \mathbb{R}^{d+1} of the famous Korteweg-de Vries equation

$$z_t + az_x + z_{xxx} + zz_x = 0, (1.2)$$

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which has been extensively studied from the control point of view (see e.g. the surveys [2, 17]). If we focus on the situation where (1.2) is supplemented with the following boundary conditions

$$z(0,t) = h(t), \quad z(L,t) = z_x(L,t) = 0,$$
(1.3)

where L > 0 is a given number and h is the control input, then it was proved in [8, 16] that (1.2)-(1.3) was null controllable on the domain (0, L). Due to the smoothing effect, with such a control at the left endpoint the exact controllability can only hold in a space of analytic functions.

More recently, a space of analytic reachable states was provided in [13] for the linearized KdV equation

$$z_t + z_x + z_{xxx} = 0$$

with the same boundary conditions as in (1.3). The method of proof was based on the flatness approach, as introduced in [12] to study the reachable states of the heat equation. The aim of the paper is to extend the results given in [13] to the ZK equation.

The wellposedness of various initial boundary value problems for ZK were studied in [6, 7, 10, 11, 18, 19]. Some unique continuation property for ZK derived with a Carleman estimate was done in [3]. Exact controllability results for ZK in the same spirit as those for KdV in [15] are given in [7, 14].

Here, we limit ourselves to the case d = 1, so that $y \in \mathbb{R}$. By a translation, we can assume without loss of generality that $x \in (-1,0)$ (this will be more convenient when using series to represent the solutions). We set $\Omega := (-1,0) \times (0,1)$. The paper is concerned with the control properties of the system:

$$u_t + u_{xxx} + u_{xyy} + au_x = 0, \qquad (x, y) \in \Omega, \ t \in (0, T), \tag{1.4}$$

$$u(0, y, t) = u_x(0, y, t) = 0, \qquad y \in (0, 1), \ t \in (0, T),$$

$$(1.5)$$

$$u(-1, y, t) = h(y, t), \qquad y \in (0, 1), \ t \in (0, T),$$
(1.6)

$$u(x,0,t) = u(x,1,t) = 0, \qquad x \in (-1,0), \ t \in (0,T),$$

$$(1.7)$$

$$u(x, y, 0) = u_0(x, y), \quad (x, y) \in \Omega,$$
(1.8)

where $u_0 = u_0(x, y)$ is the initial data and h = h(y, t) is the control input.

We shall address the following issues:

1. (Null controllability) Given any $u_0 \in L^2(\Omega)$, can we find a control h such that the solution u of (1.4)-(1.8) satisfies u(.,T) = 0?

2. (Reachable states) Given any $u_1 \in \mathcal{R}$ (a subspace of $L^2(\Omega)$ defined thereafter), can we find a control h such that the solution u of (1.4)-(1.8) with $u_0 = 0$ satisfies $u(., T) = u_1$?

We shall investigate both issues by the flatness approach and derive an exact controllability in \mathcal{R} by combining our results.

To state our result, we need introduce notations. A function $u \in C^{\infty}([t_1, t_2])$ is said to be Gevrey of order $s \ge 0$ on $[t_1, t_2]$ if there exist some constant $C, R \ge 0$ such that

$$|\partial_t^n u(t)| \le C \frac{(n!)^s}{R^n} \quad \forall n \in \mathbb{N}, \ \forall t \in [t_1, t_2].$$

The set of functions Gevrey of order s on $[t_1, t_2]$ is denoted by $G^s([t_1, t_2])$. A function $u \in C^{\infty}([x_1, x_2] \times [y_1, y_2] \times [t_1, t_2])$ is said to be Gevrey of order s_1 in x, s_2 in y and s_3 in t on $[x_1, x_2] \times [y_1, y_2] \times [t_1, t_2]$ if there exist some constants $C, R_1, R_2, R_3 > 0$ such that

$$|\partial_x^{n_1}\partial_y^{n_2}\partial_t^{n_3}u(x,y,t)| \le C \frac{(n_1!)^{s_1}(n_2!)^{s_2}(n_3!)^{s_3}}{R_1^{n_1}R_2^{n_2}R_3^{n_3}} \quad \forall n_1, n_2, n_3 \in \mathbb{N}, \ \forall (x,y,t) \in [x_1, x_2] \times [y_1, y_2] \times [t_1, t_2].$$

The set of functions Gevrey of order s_1 in x, s_2 in y and s_3 in t on $[x_1, x_2] \times [y_1, y_2] \times [t_1, t_2]$ is denoted by $G^{s_1, s_2, s_3}([x_1, x_2] \times [y_1, y_2] \times [t_1, t_2])$.

The first main result in this paper is a null controllability result with a control input in a Gevrey class.

Theorem 1.1. Let $u_0 \in L^2(\Omega)$ and $s \in [\frac{3}{2}, 2)$. Then there exists a control input $h \in G^{\frac{s}{2},s}([0,1] \times [0,T])$ such that the solution u of (1.4)-(1.8) satisfies $u(\cdot, \cdot, T)=0$. Furthermore, it holds that

$$u \in C([0,T]; L^2(\Omega)) \cap G^{\frac{s}{2}, \frac{s}{2}, s}([-1,0] \times [0,1] \times [\varepsilon,T]), \quad \forall \ \varepsilon \in (0,T).$$

Introduce the differential operator

$$Pu := \triangle u_x + au_x$$

and the following space

 $\mathcal{R}_{R_1,R_2} := \{ u \in C^{\infty}([-1,0] \times [0,1]); \ \exists C > 0, \ |\partial_x^p \partial_y^q u(x,y)| \le C \frac{(p!)^{\frac{2}{3}}(q!)^{\frac{2}{3}}}{R_1^p R_2^q} \quad \forall p,q \in \mathbb{N}, \ \forall (x,y) \in \overline{\Omega}, \\ \text{and} \ P^n u(0,y) = \partial_x P^n u(0,y) = P^n u(x,0) = P^n u(x,1) = 0, \ \forall n \in \mathbb{N}, \ \forall x \in [-1,0], \ \forall y \in [0,1] \}.$

Our second main result provides a set of reachable states for system (1.4)-(1.8).

Theorem 1.2. Let $R_0 := \sqrt[3]{9(a+2)}e^{(3e)^{-1}}$, and let $R_1, R_2 \in (R_0, +\infty)$. Then for any $u_1 \in \mathcal{R}_{R_1,R_2}$, there exists a control input $h \in G^{1,2}([0,1] \times [0,T])$ such that the solution u of (1.4)-(1.8) with $u_0 = 0$ satisfies $u(\cdot, \cdot, T) = u_1$. Furthermore, $u \in G^{1,1,2}([-1,0] \times [0,1] \times [0,T])$, and the trajectory u = u(x, y, t) and the control h = h(y, t) can be expanded as series:

$$u(x, y, t) = \sum_{j=1}^{\infty} \sum_{i=0}^{\infty} g_{i,j}(x) z_j^{(i)}(t) e_j(y), \qquad (1.9)$$

$$h(y,t) = \sum_{j=1}^{\infty} \sum_{i=0}^{\infty} g_{i,j}(-1) z_j^{(i)}(t) e_j(y).$$
(1.10)

We refer the reader to Section 2 for the definitions of the functions $g_{i,j}$ $(i \ge 0, j \ge 1)$ and of the functions e_j $(j \ge 1)$.

Combining Theorem 1.1 and Theorem 1.2, we obtain the following result which implies the exact controllability of (1.4)-(1.8) in \mathcal{R}_{R_1,R_2} for $R_1 > R_0$ and $R_2 > R_0$.

Corollary 1.1. Let $R_0 := \sqrt[3]{9(a+2)}e^{(3e)^{-1}}$, and let $R_1, R_2 \in (R_0, +\infty)$. Let $u_0 \in L^2(\Omega)$ and $u_1 \in \mathcal{R}_{R_1,R_2}$. Then there exists $h \in G^{1,2}([0,1] \times [0,T])$ such that the solution of (1.4)-(1.8) satisfies $u(.,T) = u_1$.

The paper is outlined as follows. Section 2 introduces the eigenfunctions e_j , the generating functions $g_{i,j}$, and provide some estimates needed in the sequel. The null controllability of ZK is established in Section 3, while the reachable states of ZK are investigated in Section 4.

2 Preliminaries

First we introduce the operator

$$Au := -Pu = -\triangle u_x - au_x$$

with domain

$$\mathcal{D}(A) = \{ u \in L^2(\Omega); \ Pu \in L^2(\Omega), \ u(-1, y) = u(0, y) = u_x(0, y) = 0 \text{ for a.e. } y \in (0, 1) \text{ and} u(x, 0) = u(x, 1) = 0 \text{ for a.e. } x \in (-1, 0) \}.$$

It is well-known (see e.g. [19]) that the operator A generates a semigroup of contractions in $L^2(\Omega)$. In what follows, we denote $||f||_{\mathcal{D}(A)} = ||f||_{L^2(\Omega)} + ||Af||_{L^2(\Omega)}$ for all $f \in \mathcal{D}(A)$.

It would be natural to expect, as for KdV, that the domain $\mathcal{D}(A)$ coincide with the set

$$\{u \in H^3(\Omega) \cap H^1_0(\Omega); u_x(0,y) = 0 \text{ for a.e. } y \in (0,1)\},\$$

but this is not the case. The best description (up to date) of $\mathcal{D}(A)$ is given in the following lemma.

Lemma 2.1. We have the following inclusions:

$$\{ u \in H^2(\Omega) \cap H^1_0(\Omega); \ u_x \in H^2(\Omega) \ and \ u_x(0,y) = 0 \ for \ a.e. \ y \in (0,1) \} \subset \mathcal{D}(A),$$
(2.1)
$$\mathcal{D}(A) \subset \{ u \in H^2(\Omega) \cap H^1_0(\Omega); \ (x+1)u_x \in H^2(\Omega) \cap H^1_0(\Omega) \}.$$
(2.2)

Proof. The inclusion (2.1) is obvious. For (2.2), it follows from [19, Proposition 2] that $\mathcal{D}(A) \subset H^2(\Omega) \cap H^1_0(\Omega)$. If $u \in \mathcal{D}(A)$, then $f := \Delta u_x + au_x \in L^2(\Omega)$ and hence

$$\Delta((x+1)u_x) = (x+1)\Delta u_x + 2u_{xx} = (x+1)(f - au_x) + 2u_{xx} \in L^2(\Omega).$$

On the other hand, we claim that $(x + 1)u_x \in H_0^1(\Omega)$. Indeed, $u_x \in H^1(\Omega)$ and hence $(x + 1)u_x \in H^1(\Omega)$. Moreover, u(.,0) = u(.,1) = 0 in $H^{\frac{3}{2}}(-1,0)$ gives $u_x(.,0) = u_x(.,1) = 0$ in $H^{\frac{1}{2}}(-1,0)$, and finally $((x + 1)u_x)(-1,.) = u_x(0,.) = 0$ in $H^{\frac{1}{2}}(0,1)$. By the classical boundary H^2 regularity result for the Dirichlet problem on a Lipschitz domain, we infer that $(x + 1)u_x \in H^2(\Omega) \cap H_0^1(\Omega)$.

Remark 2.1. It can be shown that the inclusion (2.1) is strict.

The following lemmas will be used several times thereafter.

Lemma 2.2. For any $n \in \mathbb{N}^*$ and any $f \in \mathcal{D}(A^n)$ with $A^i f \in H^{2(n-i)}(\Omega)$ for i = 0, 1, ..., n, we have

$$\partial_y^{2p} f(x,0) = \partial_y^{2p} f(x,1) = 0, \quad \forall x \in [-1,0], \ \forall p \in \{0,...,n-1\}.$$
(2.3)

Proof. We proceed by induction on n. For n = 1, the property (2.3) is obvious since $f \in \mathcal{D}(A)$. Assume now that (2.3) is true for $n-1 \geq 1$. If $f \in \mathcal{D}(A^n)$ with $A^i f \in H^{2(n-i)}(\Omega)$ for i = 0, 1, ..., n, then $Pf = -Af \in \mathcal{D}(A^{n-1})$ with $A^i Pf = -A^{i+1}f \in H^{2(n-i-1)}(\Omega)$ for i = 0, 1, ..., n-1, so that by (2.3) applied to Pf and p = n-2

$$\partial_y^{2n-4} Pf(x,0) = \partial_y^{2n-4} Pf(x,1) = 0.$$

This implies

$$\partial_x^3 \partial_y^{2n-4} f(x,0) + \partial_x \partial_y^{2n-2} f(x,0) + a \partial_x \partial_y^{2n-4} f(x,0) = 0,$$
(2.4)

$$\partial_x^3 \partial_y^{2n-4} f(x,1) + \partial_x \partial_y^{2n-2} f(x,1) + a \partial_x \partial_y^{2n-4} f(x,1) = 0.$$
 (2.5)

Since (2.3) is true for n - 1, we obtain that $\partial_y^{2p} f(x, 0) = \partial_y^{2p} f(x, 1) = 0$ for p = 0, 1, ..., n - 2, and hence (taking p = n - 2 and using (2.4)-(2.5))

$$\partial_x \partial_y^{2n-2} f(x,0) = \partial_x \partial_y^{2n-2} f(x,1) = 0.$$

This means that we have for some constants C_1 and C_2

$$\partial_y^{2n-2} f(x,0) = C_1, \ \partial_y^{2n-2} f(x,1) = C_2 \quad \forall x \in [-1,0].$$

Note that $\partial_y^{2n-2} f \in H^2(\Omega) \subset C(\overline{\Omega})$. On the other hand, it follows from the assumption $f \in \mathcal{D}(A)$ that

$$\partial_y^{2n-2} f(0,y) = 0 \quad \forall y \in [0,1].$$

Taking y = 0 and next y = 1, we see that $C_1 = C_2 = 0$. The proof of Lemma 2.2 is complete. \Box

Remark 2.2. It will be proved in Proposition 2.1 (see below) that $\mathcal{D}(A^n) \subset H^{2n}(\Omega)$ for all $n \in \mathbb{N}$, so that the conclusion of Lemma 2.2 will be still valid when assuming solely that $f \in \mathcal{D}(A^n)$.

