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Analysis of the magnetic field and torque in radially magnetized permanent magnet Couplings: Exact 2D formulation

ABSTRACT

A two-dimensional formulation for the magnetic field and torque in radially magnetized, permanent magnet couplings is given. Expressions for the field quantities are developed from the magnetic scalar potential in the air gap between the shaft and cylinder. The cogging torque in such multi-pole couplings is derived and compared with experimental results.

INTRODUCTION

Magnetic couplings are interesting because they can transmit torque without mechanical contact. A radial magnetic coupling consists of a shaft, fitted with a circular array of permanent magnet arc segments, which is separated by an air gap from a similar array of permanent magnet arc segments attached to the bore of a cylinder. The magnetization of each arc segment is assumed to be in a purely radial direction, either positive or negative, with the number and arrangement determining the number of poles in the coupling.

In the case of axial magnetic couplings, closed-form expressions for the force and torque have been derived from the magnetic vector potential obtained by solving Laplace's and Poisson's equation for a two-dimensional (2D) analytical model [1]. The approach is well-suited to parametric studies often carried out in early engineering investigations to size geometry and lead to simpler expressions than other approaches such as a Coulombian formulation [2,3,4].

In this paper, the magnetic field and torque in radially magnetized couplings is derived from a two-dimensional analytical model. Exact closed-form expressions are obtained by solving Poisson's equation for the magnetic scalar potential in the permanent magnet regions and Laplace's equation in the air gap region.

PROBLEM DESCRIPTION AND ASSUMPTIONS

A cross-section of an ideal, radially magnetized coupling is shown in Fig. 1. A circular array of permanent magnet arc segments extends radially to radius R_2 , from a shaft of radius R_1 . An air gap separates a corresponding array of permanent magnet arc segments attached to the bore of a cylinder with bore radius

R₄. The air gap radial clearance is R₃-R₂. The pole arc to pole pitch ratio is denoted by α . The alternating

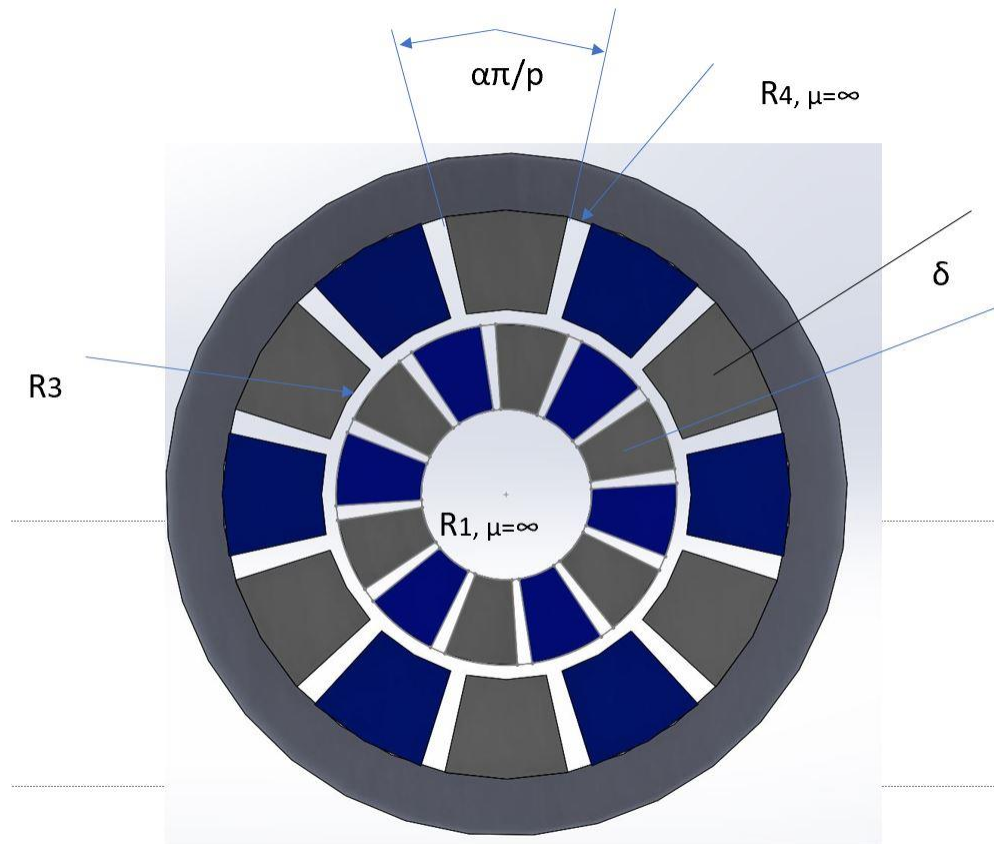


Fig 1. Cross-section of a radially magnetized coupling. The pole arc to pole pitch ratio is α , the number of poles is p ($p=6$). The shaft and housing have infinite permeability ($\mu=\infty$, at $r= R_1$ and R_4). δ is an angular offset (torque angle) used as a parameter in the torque expression.

colors denote the alternating positive and negative radial direction of the magnetization in each of the adjacent arc segments. With integer p denoting the number of pole pairs, Fig. 1 shows a $p=6$ pole-pair, radial, magnetic coupling. The cylinder is assumed to be sufficiently long and losses due to end effects and fringing negligible.

In order to simplify the two-dimensional analysis, the following additional assumptions are made:

1. The iron shaft and cylinder have infinite magnetic permeability, i.e., $\mu=\infty$
2. The radially magnetized permanent magnets have relative recoil permeability $\mu_r=1$.¹

The analysis domain will be separated into three regions. Region 1 is the region occupied by permanent magnets attached to the shaft, $R_2 \leq r \leq R_1$, region 2 is the air gap region, $R_3 \leq r \leq R_2$, and region 3 is occupied by the outer constellation of permanent magnets, $R_4 \leq r \leq R_3$. The magnets of region 3 are

¹ This assumption is reasonable for the neodymium-iron-boron (NdFeB) magnets which have $\mu_r=1.05$.

shown rotated by angle δ (torque angle) relative to the magnets in region 1. It is clear from the geometry that the magnetic field distribution is periodic with period $T = 2\pi/p$.

