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E. Landi Degl’Innocenti, V. Bommier, S. Sahal-Bréchet. Resonance line polarization for arbitrary magnetic fields in optically thick media. I - Basic formalism for a 3-dimensional medium.. Astronomy and Astrophysics - A&A, 1991, 244, pp.391-400. hal-02393560

HAL Id: hal-02393560

<https://hal.science/hal-02393560>

Submitted on 2 Feb 2021

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Resonance line polarization for arbitrary magnetic fields in optically thick media

I. Basic formalism for a 3-dimensional medium

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Received July 26, accepted September 28, 1990

Abstract. We present in this paper a general formalism to describe resonance line polarization for a two-level atom in an optically thick three-dimensional medium embedded in a magnetic field of arbitrary strength and irradiated by an arbitrary radiation field. The results contained in the present paper generalize those derived in a previous paper (Landi Degl'Innocenti et al. 1990) that were restricted to weak magnetic fields ($v_L \ll \Delta v_D$, with v_L the Larmor frequency and Δv_D the Doppler broadening of the line). Here the restriction $v_L \ll \Delta v_D$ is released, which makes the present formulation capable of describing, in a unified framework, resonance polarization, the Hanle effect and the Zeeman effect. Analogously to our previous paper, neglecting atomic polarization in the lower level and stimulated emission, we derive an integral equation for the multipole moments of the density matrix of the upper level by coupling the statistical equilibrium equations for the density matrix with the radiative transfer equations for polarized radiation. The final equation accounts for the effect of the magnetic field and for the role of inelastic and elastic (or depolarizing) collisions. Obviously, in the limiting case of weak magnetic fields ($v_L \ll \Delta v_D$), the results of our previous paper are recovered.

Key words: lines formation – line polarization – Hanle effect – Zeeman effect – radiation transfer – solar magnetic fields

1. Introduction

In a recent paper (Landi Degl'Innocenti et al. 1990, hereafter referred to as Paper V), a general formalism has been presented to describe the physics of resonance polarization and the Hanle effect for a two-level atom embedded in an optically thick, magnetized medium of arbitrary geometrical shape. The formalism developed in Paper V is valid only in the limiting case of weak magnetic fields, namely for magnetic fields such that the Zeeman splitting is much smaller than the typical width of the line absorption profile ($v_L \ll v_D$). The aim of the present paper is the one of generalizing the results of Paper V for magnetic fields of arbitrary strength in order to obtain a theoretical framework

capable of describing resonance polarization, the Hanle effect, and the Zeeman effect, in a unified and self-contained formalism.

In this paper, we will follow an approach that is completely analogous to the one followed in Paper V, with the only difference that the magnetic field may now have an arbitrary intensity. In particular, we consider, as in Paper V, a two-level atom having an unpolarized ground level and we neglect the effect of stimulated emission. The atomic system is described through its density-matrix operator which is, in general, variable from point to point in the optically thick medium. The local value of the density-matrix, which is obtained by solving the statistical equilibrium equations, is coupled to the local polarized radiation field and, through the effect of isotropic collisions, to the local thermodynamical properties of the medium, like temperature and density. On the other hand, the polarized radiation field propagating in the medium along any given direction is coupled, through the equations of radiative transfer, to the atomic density-matrix at the points that are crossed by the relative ray-path. As a result, we have again a typical non-LTE problem of the 2nd kind (see Landi Degl'Innocenti 1989, for an introduction to this concept), with the difference, with respect to Paper V, that the magnetic field is now responsible not only for relaxing coherences between magnetic sublevels (Hanle effect), but also for inducing relevant splittings between the different Zeeman components of the spectral line (Zeeman effect).

The formalism that will be used in the present paper is the one developed by Landi Degl'Innocenti (1983a, 1983b, 1984, 1985; hereafter referred to as Papers I, II, III, and IV), which assumes that the scattering process is described within the approximation of complete redistribution in frequency. As already outlined in Paper V, this is not a severe limitation for weak and medium lines, and even for the core of the strongest lines, as shown by Dumont et al. (1977).

In Sect. 2, the statistical equilibrium equations and the equations of radiative transfer are formulated, using the standard representation $\rho_J(M, M')$ for the density-matrix of the two-level atom. The same equations are then reformulated in Sect. 3 by introducing the representation of the irreducible tensors $\rho_Q^K(J)$. In Sect. 4, the statistical equilibrium equations are coupled with radiative transfer equations to get a self-contained integral equation, that is further generalized in Sect. 5 to allow for the presence

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of elastic and inelastic collisions. The main conclusions of the present paper are finally collected in Sect. 6.

2. Formulation in the standard representation

We consider a two-level atom having a lower level with angular momentum J and Landé factor g_J and an upper level with angular momentum J' and Landé factor $g_{J'}$, and we suppose that the lower level is unpolarized, so that the different Zeeman sublevels have the same populations and there are no coherences between them.

2.1. The statistical equilibrium equations

The statistical equilibrium equations for the atomic density-matrix elements can be derived through the theory developed in Paper I. To this aim we perform the following transformations on the notations employed in that paper:

(i) for the eigenvectors:

$$|n\rangle \rightarrow |JM\rangle; \quad (1)$$

(ii) for the eigenvalues:

$$E_n \rightarrow E_{JM} = E_J + h\nu_L g_J M, \quad (2)$$

where $\nu_L = (e_0 B_0)/(4\pi mc)$ is the Larmor frequency expressed in terms of the magnetic field intensity B_0 ;

(iii) for the density matrix:

$$\rho_{nm} = \langle n|\rho|m\rangle \rightarrow \langle JM|\rho|JM'\rangle = \rho_J(M, M'); \quad (3)$$

(iv) for the Bohr frequencies:

$$\begin{aligned} \nu_{nm} &= (E_n - E_m)/h \rightarrow E_{J'M'} - E_{JM} \\ &= \nu_0 + \nu_L(g_{J'}M' - g_J M) = \nu_{J'M', JM}, \end{aligned} \quad (4)$$

where ν_0 is the unperturbed frequency of the transition;

(v) for the dipole matrix elements:

$$\begin{aligned} (r_q)_{nm} &= \langle n|r_q|m\rangle \rightarrow \langle JM|r_q|J'M'\rangle \\ &= (-1)^{J'+M+1} \langle J\|r\|J'\rangle \sqrt{2J+1} \begin{pmatrix} J & J' & 1 \\ -M & M' & q \end{pmatrix}, \end{aligned} \quad (5)$$

where we have used the Wigner-Eckart theorem and introduced the reduced matrix element of the dipole operator (see Brink & Satchler 1968).

In these new notations, neglecting stimulated emission, the statistical equilibrium equations for the density matrix elements of the upper level can be written (in the reference system having its z -axis directed along the magnetic field) in the form (see Eq. (77) of Paper I):

$$\begin{aligned} \frac{d}{dt} \rho_{J'}(M, M') + 2\pi i \nu_L g_{J'}(M - M') \rho_{J'}(M, M') \\ = -R_{J'M, J'M'}^{(1)} + R_{J'M, J'M'}^{(3)}, \end{aligned} \quad (6)$$

where the two terms $R^{(1)}$ and $R^{(3)}$ describing, respectively, the contribution of spontaneous deexcitation and of absorption are given by (see Eqs. (73) to (78) of Paper I):

$$\begin{aligned} R_{J'M, J'M'}^{(1)} &= \sum_{M''Nq} \frac{32}{3} \pi^4 \frac{e_0^2}{hc^3} \nu_0^3 \langle J'M|r_q|JN\rangle \langle J'M''|r_q|JN\rangle^* \\ &\quad \times \rho_{J'}(M'', M') + \text{c.c.}(M \rightleftharpoons M'), \end{aligned} \quad (7)$$

$$\begin{aligned} R_{J'M, J'M'}^{(3)} &= \sum_{qq'N} 16\pi^4 \frac{e_0^2}{h^2 c} (-1)^{q+q'} \langle J'M|r_q|JN\rangle \langle J'M'|r_{q'}|JN\rangle^* \\ &\quad \times [\bar{J}_{-q-q'}(\nu_{J'M', JN}) + i\bar{F}_{-q-q'}(\nu_{J'M', JN})] \\ &\quad \times \rho_J(N, N) + \text{c.c.}(M \rightleftharpoons M'), \end{aligned} \quad (8)$$

where the symbol c.c. ($M \rightleftharpoons M'$) means the quantity which is obtained by taking the complex conjugate of the previous term and interchanging the indexes M and M' , and where the symbols \bar{J} and \bar{F} are given, in terms of the radiation tensor $J_{qq'}(\nu)$ defined in Paper I, by:

$$\begin{aligned} \bar{J}_{qq'}(\nu_0) &= \int_0^\infty d\nu J_{qq'}(\nu) \phi(\nu_0 - \nu) \\ \bar{F}_{qq'}(\nu_0) &= \int_0^\infty d\nu J_{qq'}(\nu) \psi(\nu_0 - \nu), \end{aligned} \quad (9)$$

where ϕ and ψ are the real and imaginary parts of the absorption profile.

