

On Stably Free Ideal Domains

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Abstract

We define a stably free ideal domain to be a Noetherian domain whose left and right ideals are all stably free. Every stably free ideal domain is a (possibly noncommutative) Dedekind domain, but the converse does not hold. The first Weyl algebra over a field of characteristic 0 is a typical example of stably free ideal domain. Some properties of these rings are studied. A ring is a principal ideal domain if, and only if it is both a stably free ideal domain and an Hermite ring.

1 Introduction

In a principal ideal domain (resp. a Dedekind domain), every left or right ideal is free (resp. projective). An intermediate situation is the one where every left or right ideal is stably free. A Noetherian domain with this property is called a *stably free ideal domain* in what follows. In a Bézout domain, every finitely generated (f.g.) left or right ideal is free. An Ore domain in which every f.g. left or right ideal is stably free is called a *semistably free ideal domain* in what follows. Stably free ideal domains and semistably free ideal domains are briefly studied in this paper.

2 Free ideal domains and semistably free ideal domains

Theorem and Definition 1 *Let \mathbf{A} be a ring and consider the following conditions.*

- (i) *Every left or right ideal in \mathbf{A} is stably-free.*
- (ii) *Every f.g. torsion-free \mathbf{A} -module is stably-free.*
- (iii) *Every f.g. left or right ideal in \mathbf{A} is stably-free.*
- (1) *If \mathbf{A} is a Noetherian domain, then (i) \Leftrightarrow (ii) \Leftrightarrow (iii). If these equivalent conditions hold, \mathbf{A} is called a stably-free ideal domain.*

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(2) If \mathbf{A} is an Ore domain, then (ii) \Leftrightarrow (iii). If these equivalent conditions hold, \mathbf{A} is called a semistably-free ideal domain.

Proof. (1) (ii) \Rightarrow (i): Assume that (ii) holds and let \mathfrak{J} be a left ideal in \mathbf{A} . Then \mathfrak{J} is a f.g. torsion-free module, therefore it is stably-free.

(i) \Rightarrow (ii): Assume that (i) holds and let P be a f.g. torsion-free \mathbf{A} -module. Since every left or right ideal is projective, \mathbf{A} is a Dedekind domain. Therefore, $P \cong \mathbf{A}^n \oplus \mathfrak{J}$ where \mathfrak{J} is a left ideal and n is an integer ([5], 5.7.8). Since \mathfrak{J} is stably-free, say of rank $r \geq 0$, there exists an integer $q \geq 0$ such that $\mathfrak{J} \oplus \mathbf{A}^q \cong \mathbf{A}^{q+r}$. Therefore, $P \oplus \mathbf{A}^q \cong \mathbf{A}^{n+q+r}$ and P is stably-free of rank $n+r$. (i) \Leftrightarrow (iii) is clear.

(2) (ii) \Rightarrow (iii) is clear.

(iii) \Rightarrow (ii): If (iii) holds, \mathbf{A} is semihereditary. Let P be a torsion-free left \mathbf{A} -module. Since \mathbf{A} is an Ore domain, there exists an integer $n > 0$ and an embedding $P \hookrightarrow \mathbf{A}^n$ [3]. Therefore, there exists a finite sequence of f.g. left ideals $(\mathfrak{J}_i)_{1 \leq i \leq k}$ such that $P \cong \bigoplus_{i=1}^k \mathfrak{J}_i$ ([4], Thm. (2.29)). For every index $i \in \{1, \dots, k\}$, \mathfrak{J}_i is stably-free, therefore there exist non-negative integers q_i and r_i such that $\mathfrak{J}_i \oplus \mathbf{A}^{q_i} \cong \mathbf{A}^{q_i+r_i}$. As a consequence,

$$P \oplus \mathbf{A}^q \cong \mathbf{A}^{q+r}$$

where $q = \sum_{1 \leq i \leq k} q_i$ and $r = \sum_{1 \leq i \leq k} r_i$, and P is stably-free. ■

3 Examples of stably free ideal domains

The examples below involve skew polynomials.