The following lemma is classical. Its proof is omitted.

Lemma 2.3. Let $A' = \partial_y^2$ with domain $\mathcal{D}(A') = H^2(0,1) \cap H_0^1(0,1)$. Then for any $m \in \mathbb{N}^*$, it holds

$$\mathcal{D}(A'^{\frac{m}{2}}) = \{g \in H^m(0,1); \ g^{(2p)}(0) = g^{(2p)}(1) = 0 \ for \ 0 \le p \le \frac{m-1}{2}\}.$$

Let $h \in L^2(0,1)$ be decomposed as $h(y) = \sum_{j=1}^{\infty} c_j e_j(y)$, and let $m \in \mathbb{N}^*$. Then

$$h \in \mathcal{D}(A'^{\frac{m}{2}}) \iff \sum_{j=1}^{\infty} |\lambda_j^{\frac{m}{2}} c_j|^2 < \infty.$$

Furthermore, for any $h \in \mathcal{D}(A'^{\frac{m}{2}})$, we have

$$\|h^{(q)}\|_{L^{2}(0,1)}^{2} = \sum_{j=1}^{\infty} \lambda_{j}^{q} |c_{j}|^{2} \quad \forall q \in \{0,...,m\}.$$

We are in a position to state the main result in this section.

Proposition 2.1. For any $n \in \mathbb{N}$, it holds $\mathcal{D}(A^n) \subset H^{2n}(\Omega)$. Furthermore, there exists a constant $B \geq 1$ such that

$$\|u\|_{H^{2n}(\Omega)} \le B^n \sum_{i=0}^n \|P^i u\|_{L^2(\Omega)}, \quad \forall n \in \mathbb{N}, \ \forall u \in \mathcal{D}(A^n).$$

$$(2.6)$$

Proof. Let $\{e_j\}_{j\geq 1}$ be an orthonormal basis in $L^2(0,1)$ such that e_j is an eigenfunction for the Dirichlet Laplacian on (0,1), λ_j being the corresponding eigenvalue; that is

$$-e_j''(y) = \lambda_k e_j(y),$$

$$e_j(0) = e_j(1) = 0.$$

A classical choice is $e_j(y) = \sqrt{2} \sin(j\pi y)$ and $\lambda_j = (j\pi)^2$ for $j \ge 1$. Following [19], we decompose any function $u \in L^2(\Omega)$ as

$$u(x,y) = \sum_{j=1}^{\infty} \hat{u}_j(x) e_j(y).$$

Note that $||u||_{L^2(\Omega)}^2 = \sum_{j=1}^{\infty} ||\hat{u}_j||^2$, where we denote $||h|| = ||h||_{L^2(-1,0)}$ for all $h \in L^2(-1,0)$ for the sake of simplicity. If $u \in \mathcal{D}(A)$ and $g := \Delta u_x + au_x$, then for any $j \ge 1$

$$\hat{u}_{j}^{\prime\prime\prime} + (a - \lambda_j)\hat{u}_{j}^{\prime} = \hat{g}_j \text{ in } L^2(-1, 0)$$
(2.7)

where ' = d/dx. For n = 0, (2.6) is obvious if we pick $C_0 \ge 1$. Let us assume first that n = 1. Note that $\hat{u}_j \in H^3(-1,0)$ by (2.7). Multiplying (2.7) by $\lambda_j(x+1)\hat{u}_j$, we obtain

$$\frac{3}{2}\lambda_j \int_{-1}^0 |\hat{u}_j'|^2 dx - (a - \lambda_j) \frac{\lambda_j}{2} \int_{-1}^0 |\hat{u}_j|^2 dx = \lambda_j \int_{-1}^0 (x + 1) \hat{u}_j \hat{g}_j dx.$$

Let $j_0 := \left[\frac{\sqrt{2a}}{\pi}\right]$. Then for $j > j_0$, we have $a \le \lambda_j/2$ and hence $|a - \lambda_j|\lambda_j/2 \ge \lambda_j^2/4$. Using

$$\left|\lambda_j \int_{-1}^0 (x+1)\hat{u}_j \hat{g}_j dx\right| \le \frac{\lambda_j^2}{8} \int_{-1}^0 |\hat{u}_j|^2 dx + 2 \int_{-1}^0 |\hat{g}_j|^2 dx,$$

we infer that for $j > j_0$

$$\frac{3}{2}\lambda_j \int_{-1}^0 |\hat{u}_j'|^2 dx + \frac{\lambda_j^2}{8} \int_{-1}^0 |\hat{u}_j|^2 dx \le 2 \int_{-1}^0 |\hat{g}_j|^2 dx,$$

and that for $1 \leq j \leq j_0$

$$\frac{3}{2}\lambda_j \int_{-1}^0 |\hat{u}_j'|^2 dx + \frac{\lambda_j^2}{8} \int_{-1}^0 |\hat{u}_j|^2 dx \le \Lambda \int_{-1}^0 |\hat{u}_j|^2 dx + 2\int_{-1}^0 |\hat{g}_j|^2 dx,$$
(2.8)

where $\Lambda := \max_{1 \le j \le j_0} \left| \frac{\lambda_j^2}{8} + (a - \lambda_j) \frac{\lambda_j}{2} \right|$. Obviously, (2.8) is valid for any $j \ge 1$. Summing in j, we obtain

$$\frac{3}{2} \int_{\Omega} |u_{xy}|^2 dx dy + \frac{1}{8} \int_{\Omega} |u_{yy}|^2 dx dy \le \Lambda ||u||^2 + 2||g||^2.$$
(2.9)

Dividing in (2.8) by $\lambda_j \ge \pi^2$ and summing in j, we obtain

$$\frac{3}{2} \int_{\Omega} |u_x|^2 dx dy + \frac{1}{8} \int_{\Omega} |u_y|^2 dx dy \le \frac{\Lambda}{\pi^2} ||u||^2 + \frac{2}{\pi^2} ||g||^2.$$
(2.10)

It remains to estimate $\int_{\Omega} |u_{xx}|^2 dx dy$. Multiplying in (2.7) by \hat{u}'_j , we obtain

$$-\int_{-1}^{0} |\hat{u}_{j}''|^{2} dx + \hat{u}_{j}' \hat{u}_{j}''|_{-1}^{0} + (a - \lambda_{j}) \int_{-1}^{0} |\hat{u}_{j}'|^{2} dx = \int_{-1}^{0} \hat{g}_{j} \hat{u}_{j}' dx,$$

and hence

$$\int_{-1}^{0} |\hat{u}_{j}''|^{2} dx \leq |\lambda_{j} - a| \int_{-1}^{0} |\hat{u}_{j}'|^{2} dx + \|\hat{g}_{j}\| \|\hat{u}_{j}'\| + |\hat{u}_{j}'(-1)\hat{u}_{j}''(-1)|$$

We are let to estimate $\hat{u}'_i(-1)$ and $\hat{u}''_i(-1)$. Multiplying in (2.7) by $\lambda_j \hat{u}_j$ results in

$$\lambda_j \frac{\hat{u}_j'(-1)^2}{2} = \lambda_j \int_{-1}^0 \hat{u}_j \hat{g}_j dx.$$

Combined with (2.8), this yields

$$\frac{\lambda_j}{2}\hat{u}_j'(-1)^2 \le \|\lambda_j\hat{u}_j\| \cdot \|\hat{g}_j\| \le \frac{1}{4}\|\lambda_j\hat{u}_j\|^2 + \|\hat{g}_j\|^2 \le 5\|\hat{g}_j\|^2 + 2\Lambda\|\hat{u}_j\|^2.$$
(2.11)

Multiplying in (2.7) by x, we obtain

$$-\int_{-1}^{0} \hat{u}_{j}'' dx + x \hat{u}_{j}''|_{-1}^{0} + (a - \lambda_{j}) \left(-\int_{-1}^{0} \hat{u}_{j} dx + x \hat{u}_{j}|_{-1}^{0} \right) = \int_{-1}^{0} x \hat{g}_{j} dx$$

which yields

$$\hat{u}_{j}''(-1) = -\hat{u}_{j}'(-1) + (a - \lambda_{j}) \int_{-1}^{0} \hat{u}_{j} dx + \int_{-1}^{0} x \hat{g}_{j} dx,$$

so that

$$|\hat{u}_j''(-1)|^2 \le 3\left(|\hat{u}_j'(-1)|^2 + 2\left(a^2 + |\lambda_j|^2\right)\|\hat{u}_j\|^2 + \|\hat{g}_j\|^2\right).$$

Using (2.8) and (2.11), we conclude that $|\hat{u}_j'(-1)|^2 = O(||\hat{u}_j||^2 + ||\hat{g}_j||^2)$. The same is true for $||\hat{u}_j''||^2$. Gathering together the above estimates, we arrive at

$$||u||_{H^{2}(\Omega)}^{2} \leq C_{1} \left(||u||_{L^{2}(\Omega)}^{2} + ||Pu||_{L^{2}(\Omega)}^{2} \right)$$

for some constant $C_1 = C_1(a) > 0$.

Let us check that $\mathcal{D}(A^n) \subset H^{2n}(\Omega)$ for $n \geq 2$. We proceed by induction on n. Assume that $\mathcal{D}(A^p) \subset H^{2p}(\Omega)$ for p = 0, 1, ..., n-1 (with $n-1 \geq 1$), and pick any $u \in \mathcal{D}(A^n)$. Then $g = Au \in \mathcal{D}(A^{n-1}) \subset H^{2(n-1)}(\Omega)$. Let $h := (-1)^{n-1} \partial_y^{2(n-1)} g \in L^2(\Omega)$. Then, using Lemmas 2.2 and 2.3, we have that for all $j \geq 1$

$$\lambda_j^{n-1}(\hat{u}_j''' + (a - \lambda_j)\hat{u}_j') = \hat{h}_j.$$
(2.12)

Multiplying in (2.12) by $\lambda_j^n(x+1)\hat{u}_j$, we obtain

$$\frac{3}{2}\lambda_j^{2n-1}\int_{-1}^0 |\hat{u}_j'|^2 dx - (a-\lambda_j)\frac{\lambda_j^{2n-1}}{2}\int_{-1}^0 |\hat{u}_j|^2 dx = \lambda_j^n \int_{-1}^0 (x+1)\hat{u}_j\hat{h}_j dx.$$

This yields

$$\lambda_j^{2n-1} \|\hat{u}_j'\|^2 + \lambda_j^{2n} \|\hat{u}_j\|^2 = O(\|\hat{u}_j\|^2 + \|\hat{h}_j\|^2), \qquad (2.13)$$

Multiplying in (2.12) by $\lambda_j^n \hat{u}_j$ gives

$$\lambda_j^{2n-1} \frac{\hat{u}_j'(-1)^2}{2} = \lambda_j^n \int_{-1}^0 \hat{u}_j \hat{h}_j dx$$

and

$$\lambda_j^{2n-1} |\hat{u}_j'(-1)|^2 = O(||\hat{u}_j||^2 + ||\hat{h}_j||^2).$$

From

$$\lambda_j^{n-1}\hat{u}_j''(-1) = -\lambda_j^{n-1}\hat{u}_j'(-1) + (a-\lambda_j)\lambda_j^{n-1}\int_{-1}^0 \hat{u}_j dx + \int_{-1}^0 x\hat{h}_j dx,$$

we infer that

$$\lambda_j^{2n-2} |\hat{u}_j''(-1)|^2 = O(||\hat{u}_j||^2 + ||\hat{h}_j||^2).$$

It follows from

$$-\lambda_j^{2n-2} \int_{-1}^0 |\hat{u}_j''|^2 dx + \lambda_j^{2n-2} \hat{u}_j' \hat{u}_j''|_{-1}^0 + (a-\lambda_j) \lambda_j^{2n-2} \int_{-1}^0 |\hat{u}_j'|^2 dx = \lambda_j^{n-1} \int_{-1}^0 \hat{h}_j \hat{u}_j' dx,$$

that

$$\lambda_j^{2n-2} \|\hat{u}_j''\|^2 = O(\|\hat{u}_j\|^2 + \|\hat{h}_j\|^2).$$
(2.14)

So far, we have proved that

$$\sum_{j=1}^{\infty} \left(\lambda_j^{2n} \| \hat{u}_j \|^2 + \lambda_j^{2n-1} \| \hat{u}_j' \|^2 + \lambda_j^{2n-2} \| \hat{u}_j'' \|^2 \right) < +\infty.$$

Using Lemma 2.3, this gives that $\partial_y^{2n} u$, $\partial_y^{2n-1} \partial_x u$, and $\partial_y^{2n-2} \partial_x^2 u$ belong to $L^2(\Omega)$. For the other derivatives of order 2n, we apply the operator ∂_x^{2k} (for $k \in \mathbb{N}$ with $2k + 3 \leq 2n$) to each term in (2.7) to obtain

$$\hat{u}_j^{(2k+3)} + (a - \lambda_j)\hat{u}_j^{(2k+1)} = \hat{g}_j^{(2k)}.$$

This yields

$$\lambda_j^{(2n-3-2k)} \|\hat{u}_j^{(2k+3)}\|^2 = O(\|\hat{u}_j\|^2 + \|\hat{h}_j\|^2 + \lambda_j^{2n-3-2k} \|\hat{g}_j^{(2k)}\|^2).$$

On the other hand, (2.7) gives by differentiation with respect to x that

$$\hat{u}_{j}^{(4)} + (a - \lambda_j)\hat{u}_{j}'' = \hat{g}_{j}',$$

and we obtain in a similar way that

$$\lambda_j^{(2n-4-2k)} \|\hat{u}_j^{(2k+4)}\|^2 = O(\|\hat{u}_j\|^2 + \|\hat{h}_j\|^2 + \lambda_j^{2n-4-2k} \|\hat{g}_j^{(2k+1)}\|^2).$$

for $k \in \mathbb{N}$ with $2k + 4 \leq 2n$. Thus we conclude that

$$\sum_{q=0}^{2n} \sum_{j=1}^{\infty} \lambda_j^{(2n-q)} \|\hat{u}_j^{(q)}\|^2 < +\infty.$$

Using Lemma 2.3, we infer that for $q \in \{0, ..., 2n\}$, $\partial_x^q u \in L^2(-1, 0, H^{2n-q}(0, 1))$, and hence that $\partial_y^{2n-q} \partial_x^q u \in L^2(\Omega)$. We also have that $\partial_y^{2n-1-q} \partial_x^q u \in L^2(\Omega)$ for $q \in \{0, ..., 2n-1\}$. Taking into account the fact that $u \in \mathcal{D}(A^{n-1}) \subset H^{2(n-1)}(\Omega)$, we conclude that $u \in H^{2n}(\Omega)$. The proof of the inclusion $\mathcal{D}(A^n) \subset H^{2n}(\Omega)$ is complete.