Evaluating the matrix elements appearing in Eq. (6) by means of the Wigner-Eckart theorem and remembering the expression of the Einstein coefficient for spontaneous deexcitation:

$$A = \frac{64\pi^4 e_0^2 \nu_0^3}{3hc^3} |\langle J'\|r\|J\rangle|^2, \quad (10)$$

one gets:

$$\begin{aligned} R_{J'M, J'M'}^{(1)} &= \sum_{M''Nq} \frac{1}{2} A (2J'+1) (-1)^{M-M''} \begin{pmatrix} J' & J & 1 \\ -M & N & q \end{pmatrix} \\ &\quad \times \begin{pmatrix} J' & J & 1 \\ -M'' & N & q \end{pmatrix} \rho_{J'}(M'', M') + \text{c.c.}(M \rightleftharpoons M'), \end{aligned} \quad (11)$$

and performing the sum over N and q :

$$R_{J'M, J'M'}^{(1)} = A \rho_{J'}(M, M'). \quad (12)$$

Analogously, remembering the expression of the Einstein coefficient for absorption:

$$B = \frac{32\pi^4 e_0^2}{3h^2 c} \frac{2J'+1}{2J+1} |\langle J'\|r\|J\rangle|^2, \quad (13)$$

introducing the complex absorption profile $\Phi(\nu_0 - \nu)$ through the equation:

$$\Phi(\nu_0 - \nu) = \phi(\nu_0 - \nu) + i\psi(\nu_0 - \nu), \quad (14)$$

and, finally, normalizing the density matrix to the overall population of the ground level

$$(2J+1)\rho_J(N, N) = 1, \quad (15)$$

we obtain:

$$\begin{aligned} R_{J'M, J'M'}^{(3)} &= B \sum_{qq'N} 3 \begin{pmatrix} J' & J & 1 \\ -M & N & q \end{pmatrix} \begin{pmatrix} J' & J & 1 \\ -M' & N & q' \end{pmatrix} \\ &\quad \times \int_0^\infty d\nu J_{-q-q'}(\nu) \left[\frac{1}{2} \Phi(\nu_{J'M', JN} - \nu) \right. \\ &\quad \left. + \frac{1}{2} \Phi^*(\nu_{J'M, JN} - \nu) \right]. \end{aligned} \quad (16)$$

Substituting Eqs. (12) and (16) into Eq. (6), we obtain the

statistical equilibrium equations in the standard representation:

$$\begin{aligned} \frac{d}{dt} \rho_{J'}(M, M') + 2\pi i v_L g_{J'}(M - M') \rho_{J'}(M, M') = \\ -A \rho_{J'}(M, M') + B \sum_{qq'N} 3 \begin{pmatrix} J' & J & 1 \\ -M & N & q \end{pmatrix} \\ \times \begin{pmatrix} J' & J & 1 \\ -M' & N & q' \end{pmatrix} \int_0^\infty dv J_{-q-q'}(v) \\ \times \left[\frac{1}{2} \Phi(v_{J'M', JN-v}) + \frac{1}{2} \Phi^*(v_{J'M', JN-v}) \right]. \end{aligned} \quad (17)$$

2.2. Radiative transfer equations

The transfer of polarized radiation in the medium is described by the usual equation (see Eq. (65) of Paper I):

$$\frac{d}{ds} S_i(v, \Omega) = -\sum_j K_{ij}(v, \Omega) S_j(v, \Omega) + \varepsilon_i(v, \Omega) \quad (i=0, \dots, 3), \quad (18)$$

where $S_i(v, \Omega)$, with the index i running from 0 to 3, is the Stokes vector relative to frequency v and direction Ω , and where the propagation matrix K_{ij} and emission vector ε_i are given by:

$$\begin{aligned} \mathbf{K} &= \begin{pmatrix} \eta_I & \eta_Q & \eta_U & \eta_V \\ \eta_Q & \eta_I & \rho_V & -\rho_U \\ \eta_U & -\rho_V & \eta_I & \rho_Q \\ \eta_V & \rho_U & -\rho_Q & \eta_I \end{pmatrix} \\ &= \begin{pmatrix} \eta_0 & \eta_1 & \eta_2 & \eta_3 \\ \eta_1 & \eta_0 & \rho_3 & -\rho_2 \\ \eta_2 & -\rho_3 & \eta_0 & \rho_1 \\ \eta_3 & \rho_2 & -\rho_1 & \eta_0 \end{pmatrix}, \\ \boldsymbol{\varepsilon} &= \begin{pmatrix} \varepsilon_I \\ \varepsilon_Q \\ \varepsilon_U \\ \varepsilon_V \end{pmatrix} = \begin{pmatrix} \varepsilon_0 \\ \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{pmatrix}. \end{aligned} \quad (19)$$

From Eqs. (80) and (81) of Paper I, we get the explicit expressions for the various coefficients η_i , ρ_i , ε_i in terms of the density-matrix elements. Remembering that the ground level is unpolarized and the normalization that we have chosen for the density matrix, we have:

$$\begin{aligned} \eta_i(v, \Omega) &= \sum_{MM'Nqq'} (-1)^{q+q'} \frac{8\pi^3 e_0^2 v_0}{hc} \frac{N_J}{2J+1} \\ &\times \text{Re} \{ \langle J'M|r_q|JN \rangle \times \langle J'M|r_{q'}|JN \rangle^* \\ &\times \Phi(v_{J'M', JN-v}) \mathcal{F}_{-q-q'}(i, \Omega) \} \\ &\quad (i=0, \dots, 3), \end{aligned} \quad (20)$$

where N_J is the number density of atoms in the lower level, and where \mathcal{F} is the tensor introduced in Paper I.

$$\rho_i(v, \Omega) = \eta_i(v, \Omega) \{ \text{Re} \rightarrow \text{Im} \} \quad (i=1, 2, 3), \quad (21)$$

or, in other words, the same expression as for $\eta_i(v, \Omega)$ with the real part substituted with the imaginary part.

$$\begin{aligned} \varepsilon_i(v, \Omega) &= \sum_{MM'Nqq'} (-1)^{q+q'} \frac{16\pi^3 e_0^2 v_0^4}{c^3} N_J \\ &\times \text{Re} \{ \langle J'M|r_q|JN \rangle \langle J'M'|r_{q'}|JN \rangle^* \\ &\times \rho_{J'}(M', M) \Phi(v_{J'M', JN-v}) \mathcal{F}_{-q-q'}(i, \Omega) \} \\ &\quad (i=0, \dots, 3). \end{aligned} \quad (22)$$

we now evaluate the matrix elements contained in these expressions by means of the Wigner-Eckart theorem and we obtain, after some algebra:

$$\begin{aligned} \eta_i(v, \Omega) &= \eta^{(0)} \sum_{MNq} 3 \begin{pmatrix} J' & J & 1 \\ -M & N & q \end{pmatrix}^2 \phi(v_{J'M', JN-v}) \\ &\times \mathcal{F}_{-q-q'}(i, \Omega), \end{aligned} \quad (23)$$

$$\begin{aligned} \rho_i(v, \Omega) &= \eta^{(0)} \sum_{MNq} 3 \begin{pmatrix} J' & J & 1 \\ -M & N & q \end{pmatrix}^2 \psi(v_{J'M', JN-v}) \\ &\times \mathcal{F}_{-q-q'}(i, \Omega), \end{aligned} \quad (24)$$

