On Stably Free Ideal Domains

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Abstract

We define a stably free ideal domain to be a Noetherian domain whose left and right ideals ideals are all stably free. Every stably free ideal domain is a (possibly noncommutative) Dedekind domain, but the converse does not hold. The first Weyl algebra over a field of characteristic 0 is a typical example of stably free ideal domain. Some properties of these rings are studied. A ring is a principal ideal domain if, and only if it is both a stably free ideal domain and an Hermite ring.

1 Introduction

In a principal ideal domain (resp. a Dedekind domain), every left or right ideal is free (resp. projective). An intermediate situation is the one where every left or right ideal is stably free. A Noetherian domain with this property is called a *stably free ideal domain* in what follows. In a Bézout domain, every finitely generated (f.g.) left or right ideal is free. An Ore domain in which every f.g. left or right ideal is stably free is called a *semistably free ideal domain* in what follows. Stably free ideal domains and semistably free ideal domains are briefly studied in this paper.

2 Free ideal domains and semistably free ideal domains

Theorem and Definition 1 *Let* $\mathbf{A}$ *be a ring and consider the following conditions.*
(i) Every left or right ideal in $\mathbf{A}$ *is stably-free.*
(ii) Every f.g. torsion-free $\mathbf{A}$*-module is stably-free.*
(iii) Every f.g. left or right ideal in $\mathbf{A}$ *is stably-free.*
(1) If $\mathbf{A}$ *is a Noetherian domain, then (i)*$\Leftrightarrow$*(ii)*$\Leftrightarrow$*(iii). If these equivalent conditions hold,* $\mathbf{A}$ *is called a* stably-free ideal domain.

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(2) If $\mathbf{A}$ *is an Ore domain, then (ii)*$\Leftrightarrow$*(iii). If these equivalent conditions hold,* $\mathbf{A}$ *is called a* semistably-free ideal domain.

Proof. (1) (ii)$\Rightarrow$(i): Assume that (ii) holds and let $\mathfrak{I}$ be a left ideal in $\mathbf{A}$. Then $\mathfrak{I}$ is a f.g. torsion-free module, therefore it is stably-free.

(i)$\Rightarrow$(ii): Assume that (i) holds and let P be a f.g. torsion-free $\mathbf{A}$-module. Since every left or right ideal is projective, $\mathbf{A}$ is a Dedekind domain. Therefore, $P \cong \mathbf{A}^n \oplus \mathfrak{I}$ where $\mathfrak{I}$ is a left ideal an n is an integer ([5], 5.7.8). Since $\mathfrak{I}$ is stably-free, say of rank $r \geq 0$, there exists an integer $q \geq 0$ such that $\mathfrak{I} \oplus \mathbf{A}^q \cong \mathbf{A}^{q+r}$. Therefore, $P \oplus \mathbf{A}^q \cong \mathbf{A}^{n+q+r}$ and P is stably-free of rank $n+r$. (i)$\Leftrightarrow$(iii) is clear.

(2) (ii)$\Rightarrow$(iii) is clear.

(iii)$\Rightarrow$(ii): If (iii) holds, $\mathbf{A}$ is semihereditary. Let P be a torsion-free left $\mathbf{A}$-module. Since $\mathbf{A}$ is an Ore domain, there exists an integer $n > 0$ and an embedding $P \hookrightarrow \mathbf{A}^n$ [3]. Therefore, there exists a finite sequence of f.g. left ideals $(\mathfrak{I}_i)_{1 \leq i \leq k}$ such that $P \cong \bigoplus_{i=1}^{k} \mathfrak{I}_i$ ([4], Thm. (2.29)). For every index $i \in \{1, ..., k\}$, $\mathfrak{I}_i$ is stably-free, therefore there exist non-negative integers q_i and r_i such that $\mathfrak{I}_i \oplus \mathbf{A}^{q_i} \cong \mathbf{A}^{q_i + r_i}$. As a consequence,

$$P \oplus \mathbf{A}^q \cong \mathbf{A}^{q+r}$$

where $q = \sum_{1 \leq i \leq k} q_i$ and $r = \sum_{1 \leq i \leq k} r_i$, and P is stably-free. ■

3 Examples of stably free ideal domains

The examples below involve skew polynomials.