It remains to prove that the constant in the r.h.s. of (2.6) is indeed of the form B^n . This will require a series of lemmas.

Lemma 2.4. For any $\varepsilon_0 > 0$, there exists a constant $K = K(\varepsilon_0) > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and all $f \in H^2(-1, 0)$,

$$\int_{-1}^{0} |f'(t)|^2 dt \le K\varepsilon \int_{-1}^{0} |f''(t)|^2 dt + K\varepsilon^{-1} \int_{-1}^{0} |f(t)|^2 dt.$$
(2.15)

Lemma 2.4 is a direct consequence of [1, Lemma 4.10] (which is concerned with twice continuously functions) by density of $C^2([-1,0])$ in $H^2(-1,0)$.

For any $j \in \mathbb{N}^*$, we define the operator P_j by

$$P_j f := f''' - (\lambda_j - a) f', \quad \forall f \in H^3(-1, 0).$$

Lemma 2.5. There exists a constant $C_1 \ge 1$ such that

$$\|f\|_{H^{2n}(-1,0)}^2 \le C_1^n \sum_{i=0}^n \lambda_j^{2n-2i} \|P_j^i f\|^2, \quad \forall n \in \mathbb{N}, \ \forall j \in \mathbb{N}^*, \ \forall f \in H^{3n}(-1,0).$$
(2.16)

Proof. For n = 0, (2.16) is obvious. For n = 1, it follows from the definition of P_j and Lemma 2.4 that

$$\begin{split} \|f\|_{H^{2}(-1,0)}^{2} &= \|f\|^{2} + \|f'\|^{2} + \|f''\|^{2} \\ &\leq C(\|f\|^{2} + \|f''\|^{2}) \\ &\leq C(\|f\|^{2} + \frac{1}{\lambda_{j}}\|f'''\|^{2} + \lambda_{j}\|f'\|^{2}) \\ &\leq C(\|f\|^{2} + \frac{1}{\lambda_{j}}\|P_{j}f\|^{2} + \lambda_{j}\|f'\|^{2}) \\ &\leq C(\|f\|^{2} + \|P_{j}f\|^{2}) + \frac{1}{2}\|f''\|^{2} + C\lambda_{j}^{2}\|f\|^{2} \\ &\leq C(\lambda_{j}^{2}\|f\|^{2} + \|P_{j}f\|^{2}) + \frac{1}{2}\|f''\|^{2}. \end{split}$$

This shows that we can find a constant $C_2 \ge 1$ such that

$$||f||_{H^2(-1,0)}^2 \le C_2(\lambda_j^2 ||f||^2 + ||P_j f||^2).$$

Let us prove (2.16) for $n \ge 2$ by induction on n. Assume (2.16) to be true for $n-1 \ge 0$. It

follows that

$$\begin{split} \|f\|_{H^{2n}(-1,0)}^{2} &= \|f\|_{H^{2n-2}(-1,0)}^{2} + \|f^{(2n-1)}\|^{2} + \|f^{(2n)}\|^{2} \\ &\leq \|f\|_{H^{2n-2}(-1,0)}^{2} + \|f^{(2n-2)}\|_{H^{2}(-1,0)}^{2} \\ &\leq \|f\|_{H^{2n-2}(-1,0)}^{2} + C_{2}(\lambda_{j}^{2}\|f^{(2n-2)}\|^{2} + \|P_{j}f^{(2n-2)}\|^{2}) \\ &\leq 2C_{2}\lambda_{j}^{2}\|f\|_{H^{2n-2}(-1,0)}^{2} + C_{2}\|P_{j}f\|_{H^{2n-2}(-1,0)}^{2} \\ &\leq 2C_{2}\lambda_{j}^{2}C_{1}^{n-1}\sum_{i=0}^{n-1}\lambda_{j}^{2n-2-2i}\|P_{j}^{i}f\|^{2} + C_{2}C_{1}^{n-1}\sum_{i=0}^{n-1}\lambda_{j}^{2n-2-2i}\|P_{j}^{i+1}f\|^{2} \\ &\leq 2C_{2}C_{1}^{n-1}\sum_{i=0}^{n-1}\lambda_{j}^{2n-2i}\|P_{j}^{i}f\|^{2} + C_{2}C_{1}^{n-1}\sum_{i=1}^{n}\lambda_{j}^{2n-2i}\|P_{j}^{i}f\|^{2} \\ &\leq 3C_{2}C_{1}^{n-1}\sum_{i=0}^{n}\lambda_{j}^{2n-2i}\|P_{j}^{i}f\|^{2}. \end{split}$$

If we pick $C_1 = 3C_2$, (2.16) is true for n.

Lemma 2.6. There exists a positive constant C_3 such that

$$\|u\|_{H^{2n}(\Omega)}^2 \le C_3 \sum_{m=0}^n \sum_{k=0}^m \|\partial_x^{2k} \partial_y^{2m-2k} u\|_{L^2(\Omega)}^2, \quad \forall n \in \mathbb{N}, \ \forall u \in \mathcal{D}(A^n).$$

Proof. For any $p \in \mathbb{N}$, we set

$$I_p := \sum_{a,b \in \mathbb{N}, \ a+b=p} \|\partial_x^a \partial_y^b u\|_{L^2(\Omega)}^2.$$

Decompose u as

$$u(x,y) = \sum_{j=1}^{\infty} \hat{u}_j(x) e_j(y).$$
(2.17)

Let us go back to the proof of Lemma 2.6. Pick any $u \in \mathcal{D}(A^n)$, for some $n \in \mathbb{N}$. Using Lemma 2.2 and applying Lemma 2.3 to the functions $\partial_x^{2m+1-k}u(x,.)$ for $0 \leq m \leq n-1$,

 $0 \leq k \leq 2m+1,$ and $x \in (-1,0),$ we obtain that

$$\begin{split} I_{2m+1} &= \sum_{k=0}^{2m+1} \|\partial_x^{2m+1-k} \partial_y^k u\|_{L^2(\Omega)}^2 \\ &= \sum_{k=0}^{2m+1} \sum_{j=1}^{\infty} \lambda_j^k \|\hat{u}_j^{(2m+1-k)}\|^2 \\ &= \sum_{j=1}^{\infty} \|\hat{u}_j^{(2m+1)}\|^2 + \sum_{k=1}^{2m+1} \sum_{j=1}^{\infty} \lambda_j^k \|\hat{u}_j^{(2m+1-k)}\|^2 \\ &\leq \sum_{j=1}^{\infty} \lambda_j \|\hat{u}_j^{(2m+1)}\|^2 + \frac{1}{2} \sum_{k=1}^{2m+1} \sum_{j=1}^{\infty} \lambda_j^{k-1} \|\hat{u}_j^{(2m+1-k)}\|^2 + \frac{1}{2} \sum_{k=1}^{2m+1} \sum_{j=1}^{\infty} \lambda_j^{k+1} \|\hat{u}_j^{(2m+1-k)}\|^2 \\ &= \|\partial_x^{2m+1} \partial_y u\|_{L^2(\Omega)}^2 + \frac{1}{2} \sum_{k=1}^{2m+1} \|\partial_x^{2m+1-k} \partial_y^{k-1} u\|_{L^2(\Omega)}^2 + \frac{1}{2} \sum_{k=1}^{2m+1} \|\partial_x^{2m+1-k} \partial_y^{k+1} u\|_{L^2(\Omega)}^2 \\ &\leq \frac{3}{2} I_{2m+2} + \frac{1}{2} I_{2m}, \end{split}$$

where we used Young's estimate. Thus, we have

$$\|u\|_{H^{2n}(\Omega)}^{2} = \sum_{m=0}^{n} I_{2m} + \sum_{m=0}^{n-1} I_{2m+1}$$

$$\leq \sum_{m=0}^{n} I_{2m} + \sum_{m=0}^{n-1} (\frac{3}{2}I_{2m+2} + \frac{1}{2}I_{2m})$$

$$\leq 3\sum_{m=0}^{n} I_{2m}.$$
(2.18)

Next, we consider I_{2m} . For m = 0, $I_0 = ||u||_{L^2(\Omega)}^2$. For $m \ge 1$, we have

$$I_{2m} = \sum_{k=0}^{m} \|\partial_x^{2k} \partial_y^{2m-2k} u\|_{L^2(\Omega)}^2 + \sum_{k=0}^{m-1} \|\partial_x^{2k+1} \partial_y^{2m-2k-1} u\|_{L^2(\Omega)}^2,$$
(2.19)

and it remains to estimate the second term in the r.h.s. of (2.19). Applying Lemma 2.4, we

obtain

$$\sum_{k=0}^{m-1} \|\partial_x^{2k+1} \partial_y^{2m-2k-1} u\|_{L^2(\Omega)}^2 = \sum_{k=0}^{m-1} \sum_{j=1}^{\infty} \lambda_j^{2m-2k-1} \|\hat{u}_j^{(2k+1)}\|^2$$

$$\leq C \Big(\sum_{k=0}^{m-1} \sum_{j=1}^{\infty} \lambda_j^{2m-2k-2} \|\hat{u}_j^{(2k+2)}\|^2 + \sum_{k=0}^{m-1} \sum_{j=1}^{\infty} \lambda_j^{2m-2k} \|\hat{u}_j^{(2k)}\|^2 \Big)$$

$$= C \Big(\sum_{k=1}^{m} \sum_{j=1}^{\infty} \lambda_j^{2m-2k} \|\hat{u}_j^{(2k)}\|^2 + \sum_{k=0}^{m-1} \sum_{j=1}^{\infty} \lambda_j^{2m-2k} \|\hat{u}_j^{(2k)}\|^2 \Big)$$

$$\leq C \sum_{k=0}^{m} \sum_{j=1}^{\infty} \lambda_j^{2m-2k} \|\hat{u}_j^{(2k)}\|^2$$

$$= C \sum_{k=0}^{m} \|\partial_x^{2k} \partial_y^{2m-2k} u\|_{L^2(\Omega)}^2.$$
(2.20)

Combining (2.18)-(2.20), the conclusion of Lemma 2.6 follows.

Lemma 2.7. There exists a constant $C_4 \ge 1$ such that

$$\lambda_j^{2m} \|P_j^i \hat{u}_j\|^2 \le C_4^m \sum_{l=0}^m \binom{m}{l} \|P_j^{i+l} \hat{u}_j\|^2, \quad \forall m, i \in \mathbb{N}, \forall j \in \mathbb{N}^*, \forall u \in \mathcal{D}(A^{m+i}),$$
(2.21)

where \hat{u}_j is the Fourier coefficients of u as in (2.17).