The constitutive law relating the magnetic flux density vector, \mathbf{B} , to the magnetic field intensity vector, \mathbf{H} , and the magnetization vector, \mathbf{M} , in the permanent magnet regions is

$$\mathbf{B} = \mu(\mathbf{H} + \mathbf{M}) \quad (1)$$

where $\mu = 4\pi \times 10^{-7}$ henrys/meter, is the permeability of space. As explained in [6], this material model,

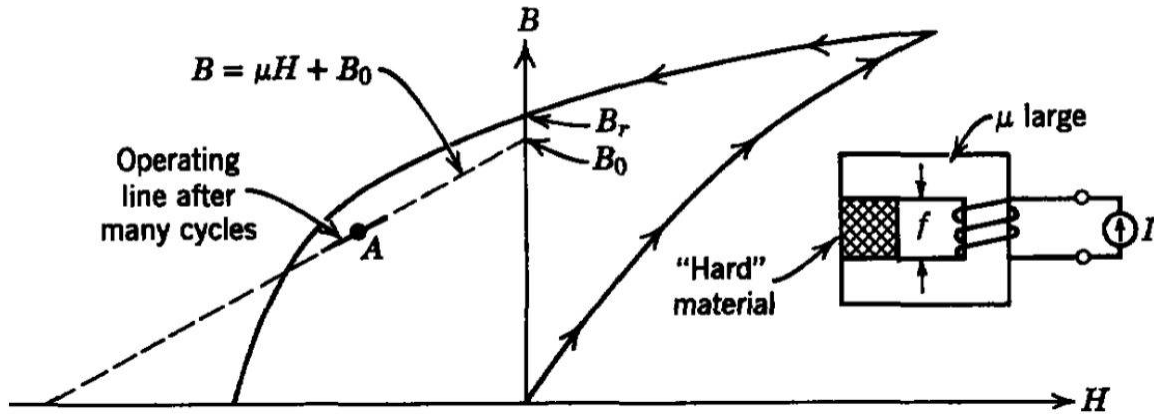


Fig. 2. (Reproduced from [6]). Hard magnetic material magnetized and then demagnetized by the current I.

shown in Fig. 2., represents “hard” magnetic materials with linear demagnetizing curve. The curve results when these materials are subjected to a field intensity \mathbf{H} by means of an extremely large current as indicated by the inset. They explain that when the current (i.e., \mathbf{H}) is removed, there is a residual flux density B_r . Then, if the sample is removed from the circuit and subjected to other magnetic fields, the \mathbf{B} - \mathbf{H} curve settles down to operate at some point, such as “A” shown in the figure. The straight line indicated by the dotted line is the representative model of the behavior.

ANALYSIS

The magnetic field problem will be formulated in terms of the scalar magnetic potential. The field intensity is defined as the negative gradient of a magnetic scalar potential, ψ

$$\mathbf{H} = -\nabla\psi \quad (2)$$

Gauss’s law of magnetism states that the magnetic field has divergence equal to zero,

$$\nabla \cdot \mathbf{B} = 0 \quad (3)$$

Substituting (2) into (1) and then imposing condition (3) results in the governing equations in each region. Thus, in the permanent magnet regions, 1 and 3, the governing equation is Poisson’s equation, which, expressed in the two cylindrical coordinates (r, θ) , is

$$\begin{aligned}\nabla \cdot \nabla \psi(r, \theta) &= \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial \theta^2} \\ &= \nabla \cdot \mathbf{M}\end{aligned}\quad (4)$$

In the air gap region, region 2, $\mathbf{M}=0$, and there is no current source, so the governing equation is Laplace's equation,

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial \theta^2} = 0 \quad (5)$$

The magnetization vector to be considered has only a radial component, ($\mathbf{M} = M_r \mathbf{e}_r$), where \mathbf{e}_r is a unit vector in the radial direction, and is distributed ideally as shown in Fig. 3, for region 3. The square wave

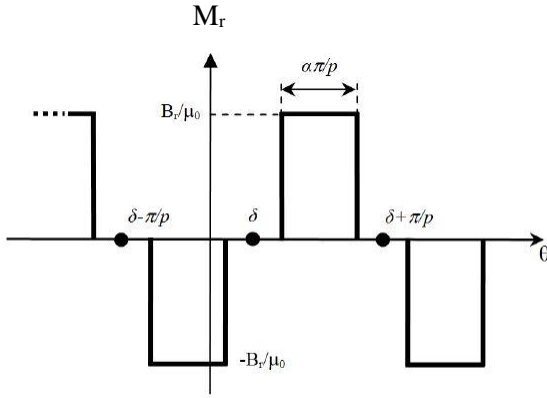


Fig.3. Radial component of magnetization vector in region 3.

can be represented as a Fourier series and substituted into the right side of Eq. (4); the result is

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial \theta^2} = \frac{1}{r} M_r \quad (6)$$

$$M_r = \sum_{k=1,3,5,\dots}^{\infty} M_k^* \sin kp (\theta - \delta) \quad (7)$$

where

$$M_k = \frac{4 B_r}{k \pi \mu} * \cos k \frac{\pi}{2} (1 - \alpha) \quad (8)$$

M_r consists of an infinite sum of sine waves of odd harmonics. The governing equation in region 1 is similar and is obtained by setting δ equal to zero. To complete the analytical model, certain boundary conditions must be satisfied. The tangential component of \mathbf{H} must be zero at the soft iron boundaries (since $\mu = \infty$), and the tangential component of \mathbf{H} and the normal component of \mathbf{B} must be continuous at the interfaces.

Using superscripts and subscripts to identify the regions and components, these boundary conditions are summarized in Table 1.

Table 1. Boundary Conditions at Interface

| Domain1 | Domain2 | Relation | Interface | Equation |
|---------|---------|--|-----------|----------|
| Iron | Magnet | $H_{\theta}^3 = -\frac{1}{r} \frac{\partial \psi^3}{\partial \theta} = 0$ | $r=R_4$ | 9 |
| Magnet | Air gap | $H_{\theta}^3 = H_{\theta}^2$ $(-\frac{1}{r} \frac{\partial \psi^3}{\partial \theta} = -\frac{1}{r} \frac{\partial \psi^2}{\partial \theta})$ | $r=R_3$ | 10 |
| Magnet | Air gap | $B_r^3 = B_r^2$ | $r=R_3$ | 11 |
| Air gap | Magnet | $H_{\theta}^2 = H_{\theta}^1$ $(-\frac{1}{r} \frac{\partial \psi^2}{\partial \theta} = -\frac{1}{r} \frac{\partial \psi^1}{\partial \theta})$ | $r=R_2$ | 12 |
| Air gap | Magnet | $B_r^2 = B_r^1$ | $r=R_2$ | 13 |
| Magnet | Iron | $H_{\theta}^1 = -\frac{1}{r} \frac{\partial \psi^1}{\partial \theta} = 0$ | $r=R_1$ | 14 |

General solution of the field equations.