where we have introduced the line absorption coefficient $\eta^{(0)}$ defined by:

$$\eta^{(0)} = \frac{h\nu_0}{4\pi} B N_J. \quad (25)$$

Analogously, we have for the emission coefficient:

$$\begin{aligned} \varepsilon_i(v, \Omega) &= \frac{h\nu_0}{4\pi} A(2J'+1)N_J \sum_{MM'Nqq'} 3 \begin{pmatrix} J' & J & 1 \\ -M & N & q \end{pmatrix} \\ &\times \begin{pmatrix} J' & J & 1 \\ -M' & N & q' \end{pmatrix} \text{Re} \{ \rho_{J'}(M', M) \\ &\times \Phi(v_{J'M', JN-v}) \mathcal{F}_{-q-q'}(i, \Omega) \}, \end{aligned} \quad (26)$$

an expression that can be transformed, with some algebra, to the following one:

$$\begin{aligned} \varepsilon_i(v, \Omega) &= \eta^{(0)} \frac{A}{B} (2J'+1)N_J \sum_{MM'Nqq'} 3 \begin{pmatrix} J' & J & 1 \\ -M & N & q \end{pmatrix} \\ &\times \begin{pmatrix} J' & J & 1 \\ -M' & N & q' \end{pmatrix} \rho_{J'}(M', M) \mathcal{F}_{-q-q'}(i, \Omega) \\ &\times \left[\frac{1}{2} \Phi(v_{J'M', JN-v}) + \frac{1}{2} \Phi^*(v_{J'M', JN-v}) \right]. \end{aligned} \quad (27)$$

2.3. Limiting case of negligible Zeeman splitting

When the Zeeman splitting is much smaller than the typical width of the profiles, we can perform the formal substitution:

$$\Phi(v_{J'M', JN-v}) = \Phi(v_0 - v), \quad (28)$$

and some important simplifications appear in the equations derived in the former subsections. For the statistical equilibrium equations we have:

$$\begin{aligned} \frac{d}{dt} \rho_{J'}(M, M') + 2\pi i v_L g_{J'}(M - M') \rho_{J'}(M, M') = -A \rho_{J'}(M, M') \\ + B \sum_{qq'N} 3 \begin{pmatrix} J' & J & 1 \\ -M & N & q \end{pmatrix} \begin{pmatrix} J' & J & 1 \\ -M' & N & q' \end{pmatrix} \\ \times \int_0^\infty dv J_{-q-q'}(v) \phi(v_0 - v) \end{aligned} \quad (29)$$

and, for the coefficients η_i , ρ_i , ε_i appearing in the radiative transfer equations we have, taking into account that $\sum_q \mathcal{T}_{qq}(i, \Omega) = \delta_{i,0}$:

$$\begin{aligned} \eta_i(v, \Omega) &= \delta_{i,0} \eta^{(0)} \phi(v_0 - v) \quad (i=0, \dots, 3) \\ \rho_i(v, \Omega) &= 0 \quad (i=1, 2, 3) \\ \varepsilon_i(v, \Omega) &= \eta^{(0)} \frac{A}{B} (2J'+1) \phi(v_0 - v) \\ &\quad \times \sum_{MM'Nqq'} 3 \begin{pmatrix} J' & J & 1 \\ -M & N & q \end{pmatrix} \begin{pmatrix} J' & J & 1 \\ -M' & N & q' \end{pmatrix} \\ &\quad \times \rho_{J'}(M', M) \mathcal{T}_{-q-q'}(i, \Omega). \end{aligned} \quad (30)$$

In this case, the propagation matrix for the Stokes parameters results in being simply proportional to the identity matrix. This is a consequence of the fact that we have supposed the ground level unpolarized.

3. Formulation in the representation of the irreducible tensors

The equations derived in the former section acquire a more compact expression by introducing the formalism of the irreducible spherical tensors. This is done through the following transformations (see Paper III):

a) for the density matrix:

$$\rho_Q^K(J') = \sum_{MM'} (-1)^{J'-M} \sqrt{2K+1} \begin{pmatrix} J' & K & J' \\ -M & Q & M' \end{pmatrix} \rho_{J'}(M, M'), \quad (31)$$

with the inverse formula:

$$\rho_{J'}(M, M') = \sum_{KQ} (-1)^{J'-M} \sqrt{2K+1} \begin{pmatrix} J' & K & J' \\ -M & Q & M' \end{pmatrix} \rho_Q^K(J'); \quad (32)$$

b) for the tensors $\mathcal{T}_{qq'}$ (analogous formulae hold for $J_{qq'}$):

$$\mathcal{T}_Q^K(i, \Omega) = \sum_{qq'} (-1)^{1+q} \sqrt{3(2K+1)} \begin{pmatrix} 1 & 1 & K \\ q & -q' & -Q \end{pmatrix} \mathcal{T}_{qq'}(i, \Omega), \quad (33)$$

with the inverse formula:

$$\mathcal{T}_{qq'}(i, \Omega) = \sum_{KQ} (-1)^{1+q} \sqrt{(2K+1)/3} \begin{pmatrix} 1 & 1 & K \\ q & -q' & -Q \end{pmatrix} \mathcal{T}_Q^K(i, \Omega). \quad (34)$$

3.1. Statistical equilibrium equations

Multiplying Eq. (17) by:

$$(-1)^{J'-M} \sqrt{2K+1} \begin{pmatrix} J' & K & J' \\ -M & Q & M' \end{pmatrix}$$

and then summing over M and M' , we obtain:

$$\begin{aligned} \frac{d}{dt} \rho_Q^K(J') + 2\pi i v_L g_J Q \rho_Q^K(J') = \\ -A \rho_Q^K(J') + B \sum_{MM'Nqq'} \sum_{K'Q'} (-1)^{J'-M+1-q} \sqrt{3(2K+1)(2K'+1)} \\ \times \begin{pmatrix} J' & K & J' \\ -M & Q & M' \end{pmatrix} \begin{pmatrix} J' & J & 1 \\ -M & N & q \end{pmatrix} \begin{pmatrix} J' & J & 1 \\ -M' & N & q' \end{pmatrix} \begin{pmatrix} 1 & 1 & K' \\ -q & q' & Q' \end{pmatrix} \\ \times \int_0^\infty dv J_{-Q'}^{K'}(v) \left[\frac{1}{2} \Phi(v_{J'M', JN-v}) + \frac{1}{2} \Phi^*(v_{J'M, JN-v}) \right]. \end{aligned} \quad (35)$$

We next observe that, due to the presence of the various 3- j symbols, we have $Q=Q'$. We can then define a *generalized profile* $\Phi_Q^{K,K'}(J, J'; v)$, whose main properties are summarized in Appendix A, through the equation:

$$\begin{aligned} \Phi_Q^{K,K'}(J, J'; v) = \sum_{MM'Nqq'} (-1)^{J'-N-1+Q} \\ \times \sqrt{3(2K+1)(2K'+1)(2J'+1)} \\ \times \begin{pmatrix} J' & K & J' \\ -M & Q & M' \end{pmatrix} \begin{pmatrix} J' & J & 1 \\ -M & N & q \end{pmatrix} \begin{pmatrix} J' & J & 1 \\ -M' & N & q' \end{pmatrix} \\ \times \begin{pmatrix} 1 & 1 & K' \\ -q & q' & Q \end{pmatrix} \left[\frac{1}{2} \Phi(v_{J'M', JN-v}) + \frac{1}{2} \Phi^*(v_{J'M, JN-v}) \right]. \end{aligned} \quad (36)$$

Through the introduction of the generalized profile, the statistical equilibrium equations can be written in the more condensed form:

$$\begin{aligned} \frac{d}{dt} \rho_Q^K(J') + 2\pi i v_L g_J Q \rho_Q^K(J') = -A \rho_Q^K(J') + B \sqrt{2J'+1} \\ \times (-1)^Q \sum_{K'} \int_0^\infty dv \Phi_Q^{K,K'}(J, J'; v) J_{-Q}^{K'}(v). \end{aligned} \quad (37)$$

This equation makes clear the physical meaning of the generalized profile. $\Phi_Q^{K,K'}$ can be interpreted as the v -profile by which we have to integrate the v -dependent K' -multipole of the radiation field tensor to obtain the contribution (due to absorption of radiation) to the statistical tensor of order K of the upper level.