Proof. The proof is by induction on m. For m = 0, (2.21) is obvious for any $C_4 \ge 1$. For m = 1 and $u \in \mathcal{D}(A^{1+i})$, we have that $P^i u \in \mathcal{D}(A)$ and, by [19, Lemma 4.1],

$$(P^{i}u)(x,y) = \sum_{j=1}^{\infty} (P_{j}^{i}\hat{u}_{j})(x)e_{j}(y)$$

where the function $P^i_j \hat{u}_j$ satisfies for each $j \in \mathbb{N}^*$

$$\begin{cases} (P_j^i \hat{u}_j)''' - (\lambda_j - a)(P_j^i \hat{u}_j)' = P_j^{i+1} \hat{u}_j, & x \in (-1, 0), \\ (P_j^i \hat{u}_j)(-1) = (P_j^i \hat{u}_j)(0) = (P_j^i \hat{u}_j)'(0) = 0. \end{cases}$$
(2.22)

Multiplying the first equation in (2.22) by $\lambda_j(x+1)P_j^i\hat{u}_j$ and integrating over (-1,0) results in

$$\frac{3}{2}\lambda_j \int_{-1}^0 |(P_j^i \hat{u}_j)'|^2 dx + (\lambda_j - a)\frac{\lambda_j}{2} \int_{-1}^0 |P_j^i \hat{u}_j|^2 dx = \lambda_j \int_{-1}^0 (x+1)(P_j^i \hat{u}_j)(P_j^{i+1} \hat{u}_j) dx.$$

After some elementary calculations, we can find a constant $C_4 = C_4(a) \ge 1$ such that

$$\lambda_j \| (P_j^i \hat{u}_j)' \|^2 + \lambda_j^2 \| P_j^i \hat{u}_j \|^2 \le C_4 (\| P_j^i \hat{u}_j \|^2 + \| P_j^{i+1} \hat{u}_j \|^2).$$

Therefore, (2.21) holds for m = 1. Pick now any $m \ge 2$, and assume that (2.21) is true for $m-1 \ge 0$. For any $u \in \mathcal{D}(A^{m+i})$, we have

$$\lambda_j^{2m} \|P_j^i \hat{u}_j\|^2 = \lambda_j^2 \lambda_j^{2m-2} \|P_j^i \hat{u}_j\|^2 \le \lambda_j^2 C_4^{m-1} \sum_{l=0}^{m-1} \binom{m-1}{l} \|P_j^{i+l} \hat{u}_j\|^2.$$

Since $u \in \mathcal{D}(A^{m+i})$, for any l = 0, 1, ..., m-1, system (2.22) is satisfied with $P_j^{i+l}\hat{u}_j$ substituted to $P_j^i\hat{u}_j$, and it follows as above that

$$\lambda_j^2 \|P_j^{i+l} \hat{u}_j\|^2 \le C_4(\|P_j^{i+l} \hat{u}_j\|^2 + \|P_j^{i+l+1} \hat{u}_j\|^2).$$

We infer that

$$\begin{split} \lambda_{j}^{2m} \|P_{j}^{i} \hat{u}_{j}\|^{2} &\leq C_{4}^{m} \sum_{l=0}^{m-1} \left(\begin{array}{c} m-1\\ l \end{array} \right) (\|P_{j}^{i+l} \hat{u}_{j}\|^{2} + \|P_{j}^{i+l+1} \hat{u}_{j}\|^{2}) \\ &= C_{4}^{m} (\|P_{j}^{i} \hat{u}_{j}\|^{2} + \sum_{l=1}^{m-1} \left(\begin{array}{c} m-1\\ l \end{array} \right) \|P_{j}^{i+l} \hat{u}_{j}\|^{2} + \sum_{l=1}^{m-1} \left(\begin{array}{c} m-1\\ l \end{array} \right) \|P_{j}^{i+l} \hat{u}_{j}\|^{2} + \|P_{j}^{i+m} \hat{u}_{j}\|^{2}) \\ &= C_{4}^{m} (\|P_{j}^{i} \hat{u}_{j}\|^{2} + \sum_{l=1}^{m-1} \left(\begin{array}{c} m\\ l \end{array} \right) \|P_{j}^{i+l} \hat{u}_{j}\|^{2} + \|P_{j}^{i+m} \hat{u}_{j}\|^{2}) \\ &= C_{4}^{m} \sum_{l=0}^{m} \left(\begin{array}{c} m\\ l \end{array} \right) \|P_{j}^{i+l} \hat{u}_{j}\|^{2} \end{split}$$

where we used Pascal's Rule. The proof of Lemma 2.7 is achieved.

We are in a position to complete the proof of Proposition 2.1. The estimate (2.6) is obvious for n = 0. Let $n \ge 1$. Using Lemmas 2.5, 2.6, and 2.7, we obtain that

$$\begin{aligned} \|u\|_{H^{2n}(\Omega)}^{2} &\leq C_{3} \sum_{m=0}^{n} \sum_{k=0}^{m} \|\partial_{x}^{2k} \partial_{y}^{2m-2k} u\|_{L^{2}(\Omega)}^{2} \\ &= C_{3} \sum_{m=0}^{n} \sum_{k=0}^{m} \sum_{j=1}^{\infty} \lambda_{j}^{2m-2k} \|\hat{u}_{j}^{(2k)}\|^{2} \\ &\leq C_{3} \sum_{m=0}^{n} \sum_{k=0}^{m} \sum_{j=1}^{\infty} \lambda_{j}^{2m-2k} C_{1}^{k} \sum_{i=0}^{k} \lambda_{j}^{2k-2i} \|P_{j}^{i} \hat{u}_{j}\|^{2} \\ &\leq C_{3} C_{1}^{n} \sum_{m=0}^{n} \sum_{k=0}^{m} \sum_{j=1}^{\infty} \sum_{i=0}^{k} \lambda_{j}^{2m-2i} \|P_{j}^{i} \hat{u}_{j}\|^{2}. \end{aligned}$$

Using the fact that $i \leq k \leq m \leq n$ in the sum above, we obtain

$$\begin{split} \|u\|_{H^{2n}(\Omega)}^{2} &\leq C_{3}C_{1}^{n} \sum_{m=0}^{n} \sum_{k=0}^{n} \sum_{j=1}^{\infty} \sum_{i=0}^{n} \lambda_{j}^{2n-2i} \|P_{j}^{i}\hat{u}_{j}\|^{2} \\ &\leq C_{3}C_{1}^{n}(n+1)^{2} \sum_{j=1}^{\infty} \sum_{i=0}^{n} \lambda_{j}^{2n-2i} \|P_{j}^{i}\hat{u}_{j}\|^{2} \\ &\leq C_{3}C_{1}^{n}(n+1)^{2} \sum_{j=1}^{\infty} \sum_{i=0}^{n} C_{4}^{n-i} \sum_{l=0}^{n-i} \binom{n-i}{l} \|P_{j}^{i+l}\hat{u}_{j}\|^{2} \\ &\leq C_{3}C_{1}^{n}(n+1)^{2}C_{4}^{n}2^{n} \sum_{j=1}^{n} \sum_{i=0}^{n} \sum_{l=i}^{n} \|P_{j}^{l}\hat{u}_{j}\|^{2} \\ &\leq C_{3}C_{1}^{n}(n+1)^{2}C_{4}^{n}2^{n} \sum_{j=1}^{\infty} \sum_{i=0}^{n} \sum_{l=0}^{n} \|P_{j}^{l}\hat{u}_{j}\|^{2} \\ &\leq C_{3}C_{1}^{n}(n+1)^{3}C_{4}^{n}2^{n} \sum_{l=0}^{\infty} \sum_{l=0}^{n} \|P_{j}^{l}\hat{u}_{j}\|^{2} \\ &\leq B^{n} \sum_{l=0}^{n} \|P^{l}u\|_{L^{2}(\Omega)}^{2} \end{split}$$

with $B := 16C_1C_3C_4$. Indeed, it is easy to see that $(n+1)^3 \leq 8^n$ for all $n \in \mathbb{N}$. The proof of Proposition 2.1 is achieved.

Recall that $\lambda_j = (j\pi)^2$ for $j \ge 1$. For any $j \ge 1$, we consider a sequence of generating functions $g_{i,j}$ $(i \ge 0)$, where $g_{0,j}$ is the solution of the Cauchy problem

$$\begin{cases} g_{0,j}''(x) - (\lambda_j - a)g_{0,j}'(x) = 0, & x \in (-1,0), \\ g_{0,j}(0) = g_{0,j}'(0) = 0, & g_{0,j}''(0) = 1, \end{cases}$$
(2.23)

while $g_{i,j}$ for $i \ge 1$ is defined inductively as the solution of the Cauchy problem

$$\begin{cases} g_{i,j}''(x) - (\lambda_j - a)g_{i,j}'(x) = -g_{i-1,j}(x), & x \in (-1,0), \\ g_{i,j}(0) = g_{i,j}'(0) = g_{i,j}''(0) = 0. \end{cases}$$
(2.24)

Proposition 2.2. For any $i \ge 0, j \ge 1$ and $x \in [-1, 0]$, we have

$$|g_{i,j}(x)| \le e^{\sqrt{\lambda_j}} \frac{3^i i!}{(3i+2)!}.$$
(2.25)

Proof. It follows from (2.23) and (2.24) that

$$g_{i,j}(x) = -\int_0^x g_{0,j}(x-\xi)g_{i-1,j}(\xi)d\xi$$

= $-\int_0^x g_{0,j}''(x-\xi) \Big(\int_0^\xi (\int_0^\zeta g_{i-1,j}(\sigma)d\sigma)d\zeta\Big)d\xi, \quad i,j \ge 1.$

(1) if $\lambda_j \leq a$, it is not difficult to obtain that

$$g_{0,j}(x) = \begin{cases} \frac{1}{a - \lambda_j} (1 - \cos(\sqrt{a - \lambda_j}x)), & \lambda_j < a; \\ \frac{1}{2}x^2, & \lambda_j = a, \end{cases}$$

this implies

$$0 \le g_{0,j}(x) \le \frac{x^2}{2}, \quad \forall \ j \ge 1, \ x \in [-1,0].$$

Then it follows from [13, Lemma 2.1] that

$$|g_{i,j}(x)| \le \frac{|x|^{3i+2}}{(3i+2)!} \le e^{\sqrt{\lambda_j}} \frac{3^i i!}{(3i+2)!}, \quad \forall i \ge 0, \ \forall j \ge 1, \ \forall x \in [-1,0].$$

(2) if $\lambda_j > a$, we claim that

$$g_{i,j}(x) \le \cosh(\sqrt{\lambda_j - ax}) \frac{(-x)^{3i+2} 3^i i!}{(3i+2)!}, \quad \forall i \ge 0, \ \forall j \ge 1, \ \forall x \in [-1,0]$$
(2.26)

which implies (2.25).

Let us prove (2.26) by induction on *i*. For i = 0,

$$0 \le g_{0,j}(x) = \frac{1}{\lambda_j - a} (\cosh(\sqrt{\lambda_j - ax}) - 1)$$
$$= \sum_{q=1}^{\infty} \frac{(\lambda_j - a)^{q-1} x^{2q}}{(2q)!}$$
$$\le \sum_{q=1}^{\infty} \frac{(\lambda_j - a)^{q-1} x^{2q-2}}{(2q-2)!} \frac{x^2}{2!}$$
$$= \cosh(\sqrt{\lambda_j - ax}) \frac{x^2}{2!},$$

so that (2.26) is true for i = 0.

Assume now that (2.26) is true for $i - 1 \ge 0$. We can deduce that for $x \in [-1, 0]$

$$\begin{split} |g_{i,j}(x)| &\leq -\int_0^x g_{0,j}''(x-\xi) \Big(\int_0^{\xi} \Big(\int_0^{\zeta} |g_{i-1,j}(\sigma)| d\sigma \Big) d\zeta \Big) d\xi \\ &\leq -\int_0^x \sum_{p=0}^\infty \frac{(\lambda_j-a)^p (x-\xi)^{2p}}{(2p)!} \Big(\int_0^{\xi} \Big(\int_0^{\zeta} 3^{i-1} (i-1)! \sum_{q=0}^\infty \frac{(\lambda_j-a)^q (-\sigma)^{3i-1+2q}}{(2q)! (3i-1)!} d\sigma \Big) d\zeta \Big) d\xi \\ &= -3^{i-1} (i-1)! \int_0^x \sum_{p=0}^\infty \frac{(\lambda_j-a)^p (x-\xi)^{2p}}{(2p)!} \sum_{q=0}^\infty \frac{(\lambda_j-a)^q (-\xi)^{3i+1+2q}}{(2q)! (3i-1)! (3i+2q) (3i+2q+1)} d\xi \\ &= -3^{i-1} (i-1)! \int_0^x \sum_{p=0}^\infty \sum_{q=0}^\infty \frac{(\lambda_j-a)^{p+q} (x-\xi)^{2p} (-\xi)^{3i+1+2q}}{(2p)! (2q)! (3i-1)! (3i+2q) (3i+2q+1)} d\xi. \end{split}$$

Then, integrating by parts 2p times, we obtain

$$|g_{i,j}(x)| \leq -3^{i-1}(i-1)! \int_0^x \sum_{p=0}^\infty \sum_{q=0}^\infty \frac{(\lambda_j - a)^{p+q}(-\xi)^{3i+1+2q+2p}(3i+2q-1)!}{(2q)!(3i-1)!(3i+1+2q+2p)!} d\xi$$

=3ⁱ⁻¹(i-1)! $\sum_{p=0}^\infty \sum_{q=0}^\infty \frac{(\lambda_j - a)^{p+q}(-x)^{3i+2+2q+2p}(3i+2q-1)!}{(2q)!(3i-1)!(3i+2+2q+2p)!}$.