The general solution for the potential will be obtained by superposition. The analytic model will be split into two parts which, when solved and summed together, provide the solution. The geometry of the first excludes the magnets of region 1, and the second excludes the magnets of region 3. The two cases are shown in Fig. 4. In each case, a preliminary solution is first obtained by considering a single term in the driving function. The complete solution will be obtained by summation of additional terms as indicated by Eq. (7) and Eq. (8).

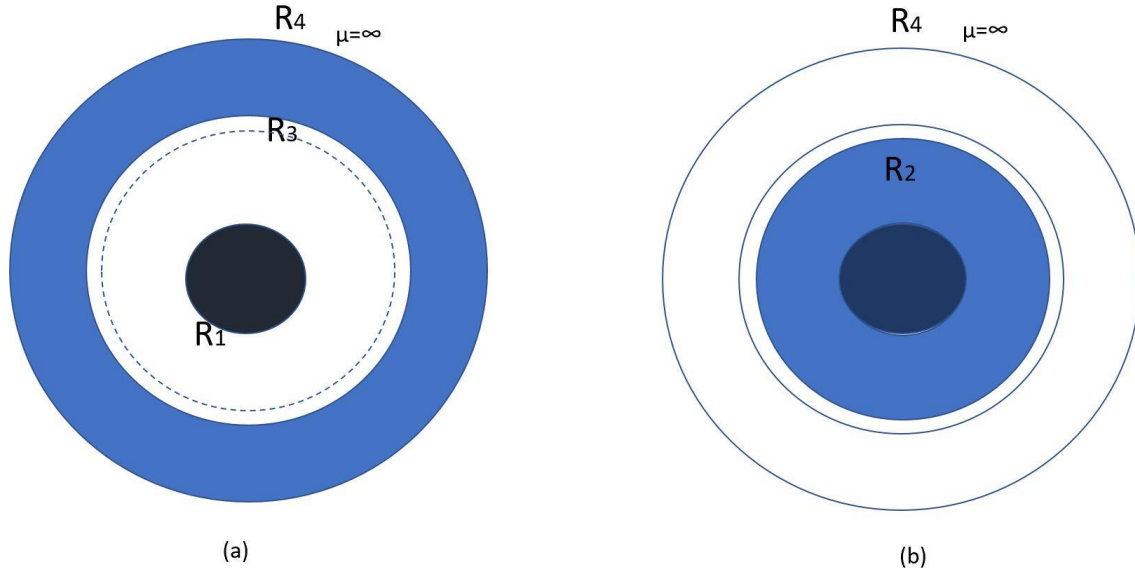


Fig. 4. The analytic model split into two parts: (a) the model geometry without the inner magnets of region 1, and (b), the model geometry without the outer magnets of region 3. The iron shaft has radius $r= R_1$, and the outer boundary is at $r= R_4$ for both.

Solution for case (a) and $kp \neq 1$

Referring to Fig. 4(a), and considering a single term in the driving function, the governing equation in the permanent magnet outer ring is Poisson's equation,

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial \theta^2} = \frac{1}{r} M_k \sin kp(\theta - \delta) \quad (15)$$

The solution is

$$\psi^3(r, \theta) = \left[a \left(\frac{r}{R_4} \right)^{kp} + b \left(\frac{R_3}{r} \right)^{kp} + \frac{M_k * r}{1 - (kp)^2} \right] \sin kp(\theta - \delta) \quad (16)$$

where a superscript on the scalar potential is used to denote the region, a and b are two arbitrary constants in the complementary solution, and the last term is the particular solution. This particular solution is valid for $kp \neq 1$. The solution for $kp=1$ will be given later as a special case.

In the air region, ($R_3 \leq r \leq R_1$), the governing equation is Laplace's equation, Eq. (5), whose general solution is well-known. The solution must also be chosen to satisfy the boundary conditions, which for case (a) is given by Eqs. (9-11), and Eq. (14), in Table 1, adjusting the superscripts for the interface boundaries. Accordingly, the solution is of the form

$$\psi^{(a)}(r, \theta) = \left[A \left(\frac{r}{R_1} \right)^{kp} + B \left(\frac{R_3}{r} \right)^{kp} \right] [C \sin(kp * \theta) + D \cos(kp * \theta)] \quad (17)$$

where a superscript (a) is used in this case to represent the scalar potential in the air region for case (a), and A, B, C, and D, are arbitrary constants. All the constants in Eq. (16) and Eq. (17) can be determined by applying the boundary conditions. The details are provided in the Appendix. The result for the scalar potential in the permanent magnet region is

$$\psi^3(r, \theta) = J(r) * \sin kp(\theta - \delta) \quad (18)$$

where

$$J(r) = b \left[\left(\frac{R_3}{r} \right)^{kp} - \left(\frac{R_3}{R_4} \right)^{kp} \left(\frac{r}{R_4} \right)^{kp} \right] - R_4 \left(\frac{r}{R_4} \right)^{kp} * \frac{M_k}{1-(kp)^2} + \frac{M_k * r}{1-(kp)^2} \quad (19)$$

and

$$b = \frac{M_k * R_3}{2 [1-(kp)^2] \left[\left(\frac{R_3}{R_1} \right)^{2kp} - \left(\frac{R_3}{R_4} \right)^{2kp} \right]} * \left[2 \left(\frac{R_3}{R_4} \right)^{kp-1} - \left(1 + \left(\frac{R_3}{R_1} \right)^{2kp} \right) - \left(1 - \left(\frac{R_3}{R_1} \right)^{2kp} \right) (kp) \right] \quad (20)$$

The result for the scalar potential in the air gap is

$$\psi^{(a)}(r, \theta) = \frac{J(r = R_3)}{\left(1 - \left(\frac{R_3}{R_1} \right)^{2kp} \right)} \left[\left(\frac{R_3}{r} \right)^{kp} - \left(\frac{R_3}{R_1} \right)^{kp} \left(\frac{r}{R_1} \right)^{kp} \right] \sin kp(\theta - \delta) \quad (21)$$