3.2. Radiative transfer equations

Introducing the formalism of the irreducible tensors in Eq. (23) for η_i , we obtain:

$$\begin{aligned} \eta_i(v, \Omega) = \eta^{(0)} \sum_{MNq} \sum_K (-1)^{1-q} \sqrt{3(2K+1)} \\ \times \begin{pmatrix} J' & J & 1 \\ -M & N & q \end{pmatrix}^2 \begin{pmatrix} 1 & 1 & K \\ -q & q & 0 \end{pmatrix} \mathcal{T}_0^K(i, \Omega) \phi(v_{J'M, JN-v}). \end{aligned} \quad (38)$$

Taking into account Eq. (A4) of Appendix A, this can be reduced to the more compact form:

$$\eta_i(v, \Omega) = \eta^{(0)} \sum_K \mathcal{T}_0^K(i, \Omega) \Phi_0^{0,K}(J, J'; v). \quad (39)$$

Analogously, performing the same transformations on Eq. (24) for ρ_i , we get, through Eq. (A11):

$$\rho_i(v, \Omega) = \eta^{(0)} \sum_K \mathcal{T}_0^K(i, \Omega) \Psi_0^{0,K}(J, J'; v). \quad (40)$$

Finally, working on the expression for ε_i , we obtain for the emission coefficient:

$$\varepsilon_i(v, \Omega) = \eta^{(0)} \frac{A}{B} \sqrt{2J'+1} \sum_{KK'Q} \rho_Q^K(J') \mathcal{T}_Q^{K'}(i, \Omega) \Phi_Q^{K,K'}(J, J'; v). \quad (41)$$

From this equation, we get a physical interpretation of the generalized profile that is strictly complementary of the one discussed in the previous section. $\Phi_Q^{K,K'}$ is the v -profile of the radiation emitted in the multipole K' from the statistical tensor of rank K .

3.3. Limiting case of negligible Zeeman splitting

In the limiting case of negligible Zeeman splitting, we can substitute for the generalized profiles their asymptotic expression given in Eq. (A2) of Appendix A. The statistical equilibrium equations then reduce to:

$$\frac{d}{dt} \rho_Q^K(J') + 2\pi i v_L g_J Q \rho_Q^K(J') = -A \rho_Q^K(J') + B \frac{1}{\sqrt{2J'+1}} \times w_{J'J}^{(K)}(-1)^Q \int_0^\infty dv \phi(v_0 - v) J_{-Q}^K(v), \quad (42)$$

which is formally coincident with our Eq. (3) of Paper V. Analogously, for the various coefficients η_i , ρ_i , ε_i appearing in the radiative transfer equations we have, remembering that $\mathcal{F}_0^0(i, \Omega) = \delta_{i,0}$:

$$\eta_i(v, \Omega) = \delta_{i,0} \eta^{(0)} \phi(v_0 - v) \quad (i=0, \dots, 3)$$

$$\rho_i(v, \Omega) = 0 \quad (i=1, 2, 3)$$

$$\varepsilon_i(v, \Omega) = \eta^{(0)} \frac{A}{B} \sqrt{2J'+1} \sum_{KQ} w_{J'J}^{(K)} \rho_Q^K(J') \mathcal{F}_Q^K(i, \Omega) \phi(v_0 - v). \quad (43)$$

Again, we find here an expression that is formally coincident with Eq. (14) of Paper V.

4. Coupling of radiative transfer and statistical equilibrium equations

Referring to Fig. 1, we consider a point P , having coordinate \mathbf{x} , in a medium of arbitrary geometrical shape with a number density N_J (that may be variable from point to point) of model-atoms. Any assigned direction Ω , passing through P , defines a ray-path which crosses the boundary of the medium in the point P_0 having coordinate \mathbf{x}_0 . Denoting by s the coordinate along this ray-path (counted from P_0), we can express the line optical depth between any given point P' , having coordinate \mathbf{x}' , and P , through the expression:

$$\tau(\mathbf{x}, \mathbf{x}') = \int_{s(\mathbf{x}') }^{s(\mathbf{x})} \eta^{(0)}(s) ds, \quad (44)$$

where $s(\mathbf{x})$ is the value of s corresponding to the point having coordinate \mathbf{x} .

With these notations, we can now give a formal solution for the radiative transfer Eq. (18). For the Stokes parameters of the radiation propagating at point P through the direction Ω , we have:

$$[S_i(v, \Omega)]_{\mathbf{x}} = \int_0^{s(\mathbf{x})} \sum_{j=0}^3 O_{ij}(\mathbf{x}, \mathbf{x}'; v) [\varepsilon_j(v, \Omega)]_{\mathbf{x}'} ds + \sum_{j=0}^3 O_{ij}(\mathbf{x}, \mathbf{x}_0; v) S_j^{(0)}(v, \Omega), \quad (45)$$

where $O_{ij}(\mathbf{x}, \mathbf{x}'; v)$ is the (frequency dependent) ‘‘attenuation operator’’ for the Stokes parameters defined in Landi Degl’Innocenti (1987), and where $S_j^{(0)}(v, \Omega)$ represents the boundary value Stokes parameters entering the medium at point P_0 along the direction Ω .

In general, the operator O_{ij} can be computed only through numerical algorithms. However, if the magnetic field is constant throughout all the medium and if, moreover, the various profiles

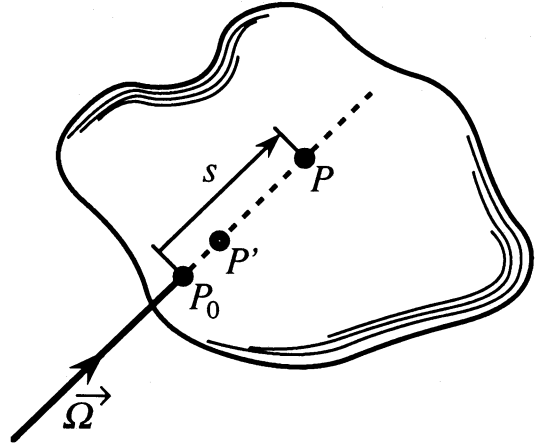


Fig. 1. Explanation of the symbols introduced in Eq. (45)

are constant and there are no velocity gradients (or, in other words, if the medium is static), the operator O_{ij} can be expressed in an analytical form. In fact, when all these requirements are fulfilled, we can write the matrix \mathbf{K} in the form:

$$\mathbf{K} = \eta^{(0)} \mathbf{H}(v, \Omega), \quad (46)$$

where the matrix $\mathbf{H}(v, \Omega)$ is constant throughout the medium. The operator O_{ij} can then be written in the form:

$$O_{ij}(\mathbf{x}, \mathbf{x}'; v) = [\exp\{-\tau(\mathbf{x}, \mathbf{x}') \mathbf{H}(v, \Omega)\}]_{ij}, \quad (47)$$

and, taking into account the algebraic properties of the matrix \mathbf{H} , this expression can be reduced to the sum of four 4×4 matrices (see Landi Degl’Innocenti & Landi Degl’Innocenti 1985). The final equation expressing O_{ij} in terms of the quantities η_i and ρ_i is rather cumbersome and will not be written down explicitly here. For details, the reader is referred to the paper quoted above.