Next, we will show that

$$\frac{3^{i-1}(i-1)!(3i+2q-1)!}{(2q)!(3i-1)!(3i+2+2q+2p)!} \le \frac{3^{i}i!}{p+q+1} \frac{1}{(2p+2q)!(3i+2)!} \quad \forall \ p,q \ge 0, \ i \ge 1.$$
(2.27)

It is easy to see that (2.27) is equivalent to

$$\frac{(3i+2q-1)!}{(2q)!(3i-1)!} \le \frac{3i}{p+q+1} \frac{(3i+2+2q+2p)!}{(2p+2q)!(3i+2)!} = 6i \frac{(2p+2q+1)(2p+2q+3)(2p+2q+4)\cdots(2p+2q+3i+2)}{(3i+2)!}.$$
(2.28)

Since the left hand side of (2.28) is independent of p and the right hand side of (2.28) is increasing in p, we only need to prove (2.27) for p = 0, namely, we need to show that

$$\frac{(3i+2q-1)!}{(3i-1)!} \le \frac{3i}{q+1} \frac{(3i+2+2q)!}{(3i+2)!} \quad \forall q \ge 0, \ \forall i \ge 1,$$

this is obvious due to the fact that

$$\frac{(3i+2)!(3i+2q-1)!}{3i(3i-1)!(3i+2+2q)!} = \frac{(3i+1)(3i+2)}{(3i+2q)(3i+2q+1)(3i+2q+2)} \\ \leq \frac{1}{3i+2q} \\ \leq \frac{1}{q+1}.$$

Applying (2.27), we infer that

$$\begin{aligned} |g_{i,j}(x)| &\leq \frac{(-x)^{3i+2}3^{i}i!}{(3i+2)!} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{(\lambda_j - a)^{p+q}x^{2p+2q}}{(p+q+1)(2p+2q)!} \\ &= \frac{(-x)^{3i+2}3^{i}i!}{(3i+2)!} \sum_{k=0}^{\infty} \frac{(\lambda_j - a)^k x^{2k}}{(2k)!} \\ &= \cosh(\sqrt{\lambda_j - a}x) \frac{(-x)^{3i+2}3^{i}i!}{(3i+2)!}, \end{aligned}$$

where we have used the fact that for any function $f: \mathbb{N} \to \mathbb{R}_+$, it holds

$$\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} f(p+q) = \sum_{k=0}^{\infty} (k+1)f(k).$$

This ends the proof of Proposition 2.2.

Remark 2.3. Compared with the result in [13, Lemma 2.1], it seems that a more natural estimate of $g_{i,j}$ is

$$|g_{i,j}(x)| \le \cosh(\sqrt{\lambda_j - a}) \frac{R^i (-x)^{3i+2}}{(3i+2)!}$$

for some constant R > 0. According to the proof of Proposition 2.2, to prove this result, we need to obtain that

$$\frac{(3i+2q-1)!}{(3i-1)!} \le \frac{R}{q+1} \frac{(3i+2+2q)!}{(3i+2)!} \quad \forall q \ge 0, \ \forall i \ge 1.$$

This is equivalent to

$$\frac{(q+1)(3i)(3i+1)(3i+2)}{(3i+2q)(3i+2q+1)(3i+2q+2)} \le R \quad \forall q \ge 0, \ \forall i \ge 1.$$

However, this is impossible if we pick q = 3i.

Using Proposition 2.2, we can obtain the following corollary which will be used in the proof of the main results.

Corollary 2.1. For any $i \ge 0, j \ge 1$ and $x \in [-1, 0]$, we have

$$|g_{i,j}(x)| \le Ce^{\sqrt{\lambda_j}} \frac{1}{(2i)!},$$
(2.29)

where the constant C is independent of i and j.

Proof. By Stirling's formula $i! \sim (i/e)^i \sqrt{2\pi i}$, and it follows from (2.25) that for $i \ge 1$ and $j \ge 1$ we have

$$\begin{aligned} |g_{i,j}(x)| &\leq e^{\sqrt{\lambda_j}} \frac{3^i i!}{(3i+2)!} \\ &\leq C e^{\sqrt{\lambda_j}} \frac{3^i i!}{(3i+1)(3i+2)\frac{3^{3i}}{2^{2i}}\frac{\sqrt{6\pi i}}{\sqrt{2\pi i}\sqrt{4\pi i}}(2i)! i!} \\ &\leq C e^{\sqrt{\lambda_j}} \frac{1}{(2i)!}. \end{aligned}$$

3 Null controllability

Proposition 3.1. Let $s \in [0,2)$, $0 < t_1 < t_2 \le T$ and $z_j \in G^s([t_1, t_2])$ satisfy

$$|z_j^{(i)}(t)| \le M_j \frac{(i!)^s}{R^i},$$

where R is a positive constant and the positive constants M_j are such that

$$\sum_{j=1}^{\infty} M_j e^{\sqrt{\lambda_j}} < \infty.$$
(3.1)

Then the function u defined by (1.9) solves system (1.4)-(1.8) and $u \in G^{\frac{s}{2},\frac{s}{2},s}([-1,0] \times [0,1] \times [t_1,t_2]).$

Proof. As the proof is similar to that of [13, Proposition 2.1], it is only sketched.

Let $m, p, q \in \mathbb{N}$. By applying Proposition 2.1 and (1.9), we obtain that

$$\begin{split} |\partial_t^m \partial_x^p \partial_y^q u(x, y, t)| &\leq C \|\partial_t^m u(\cdot, \cdot, t)\|_{H^{p+q+2}(\Omega)} \\ &\leq C B^{[\frac{p+q+2}{2}]+1} \sum_{n=0}^{[\frac{p+q+2}{2}]+1} \|P^n \partial_t^m u(\cdot, \cdot, t)\|_{L^2(\Omega)} \\ &\leq C B^{[\frac{p+q+2}{2}]+1} \sum_{n=0}^{[\frac{p+q+2}{2}]+1} \sup_{(x,y)\in\Omega} |\partial_t^m P^n u(x, y, t)| \\ &\leq C B^{[\frac{p+q+2}{2}]+1} \sum_{n=0}^{[\frac{p+q+2}{2}]+1} \sup_{(x,y)\in\Omega} \sum_{j=1}^{\infty} \sum_{i=0}^{\infty} |\partial_t^m P^n (g_{i,j}(x) z_j^{(i)}(t) e_j(y))|. \end{split}$$

By the definitions of $g_{i,j}$ and e_j , it is clear that

$$\partial_t^m P^n(g_{i,j}(x)z_j^{(i)}(t)e_i(y)) = \begin{cases} z_j^{(i+m)}(t)(-1)^n g_{i-n,j}(x)e_j(y), & i \ge n; \\ 0, & i < n. \end{cases}$$

Setting k = i - n and N = n + m, arguing as in [13, Proposition 2.1], we infer from Corollary 2.1 that

$$\begin{split} \sum_{j=1}^{\infty} \sum_{i=0}^{\infty} |\partial_t^m P^n(g_{i,j}(x) z_j^{(i)}(t) e_i(y))| &= \sum_{j=1}^{\infty} \sum_{i=n}^{\infty} |z_j^{(i+m)}(t) g_{i-n,j}(x) e_j(y)| \\ &\leq C \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} M_j \frac{(k+N)!^s}{R^{k+N}} e^{\sqrt{\lambda_j}} \frac{1}{(2k)!} \\ &\leq C \frac{(N!)^s}{(\frac{R}{2^s})^N} \\ &\leq C \frac{(n!)^s (m!)^s}{R_1^n R_2^m}, \end{split}$$

where $R_1 = R_2 = R/4^s$.

Gathering the above estimates together, we obtain that

$$\begin{split} |\partial_t^m \partial_x^p \partial_y^q u(x, y, t)| \leq & CB^{[\frac{p+q+2}{2}]+1} \sum_{n=0}^{[\frac{p+q+2}{2}]+1} \sup_{(x,y)\in\Omega} \sum_{j=1}^\infty \sum_{i=0}^\infty |\partial_t^m P^n(g_{i,j}(x) z_j^{(i)}(t) e_i(y))| \\ \leq & CB^{[\frac{p+q+2}{2}]+1} \sum_{n=0}^{[\frac{p+q+2}{2}]+1} \frac{(n!)^s (m!)^s}{R_1^n R_2^m} \\ \leq & C\frac{(p!)^{\frac{s}{2}}(q!)^{\frac{s}{2}}(m!)^s}{R_1^p \overline{R}_2^q \overline{R}_3^m} \end{split}$$

for some positive constants $\overline{R}_1, \overline{R}_2, \overline{R}_3$. Finally, it is easily seen that u is indeed a solution of the ZK system.

Let \overline{u} denote the solution of the free evolution for the ZK system:

$$\begin{cases} \overline{u}_t + a\overline{u}_x + \Delta \overline{u}_x = 0, & (x,y) \in (-1,0) \times (0,1), \ t \in (0,T), \\ \overline{u}(-1,y,t) = \overline{u}(0,y,t) = \overline{u}_x(0,y,t) = 0, & y \in (0,1), \ t \in (0,T), \\ \overline{u}(x,0,t) = \overline{u}(x,1,t) = 0, & x \in (-1,0), \ t \in (0,T), \\ \overline{u}(x,y,0) = u_0(x,y), & x \in (-1,0), \ y \in (0,1). \end{cases}$$
(3.2)

As for KdV, we have a Kato smoothing effect.

Proposition 3.2. Let $u_0 \in L^2(\Omega)$.

1. System (3.2) admits a unique solution $\overline{u} \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1_0(\Omega))$ and we have

$$\sup_{t \in [0,T]} \|\overline{u}(\cdot, \cdot, t)\|_{L^2(\Omega)}^2 + \int_0^T \|\overline{u}(\cdot, \cdot, t)\|_{H^1(\Omega)}^2 dt \le C \|u_0\|_{L^2(\Omega)}^2.$$
(3.3)

2. If, in addition, $u_0 \in \mathcal{D}(A) \cap H^3(\Omega)$, then $\overline{u} \in C([0,T]; H^3(\Omega)) \cap L^2(0,T; H^4(\Omega))$ and we have

$$\sup_{t \in [0,T]} \|\overline{u}(\cdot, \cdot, t)\|_{H^3(\Omega)}^2 + \int_0^T \|\overline{u}(\cdot, \cdot, t)\|_{H^4(\Omega)}^2 dt \le C \|u_0\|_{H^3(\Omega)}^2.$$
(3.4)

Proof. (i) comes from [19]. Let us proceed with the proof of (ii). For any $u_0 \in \mathcal{D}(A) \cap H^3(\Omega)$, we have that $\overline{u} \in C([0,T]; \mathcal{D}(A))$ by the semigroup theory, and hence $\overline{u} \in C([0,T]; H^2(\Omega) \cap H_0^1(\Omega))$. Let $w_0 = Au_0$ and $w = A\overline{u}$. It is well known that w is the solution of (3.2) with initial value $w_0 \in L^2(\Omega)$. According to (i), we have

$$-\Delta \overline{u}_x - a\overline{u}_x = A\overline{u} = w \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1_0(\Omega))$$

Therefore $\Delta \overline{u}_x \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega))$. Assume finally that $u_0 \in \mathcal{D}(A) \cap H^3(\Omega)$, and let us prove that $u \in C([0,T], H^3(\Omega)) \cap L^2(0,T, H^4(\Omega))$. Decompose u as $u(x,y,t) = \sum_{j=1}^{\infty} \hat{u}_j(x,t) e_j(y)$. Then for $j \ge 1$, \hat{u}_j solves

$$\frac{d\hat{u}_j}{dt} + \hat{u}_j''' + (a - \lambda_j)\hat{u}_j' = 0, \qquad (3.5)$$

$$\hat{u}_j(-1,t) = \hat{u}_j(0,t) = \hat{u}'_j(0,t) = 0,$$
(3.6)

$$\hat{u}_j(.,0) = \hat{u}_j^0, \tag{3.7}$$

where $u_0(x,y) = \sum_{j=1}^{\infty} \hat{u}_j^0(x) e_j(y)$. Multiplying in (3.5) by \hat{u}_j (resp. by $(x+1)\hat{u}_j$) and integrating over $(-1,0)_x \times (0,T)_t$, we obtain respectively

$$\int_{-1}^{0} |\hat{u}_{j}(x,T)|^{2} dx + \int_{0}^{T} |\hat{u}_{j}'(-1,t)|^{2} dt = \int_{-1}^{0} |\hat{u}_{j}^{0}(x)|^{2} dx, \qquad (3.8)$$
$$\int_{-1}^{0} (x+1)|\hat{u}_{j}(x,T)|^{2} dx + 3 \int_{0}^{T} \int_{-1}^{0} |\hat{u}_{j}'|^{2} dx dt + (\lambda_{j}-a) \int_{0}^{T} \int_{-1}^{0} |\hat{u}_{j}|^{2} dx dt$$

$$= \int_{-1}^{0} (x+1) |\hat{u}_{j}^{0}(x)|^{2} dx.$$
(3.9)

It follows from (3.8) that for any $k \in \mathbb{N}$

$$\sum_{j=1}^{\infty} \lambda_j^k \| \hat{u}_j(.,t) \|^2 \le \sum_{j=1}^{\infty} \lambda_j^k \| \hat{u}_j^0 \|^2, \qquad \forall t \in \mathbb{R}_+$$
(3.10)

(that is, $\|\partial_y^k u(.,.,t)\|_{L^2(\Omega)}^2 \le \|\partial_y^k u_0\|_{L^2(\Omega)}^2$ for all $t \in \mathbb{R}_+$), and from (3.9) that

$$\int_{0}^{T} \sum_{j=1}^{\infty} (\lambda_{j}^{k} \| \hat{u}_{j}'(.,t) \|^{2} + \lambda_{j}^{k+1} \| \hat{u}_{j}(.,t) \|^{2}) dt \leq (1+aT) \sum_{j=1}^{\infty} \lambda_{j}^{k} \| \hat{u}_{j}^{0} \|^{2}, \qquad \forall T > 0$$
(3.11)

(that is, $\int_0^T \|\nabla \partial_y^k u(.,.,t)\|_{L^2(\Omega)}^2 dt \leq (1+aT) \|\partial_y^k u_0\|_{L^2(\Omega)}^2$ for all T > 0). We need the following lemma.