The two components of the magnetic flux density vector in the air region can be computed from Eq. (21). The radial component is

$$\begin{aligned} B_r^{(a)}(r, \theta) &= -\mu \frac{\partial \psi^{(a)}}{\partial r} \\ &= \mu \frac{J(r = R_3) * (kp)}{\left(1 - \left(\frac{R_3}{R_1} \right)^{2kp} \right) * R_3} \left[\left(\frac{R_3}{r} \right)^{kp+1} + \left(\frac{R_3}{R_1} \right)^{kp+1} \left(\frac{r}{R_1} \right)^{kp-1} \right] \sin kp(\theta - \delta) \end{aligned} \quad (22)$$

and the tangential component is

$$\begin{aligned}
B_{\theta}^{(a)}(r, \theta) &= -\mu \frac{\partial \psi^{(a)}}{r \partial \theta} \\
&= -\mu \frac{J(r = R_3) * (kp)}{\left(1 - \left(\frac{R_3}{R_1}\right)^{2kp}\right) * r} \left[\left(\frac{R_3}{r}\right)^{kp} - \left(\frac{R_3}{R_1}\right)^{kp} \left(\frac{r}{R_1}\right)^{kp} \right] \cos kp(\theta - \delta)
\end{aligned} \tag{23}$$

Solution for case (b) and $kp \neq 1$

For case (b), set $\delta=0$ in Eq. (15) to obtain the governing equation in the permanent magnet inner ring of region 1,

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial \theta^2} = \frac{1}{r} M_k \sin(kp\theta) \tag{24}$$

The solution is

$$\psi^1(r, \theta) = \left[a \left(\frac{r}{R_1}\right)^{kp} + b \left(\frac{R_2}{r}\right)^{kp} + \frac{M_k * r}{1 - (kp)^2} \right] \sin(kp\theta) \tag{25}$$

where a and b are two arbitrary constants (different from case (a)). In the air region, ($R_2 \leq r \leq R_4$), the solution to Laplace's equation must be chosen to satisfy the boundary conditions, which for case (b) is given by Eqs. (12-14), and Eq. (9) in Table 1. The solution is

$$\psi^{(b)}(r, \theta) = \left[A \left(\frac{r}{R_4}\right)^{kp} + B \left(\frac{R_2}{r}\right)^{kp} \right] [C \sin(kp * \theta) + D \cos(kp * \theta)] \tag{26}$$

where the superscript (b) is used to represent the scalar potential in the air region for case (b), and A, B, C, and D, are arbitrary constants. All the constants in Eq. (25) and Eq. (26) can be determined by applying the boundary conditions as before. Considerable work can be avoided by noting similarities in the boundary conditions and equations compared with case (a). For example, we can replace R_4 and R_3 in the magnet region for case (a) with R_1 and R_2 , to obtain the corresponding solution for case (b). Similarly, by replacing R_3 and R_1 in the air region for case (a) with R_2 and R_4 , respectively, we can obtain the corresponding solution for case (b). With this in mind, we can write the solution for the scalar potential in the permanent magnet region

$$\psi^1(r, \theta) = L(r) * \sin(kp\theta) \tag{27}$$

where

$$L(r) = b \left[\left(\frac{R_2}{r}\right)^{kp} - \left(\frac{R_2}{R_1}\right)^{kp} \left(\frac{r}{R_1}\right)^{kp} \right] - R_1 \left(\frac{r}{R_1}\right)^{kp} * \frac{M_k}{1 - (kp)^2} + \frac{M_k * r}{1 - (kp)^2} \tag{28}$$

and

$$b = \frac{M_k * R_2}{2 [1 - (kp)^2] \left[\left(\frac{R_2}{R_4} \right)^{2kp} - \left(\frac{R_2}{R_1} \right)^{2kp} \right]} * \left[2 \left(\frac{R_2}{R_1} \right)^{kp-1} - \left(1 + \left(\frac{R_2}{R_4} \right)^{2kp} \right) - \left(1 - \left(\frac{R_2}{R_4} \right)^{2kp} \right) (kp) \right] \quad (29)$$

The scalar potential in the air gap is

$$\psi^{(b)}(r, \theta) = \frac{L(r = R_2)}{\left(1 - \left(\frac{R_2}{R_4} \right)^{2kp} \right)} \left[\left(\frac{R_2}{r} \right)^{kp} - \left(\frac{R_2}{R_4} \right)^{kp} \left(\frac{r}{R_4} \right)^{kp} \right] \sin(kp\theta) \quad (30)$$

For the magnetic flux vector in air, we have for the radial component

$$\begin{aligned} B_r^{(b)}(r, \theta) &= -\mu \frac{\partial \psi^{(b)}}{\partial r} \\ &= \mu \frac{L(r = R_2) * (kp)}{\left(1 - \left(\frac{R_2}{R_4} \right)^{2kp} \right) * R_2} \left[\left(\frac{R_2}{r} \right)^{kp+1} + \left(\frac{R_2}{R_4} \right)^{kp+1} \left(\frac{r}{R_4} \right)^{kp-1} \right] \sin(kp\theta) \end{aligned} \quad (31)$$

and for the tangential component

$$\begin{aligned} B_\theta^{(b)}(r, \theta) &= -\mu \frac{\partial \psi^{(b)}}{r \partial \theta} \\ &= -\mu \frac{L(r = R_2) * (kp)}{\left(1 - \left(\frac{R_2}{R_4} \right)^{2kp} \right) * r} \left[\left(\frac{R_2}{r} \right)^{kp} - \left(\frac{R_2}{R_4} \right)^{kp} \left(\frac{r}{R_4} \right)^{kp} \right] \cos(kp\theta) \end{aligned} \quad (32)$$

Scalar potential and torque, $kp \neq 1$

The solution for all field quantities is obtained by superposition. For example, the magnetic scalar potential in the air gap region of the coupling is

$$\psi(r, \theta) = \sum_{k=1,3,5,\dots}^{\infty} (\psi^{(a)} + \psi^{(b)}) \quad (33)$$

where Eqs. (21) and (30) can be substituted in the above, and as many harmonic terms of the forcing function can be included as desired, in accordance with Eqs. (7) and (8).