We can now substitute the formal solution of the transfer equation [Eq. (44)] into the statistical equilibrium equations (Eq. (37)). Remembering that the tensor $J_Q^K(v)$ is defined by:

$$J_Q^K(v) = \oint \frac{d\Omega}{4\pi} \sum_{i=0}^3 \mathcal{F}_Q^K(i, \Omega) S_i(v, \Omega), \quad (48)$$

and indicating with $\rho_Q^K(\mathbf{x})$ the statistical tensor of the upper level evaluated at point P , we have, in stationary situations:

$$\rho_Q^K(\mathbf{x}) = -i\Gamma Q \rho_Q^K(\mathbf{x}) + \frac{B}{A} \frac{1}{\sqrt{2J'+1}} (-1)^Q \int_0^\infty dv \oint \frac{d\Omega}{4\pi} \sum_{K'} \Phi_Q^{K, K'}(J, J'; v) \times \sum_{i=0}^3 \mathcal{F}_{-Q}^{K'}(i, \Omega) \left\{ \int_0^{s(\mathbf{x})} \sum_{j=0}^3 O_{ij}(\mathbf{x}, \mathbf{x}'; v) [\varepsilon_j(v, \Omega)]_{\mathbf{x}'} ds + \sum_{j=0}^3 O_{ij}(\mathbf{x}, \mathbf{x}_0; v) S_j^{(0)}(v, \Omega) \right\}, \quad (49)$$

where we have defined, as in Paper I:

$$\Gamma = g_J \frac{2\pi v_L}{A} = g_J \frac{e_0 B_0}{2mcA}. \quad (50)$$

The contribution to this equation due to the boundary conditions Stokes parameters will be called, as in Paper I, the ‘‘external

rate”, and it will be here indicated with the symbol $E_Q^K(x)$, while the other contribution, due to the emission coefficient at point x' will be called the “internal rate”. This contribution can be calculated by substituting the expression for the emission coefficient given in Eq. (41). Observing that:

$$\oint \frac{d\Omega}{4\pi} \int_0^{s(x)} ds = \frac{1}{4\pi} \int_V d^3x' \frac{1}{(x-x')^2}, \quad (51)$$

we have:

$$\begin{aligned} \rho_Q^K(x) = & -i\Gamma_Q \rho_Q^K(x) \\ & + \sum_{K'Q'} \frac{1}{4\pi} \int_V \eta^{(0)}(x') \frac{\tilde{G}_{KQ,K'Q'}(x,x')}{(x-x')^2} \rho_{Q'}^{K'}(x') d^3x' + E_Q^K(x), \end{aligned} \quad (52)$$

where the generalized multipole coupling coefficients, $\tilde{G}_{KQ,K'Q'}$ are defined by:

$$\begin{aligned} \tilde{G}_{KQ,K'Q'}(x,x') = & (-1)^Q \sum_{i,j=0}^3 \sum_{LL'} \int_0^\infty dv \Phi_Q^{K,L}(J,J';v) \Phi_{Q'}^{K',L'}(J',J';v) \\ & \times \mathcal{T}_{-Q}^L(i,\Omega) O_{ij}(x,x';v) \mathcal{T}_{Q'}^{L'}(j,\Omega), \end{aligned} \quad (53)$$

and where:

$$\begin{aligned} E_Q^K(x) = & \frac{\mathcal{B}}{A} \frac{1}{\sqrt{2J'+1}} (-1)^Q \sum_{i,j=0}^3 \sum_L \int_0^\infty dv \oint \frac{d\Omega}{4\pi} \\ & \times \Phi_Q^{K,L}(J,J';v) \mathcal{T}_{-Q}^L(i,\Omega) O_{ij}(x,x_0;v) S_j^{(0)}(v,\Omega). \end{aligned} \quad (54)$$

Some properties of the generalized multipole coupling coefficients and of the external rate are presented in Appendix B. Equation (B5) shows that, in the limiting case of weak magnetic fields (v_L much smaller than the typical width of the line profile), the results of Paper I are recovered. Moreover, Eq. (B17) shows that, analogously to the case presented in Paper V, the multipole coupling coefficients connecting multipoles of different parity are identically zero. It has however to be stressed that this property is valid only for a static medium.

5. The role of collisions

Analogously to what has been done in Paper I, we can easily add the contribution of collisional rates to Eq. (52). Supposing:

- that the velocity distribution of the collider is Maxwellian;
- that the collision time is much smaller than the time between successive collisions (impact approximation);
- that the collision time is also much smaller than the lifetime of the excited level,

we obtain the following general equation (see Paper I for further details):

$$\begin{aligned} (1 + \varepsilon + \delta^{(K)}) \rho_Q^K(x) = & -i\Gamma_Q \rho_Q^K(x) \\ & + \sum_{K'Q'} \frac{1}{4\pi} \int_V \eta^{(0)}(x') \frac{\tilde{G}_{KQ,K'Q'}(x,x')}{(x-x')^2} \rho_{Q'}^{K'}(x') d^3x' \\ & + E_Q^K(x) + \varepsilon \beta_0 \delta_{K,0} \delta_{Q,0}, \end{aligned} \quad (55)$$

where $\varepsilon = C_{J,J}/A$, with $C_{J,J}$ the collisional deexcitation rate; $\delta^{(K)} = D^{(K)}/A$, with $D^{(K)}$ the relaxation rate of the density-matrix

multipole element of rank K due to elastic collisions; and, finally:

$$\beta_0 = \frac{\sqrt{2J'+1}}{2J+1} e^{-h\nu_0/kT},$$

where T is the temperature characterizing the Maxwellian distribution of the colliders.

This equation shows that isotropic collisions provide a source term only for the $K=0, Q=0$ density-matrix multipole element (the total population of the upper level). In particular, in the limiting case $\varepsilon \gg 1$, Eq. (55) can be immediately solved to give:

$$\rho_Q^K(x) = \beta_0 \delta_{K,0} \delta_{Q,0}, \quad (56)$$

which is the obvious result valid in the LTE approximation.

6. Conclusion

In this paper, we have presented a very general formalism for describing resonance scattering in an optically thick, three-dimensional medium of arbitrary geometrical shape, embedded in a magnetic field of arbitrary intensity and direction. The formalism has been developed for a two-level atom with arbitrary values of quantum numbers J and J' and Landé factors g_J and $g_{J'}$ for the lower and upper level, respectively. By introducing the formalism of the density-matrix in the representation of the irreducible tensors, a very general integral equation is obtained, which relates the value of the density-matrix multipole element $\rho_Q^K(x)$ at point x with the density-matrix multipole element $\rho_{Q'}^{K'}(x')$ at different points x' in the medium. With respect to Paper V, the shifts of the Zeeman components have been taken into account, through the definition of a “generalized profile” in the formalism of the density-matrix multipole development on the basis of the irreducible tensors. This “generalized profile” enters the statistical equilibrium equations for the atomic density matrix, and the transfer equations for polarized radiation. Its properties are studied in Appendix A.

Once the geometry of the scattering medium, its physical properties and the relevant boundary conditions are specified, the integral equation can, in principle, be solved. From the solution, the Stokes parameters propagating along any line-of-sight can be recovered.

For a static medium, when the radiation which is illuminating the scattering medium from the exterior is either zero or satisfies a well precise symmetry requirement with respect to the line center (I, Q, U symmetric and V antisymmetric), the density-matrix multipole elements of odd rank, having no source terms in the integral equation, and being decoupled from those of even rank, are identically zero. In other words, no “orientation” can be found in the scattering atoms in this situation. The only non-vanishing multipoles are those of even rank, and, differently from the case of a weak magnetic field considered in Paper V, all the density-matrix multipole elements with K even can be excited (up to the maximum allowed, which is $2J'$), and not only the $K=0$ and $K=2$ multipole elements.

The theory presented in this paper is capable of reuniting, in a unique approach, the basic physical facts underlying the phenomena of resonance scattering, the Hanle effect and the Zeeman effect. It can then be considered of fundamental importance for the interpretation of spectro-polarimetric observations of resonance lines formed in a magnetized stellar atmosphere.