Lemma 3.1. Let $a \ge 0$ and $\lambda > 0$ be given. Let H^k $(k \in \mathbb{N})$ denote the Sobolev space $H^k(-1,0)$, and let $\mathcal{H}^3 := \{u \in H^3(-1,0); u(-1) = u(0) = u'(0) = 0\}$. Let $\|\cdot\|$ denote the norm $\|\cdot\|_{L^2(-1,0)}$. 1. There exists a constant C > 0 such that

$$\sum_{k=0}^{3} \lambda^{k} \|\partial_{x}^{3-k}y\|^{2} \leq C\left(\|y^{\prime\prime\prime}+(a-\lambda)y^{\prime}\|^{2}+\lambda^{3}\|y\|^{2}\right) \qquad \forall y \in \mathcal{H}^{3}, \ \forall \lambda \geq \lambda_{0}.$$
(3.12)

2. There exists a constant C' > 0 such that

$$\sum_{k=0}^{4} \lambda^{k} \|\partial_{x}^{4-k}y\|^{2} \leq C' \left(\|y^{(4)} + (a-\lambda)y''\|^{2} + \|y''' + (a-\lambda)y'\|^{2} + \lambda^{4} \|y\|^{2} \right)$$

$$\forall y \in \mathcal{H}^{3} \cap H^{4}, \ \forall \lambda > \lambda_{0}.$$
(3.13)

Proof of Lemma 3.1: 1. Pick any $y \in \mathcal{H}^3$ and any $\lambda \geq 0$. By the Interpolation Theorem and Young inequality, we have that

$$\lambda^{2} \|y'\|^{2} \leq C\lambda^{2} \|y\|^{\frac{4}{3}} \|y'''\|^{\frac{2}{3}} \leq \varepsilon \|y'''\|^{2} + C_{\varepsilon}\lambda^{3} \|y\|^{2},$$

$$\lambda \|y''\|^{2} \leq C\lambda \|y\|^{\frac{2}{3}} \|y'''\|^{\frac{4}{3}} \leq \varepsilon \|y'''\|^{2} + C_{\varepsilon}'\lambda^{3} \|y\|^{2}.$$

We infer that if $\lambda \geq \lambda_0 > 0$

$$\begin{aligned} \|y'''\|^2 &\leq 2\|y''' + (a-\lambda)y'\|^2 + 2(a-\lambda)^2\|y'\|^2 \\ &\leq 2\|y''' + (a-\lambda)y'\|^2 + 2\varepsilon\|y'''\|^2 + 2C_{\varepsilon}|a-\lambda|^3\|y\|^2 \\ &\leq 2\|y''' + (a-\lambda)y'\|^2 + 2\varepsilon\|y'''\|^2 + C_{\varepsilon}''\lambda^3\|y\|^2 \end{aligned}$$

and (3.12) follows by picking $\varepsilon < 1/4$. 2. Pick now any $y \in \mathcal{H}^3 \cap H^4$ and any $\lambda \ge 0$. Then we have

$$\lambda^{3} \|y'\|^{2} \leq C\lambda^{3} \|y\|^{\frac{3}{2}} \|y\|^{\frac{1}{2}}_{H^{4}} \leq \varepsilon(\|y^{(4)}\|^{2} + \|y'''\|^{2}) + C_{\varepsilon}\lambda^{4} \|y\|^{2},$$

$$\begin{split} \lambda^2 \|y''\|^2 &\leq C\lambda^2 \|y\| \|y\|_{H^4} \leq \varepsilon (\|y^{(4)}\|^2 + \|y'''\|^2) + C'_{\varepsilon}\lambda^4 \|y\|^2, \\ \lambda \|y'''\|^2 &\leq C\lambda \|y\|^{\frac{1}{2}} \|y\|^{\frac{3}{2}}_{H^4} \leq \varepsilon (\|y^{(4)}\|^2 + \|y'''\|^2) + C''_{\varepsilon}\lambda^4 \|y\|^2. \end{split}$$

On the other hand, we have that for $\lambda \geq \lambda_0 > 0$

$$\begin{aligned} \|y^{(4)}\|^2 &\leq 2\|y^{(4)} + (a-\lambda)y''\|^2 + 2(a-\lambda)^2\|y''\|^2 \\ &\leq 2\|y^{(4)} + (a-\lambda)y''\|^2 + 2\varepsilon(\|y^{(4)}\|^2 + \|y'''\|^2) + C_{\varepsilon}'''\lambda^4\|y\|^2, \end{aligned}$$

and (3.13) follows by picking $\varepsilon < 1/4$ and by using (3.12).

Assuming that $u_0 \in \mathcal{D}(A) \cap H^3(\Omega)$ and using (3.10) and (3.12), we obtain that for any $t \in [0,T]$ (with a constant C that may vary from line to line)

$$\begin{aligned} \|u(.,.,t)\|_{H^{3}(\Omega)}^{2} &= \|u(.,.,t)\|_{H^{2}(\Omega)}^{2} + \sum_{k=0}^{3} \|\partial_{y}^{k} \partial_{x}^{3-k} u(.,.,t)\|_{L^{2}(\Omega)}^{2} \\ &\leq C \|u_{0}\|_{\mathcal{D}(A)}^{2} + \sum_{k=0}^{3} \sum_{j=1}^{\infty} \lambda_{j}^{k} \|\partial_{x}^{3-k} \hat{u}_{j}(.,t)\|^{2} \\ &\leq C \|u_{0}\|_{\mathcal{D}(A)}^{2} + C \sum_{k=0}^{3} \sum_{j=1}^{\infty} \left(\|\hat{u}_{j}^{\prime\prime\prime}(.,t) + (a-\lambda_{j})\hat{u}_{j}^{\prime}(.,t)\|^{2} + \lambda_{j}^{3} \|\hat{u}_{j}(.,t)\|^{2} \right) \\ &\leq C \|u_{0}\|_{\mathcal{D}(A)}^{2} + C \|\partial_{y}^{3} u_{0}\|_{L^{2}(\Omega)}^{2} \\ &\leq C \|u_{0}\|_{\mathcal{H}^{3}(\Omega)}^{2}. \end{aligned}$$

On the other hand $\|u(.,.,t)\|_{H^4(\Omega)}^2 = \|u(.,.,t)\|_{H^3(\Omega)}^2 + \sum_{k=0}^4 \|\partial_y^k \partial_x^{4-k} u(.,.,t)\|_{L^2(\Omega)}^2$ and it is clear that $\int_0^T \|u(.,.,t)\|_{H^3(\Omega)}^2 dt \le C \|u_0\|_{H^3(\Omega)}^2$. Using (3.13), we obtain $\int_0^T \sum_{k=0}^4 \|\partial_y^k \partial_x^{4-k} u(.,.,t)\|_{L^2(\Omega)}^2 dt$

$$\begin{split} &= \int_0^T \sum_{k=0}^4 \sum_{j=1}^\infty \lambda_j^k \|\partial_x^{4-k} \hat{u}_j(.,t)\|^2 dt \\ &\leq C \int_0^T \sum_{j=1}^\infty \left(\|\hat{u}_j^{(4)} + (a-\lambda_j) \hat{u}_j''\|^2 + \|\hat{u}_j''' + (a-\lambda_j) \hat{u}_j'\|^2 + \lambda_j^4 \|\hat{u}_j\|^2 \right) dt \\ &\leq C \int_0^T \left(\|Au(.,.,t)\|_{H^1(\Omega)}^2 + \|\partial_y^4 u(.,.,t)\|_{L^2(\Omega)}^2 \right) dt \\ &\leq C \|u_0\|_{H^3(\Omega)}^2 \end{split}$$

where we used (3.11) with k = 3. This completes the proof of the proposition.

Interpolating between (3.3) and (3.4), we obtain

$$\sup_{t\in[0,T]} \|\overline{u}(\cdot,\cdot,t)\|_{H^{1}(\Omega)}^{2} + \int_{0}^{T} \|\overline{u}(\cdot,\cdot,t)\|_{H^{2}(\Omega)}^{2} dt \leq C \|u_{0}\|_{H^{1}(\Omega)}^{2},$$
$$\sup_{t\in[0,T]} \|\overline{u}(\cdot,\cdot,t)\|_{H^{2}(\Omega)}^{2} + \int_{0}^{T} \|\overline{u}(\cdot,\cdot,t)\|_{H^{3}(\Omega)}^{2} dt \leq C \|u_{0}\|_{H^{2}(\Omega)}^{2}.$$

This gives

$$\|\overline{u}(\cdot, \cdot, t)\|_{H^{n+1}(\Omega)} \le \frac{C}{\sqrt{t}} \|u_0\|_{H^n(\Omega)}, \text{ for } n \in \{0, 1, 2, 3\}.$$

Proceeding as in [13, Proposition 2.2], we can show that if $u_0 \in L^2(\Omega)$, then $\overline{u}(t) \in \mathcal{D}(A^n)$ for any $t \in (0,T]$ and $n \in \mathbb{N}$, and it holds

$$\|A^{n}\overline{u}(\cdot,\cdot,t)\|_{L^{2}(\Omega)} \leq \frac{C^{n}}{t^{\frac{3n}{2}}} n^{\frac{3n}{2}} \|u_{0}\|_{L^{2}(\Omega)}.$$
(3.14)

Without loss of generality, we assume that T = 1. Then for any $p, q \in \mathbb{N}$, we infer from Proposition 2.1 that

$$\begin{split} |\partial_x^p \partial_y^q \overline{u}(x, y, t)| &\leq \|\overline{u}(\cdot, \cdot, t)\|_{H^{p+q+2}(\Omega)} \\ &\leq C_0 B^{[\frac{p+q+2}{2}]+1} \sum_{n=0}^{[\frac{p+q+2}{2}]+1} \|P^n \overline{u}(\cdot, \cdot, t)\|_{L^2(\Omega)} \\ &\leq C_0 B^{[\frac{p+q+2}{2}]+1} \sum_{n=0}^{[\frac{p+q+2}{2}]+1} \frac{C^n}{t^{\frac{3n}{2}}} n^{\frac{3n}{2}} \|u_0\|_{L^2(\Omega)} \\ &\leq C t^{-\frac{3}{2}[\frac{p+q}{2}]-3} \frac{(p!)^{\frac{3}{4}}(q!)^{\frac{3}{4}}}{R_1^n R_2^q} \end{split}$$

for some $R_1, R_2 > 0$. This means that $\overline{u}(\cdot, \cdot, t) \in G^{\frac{3}{4}, \frac{3}{4}}([-1, 0] \times [0, 1])$ for any $t \in (0, T]$. Let

$$f_j(t) := \int_0^1 e_j(y) \partial_x^2 \overline{u}(0, y, t) dy.$$

Lemma 3.2. For any $j \ge 1$ and $n \ge 0$, there exist positive constants R_1, R_2 and C such that

$$|f_j^{(n)}(t)| \le \frac{C}{(j\pi)^j} t^{-\frac{3}{2}(n+[\frac{j}{2}]+3)} \frac{(n!)^{\frac{3}{2}}(j!)^{\frac{3}{4}}}{R_1^n R_2^j}.$$

Proof. Without loss of generality, we can assume that T = 1. Since $\overline{u}(\cdot, \cdot, t) \in \mathcal{D}(A^n)$ for any $t \in (0, T]$ and $n \in \mathbb{N}$, it follows from Lemma 2.2 that

$$\partial_x^2 \partial_y^{2n} \overline{u}(x,0,t) = \partial_x^2 \partial_y^{2n} \overline{u}(x,1,t) = 0, \quad \forall x \in [-1,0], \ \forall t \in (0,T], \ \forall n \in \mathbb{N}.$$

Then, integrating by parts j-times, we deduce that

$$\begin{split} f_{j}(t) &= \sqrt{2} \int_{0}^{1} \sin(j\pi y) \partial_{x}^{2} \overline{u}(0, y, t) dy \\ &= \frac{\sqrt{2}}{j\pi} \int_{0}^{1} \cos(j\pi y) \partial_{x}^{2} \partial_{y} \overline{u}(0, y, t) dy \\ &= \frac{\sqrt{2}}{(j\pi)^{2}} \int_{0}^{1} \sin(j\pi y) \partial_{x}^{2} \partial_{y}^{2} \overline{u}(0, y, t) dy \\ &= \begin{cases} \frac{\sqrt{2}}{(j\pi)^{j}} \int_{0}^{1} \sin(j\pi y) \partial_{x}^{2} \partial_{y}^{j} \overline{u}(0, y, t) dy, & \text{if } j \text{ is even;} \\ -\frac{\sqrt{2}}{(j\pi)^{j}} \int_{0}^{1} \cos(j\pi y) \partial_{x}^{2} \partial_{y}^{j} \overline{u}(0, y, t) dy, & \text{if } j \text{ is odd.} \end{cases} \end{split}$$
(3.15)

To estimate $|f_j^{(n)}(t)|(n \in \mathbb{N})$, it remains to estimate $|\partial_t^n \partial_x^2 \partial_y^j \overline{u}(0, y, t)|$. Let

$$l = [\frac{j+4}{2}] + 1.$$

Taking (2.6) (with $u = P^i \overline{u}$) and (3.14) into account, we obtain that

$$\begin{aligned} |\partial_{t}^{n}\partial_{x}^{2}\partial_{y}^{j}\overline{u}(x,y,t)| &= |P^{n}\partial_{x}^{2}\partial_{y}^{j}\overline{u}(x,y,t)| \\ &\leq C\|P^{n}\overline{u}(\cdot,\cdot,t)\|_{H^{j+4}(\Omega)} \\ &\leq CB^{l}\sum_{k=0}^{l}\|P^{n+k}\overline{u}(\cdot,\cdot,t)\|_{L^{2}(\Omega)} \\ &\leq CB^{l}\sum_{k=0}^{n+l}\|P^{k}\overline{u}(\cdot,\cdot,t)\|_{L^{2}(\Omega)} \\ &\leq CB^{l}\sum_{k=0}^{n+l}\frac{C^{k}k^{\frac{3}{2}k}}{t^{\frac{3}{2}k}}\|u_{0}\|_{L^{2}(\Omega)} \\ &\leq CB^{l}\frac{C^{n+l}(n+l+1)(n+l)^{\frac{3}{2}(n+l)}}{t^{\frac{3}{2}(n+l)}}\|u_{0}\|_{L^{2}(\Omega)} \\ &\leq Ct^{-\frac{3}{2}(n+[\frac{j}{2}]+3)}\frac{(n!)^{\frac{3}{2}}(j!)^{\frac{3}{4}}}{R_{1}^{n}R_{2}^{j}}\|u_{0}\|_{L^{2}(\Omega)} \end{aligned}$$
(3.16)

for some $R_1, R_2 > 0$.