Of special interest is the torque in the coupling; this can be computed from the Maxwell stress tensor, which, in electromagnetism with only a magnetic field, takes the form

$$\sigma_{ij} = \frac{1}{\mu} B_i B_j - \frac{1}{2\mu} B^2 \delta_{ij} \quad (34)$$

where B_i, B_j are components of the flux density vector, and δ_{ij} is the Kronecker delta. The magnetic torque is obtained by considering a circular integration path of any radius, R , within the air gap around the shaft and the tangential component of stress $\sigma_{r\theta}$ from Eq. (34). The differential torque, dT , acting on an element of area $Rd\theta dz$ is $dT = R * \sigma_{r\theta} * (Rd\theta dz)$. Thus, the total magnetic torque per unit length acting on the cylindrical coupling is

$$T/L = \frac{R^2}{\mu} \int_{-\pi}^{\pi} B_r(R, \theta) * B_{\theta}(R, \theta) d\theta \quad (35)$$

where L is the axial length of the magnet segments. By adding all the contributions to the field in the same manner as was done for the scalar potential, we now express the torque per unit length as

$$T/L = \frac{R^2}{\mu} \sum_k^{\infty} \int_{-\pi}^{\pi} [B_r^{(a)}(R, \theta) + B_r^{(b)}(R, \theta)] * [B_{\theta}^{(a)}(R, \theta) + B_{\theta}^{(b)}(R, \theta)] d\theta \quad (36)$$

By expansion,

$$T/L = \frac{R^2}{\mu} \sum_k^{\infty} \int_{-\pi}^{\pi} [B_r^{(a)} B_{\theta}^{(a)} + B_r^{(a)} B_{\theta}^{(b)} + B_r^{(b)} B_{\theta}^{(a)} + B_r^{(b)} B_{\theta}^{(b)}] d\theta \quad (37)$$

An expression for the torque is obtained by first substituting for each term using Eqs. (22), (23), (31), and (32), and carrying out the integration. The result is the first and fourth integral is zero and only the two middle terms contribute to the torque; therefore, it is the interaction of the two fields that give rise to the torque. The final resulting expression for the torque in the coupling can be written compactly as

$$T/L = \frac{\pi R^2}{\mu} \sum_k^{\infty} \left(B_{r0}^{(b)} * B_{\theta 0}^{(a)} - B_{r0}^{(a)} * B_{\theta 0}^{(b)} \right) \sin \delta \quad (38)$$

where the extra subscript 0 indicates the amplitude of each respective sinusoid term, which we evaluated before; namely,

$$B_{r0}^{(a)} = \mu \frac{J(r=R_3) * (kp)}{\left(1 - \left(\frac{R_3}{R_1}\right)^{2kp}\right) * R_3} \left[\left(\frac{R_3}{R}\right)^{kp+1} + \left(\frac{R_3}{R_1}\right)^{kp+1} \left(\frac{R}{R_1}\right)^{kp-1} \right] \quad (39)$$

$$B_{\theta 0}^{(a)} = -\mu \frac{J(r = R_3) * (kp)}{\left(1 - \left(\frac{R_3}{R_1}\right)^{2kp}\right) * R} \left[\left(\frac{R_3}{R}\right)^{kp} - \left(\frac{R_3}{R_1}\right)^{kp} \left(\frac{R}{R_1}\right)^{kp} \right] \quad (40)$$

$$B_{r0}^{(b)} = \mu \frac{L(r = R_2) * (kp)}{\left(1 - \left(\frac{R_2}{R_4}\right)^{2kp}\right) * R_2} \left[\left(\frac{R_2}{R}\right)^{kp+1} + \left(\frac{R_2}{R_4}\right)^{kp+1} \left(\frac{R}{R_4}\right)^{kp-1} \right] \quad (41)$$

$$B_{\theta 0}^{(b)} = -\mu \frac{L(r = R_2) * (kp)}{\left(1 - \left(\frac{R_2}{R_4}\right)^{2kp}\right) * R} \left[\left(\frac{R_2}{R}\right)^{kp} - \left(\frac{R_2}{R_4}\right)^{kp} \left(\frac{R}{R_4}\right)^{kp} \right] \quad (42)$$

Special case: $kp=1$, case (a)

The results obtained in the preceding sections must be modified when $kp=1$. Poisson's equation for case (a) now becomes

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial \theta^2} = \frac{1}{r} M_k \sin(\theta - \delta) \quad (43)$$

The solution is

$$\psi^3(r, \theta) = \left[a \left(\frac{r}{R_4}\right) + b \left(\frac{R_3}{r}\right) + \frac{M_k * \ln r * r}{2} \right] \sin(\theta - \delta) \quad (44)$$

Thus, it is seen that the new solution is obtained from the previous by setting $kp=1$, and replacing the term $M_k/(1-(kp)^2)$ in the particular solution with $M_k/2 * \ln r$.

Similarly, in the air region the solution to Laplace's equation is now written as

$$\psi^{(a)}(r, \theta) = \left[A \left(\frac{r}{R_1} \right) + B \left(\frac{R_3}{r} \right) \right] [C \sin \theta + D \cos \theta] \quad (45)$$

All the constants are determined by applying the boundary conditions as before. The result for the scalar potential in the permanent magnet region is

$$\psi^3(r, \theta) = J(r) * \sin(\theta - \delta) \quad (46)$$

where

$$J(r) = b \left[\left(\frac{R_3}{r} \right) - \left(\frac{R_3}{R_4} \right) \left(\frac{r}{R_4} \right) \right] - R_4 \left(\frac{r}{R_4} \right) * \frac{M_k}{2} * \ln R_4 + \frac{M_k * r}{2} \ln r \quad (47)$$

and

$$b = \frac{M_k * R_3}{2 [2] \left[\left(\frac{R_3}{R_1} \right)^2 - \left(\frac{R_3}{R_4} \right)^2 \right]} * \left[\left(1 - \left(\frac{R_3}{R_1} \right)^2 \right) - 2 * \ln \left(\frac{R_3}{R_4} \right) \right] \quad (48)$$

The result for the scalar potential in the air gap is

$$\psi^{(a)}(r, \theta) = \frac{J(r = R_3)}{\left(1 - \left(\frac{R_3}{R_1} \right)^2 \right)} \left[\left(\frac{R_3}{r} \right) - \left(\frac{R_3}{R_1} \right) \left(\frac{r}{R_1} \right) \right] \sin(\theta - \delta) \quad (49)$$