Appendix A: properties of the generalized profile

The generalized profile is defined through the equation:

$$\begin{aligned} \Phi_Q^{K,K'}(J, J'; \nu) = & \sum_{MM'Nqq'} (-1)^{J'-N-1+Q} \\ & \times \sqrt{3(2K+1)(2K'+1)(2J'+1)} \\ & \times \begin{pmatrix} J' & K & J' \\ -M & Q & M' \end{pmatrix} \begin{pmatrix} J' & J & 1 \\ -M & N & q \end{pmatrix} \\ & \times \begin{pmatrix} J' & J & 1 \\ -M' & N & q' \end{pmatrix} \begin{pmatrix} 1 & 1 & K' \\ -q & q' & Q \end{pmatrix} \\ & \times \left[\frac{1}{2} \Phi(\nu_{J',M',JN}-\nu) + \frac{1}{2} \Phi^*(\nu_{J',M,JN}-\nu) \right], \quad (\text{A1}) \end{aligned}$$

with $K=0, 1, \dots, 2J'$; $K'=0, 1, 2$; $Q=-K_{\min}, \dots, 0, \dots, K_{\min}$, with $K_{\min}=\min(K, K')$. This profile satisfies the properties given in the following.

a) Limit for weak fields

When the splitting introduced by the magnetic field is much smaller than the typical width of the profiles, we can simply substitute, for any M and N , $\Phi(\nu_{J',M',JN}-\nu)$ with $\Phi(\nu_0-\nu)$. Performing the sums, we then get, after some Racah algebra:

$$\begin{aligned} \lim_{\nu_L \rightarrow 0} \Phi_Q^{K,K'}(J, J'; \nu) = & \delta_{K,K'} (-1)^{1+J+J'} \sqrt{3(2J'+1)} \\ & \times \begin{Bmatrix} 1 & 1 & K \\ J' & J' & J \end{Bmatrix} \phi(\nu_0-\nu) \\ = & \delta_{K,K'} w_{J'J}^{(K)} \phi(\nu_0-\nu), \quad (\text{A2}) \end{aligned}$$

where $w_{J'J}^{(K)}$ is the symbol introduced in Landi Degl'Innocenti (1984). Under this limit we obtain that the generalized profile is diagonal with respect to K and K' , is independent of Q , and has the frequency dependence of a normal absorption profile centered at line center ν_0 .

b) Integral with respect to frequency

As the profiles $\Phi(\nu_{J',M',JN}-\nu)$ are normalized to 1 in frequency:

$$\int_{-\infty}^{+\infty} d\nu \Phi(\nu_{J',M',JN}-\nu) = 1,$$

through the same algebra of case a) we obtain:

$$\int_{-\infty}^{+\infty} d\nu \Phi_Q^{K,K'}(J, J'; \nu) = \delta_{K,K'} w_{J'J}^{(K)}. \quad (\text{A3})$$

c) Particular cases

For $K=0$ we easily obtain, taking into account the properties of the 3- j symbols:

$$\begin{aligned} \Phi_0^{0,K'}(J, J'; \nu) = & \sum_{MNq} (-1)^{q-1} \sqrt{3(2K'+1)} \\ & \times \begin{pmatrix} J' & J & 1 \\ -M & N & q \end{pmatrix}^2 \begin{pmatrix} 1 & 1 & K' \\ -q & q & 0 \end{pmatrix} \\ & \times \phi(\nu_{J',M',JN}-\nu), \quad (\text{A4}) \end{aligned}$$

and, when also $K'=0$:

$$\Phi_0^{0,0}(J, J'; \nu) = \sum_{MNq} \begin{pmatrix} J' & J & 1 \\ -M & N & q \end{pmatrix}^2 \phi(\nu_{J',M',JN}-\nu). \quad (\text{A5})$$

This is an absorption profile, normalized to 1 in frequency, which accounts for all the Zeeman components of the transition, each Zeeman transition being multiplied for its relative strength.

d) Conjugation property

$$[\Phi_Q^{K,K'}(J, J'; \nu)]^* = \Phi_{-Q}^{K,K'}(J, J'; \nu), \quad (\text{A6})$$

as it can be easily verified directly from the definition.

e) Inversion with respect to line center

$$\Phi_Q^{K,K'}(J, J'; -\Delta\nu) = (-1)^{K+K'} \Phi_Q^{K,K'}(J, J'; \Delta\nu), \quad (\text{A7})$$

where $\Delta\nu = \nu - \nu_0$.

To prove this property, we write the argument of each profile in terms of $\Delta\nu$; we have:

$$\Phi(\nu_{J',M',JN}-\nu) = \Phi(\nu_0 + \nu_{M',N}-\nu) = \Phi(\nu_{M',N}-\Delta\nu), \quad (\text{A8})$$

where:

$$\nu_{M',N} = \nu_L(g_J, M' - g_J N). \quad (\text{A9})$$

Taking them into account the inversion property of each profile with respect to its argument:

$$\Phi(x) = \Phi^*(-x),$$

we have:

$$\Phi(\nu_{M',N} + \Delta\nu) = \Phi^*(-\nu_{M',N} - \Delta\nu) = \Phi^*(\nu_{-M',-N} - \Delta\nu). \quad (\text{A10})$$

An inversion with respect to line center corresponds then to substitute the quantity

$$\left[\frac{1}{2} \Phi(\nu_{J',M',JN}-\nu) + \frac{1}{2} \Phi^*(\nu_{J',M',JN}-\nu) \right]$$

appearing in Eq. (A1) with the quantity:

$$\left[\frac{1}{2} \Phi^*(\nu_{J',-M',J-N}-\nu) + \frac{1}{2} \Phi(\nu_{J',-M',J-N}-\nu) \right].$$

Renaming the various indices appearing in Eq. (A1), and taking into account the symmetry properties of the 3- j symbols, Eq. (A7) is easily proven.

f) Associated dispersion profile

A generalized dispersion profile associated to $\Phi_Q^{K,K'}(J, J'; \nu)$ can be defined through a similar equation:

$$\begin{aligned} i\Psi_Q^{K,K'}(J, J'; \nu) = & \sum_{MM'Nqq'} (-1)^{J'-N-1+Q} \\ & \times \sqrt{3(2K+1)(2K'+1)(2J'+1)} \\ & \times \begin{pmatrix} J' & K & J' \\ -M & Q & M' \end{pmatrix} \begin{pmatrix} J' & J & 1 \\ -M & N & q \end{pmatrix} \\ & \times \begin{pmatrix} J' & J & 1 \\ -M' & N & q' \end{pmatrix} \begin{pmatrix} 1 & 1 & K' \\ -q & q' & Q \end{pmatrix} \\ & \times \left[\frac{1}{2} \Phi(\nu_{J',M',JN}-\nu) - \frac{1}{2} \Phi^*(\nu_{J',M',JN}-\nu) \right], \quad (\text{A11}) \end{aligned}$$

this equation being the same as the equation defining $\Phi_Q^{K,K'}$, except for a minus sign in the last square bracket. In strict analogy with the previous profile, we can prove the following properties:

$$a') \lim_{v_L \rightarrow 0} \Psi_Q^{K,K'}(J, J'; v) = \delta_{K,K'} w_{J'}^{(K)} \psi(v_0 - v), \quad (\text{A12})$$

$$b') \int_{-\infty}^{+\infty} dv \Psi_Q^{K,K'}(J, J'; v) = 0, \quad (\text{A13})$$

$$c') \Psi_0^{0,K'}(J, J'; v) = \sum_{MNq} (-1)^{q-1} \sqrt{3(2K'+1)} \begin{pmatrix} J' & J & 1 \\ -M & N & q \end{pmatrix}^2 \times \begin{pmatrix} 1 & 1 & K' \\ -q & q & 0 \end{pmatrix} \psi(v_{J',M,JN-v}), \quad (\text{A14})$$

$$\Psi_0^{0,0}(J, J'; v) = \sum_{MNq} \begin{pmatrix} J' & J & 1 \\ -M & N & q \end{pmatrix}^2 \Psi(v_{J',M,JN-v}), \quad (\text{A15})$$

$$d') [\Psi_Q^{K,K'}(J, J'; v)]^* = \Psi_{-Q}^{K,K'}(J, J'; v), \quad (\text{A16})$$

$$e') \Psi_Q^{K,K'}(J, J'; -\Delta v) = (-1)^{K+K'+1} \Psi_Q^{K,K'}(J, J'; \Delta v). \quad (\text{A17})$$

Appendix B: properties of the multiple coupling coefficients and of the external rate