Combining (3.15) and (3.16), we obtain

$$\begin{split} |f_j^{(n)}(t)| \leq & \frac{C}{(j\pi)^j} \sup_{y \in [0,1]} |\partial_t^n \partial_x^2 \partial_y^j \overline{u}(0,y,t)| \\ \leq & \frac{C}{(j\pi)^j} t^{-\frac{3}{2}(n+[\frac{j}{2}]+3)} \frac{(n!)^{\frac{3}{2}}(j!)^{\frac{3}{4}}}{R_1^n R_2^j} \cdot \end{split}$$

	-	

Now, we can prove the first main result in this paper.

Proof of Theorem 1.1. Pick any $\tau \in (0,T), s \in [3/2,2)$ and let

$$z_j(t) = \phi_s\left(\frac{t-\tau}{T-\tau}\right)f_j(t), \quad 0 \le t \le T,$$

where

$$\phi_{s}(\rho) = \begin{cases} 1 & \text{if } \rho \leq 0, \\ 0 & \text{if } \rho \geq 1, \\ \frac{e^{-\frac{M}{(1-\rho)^{\sigma}}}}{e^{-\frac{M}{\rho^{\sigma}}} + e^{-\frac{M}{(1-\rho)^{\sigma}}}} & \text{if } \rho \in (0,1) \end{cases}$$

with M > 0 and $\sigma = (s - 1)^{-1}$. As ϕ_s is Gevrey of order s, there exist $R_{\phi} > 0$ such that

$$|\phi_s^{(p)}(\rho)| \le C \frac{(p!)^s}{R_{\phi}^p} \quad \forall \ p \in \mathbb{N}, \rho \in \mathbb{R}.$$

Then, applying Lemma 3.2, for any $\varepsilon \in (0,T)$ and $t \in [\varepsilon,T]$, we have

$$\begin{split} |z_{j}^{(i)}(t)| &\leq \sum_{n=0}^{i} \binom{i}{n} \left| \partial_{t}^{i-n} [\phi_{s} \left(\frac{t-\tau}{T-\tau}\right)] \right| |f_{j}^{(n)}(t)| \\ &\leq C \sum_{n=0}^{i} \binom{i}{n} \frac{i}{n} \frac{(i-n)!^{s}}{R_{\phi}^{i-n}} (\frac{1}{T-\tau})^{i-n} \frac{1}{(j\pi)^{j}} t^{-\frac{3}{2}(n+[\frac{j}{2}]+3)} \frac{(n!)^{\frac{3}{2}}(j!)^{\frac{3}{4}}}{R_{1}^{n} R_{2}^{j}} \\ &\leq C \frac{1}{(j\pi)^{j}} \varepsilon^{-\frac{3}{2}([\frac{j}{2}]+3)} \frac{(j!)^{\frac{3}{4}}}{R_{2}^{j}} \sum_{n=0}^{i} \binom{i}{n} \frac{(i-n)!^{s}}{R_{\phi}^{i-n}} (\frac{1}{T-\tau})^{i-n} \varepsilon^{-\frac{3}{2}n} \frac{(n!)^{\frac{3}{2}}}{R_{1}^{n}} \\ &\leq C \frac{1}{(j\pi)^{j}} \varepsilon^{-\frac{3}{2}([\frac{j}{2}]+3)} \frac{(j!)^{\frac{3}{4}}}{R_{2}^{j}} \frac{(i!)^{s}}{\min\{R_{\phi},R_{1}\}^{i}} \sum_{n=0}^{i} \binom{i}{n} (\frac{1}{T-\tau})^{i-n} \varepsilon^{-\frac{3}{2}n} \\ &\leq M_{j} \frac{(i!)^{s}}{\hat{R}^{i}}, \end{split}$$

where M_j satisfies (3.1). Let

$$u(x, y, t) = \begin{cases} u_0(x, y) & \text{if } x \in [-1, 0], y \in [0, 1], t = 0, \\ \sum_{j=1}^{\infty} \sum_{i=0}^{\infty} g_{i,j}(x) z_j^{(i)}(t) e_j(y) & \text{if } x \in [-1, 0], y \in [0, 1], t \in (0, T]. \end{cases}$$

Then, it is easy to see that $u(\cdot, \cdot, T) = 0$. By Proposition 3.1, $u \in G^{\frac{s}{2}, \frac{s}{2}, s}([-1, 0] \times [0, 1] \times [\varepsilon, T])$ for any $\varepsilon \in (0, T)$. Furthermore, we have

$$\begin{split} u_t + au_x + \Delta u_x &= 0 = \overline{u}_t + a\overline{u}_x + \Delta \overline{u}_x \quad \text{in } \Omega \times (0, T), \\ u(0, y, t) &= 0 = \overline{u}(0, y, t), \quad \forall y \in [0, 1], \ \forall t \in (0, \tau), \\ \partial_x u(0, y, t) &= 0 = \partial_x \overline{u}(0, y, t), \quad \forall y \in [0, 1], \ \forall t \in (0, \tau), \\ \partial_x^2 u(0, y, t) &= \sum_{j=1}^{\infty} z_j(t) e_j(y) = \partial_x^2 \overline{u}(0, y, t), \quad \forall y \in [0, 1], \ \forall t \in (0, \tau). \end{split}$$

It follows from Holmgren theorem that $u(x, y, t) = \overline{u}(x, y, t)$ for any $(x, y, t) \in [-1, 0] \times [0, 1] \times (0, \tau)$. In particular, $u \in C([0, T]; L^2(\Omega))$ and h = 0 for $t \in [0, \tau)$, so that $h \in G^{\frac{s}{2}, s}([0, 1] \times [0, T])$. The proof of Theorem 1.1 is complete.

4 Reachable states

Proposition 4.1. For any $j \ge 1$, assume that $z_j \in G^2([0,T])$ is such that

$$|z_j^{(i)}(t)| \le M_j \frac{(2i)!}{R^{2i}}, \quad \forall \ i \ge 0, \ t \in [0,T],$$

where R > 1 and M_j satisfies (3.1). Then the function u defined by (1.9) solves system (1.4)-(1.8) and $u \in G^{1,1,2}([-1,0] \times [0,1] \times [0,T])$.

Proof. According to the proof of Proposition 3.1, for any $m, p, q \in \mathbb{N}$, we have

$$|\partial_t^m \partial_x^p \partial_y^q u(x, y, t)| \le CB^{\left[\frac{p+q+2}{2}\right]+1} \sum_{n=0}^{\left[\frac{p+q+2}{2}\right]+1} \sup_{x,y)\in\Omega} \sum_{j=1}^{\infty} \sum_{i=n}^{\infty} |z_j^{(i+m)}(t)g_{i-n,j}(x)e_j(y)|.$$

Let k = 2i - 2n and N = 2n + 2m. We can obtain by the same arguments as in [13, Proposition 3.1] that

$$\begin{split} \sum_{j=1}^{\infty} \sum_{i=n}^{\infty} |z_{j}^{(i+m)}(t)g_{i-n,j}(x)e_{j}(y)| &\leq \sum_{j=1}^{\infty} \sum_{i=n}^{\infty} M_{j} \frac{(2i+2m)!}{R^{2i+2m}} \frac{Ce^{\sqrt{\lambda_{j}}}}{(2i-2n)!} \\ &= \sum_{j=1}^{\infty} CM_{j}e^{\sqrt{\lambda_{j}}} \sum_{k=0}^{\infty} \frac{(k+N)!}{R^{k+N}k!} \\ &\leq C \sum_{k=0}^{\infty} \frac{(k+N)!}{R^{k+N}k!} \\ &= C \sum_{k=0}^{\infty} \frac{(k+1)\cdots(k+N)}{R^{k+N}} \\ &\leq C(\frac{\alpha e}{R^{\sigma}})^{N}N!\sqrt{N} \\ &\leq C \frac{(2n)!(2m)!}{R_{1}^{n}R_{2}^{m}}, \end{split}$$

where R_1, R_2 are two positive constants, $\sigma \in (0, 1)$ and

$$\alpha = \sup_{k \ge 0} \frac{k+2}{(R^{1-\sigma})^{k+1}}.$$

It follows from the above estimates that

$$\begin{split} |\partial_t^m \partial_x^p \partial_y^q u(x, y, t)| \leq & CB^{[\frac{p+q+2}{2}]+1} \sum_{n=0}^{[\frac{p+q+2}{2}]+1} \frac{(2n)!(2m)!}{R_1^n R_2^m} \\ \leq & C\frac{p!q!(m!)^2}{\widehat{R}_1^p \widehat{R}_2^q \widehat{R}_3^m} \end{split}$$

for some positive constants \hat{R}_1, \hat{R}_2 and \hat{R}_3 . This ends the proof of Proposition 4.1.

As a particular case of [12, Proposition 3.6] (with $a_0 = 1, a_p = [2p(2p-1)]^{-1}$ for $p \ge 1$), we have the following result.

Proposition 4.2. Let $\{d_q\}_{q\geq 0}$ be a sequence of real numbers such that

$$|d_q| \le CH^q(2q)! \quad \forall \ q \ge 0$$

for some H > 0 and C > 0. Then for all $\tilde{H} > e^{e^{-1}}H$, there exists a function $f \in C^{\infty}(\mathbb{R})$ such that

$$\begin{aligned} f^{(q)}(0) &= d_q \quad \forall \ q \ge 0, \\ |f^{(q)}(x)| &\le C \widetilde{H}^q(2q)! \quad \forall \ q \ge 0, \ x \in \mathbb{R}. \end{aligned}$$

Let

$$\begin{aligned} \mathcal{X} &:= & \{ u \in C^{\infty}([-1,0] \times [0,1]); \\ & P^{n}u(0,y) = \partial_{x}P^{n}u(0,y) = P^{n}u(x,0) = P^{n}u(x,1) = 0, \quad \forall n \in \mathbb{N}, \ \forall x \in [-1,0], \ \forall y \in [0,1] \} \end{aligned}$$

A result similar to Lemma 2.2 can be derived.

Lemma 4.1. For any $n \in \mathbb{N}$, we have

$$\partial_y^{2n} f(x,0) = \partial_y^{2n} f(x,1) = 0, \quad \forall f \in \mathcal{X}, \ \forall x \in [-1,0].$$

$$(4.1)$$

Proof. We proceed by induction on n. For n = 0, (4.1) is obvious since $f \in \mathcal{X}$. Assume now that (4.1) is true for $n - 1 \ge 0$. If $f \in \mathcal{X}$, then $Pf \in \mathcal{X}$, so that by the induction hypothesis

$$\partial_y^{2n-2} Pf(x,0) = \partial_y^{2n-2} Pf(x,1) = 0.$$

This implies

$$\partial_x^3 \partial_y^{2n-2} f(x,0) + \partial_x \partial_y^{2n} f(x,0) + a \partial_x \partial_y^{2n-2} f(x,0) = 0,$$

$$\partial_x^3 \partial_y^{2n-2} f(x,1) + \partial_x \partial_y^{2n} f(x,1) + a \partial_x \partial_y^{2n-2} f(x,1) = 0.$$

Since (4.1) is true for n-1, we obtain that

$$\partial_x \partial_y^{2n} f(x,0) = \partial_x \partial_y^{2n} f(x,1) = 0.$$

This means that for some constants C_1 and C_2 ,

$$\partial_y^{2n} f(x,0) = C_1, \ \partial_y^{2n} f(x,1) = C_2 \quad \forall x \in [-1,0].$$

On the other hand, we infer from the assumption $f \in \mathcal{X}$ that

$$\partial_y^{2n} f(0, y) = 0 \quad \forall y \in [0, 1].$$

Taking y = 0 and next y = 1, we see that $C_1 = C_2 = 0$. The proof of Lemma 4.1 is complete. \Box

Lemma 4.2. If $f \in \mathcal{X}$ is such that

$$\int_0^1 e_l(y) P^n f(0, y) dy = \int_0^1 e_l(y) \partial_x P^n f(0, y) dy = \int_0^1 e_l(y) \partial_x^2 P^n f(0, y) dy = 0$$
(4.2)

for any $l \ge 1$ and any $n \ge 0$, then

$$\int_{0}^{1} e_{l}(y)\partial_{x}^{m}f(0,y)dy = 0$$
(4.3)

holds for any $l \geq 1$ and any $m \geq 0$.

Proof. To prove that (4.3) holds for any $l \ge 1$ and any $m \ge 0$, it is sufficient to show that for any $M \in \mathbb{N}$, (4.3) holds for any $l \ge 1$ and any $m \le 3M + 2$. We proceed by induction on M.