The two components of the magnetic flux density vector in the air region can be computed from Eq. (49). The radial component is

$$\begin{aligned} B_r^{(a)}(r, \theta) &= -\mu \frac{\partial \psi^{(a)}}{\partial r} \\ &= \mu \frac{J(r = R_3)}{\left(1 - \left(\frac{R_3}{R_1} \right)^2 \right) * R_3} \left[\left(\frac{R_3}{r} \right)^2 + \left(\frac{R_3}{R_1} \right)^2 \right] \sin(\theta - \delta) \end{aligned} \quad (50)$$

and the tangential component is

$$\begin{aligned}
B_{\theta}^{(a)}(r, \theta) &= -\mu \frac{\partial \psi^{(a)}}{r \partial \theta} \\
&= -\mu \frac{J(r = R_3)}{\left(1 - \left(\frac{R_3}{R_1}\right)^2\right) * r} \left[\left(\frac{R_3}{r}\right) - \left(\frac{R_3}{R_1}\right) \left(\frac{r}{R_1}\right) \right] \cos(\theta - \delta)
\end{aligned} \tag{51}$$

Special case: $kp=1$, case (b)

Poisson's equation for case (b) is

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial \theta^2} = \frac{1}{r} M_k \sin \theta \tag{52}$$

Obtained by letting $\delta=0$. The solution is

$$\psi^1(r, \theta) = \left[a \left(\frac{r}{R_1}\right) + b \left(\frac{R_2}{r}\right) + \frac{M_k * \ln r * r}{2} \right] \sin \theta \tag{53}$$

Similarly, in the air region the solution to Laplace's equation is now written as

$$\psi^{(b)}(r, \theta) = \left[A \left(\frac{r}{R_4}\right) + B \left(\frac{R_2}{r}\right) \right] [C \sin \theta + D \cos \theta] \tag{54}$$

All the constants are determined by applying the boundary conditions as before. The result for the scalar potential in the permanent magnet region is

$$\psi^1(r, \theta) = L(r) * \sin \theta \tag{55}$$

where

$$L(r) = b \left[\left(\frac{R_2}{r}\right) - \left(\frac{R_2}{R_1}\right) \left(\frac{r}{R_1}\right) \right] - R_1 \left(\frac{r}{R_1}\right) * \frac{M_k}{2} * \ln R_1 + \frac{M_k * r}{2} \ln r \tag{56}$$

and

$$b = \frac{M_k * R_2}{2 [2 \left[\left(\frac{R_2}{R_4}\right)^2 - \left(\frac{R_2}{R_1}\right)^2 \right]} * \left[\left(1 - \left(\frac{R_2}{R_4}\right)^2\right) - 2 * \ln\left(\frac{R_2}{R_1}\right) \right] \tag{57}$$

The result for the scalar potential in the air gap is

$$\psi^{(b)}(r, \theta) = \frac{L(r = R_2)}{\left(1 - \left(\frac{R_2}{R_4}\right)^2\right)} \left[\left(\frac{R_2}{r}\right) - \left(\frac{R_2}{R_4}\right) \left(\frac{r}{R_4}\right) \right] \sin \theta \quad (58)$$

The two components of the magnetic flux density vector in the air region can be computed from Eq. (58). The radial component is

$$B_r^{(b)}(r, \theta) = -\mu \frac{\partial \psi^{(a)}}{\partial r} = \mu \frac{L(r = R_2)}{\left(1 - \left(\frac{R_2}{R_4}\right)^2\right) * R_2} \left[\left(\frac{R_2}{r}\right)^2 + \left(\frac{R_2}{R_4}\right)^2 \right] \sin \theta \quad (59)$$

and the tangential component is

$$B_\theta^{(a)}(r, \theta) = -\mu \frac{\partial \psi^{(a)}}{r \partial \theta} = -\mu \frac{L(r = R_2)}{\left(1 - \left(\frac{R_2}{R_4}\right)^2\right) * r} \left[\left(\frac{R_2}{r}\right) - \left(\frac{R_2}{R_4}\right) \left(\frac{r}{R_4}\right) \right] \cos \theta \quad (60)$$

RESULTS

To illustrate the utility of the derived closed-form expressions, computations were done on a radially magnetized, multi-pole coupling, with parameters given in Table 2.

The torque calculations were carried out by taking a contour in the air gap with radius equal to the mean radius, (i.e., $R = (R_2 + R_3)/2$). Table 3 shows the amplitude of the cogging torque per unit length, computed from the torque expressions, for different number of pole-pairs as well as the contribution of the first four terms of k included in the summation. The sum converges rapidly and even a single term ($k=1$) approximation would yield a sufficiently accurate estimate for most preliminary sizing study. Within the range examined for this coupling, the maximum cogging torque that could be achieved is 29.37 N-m per meter of axial length of the magnet segments. This corresponds to a 3 pole-pair design, as shown in Fig. 5.

Torque measurements were taken on several couplings prepared by BJA Magnetics, [7]. The magnet segments were constructed from N52 (NdFeB) magnets and all samples correspond to the 6 pole-pair geometry of Table 2. The steel shaft and housing were made from SUS403. The magnet arc segments were magnetized diametrically (i.e., in a uniform direction) rather than in a truly radial direction. As others have noted [2], this is due to difficulty in producing a radial magnetization field. As the number of arc segments increases, the field will better approximate a true radial field.

The torque measurements are shown in Fig. 6. Measurements were made in both clockwise and counter-clockwise directions and cover one full revolution of the shaft. The average amplitude was approximately 70 Kgf-mm. From Table 3, the predicted amplitude of the cogging torque for an ideal, radially magnetized, 6 pole-pair coupling with this geometry would be 95.9 Kgf-mm. It is interesting to note that the prediction

could be adjusted and brought into better correlation with the experimental measurements by replacing the value of B_r with a reduced value B_0 , (Refer to Fig. 2 and the discussion of it).