Proposition 2 *Let* $\mathbf{R}$ *be a commutative stably free ideal domain.*
(1) Assume that $\mathbf{R}$ *is a* $\mathbb{Q}$*-algebra and let* $\mathbf{A} = \mathbf{R}[X; \delta]$ *where* δ *is an outer derivation of* $\mathbf{R}$ *and* $\mathbf{R}$ *has no proper nonzero* δ*-stable (left or right) ideals. Then* $\mathbf{A}$ *is a stably free ideal domain.*
(2) Let $\mathbf{A} = \mathbf{R}\left[X, X^{-1}; \sigma\right]$ *where* σ *is an automorphism of* $\mathbf{R}$ *such that* $\mathbf{R}$ *has no proper nonzero* σ*-stable (left or right) ideals and no power of* σ *is an inner automorphism of* $\mathbf{R}$*. Then* $\mathbf{A}$ *is a stably free ideal domain.*

Proof. The ring $\mathbf{A}$ is simple ([5], 1.8.4/5), therefore it is a noncommutative Dedekind domain ([5], 7.11.2), thus every left or right ideal of $\mathbf{A}$ is projective, and, moreover, stably free ([5], 12.3.3). ■

Thus we have the following examples:

1. Let k be a field of characteristic 0. The first Weyl algebra $A_1(k)$ and the ring $A_1'(k) = k\left[x, x^{-1}\right]\left[X; \frac{d}{dx}\right] \cong k[X]\left[x, x^{-1}; \sigma\right]$ with $\sigma(X) = X + 1$ ([5], 1.8.7) are both stably free ideal domains.

2. Likewise, let $k = \mathbb{R}$ or $\mathbb{C}$, let $k\{x\}$ be the ring of convergent power series with coefficients in k, and let $A_{1c}(k) = k\{x\}\left[X; \frac{d}{dx}\right]$. This ring is a stably free ideal domain.

3. Let Ω be a nonempty open interval of the real line and let $\mathcal{R}(\Omega)$ be the largest ring of rational functions analytic in Ω, i.e. $\mathcal{R}(\Omega) = \mathbb{C}(x) \cap \mathcal{O}(\Omega)$ where $\mathcal{O}(\Omega)$ is the ring of all $\mathbb{C}$-valued analytic functions in Ω. The ring $A(\Omega) = \mathcal{R}(\Omega)\left[X; \frac{d}{dx}\right]$ is a simple Dedekind domain [2] and, since $\mathcal{R}(\Omega)$ is a principal ideal domain, $A(\Omega)$ is a stably free ideal domain.

Note that a commutative Dedekind domain which is not a principal ideal domain is not a stably free ideal domain ([5], 11.1.5).

4 Connection with principal ideal domains, Bézout domains, and Hermite rings

Proposition 3 *(i) A ring is a principal ideal domain if, and only if it is both a stably free ideal domain and an Hermite ring.*
(ii) A ring is a Bézout domain if, and only if it is both a semistably free ideal domain and an Hermite ring.

Proof. (i): The necessary condition is clear. Let us prove the sufficient condition. Let $\mathbf{A}$ be a stably free ideal domain and let $\mathfrak{a}$ be a left ideal of $\mathbf{A}$. This ideal is stably free. If $\mathbf{A}$ is Hermite, $\mathfrak{a}$ is free, and since $\mathbf{A}$ is left Noetherian, it is a principal left ideal domain ([1], Chap. 1, Prop. 2.2).

The proof of (ii) is similar, using ([1], Chap. 1, Prop. 1.7). ■

5 Localization

Proposition 4 *Let $\mathbf{A}$ be a stably free ideal domain (resp. a semistably free ideal domain) and let S be a two-sided denominator set ([5], §2.1). Then $S^{-1}\mathbf{A}$ is a stably free ideal domain (resp. a semistably free ideal domain).*

Proof. (1) Let us consider the case of stably free ideal domains. Let $\mathbf{A}$ be a stably free ideal domain. For any left ideal $\mathfrak{a}$ of $S^{-1}\mathbf{A}$ there exists a left ideal $\mathfrak{I}$ of $\mathbf{A}$ such that $\mathfrak{a} = S^{-1}\mathfrak{I}$. Since $\mathfrak{I}$ is stably free, there exist integers q and r such that $\mathfrak{I} \oplus \mathbf{A}^q = \mathbf{A}^r$, therefore $S^{-1}\mathfrak{I} \oplus S^{-1}\mathbf{A}^q = S^{-1}\mathbf{A}^r$, and $\mathfrak{a}$ is stably free. The same rationale holds for right ideals, and this proves that $S^{-1}\mathbf{A}$ is a stably free ideal domain.

(2) The case of semistably free ideal domains is similar, considering f.g. ideals. ■

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