For M = 0, we can take n = 0 in (4.2) to see that (4.3) holds for any $l \ge 1$ and $m \le 2$.

Assume that (4.3) is true for any $l \ge 1$ and any $m \le 3M - 1$. We claim that (4.3) holds for any $l \ge 1$ and m = 3M, 3M + 1, 3M + 2. Indeed, taking n = M in (4.2), we have

$$0 = (-1)^{M} \int_{0}^{1} e_{l}(y) P^{M} f(0, y) dy$$

=
$$\int_{0}^{1} e_{l}(y) (\partial_{x}^{2} + \partial_{y}^{2} + a)^{M} \partial_{x}^{M} f(0, y) dy$$

=
$$\int_{0}^{1} e_{l}(y) \partial_{x}^{3M} f(0, y) dy$$

+
$$\int_{0}^{1} e_{l}(y) \sum_{k=0}^{M-1} {M \choose k} \sum_{i=0}^{M-k} {M-k \choose i} a^{M-k-i} \partial_{x}^{2k+M} \partial_{y}^{2i} f(0, y) dy.$$
(4.4)

Since $f \in \mathcal{X}$, it follows from Lemma 4.1 that

$$\partial_y^{2n} f(x,0) = \partial_y^{2n} f(x,1) = 0, \quad \forall x \in [-1,0], \ \forall n \in \mathbb{N}.$$

Then, we obtain by integrations by parts that for $k \in \{0, ..., M-1\}$ and $i \in \{0, ..., M-k\}$

$$\int_0^1 e_l(y) \partial_x^{2k+M} \partial_y^{2i} f(0,y) dy = (-1)^i (l\pi)^{2i} \int_0^1 e_l(y) \partial_x^{2k+M} f(0,y) dy = 0.$$

In the last step, we used the fact that $2k + M \leq 3M - 1$. Thus, we infer from (4.4) that

$$\int_0^1 e_l(y) \partial_x^{3M} f(0, y) dy = 0, \quad \forall l \ge 1.$$

We can show in the same way that (4.3) is true for m = 3M + 1, 3M + 2 by using the fact that

$$\int_{0}^{1} e_{l}(y)\partial_{x}P^{M}f(0,y)dy = \int_{0}^{1} e_{l}(y)\partial_{x}^{2}P^{M}f(0,y)dy = 0, \quad \forall l \ge 1.$$

The proof of Lemma 4.2 is complete.

Now, we are in a position to prove the second main result in this paper.

Proof of Theorem 1.2. Assume that $R := \min\{R_1, R_2\} > R_0 = \sqrt[3]{9(a+2)}e^{(3e)^{-1}}$ and pick any $u_1 \in \mathcal{R}_{R_1,R_2}$. We intend to expand u_1 in the following form:

$$u_1(x,y) = \sum_{j=1}^{\infty} \sum_{i=0}^{\infty} b_{i,j} g_{i,j}(x) e_j(y),$$

where

$$b_{i,j} = (-1)^i \int_0^1 e_j(y) \partial_x^2 P^i u_1(0,y) dy.$$

Since $u_1 \in \mathcal{R}_{R_1,R_2} \subset \mathcal{X}$, we have that $P^i u_1 \in \mathcal{X}$ for any $i \in \mathbb{N}$. By Lemma 4.1, we infer that

$$\partial_y^{2n} P^i u_1(x,0) = \partial_y^{2n} P^i u_1(x,1) = 0, \quad \forall x \in [-1,0].$$

Then, by integration by parts, we have

$$|b_{i,j}| = |\int_0^1 e_j(y)\partial_x^2 P^i u_1(0,y)dy| \le \frac{C}{(j\pi)^j} \sup_{(x,y)\in\Omega} |\partial_x^2 \partial_y^j P^i u_1(x,y)|.$$

Next, we estimate $|\partial_x^2 \partial_y^j P^i u_1(x,y)|$.

$$\begin{split} |\partial_x^2 \partial_y^j P^i u_1(x,y)| &= |\partial_x^2 \partial_y^j \sum_{n=0}^i \left(\begin{array}{c} i \\ n \end{array} \right) (\partial_x^2 + \partial_y^2)^n \partial_x^n (a\partial_x)^{i-n} u_1(x,y)| \\ &= |\partial_x^{i+2} \partial_y^j \sum_{n=0}^i \left(\begin{array}{c} i \\ n \end{array} \right) a^{i-n} (\partial_x^2 + \partial_y^2)^n u_1(x,y)| \\ &= |\partial_x^{i+2} \partial_y^j \sum_{n=0}^i \left(\begin{array}{c} i \\ n \end{array} \right) a^{i-n} \sum_{m=0}^n \left(\begin{array}{c} n \\ m \end{array} \right) \partial_x^{2m} \partial_y^{2n-2m} u_1(x,y)| \\ &\leq \sum_{n=0}^i \sum_{m=0}^n \left(\begin{array}{c} i \\ n \end{array} \right) \left(\begin{array}{c} n \\ m \end{array} \right) a^{i-n} |\partial_x^{2m+i+2} \partial_y^{2n-2m+j} u_1(x,y)| \\ &\leq C \sum_{n=0}^i \sum_{m=0}^n \left(\begin{array}{c} i \\ n \end{array} \right) \left(\begin{array}{c} n \\ m \end{array} \right) a^{i-n} \frac{(2m+i+2)!^{\frac{2}{3}}(2n-2m+j)!^{\frac{2}{3}}}{R_1^{2m+i+2}R_2^{2n-2m+j}} \\ &\leq C \sum_{n=0}^i \sum_{m=0}^n \left(\begin{array}{c} i \\ n \end{array} \right) \left(\begin{array}{c} n \\ m \end{array} \right) a^{i-n} \frac{(2m+i+2)!^{\frac{2}{3}}(2n-2m+j)!^{\frac{2}{3}}}{R^{2m+i+2}R_2^{2n-2m+j}}. \end{split}$$

We notice that

$$\begin{aligned} (2m+i+2)!(2n-2m+j)! &= \left(\begin{array}{c} 2m+i+2\\2\end{array}\right) \left(\begin{array}{c} 2n-2m+j\\j\end{array}\right) 2!j!(2m+i)!(2n-2m)!\\ &\leq \left(\begin{array}{c} 2m+i+2\\2\end{array}\right) \left(\begin{array}{c} 2n-2m+j\\j\end{array}\right) 2!j!(2n+i)!, \end{aligned}$$

where we used the fact that

$$\binom{2n+i}{2m+i} = \frac{(2n+i)!}{(2m+i)!(2n-2m)!} \ge 1.$$

According to [9, Lemma A.1], we have

$$\left(\begin{array}{c}2m+i+2\\2\end{array}\right)\left(\begin{array}{c}2n-2m+j\\j\end{array}\right) \leq \left(\begin{array}{c}2n+i+j+2\\j+2\end{array}\right).$$

This implies

$$\begin{split} (2m+i+2)!(2n-2m+j)! &\leq \left(\begin{array}{c} 2n+i+j+2\\ j+2 \end{array}\right) 2!j!(2n+i)!\\ &= \frac{(2n+i+j+2)!2!j!(2n+i)!}{(j+2)!(2n+i)!}\\ &\leq (2n+i+j+2)!. \end{split}$$

Combining the above estimates, we infer that

$$\begin{split} |b_{i,j}| &\leq \frac{C}{(j\pi)^j} \sum_{n=0}^i \sum_{m=0}^n \binom{i}{n} \binom{n}{m} a^{i-n} \frac{(2n+i+j+2)!^{\frac{2}{3}}}{R^{2n+i+j+2}} \\ &\leq \frac{C}{(j\pi)^j} \sum_{n=0}^i \binom{i}{n} 2^n a^{i-n} \frac{(3i+j+2)!^{\frac{2}{3}}}{R^{3i+j+2}} \\ &= \frac{C}{(j\pi)^j} \frac{(3i+j+2)!^{\frac{2}{3}}}{R^{3i+j+2}} (2+a)^i \\ &\leq \frac{C2^{\frac{2}{3}(3i+j+2)}(3i)!^{\frac{2}{3}}(j+2)!^{\frac{2}{3}}(2+a)^i}{(j\pi)^j R^{3i+j+2}} \\ &\leq \frac{C2^{\frac{2}{3}(j+2)}(j+2)!^{\frac{2}{3}}}{(j\pi)^j R^{j+2}} \frac{2^{2i}(3i)!^{\frac{2}{3}}(2+a)^i}{R^{3i}} \\ &\leq \frac{C2^{\frac{2}{3}(j+2)}(j+2)!^{\frac{2}{3}}}{(j\pi)^j R^{j+2}} \frac{3^{2i}(6\pi i)^{\frac{1}{3}}(4\pi i)^{-\frac{1}{2}}(2i)!(2+a)^i}{R^{3i}} \\ &\leq \frac{C2^{\frac{2}{3}(j+2)}(j+2)!^{\frac{2}{3}}}{(j\pi)^j R^{j+2}} \frac{[9(2+a)]^i(2i)!}{R^{3i}} \\ &= M_j \frac{[9(2+a)]^i(2i)!}{R^{3i}}, \end{split}$$

where M_j satisfies (3.1). By Proposition 4.2, for any $j \ge 1$, there exists a function $h_j \in G^2([0,T])$ and a number $\tilde{R} > 1$ such that

$$\begin{aligned} h_{j}^{(i)}(T) &= b_{i,j} \quad \forall \ i \ge 0, \\ |h_{j}^{(i)}(t)| &\le M_{j} \frac{(2i)!}{\tilde{R}^{2i}} \quad \forall \ i \ge 0, \ t \in [0,T]. \end{aligned}$$

$$(4.5)$$

Pick any $\tau \in (0,T), s \in (1,2)$ and let

$$g(t) = 1 - \phi_s \left(\frac{t - \tau}{T - \tau}\right) \quad \text{for } t \in [0, T].$$

Setting

$$z_j(t) = h_j(t)g(t) \quad \forall \ t \in [0, T],$$

following the method developed in [12, Theorem 3.2], and taking into account the fact that s < 2, we see that z_j satisfies

$$z_{j}^{(i)}(T) = b_{i,j} \quad \forall \ j \ge 1, \ i \ge 0,$$

$$z_{j}^{(i)}(0) = 0 \quad \forall \ j \ge 1, \ i \ge 0,$$

$$|z_{j}^{(i)}(t)| \le CM_{j} \frac{(2i)!}{\tilde{R}^{2i}} \quad \forall \ j \ge 1, \ i \ge 0, \ t \in [0,T],$$

(4.6)

where \tilde{R} is the same as in (4.5) and C is a positive constant independent of i and j.

Let u be as in (1.9). According to (4.6), we have $u_0 = 0$ and

$$u(x, y, T) = \sum_{j=1}^{\infty} \sum_{i=0}^{\infty} g_{i,j}(x) z_j^{(i)}(T) e_j(y) = \sum_{j=1}^{\infty} \sum_{i=0}^{\infty} b_{i,j} g_{i,j}(x) e_j(y).$$

By Proposition 4.1, u solves system (1.4)-(1.8) and $u \in G^{1,1,2}([-1,0] \times [0,1] \times [0,T])$. Let

$$h(y,t) = u(-1,y,t) \quad \forall y \in [0,1], \forall t \in [0,T].$$

Then $h \in G^{1,2}([0,1] \times [0,T]).$

Finally, for any $l \ge 1$ and $n \ge 0$, we have

$$\begin{split} &\int_{0}^{1} e_{l}(y)P^{n}u(0,y,T)dy = \int_{0}^{1} e_{l}(y)\sum_{j=1}^{\infty}\sum_{i=n}^{\infty}b_{i,j}(-1)^{n}g_{i-n,j}(0)e_{j}(y)dy = 0\\ &= \int_{0}^{1} e_{l}(y)P^{n}u_{1}(0,y)dy,\\ &\int_{0}^{1} e_{l}(y)\partial_{x}P^{n}u(0,y,T)dy = \int_{0}^{1} e_{l}(y)\sum_{j=1}^{\infty}\sum_{i=n}^{\infty}b_{i,j}(-1)^{n}g_{i-n,j}'(0)e_{j}(y)dy = 0\\ &= \int_{0}^{1} e_{l}(y)\partial_{x}P^{n}u_{1}(0,y)dy,\\ &\int_{0}^{1} e_{l}(y)\partial_{x}^{2}P^{n}u(0,y,T)dy = \int_{0}^{1} e_{l}(y)\sum_{j=1}^{\infty}\sum_{i=n}^{\infty}b_{i,j}(-1)^{n}g_{i-n,j}'(0)e_{j}(y)dy = b_{nl}\\ &= \int_{0}^{1} e_{l}(y)\partial_{x}^{2}P^{n}u_{1}(0,y)dy. \end{split}$$

Since $u(\cdot, \cdot, T), u_1 \in \mathcal{X}$, it follows from Lemma 4.2 that

$$\int_{0}^{1} e_{l}(y) [\partial_{x}^{m} u(0, y, T) - \partial_{x}^{m} u_{1}(0, y)] dy = 0 \quad \forall l \ge 1, \ \forall m \ge 0,$$

and hence

$$\partial_x^m u(0, y, T) - \partial_x^m u_1(0, y) = 0 \quad \forall m \ge 0, \ \forall y \in [0, 1].$$

Since the map $x \to u(x, y, T) - u_1(x, y)$ is in $G^1([-1, 0])$ (i.e. is analytic) for any $y \in [0, 1]$, we infer that

$$u(x, y, T) = u_1(x, y) \quad \forall (x, y) \in [-1, 0] \times [0, 1].$$

The proof of Theorem 1.2 is complete.

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