Table 2. Parameters of radial coupling studied

| Item | Description | Value |
|----------|-----------------------------------|-------------------------|
| R_1 | Inner radius of shaft magnets | 2.5×10^{-3} m |
| R_2 | Outer radius of shaft magnets | 3.0×10^{-3} m |
| R_3 | Inner radius of cylinder magnets | 3.25×10^{-3} m |
| R_4 | Outer radius of cylinder magnets | 5×10^{-3} m |
| L | Axial length of magnets | 37×10^{-3} m |
| B_r | Residual flux density, N52 magnet | 1.44 Tesla |
| α | Pole arc to pole pitch ratio | 0.9 |
| p | Number of pole-pairs | variable |

Table 3. Computed torque amplitude per unit length, N-m/m, on coupling of Table 2.

| Index, k | Number of pole-pairs, p | | | | | |
|-----------------|-------------------------|-------|-------|-------|-------|-------|
| | 1 | 2 | 3 | 4 | 5 | 6 |
| 1 | 15.95 | 24.60 | 27.33 | 27.27 | 26.08 | 24.48 |
| 3 | 1.76 | 2.21 | 1.76 | 1.39 | 1.09 | 0.86 |
| 5 | 0.11 | 0.37 | 0.25 | 0.17 | 0.11 | 0.07 |
| 7 | 0.07 | 0.06 | 0.03 | 0.02 | 0.01 | 0.01 |
| $\Sigma(T/L) =$ | 17.89 | 27.24 | 29.37 | 28.85 | 27.29 | 25.42 |

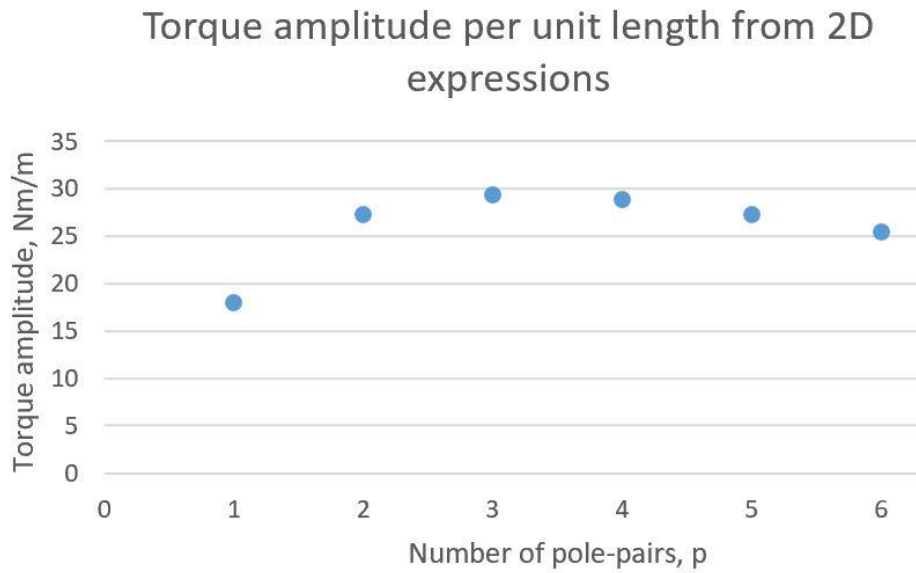


Fig. 5. Torque amplitude computed from closed-form expressions for the coupling of Table 2.

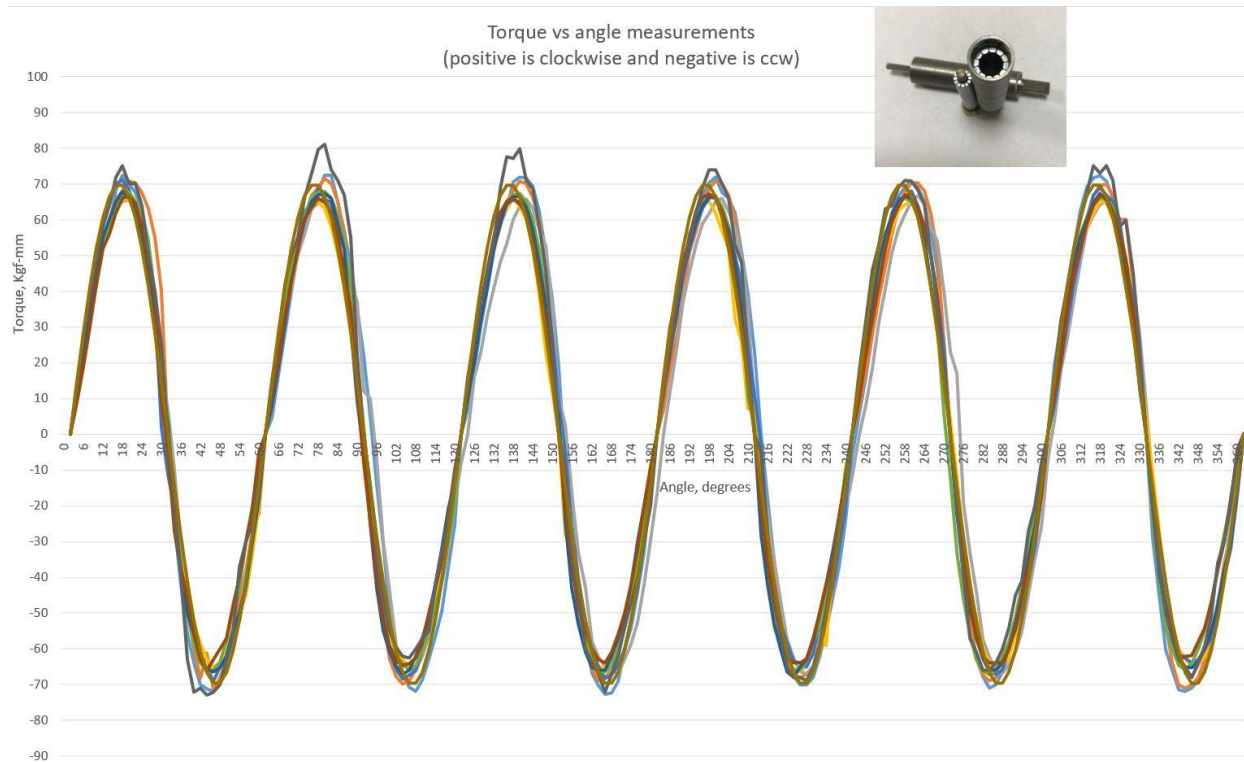


Fig. 6. Torque measurements on several samples of the 6 pole-pair, radially magnetized coupling of Table 2.

CONCLUSION

The scalar potential for multi-pole, radially magnetized couplings have been derived from the solution of Poisson's equation in the permanent magnet regions, and Laplace's equation in the air-gap region. All field quantities of interest can be obtained from the scalar potentials. In particular, the closed-form expression for the cogging torque was developed, which shows the sinusoidal behavior. The theoretical prediction for a 6 pole-pair coupling was carried out, compared with measurements taken on several samples, and the utility of the closed-form expression illustrated with a simple parametric study. Finally, it was explained how the model can be correlated to experimental findings in practice.