The multipole coupling coefficients are defined by the equation:

$$\begin{aligned} \tilde{G}_{KQ,K'Q'}(\mathbf{x}, \mathbf{x}') &= (-1)^Q \sum_{i,j=0}^3 \sum_{LL'} \int_0^\infty dv \Phi_Q^{K,L}(J, J'; v) \Phi_{Q'}^{K',L'}(J, J'; v) \\ &\quad \times \mathcal{T}_{-Q}^L(i, \Omega) O_{ij}(\mathbf{x}, \mathbf{x}'; v) \mathcal{T}_{Q'}^{L'}(j, \Omega), \end{aligned} \quad (\text{B1})$$

with:

$$K, K' = 0, 1, \dots, 2J',$$

$$|Q| \leq \min(K, 2),$$

$$|Q'| \leq \min(K', 2). \quad (\text{B2})$$

They satisfy the properties given in the following.

a) Limit for weak fields

Taking into account Eq. (A2) for the generalized profiles and considering that:

$$\lim_{v_L \rightarrow 0} O_{ij}(\mathbf{x}, \mathbf{x}'; v) = \delta_{ij} \exp[-\tau(\mathbf{x}, \mathbf{x}') \phi(v_0 - v)], \quad (\text{B3})$$

one gets:

$$\begin{aligned} \lim_{v_L \rightarrow 0} \tilde{G}_{KQ,K'Q'}(\mathbf{x}, \mathbf{x}') &= (-1)^Q w_{J'}^{(K)} w_{J'}^{(K')} \sum_i \mathcal{T}_{-Q}^K(i, \Omega) \mathcal{T}_{Q'}^{K'}(i, \Omega) \\ &\quad \times \phi^2(v_0 - v) \exp[-\tau(\mathbf{x}, \mathbf{x}') \phi(v_0 - v)]. \end{aligned} \quad (\text{B4})$$

Comparing with Eq. (20) of Paper I, one finds:

$$\begin{aligned} \lim_{v_L \rightarrow 0} \tilde{G}_{KQ,K'Q'}(\mathbf{x}, \mathbf{x}') &= G_{KQ,K'Q'}(\Omega) \\ &\quad \times \phi^2(v_0 - v) \exp[-\tau(\mathbf{x}, \mathbf{x}') \phi(v_0 - v)], \end{aligned} \quad (\text{B5})$$

where $G_{KQ,K'Q'}(\Omega)$ is the quantity there defined and whose general expression can be found in Table 2 of Paper I.

b) Expression in terms of rotation matrices

Remembering the definition of the tensor $\mathcal{T}_Q^K(i, \Omega)$ given in Eq. (A6) of Paper III:

$$\mathcal{T}_Q^K(i, \Omega) = \sum_P t_P^K(i) \mathcal{D}_{PQ}^K(\mathbf{R}), \quad (\text{B6})$$

we can write:

$$\begin{aligned} \tilde{G}_{KQ,K'Q'}(\mathbf{x}, \mathbf{x}') &= (-1)^Q \sum_{LL'PP'} \int_0^\infty dv \Phi_Q^{K,L}(J, J'; v) \Phi_{Q'}^{K',L'}(J, J'; v) \\ &\quad \times \left[\sum_{i,j=0}^3 t_P^L(i) O_{ij}(\mathbf{x}, \mathbf{x}'; v) t_{P'}^{L'}(j) \right] \mathcal{D}_{P-Q}^L(\mathbf{R}) \mathcal{D}_{P'-Q'}^{L'}(\mathbf{R}), \end{aligned} \quad (\text{B7})$$

where \mathbf{R} is the rotation matrix bringing the system of unit vectors defining the Stokes parameters for the direction Ω , ($e_a(\Omega)$, $e_b(\Omega)$, Ω), into the reference system of the magnetic field.

For the quantity in brackets, we can introduce a more compact notation, in writing:

$$\tilde{O}_{LP,L'P'}(\mathbf{x}, \mathbf{x}'; v) = \left[\sum_{i,j=0}^3 t_P^L(i) O_{ij}(\mathbf{x}, \mathbf{x}'; v) t_{P'}^{L'}(j) \right] \quad (\text{B8})$$

We then have:

$$\begin{aligned} \tilde{G}_{KQ,K'Q'}(\mathbf{x}, \mathbf{x}') &= (-1)^Q \sum_{LL'PP'} \mathcal{D}_{P-Q}^L(\mathbf{R}) \mathcal{D}_{P'-Q'}^{L'}(\mathbf{R}) \\ &\quad \times \int_0^\infty dv \Phi_Q^{K,L}(J, J'; v) \Phi_{Q'}^{K',L'}(J, J'; v) \tilde{O}_{LP,L'P'}(\mathbf{x}, \mathbf{x}'; v). \end{aligned} \quad (\text{B9})$$

The new quantity just introduced obeys the following conjugation property:

$$[\tilde{O}_{LP,L'P'}(\mathbf{x}, \mathbf{x}'; v)]^* = \tilde{O}_{L-P,L'-P'}(\mathbf{x}, \mathbf{x}'; v). \quad (\text{B10})$$

That can be easily deduced observing that O_{ij} is real and that the symbol $t_P^K(i)$ satisfies Eq. (A8) of Paper III, namely:

$$[t_P^K(i)]^* = t_{-P}^K(i).$$

c) Conjugation property

$$[\tilde{G}_{KQ,K'Q'}(\mathbf{x}, \mathbf{x}')]^* = (-1)^{Q'-Q} \tilde{G}_{K-Q,K'-Q'}(\mathbf{x}, \mathbf{x}'). \quad (\text{B11})$$

This property can be deduced directly from the definition (B1), taking into account Eq. (A6) and the conjugation property of the symbols $\mathcal{T}_Q^K(i, \Omega)$ (cf. Eq. (A9) of Paper III):

$$[\mathcal{T}_Q^K(i, \Omega)]^* = (-1)^Q \mathcal{T}_{-Q}^K(i, \Omega). \quad (\text{B12})$$

d) Vanishing values

Some of the symbols $\tilde{G}_{KQ,K'Q'}$ are zero due to the inversion property, with respect to line center, of the various frequency-dependent quantities entering the integral over v in Eq. (B9). Taking into account Eq. (A7), we have:

$$\begin{aligned} \Phi_Q^{K,L}(J, J'; -\Delta v) &= (-1)^{K+L} \Phi_Q^{K,L}(J, J'; \Delta v), \\ \Phi_{Q'}^{K',L'}(J, J'; -\Delta v) &= (-1)^{K'+L'} \Phi_{Q'}^{K',L'}(J, J'; \Delta v). \end{aligned} \quad (\text{B13})$$

The inversion properties with respect to line center of the quantity \tilde{O} involves a careful examination of the results derived in Landi Degl'Innocenti & Landi Degl'Innocenti (1985) from which it can be shown that:

$$O_{ij}(\mathbf{x}, \mathbf{x}'; -\Delta v) = \xi_i \xi_j O_{ij}(\mathbf{x}, \mathbf{x}'; \Delta v), \quad (\text{B14})$$

where ξ_i is the formal vector defined by: $\xi_i \equiv (1, 1, 1, -1)$. Taking also into account that, for the symbols $t_P^L(i)$ we have:

$$\xi_i t_P^L(i) = (-1)^L t_P^L(i), \quad (\text{B15})$$

we obtain:

$$\tilde{O}_{LP, L'P'}(\mathbf{x}, \mathbf{x}'; -\Delta v) = (-1)^{L+L'} \tilde{O}_{LP, L'P'}(\mathbf{x}, \mathbf{x}'; \Delta v). \quad (\text{B16})$$

Collecting together properties (B13) and (B16), we obtain:

$$\tilde{G}_{KQ, K'Q'}(\mathbf{x}, \mathbf{x}') = (-1)^{K+K'} \tilde{G}_{KQ, K'Q'}(\mathbf{x}, \mathbf{x}'). \quad (\text{B17})$$

This equation shows that all the multipole coupling coefficients connecting multipole tensors of different parity are identically zero. For those that are different from zero, the integral in Eq. (B7) can be obtained by limiting the frequency integration between ν_0 and ∞ and then doubling the result (the integrand is an even function of Δv).

e) *Symmetry with respect to the arguments*

$$\tilde{G}_{KQ, K'Q'}(\mathbf{x}, \mathbf{x}') = \tilde{G}_{KQ, K'Q'}(\mathbf{x}', \mathbf{x}). \quad (\text{B18})$$