Proposition 2 *Let \mathbf{R} be a commutative stably free ideal domain.*

(1) *Assume that \mathbf{R} is a \mathbb{Q} -algebra and let $\mathbf{A} = \mathbf{R}[X; \delta]$ where δ is an outer derivation of \mathbf{R} and \mathbf{R} has no proper nonzero δ -stable (left or right) ideals. Then \mathbf{A} is a stably free ideal domain.*

(2) *Let $\mathbf{A} = \mathbf{R}[X, X^{-1}; \sigma]$ where σ is an automorphism of \mathbf{R} such that \mathbf{R} has no proper nonzero σ -stable (left or right) ideals and no power of σ is an inner automorphism of \mathbf{R} . Then \mathbf{A} is a stably free ideal domain.*

Proof. The ring \mathbf{A} is simple ([5], 1.8.4/5), therefore it is a noncommutative Dedekind domain ([5], 7.11.2), thus every left or right ideal of \mathbf{A} is projective, and, moreover, stably free ([5], 12.3.3). ■

Thus we have the following examples:

1. Let k be a field of characteristic 0. The first Weyl algebra $A_1(k)$ and the ring $A'_1(k) = k[x, x^{-1}][X; \frac{d}{dx}] \cong k[X][x, x^{-1}; \sigma]$ with $\sigma(X) = X + 1$ ([5], 1.8.7) are both stably free ideal domains.
2. Likewise, let $k = \mathbb{R}$ or \mathbb{C} , let $k\{x\}$ be the ring of convergent power series with coefficients in k , and let $A_{1c}(k) = k\{x\}[X; \frac{d}{dx}]$. This ring is a stably free ideal domain.

3. Let Ω be a nonempty open interval of the real line and let $\mathcal{R}(\Omega)$ be the largest ring of rational functions analytic in Ω , i.e. $\mathcal{R}(\Omega) = \mathbb{C}(x) \cap \mathcal{O}(\Omega)$ where $\mathcal{O}(\Omega)$ is the ring of all \mathbb{C} -valued analytic functions in Ω . The ring $A(\Omega) = \mathcal{R}(\Omega)[X; \frac{d}{dx}]$ is a simple Dedekind domain [2] and, since $\mathcal{R}(\Omega)$ is a principal ideal domain, $A(\Omega)$ is a stably free ideal domain.

Note that a commutative Dedekind domain which is not a principal ideal domain is not a stably free ideal domain ([5], 11.1.5).

4 Connection with principal ideal domains, Bézout domains, and Hermite rings

Proposition 3 (i) *A ring is a principal ideal domain if, and only if it is both a stably free ideal domain and an Hermite ring.*

(ii) *A ring is a Bézout domain if, and only if it is both a semistably free ideal domain and an Hermite ring.*

Proof. (i): The necessary condition is clear. Let us prove the sufficient condition. Let \mathbf{A} be a stably free ideal domain and let \mathfrak{a} be a left ideal of \mathbf{A} . This ideal is stably free. If \mathbf{A} is Hermite, \mathfrak{a} is free, and since \mathbf{A} is left Noetherian, it is a principal left ideal domain ([1], Chap. 1, Prop. 2.2).

The proof of (ii) is similar, using ([1], Chap. 1, Prop. 1.7). ■

5 Localization

Proposition 4 *Let \mathbf{A} be a stably free ideal domain (resp. a semistably free ideal domain) and let S be a two-sided denominator set ([5], §2.1). Then $S^{-1}\mathbf{A}$ is a stably free ideal domain (resp. a semistably free ideal domain).*

Proof. (1) Let us consider the case of stably free ideal domains. Let \mathbf{A} be a stably free ideal domain. For any left ideal \mathfrak{a} of $S^{-1}\mathbf{A}$ there exists a left ideal \mathfrak{J} of \mathbf{A} such that $\mathfrak{a} = S^{-1}\mathfrak{J}$. Since \mathfrak{J} is stably free, there exist integers q and r such that $\mathfrak{J} \oplus \mathbf{A}^q = \mathbf{A}^r$, therefore $S^{-1}\mathfrak{J} \oplus S^{-1}\mathbf{A}^q = S^{-1}\mathbf{A}^r$, and \mathfrak{a} is stably free. The same rationale holds for right ideals, and this proves that $S^{-1}\mathbf{A}$ is a stably free ideal domain.

(2) The case of semistably free ideal domains is similar, considering f.g. ideals. ■

References

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