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APPENDIX

All the constants can be determined from the boundary conditions and will be illustrated for case (a), $kp \neq$

1. First, apply boundary condition, Eq. (9), ($\frac{\partial \psi^3(R_4, \theta)}{\partial \theta} = 0$), together with Eq. (16), then

$$a = -b \left(\frac{R_3}{R_4} \right)^{kp} - \frac{M_k * R_4}{1 - (kp)^2} \quad (A1)$$

Eqs. (18) and (19) follows from substituting Eq. (A1) into Eq. (16):

$$\psi^3(r, \theta) = J(r) * \sin kp(\theta - \delta) \quad (18)$$

$$J(r) = b \left[\left(\frac{R_3}{r} \right)^{kp} - \left(\frac{R_3}{R_4} \right)^{kp} \left(\frac{r}{R_4} \right)^{kp} \right] - R_4 \left(\frac{r}{R_4} \right)^{kp} * \frac{M_k}{1 - (kp)^2} + \frac{M_k * r}{1 - (kp)^2} \quad (19)$$

Apply boundary condition, Eq. (14), ($\frac{\partial \psi^a(R_1, \theta)}{\partial \theta} = 0$), together with Eq. (17), then

$$A = -B \left(\frac{R_3}{R_1} \right)^{kp} \quad (A2)$$

Substitute Eq. (A2) into Eq. (17)

$$\psi^{(a)}(r, \theta) = \left[\left(\frac{R_3}{r} \right)^{kp} - \left(\frac{R_3}{R_1} \right)^{kp} \left(\frac{r}{R_1} \right)^{kp} \right] [BC \sin(kp * \theta) + BD \cos(kp * \theta)] \quad (A3)$$

Apply boundary condition, Eq. (10), ($-\frac{1}{r} \frac{\partial \psi^3(R_3)}{\partial \theta} = -\frac{1}{r} \frac{\partial \psi^a(R_3)}{\partial \theta}$)

$$J(r = R_3) \cos kp(\theta - \delta) = \left[1 - \left(\frac{R_3}{R_1} \right)^{2kp} \right] [BC \cos(kp * \theta) - BD \sin(kp * \theta)] \quad (A4)$$

Equating coefficients of the sine and cosine terms, we obtain the two relations

$$J(r = R_3) \cos kp\delta = \left[1 - \left(\frac{R_3}{R_1} \right)^{2kp} \right] BC \quad (A5)$$

$$J(r = R_3) \sin kp\delta = - \left[1 - \left(\frac{R_3}{R_1} \right)^{2kp} \right] BD \quad (A6)$$

We can eliminate δ by squaring both sides of Eqs. (A5) and (A6) and adding to get

$$[J(r = R_3)]^2 = \left[1 - \left(\frac{R_3}{R_1} \right)^{2kp} \right]^2 [BC^2 + BD^2] \quad (A7)$$

Apply boundary condition, Eq. (11), $B_r^3 = B_r^a$ or ($-\frac{\partial \psi^3(R_3)}{\partial r} + M_r = -\frac{\partial \psi^a(R_3)}{\partial r}$)

$$\left(M_k - \frac{\partial J}{\partial r} \Big|_{r=R_3}\right) \sin kp(\theta - \delta) = \frac{kp}{R_3} \left[1 + \left(\frac{R_3}{R_1}\right)^{2kp}\right] [BC \sin(kp * \theta) + BD \cos(kp * \theta)] \quad (A8)$$

Equating coefficients of the sine and cosine terms, we obtain the two relations

$$\left(M_k - \frac{\partial J}{\partial r} \Big|_{r=R_3}\right) \cos kp\delta = \frac{kp}{R_3} \left[1 + \left(\frac{R_3}{R_1}\right)^{2kp}\right] BC \quad (A9)$$

$$-\left(M_k - \frac{\partial J}{\partial r} \Big|_{r=R_3}\right) \sin kp\delta = \frac{kp}{R_3} \left[1 + \left(\frac{R_3}{R_1}\right)^{2kp}\right] BD \quad (A10)$$

Squaring both sides and adding,

$$\left(M_k - \frac{\partial J}{\partial r} \Big|_{r=R_3}\right)^2 = (kp/R_3)^2 \left[1 + \left(\frac{R_3}{R_1}\right)^{2kp}\right]^2 [BC^2 + BD^2] \quad (A11)$$

We can now eliminate $[BC^2 + BD^2]$ using Eqs. (A7) and (A11) to get

$$\frac{J(r=R_3)}{1 - \left(\frac{R_3}{R_1}\right)^{2kp}} = \frac{\left(\frac{R_3}{kp}\right) \left(M_k - \frac{\partial J}{\partial r} \Big|_{r=R_3}\right)}{1 + \left(\frac{R_3}{R_1}\right)^{2kp}} \quad (A12)$$

By substituting Eq. (19) into Eq. (A12), the equation can be solved to determine the constant b. The result is Eq. (20).

$$b = \frac{M_k * R_3}{2 [1 - (kp)^2] \left[\left(\frac{R_3}{R_1}\right)^{2kp} - \left(\frac{R_3}{R_4}\right)^{2kp} \right]} * \left[2 \left(\frac{R_3}{R_4}\right)^{kp-1} - \left(1 + \left(\frac{R_3}{R_1}\right)^{2kp}\right) - \left(1 - \left(\frac{R_3}{R_1}\right)^{2kp}\right) (kp) \right] \quad (20)$$

With b determined, we can compute $J(r=R_3)$ from Eq. (19). Then, for every value of δ , we can compute BC and BD from Eqs. (A5) and (A6). Then we have ψ , B_r and B_θ . For example, substituting for BC and BD from Eqs. (A5) and A(6) into Eq. (A3), we arrive at Eq. (21):

$$\psi^{(a)}(r, \theta) = \frac{J(r=R_3)}{\left(1 - \left(\frac{R_3}{R_1}\right)^{2kp}\right)} \left[\left(\frac{R_3}{r}\right)^{kp} - \left(\frac{R_3}{R_1}\right)^{kp} \left(\frac{r}{R_1}\right)^{kp} \right] \sin kp(\theta - \delta) \quad (21)$$

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