To prove this identity, we observe that interchanging \mathbf{x} with \mathbf{x}' is equivalent to interchange the direction Ω with $-\Omega$. If $(e_a(\Omega), e_b(\Omega))$ is the couple of unit vectors defining the Stokes parameters with respect to Ω , we can define the Stokes parameters with respect to the opposite direction $(-\Omega)$ through the unit vectors: $e_a(-\Omega) = e_a(\Omega)$; $e_b(-\Omega) = -e_b(\Omega)$;

$$(\text{B19})$$

We then have:

$$\mathcal{F}_Q^K(i, -\Omega) = \sum_P t_P^K(i) \mathcal{D}_{PQ}^K(R'), \quad (\text{B20})$$

where R' is the rotation bringing the coordinate system $(e_a(-\Omega), e_b(-\Omega), -\Omega)$ into the reference system of the magnetic field. The rotation R' , on the other hand, can be written in the form:

$$R' \equiv (\pi, \pi, 0) \otimes R, \quad (\text{B21})$$

where R is the analogous rotation defined after Eq. (B7). Through the properties of the rotation matrices, we get:

$$\begin{aligned} \mathcal{F}_Q^K(i, -\Omega) &= \sum_{PP'} t_P^K(i) \mathcal{D}_{PP'}^K(\pi, \pi, 0) \mathcal{D}_{P'Q}^K(R) \\ &= \sum_{PP'} t_P^K(i) (-1)^K \delta_{P, -P'} \mathcal{D}_{P'Q}^K(R) \\ &= (-1)^K t_{-P}^K(i) \mathcal{D}_{PQ}^K(R). \end{aligned} \quad (\text{B22})$$

On the other hand, from the properties of the symbol t_P^K , we have:

$$(-1)^K t_{-P}^K(i) = \zeta_i t_P^K(i), \quad (\text{B23})$$

where ζ_i is the formal vector defined by: $\zeta_i \equiv (1, 1, -1, -1)$. We thus obtain the important result:

$$\mathcal{F}_Q^K(i, -\Omega) = \zeta_i \mathcal{F}_Q^K(i, \Omega). \quad (\text{B24})$$

On the other hand, remembering the definition of the matrix \mathbf{H} , we get:

$$H_{ij}(v, -\Omega) = \zeta_i \zeta_j H_{ij}(v, \Omega), \quad (\text{B25})$$

and as this symmetry property is conserved for any arbitrary function of the matrix \mathbf{H} , we also get:

$$O_{ij}(\mathbf{x}', \mathbf{x}; v) = \zeta_i \zeta_j O_{ij}(\mathbf{x}, \mathbf{x}'; v). \quad (\text{B26})$$

Taking into account Eqs. (B24) and (B26), Eq. (B18) is easily proven.

f) *Symmetry with respect to the indices (only when $Q=0$)*

$$\tilde{G}_{K0, K'0}(\mathbf{x}, \mathbf{x}') = \tilde{G}_{K'0, K0}(\mathbf{x}, \mathbf{x}'). \quad (\text{B27})$$

To prove this property, let us previously consider the quantity:

$$\mathcal{C}_{LL'} = \sum_{i,j=0}^3 \mathcal{F}_0^L(i, \Omega) O_{ij}(\mathbf{x}, \mathbf{x}'; v) \mathcal{F}_0^{L'}(j, \Omega). \quad (\text{B28})$$

Both the tensor \mathcal{F}_Q^K and the attenuation operator O_{ij} appearing in this expression depend on the particular choice of the reference direction, $e_a(\Omega)$, that has been chosen to define the Stokes parameters relative to the direction Ω . When a different choice of the reference direction is chosen, one can easily prove that the tensor $\mathcal{F}_Q^K(i, \Omega)$ transforms according to the equation:

$$[\mathcal{F}_Q^L(i, \Omega)]' = \sum_k S_{ik} \mathcal{F}_Q^L(k, \Omega), \quad (\text{B29})$$

where the primed quantities are those defined with respect to the new reference direction, and where:

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\gamma & \sin 2\gamma & 0 \\ 0 & -\sin 2\gamma & \cos 2\gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (\text{B30})$$

with γ defined in Fig. 2 and with $S^\dagger = S^{-1}$.

From the definition of the matrix \mathbf{H} , one also has:

$$\mathbf{H}' = \mathbf{H} \mathbf{S} \mathbf{H}^{-1}, \quad (\text{B31})$$

so that:

$$\mathbf{O}' = \mathbf{O} \mathbf{S} \mathbf{O}^{-1}. \quad (\text{B32})$$

Equations (B29) and (B32), once substituted into Eq. (B28), show that the quantity $\mathcal{C}_{LL'}$ is invariant with respect to the choice of the reference direction.

If, in particular, one chooses the reference direction in such a way that the operator $\mathcal{F}_0^K(2, \Omega)$ is zero (which is equivalent to put η_2 and ρ_2 equal to zero in Eq. (19)), the matrix \mathbf{H} assumes the more symmetrical form:

$$H_{ij} = \tau_i \tau_j H_{ji}, \quad (\text{B33})$$

where τ_i is the formal vector defined by: $\tau_i \equiv (1, 1, -1, 1)$. We then have:

$$\mathcal{C}_{LL'} = \sum_{i,j=0}^3 \mathcal{F}_0^L(i, \Omega) \tau_i \tau_j O_{ji}(\mathbf{x}, \mathbf{x}'; v) \mathcal{F}_0^{L'}(j, \Omega). \quad (\text{B34})$$

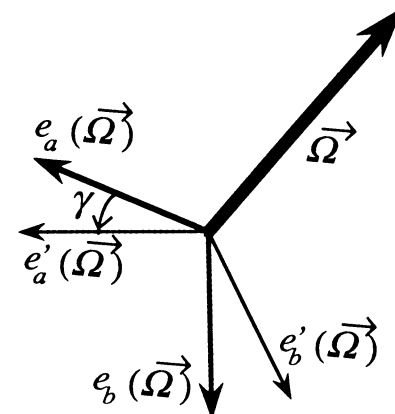


Fig. 2. Definition of γ

On the other hand, being $\mathcal{T}_0^K(2, \Omega) = 0$, we also have:

$$\tau_i \mathcal{T}_0^L(i, \Omega) = \mathcal{T}_0^L(i, \Omega), \quad (\text{B35})$$

so that:

$$\mathcal{C}_{LL'} = \sum_{i,j=0}^3 \mathcal{T}_0^{L'}(j, \Omega) O_{ji}(\mathbf{x}, \mathbf{x}'; \nu) \mathcal{T}_0^L(i, \Omega) = \mathcal{C}_{L'L}. \quad (\text{B36})$$

From this property, Eq. (B27) is easily derived.

For what the external rate $E_Q^K(\mathbf{x})$ is concerned, we can prove, through Eq. (A2) and (B3) that:

$$\begin{aligned} \lim_{\nu_L \rightarrow 0} E_Q^K(\mathbf{x}) &= \frac{B}{A} \frac{1}{\sqrt{2J'+1}} w_{J'J}^{(K)} (-1)^Q \int_0^\infty d\nu \phi(\nu_0 - \nu) \\ &\times \oint \frac{d\Omega}{4\pi} \exp[-\tau(\mathbf{x}, \mathbf{x}') \phi(\nu_0 - \nu)] \\ &\times \sum_{i=0}^3 \mathcal{T}_{-Q}^K(i, \Omega) S_i^{(0)}(\nu, \Omega), \end{aligned} \quad (\text{B37})$$

and we obtain the same result as in Paper I.

Moreover, repeating similar arguments to those leading to Eq. (B17), we can prove that, if the boundary conditions Stokes parameters present, with respect to the inversion around line center, the following symmetry property:

$$S_i^{(0)}(-\Delta\nu, \Omega) = \xi_i S_i^{(0)}(\Delta\nu, \Omega), \quad (\text{B38})$$

(or, in other words, if the Stokes parameters I , Q , U are symmetric with respect to line center, while V is antisymmetric), then we get:

$$E_Q^K(\mathbf{x}) = (-1)^K E_Q^K(\mathbf{x}), \quad (\text{B39})$$

which shows that the external rate for the odd multipoles is identically zero. However, it is worth to remember that this conclusion holds only for a static medium.

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