Resonance Line Polarization for Arbitrary Magnetic Fields in Optically Thick Media. II - Case of a Plane-Parallel Atmosphere and Absence of Zeeman Coherences

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II. Case of a plane-parallel atmosphere and absence of Zeeman coherences

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Abstract. The general problem of resonance scattering for a two-level atom in a magnetized plane-parallel atmosphere is formulated by using the irreducible tensor representation of the atomic density matrix. A more general formalism presented in a previous paper (Paper I), convenient for taking into account arbitrary magnetic field effects (Zeeman effect as well as Hanle effect), is particularized in the present paper to the case of a "strong" magnetic field (Zeeman splitting much larger than the natural width of the upper level), and to the case of a unidimensional medium. A system of coupled integral equations, relating the density matrix elements at different depths in the atmosphere, is derived, and the properties of the relative kernels are discussed. It is shown that, in the limiting case of weak magnetic fields ($\nu_L \ll \Delta\nu_D$), the presence of atomic polarization induces a breakdown in the usual formula $V(\nu)=\bar{g}\nu_L(dI/d\nu)$, that is commonly used for measuring magnetic fields from magnetograph-type observations.

Key words: lines formation – line polarization – Hanle effect – Zeeman effect – radiation transfer – solar magnetic fields – magnetic fields measurements

1. Introduction

In the previous paper of this series (Landi Degl'Innocenti et al. 1990b, hereafter referred to as Paper I), we have presented a general formalism able to describe resonance polarization for a two-level atom in an optically thick, threedimensional medium embedded in a magnetic field of arbitrary strength and irradiated by an arbitrary radiation field. In the present paper, we intend to particularize the formalism of Paper I to the case of a plane-parallel atmosphere, where all the quantities entering the relevant equations depend only on one coordinate, namely the geometrical (or optical) depth.

The problem of resonance scattering in a magnetic atmosphere is particularly relevant to attack the associated problem of the diagnostic of the magnetic field in the higher layers of a stellar atmosphere. The diagnostic of these layers is indeed primarily based on the analysis of the spectropolarimetric profiles of strong or medium strong lines (like, for instance, the Mg I b lines, the Na I D lines and the Ca II H and K lines). On the other hand, the polarization observed in these lines is due both to the Zeeman splitting and to resonance polarization (as modified by the Hanle effect). Hence, it is important to provide a theory of line formation capable of encompassing the three mechanisms altogether.

As it will be discussed more extensively in Sect. 2, the physical conditions of a typical stellar atmosphere are such that when the magnetic field is sufficiently large to induce a noticeable Zeeman splitting in the line profile, all the coherences between different Zeeman sublevels vanish. This allows to simplify the physical description of the atomic excitation by retaining only the diagonal terms of the density matrix, in the magnetic field reference frame. We then obtain a typical non-LTE problem, where the unknowns are the populations of the various Zeeman sublevels of the upper level (non-LTE of the 2nd kind, according to the denomination proposed by Landi Degl'Innocenti 1989).

The same problem has already been attacked by different authors (Rees 1969; Domke 1969, 1971a, 1971b; Domke & Staude 1973; Auer et al. 1977; Landi Degl'Innocenti 1978). The advantage of the present paper is to introduce a very compact formalism which fully employs the machinery of irreducible tensors and of Racah algebra. Through this formalism, it is possible to obtain a very general system of integral equations, which, once particularized to different special cases, reduces to the equations presented in the quoted papers.

This paper is organized as follows: in Sect. 2, the approximation of neglecting coherences between different Zeeman sublevels is discussed in detail. In Sect. 3, the general formalism presented in Paper I, for an arbitrary three-dimensional medium, is particularized to the case of a plane-parallel atmosphere. The explicit expressions for the Stokes parameters emerging from the atmosphere along any given line-of-sight are presented in Sect. 4, while in Sect. 5 the limiting case of weak magnetic fields is considered. Finally, in the two appendices, some important properties for the kernels entering the relevant equations are proved. The numerical solution of the integral equations derived in the present paper will be the subject of a further investigation.

2. Classification of the magnetic field regimes

Following the same notations introduced in Paper I, we consider a two-level atom having a lower level with angular momentum J

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and Landé factor g_J and an upper level having angular momentum J' and Landé factor $g_{J'}$. Supposing the ground level to be unpolarized, the physical properties of an atom, located at the point having coordinate $\boldsymbol{x}$ in a three-dimensional medium of arbitrary shape, can be specified by means of the density matrix elements $\rho^K_Q(\boldsymbol{x})$, describing the atomic polarization of the upper level in the reference frame having its z-axis directed along the magnetic field. By coupling the statistical equilibrium equations for the atomic density matrix elements with the radiative transfer equations for polarized radiation, the following integral equation is obtained (cf. Eq. (55.I); from now on, we will refer to equation (n) of Paper I with the symbol (n.I)):

$$(1+\varepsilon+\delta^{(K)})\rho^K_Q(\boldsymbol{x}) = -i\Gamma Q\rho^K_Q(\boldsymbol{x}) + \sum_{K'Q'}\frac{1}{4\pi}\int_V \eta^{(0)}(\boldsymbol{x}')\frac{\tilde{G}_{KQ,K'Q'}(\boldsymbol{x},\boldsymbol{x}')}{(\boldsymbol{x}-\boldsymbol{x}')^2} \times \rho^{K'}_{Q'}(\boldsymbol{x}')\mathrm{d}^3\boldsymbol{x}' + E^K_Q(\boldsymbol{x}) + \varepsilon\beta_0\delta_{K,0}\delta_{Q,0}. \quad (1)$$

The derivation of this equation, together with the definitions of the different symbols employed, has been given in Paper I, and will not be repeated here. In this paper, we will just proceed to particularize this equation to the case of a plane-parallel atmosphere, and to introduce an important simplification that will bring the same equation to a more tractable form.

Referring to Eq. (1), we observe that the magnetic field amplitude – that in Paper I has been supposed constant throughout the whole medium – is contained in two of the quantities appearing in the right-hand side. Indeed, directly from the definition, we have (see Eq. (50.I)):

$$\Gamma = g_{J'}\frac{2\pi\nu_L}{A}, \quad (2)$$

where $\nu_L = e_0 B_0/(4\pi mc)$ is the Larmor frequency, and where A is the Einstein coefficient for spontaneous deexcitation. Moreover, the magnetic field amplitude is also contained, although in a more involved form, in the multipole coupling coefficients $\tilde{G}_{KQ,K'Q'}(\boldsymbol{x},\boldsymbol{x}')$, whose definition is given in Eq. (53.I). These coefficients depend on the magnetic field only through the parameter $\gamma = \nu_L/\Delta\nu_D$, where $\Delta\nu_D$ is the typical Doppler broadening of the absorption profile. If we then consider the ratio Γ/γ, we have:

$$\frac{\Gamma}{\gamma} = g_{J'}\frac{2\pi\Delta\nu_D}{A}. \quad (3)$$

For a typical stellar atmosphere and for a line at optical wavelengths, this ratio is generally much larger than unity. Writing $\Delta\nu_D$ in the form:

$$\Delta\nu_D = \frac{\nu_T}{\lambda_0}, \quad (4)$$

where ν_T is the thermal velocity, and λ_0 the wavelength of the line, we get:

$$\frac{\Gamma}{\gamma} = g_{J'}\frac{2\pi\nu_T}{\lambda_0 A}, \quad (5)$$

and assuming as typical values: $\nu_T = 2\ \mathrm{km\,s^{-1}}$, $A = 10^8\ \mathrm{s^{-1}}$, $\lambda_0 = 5000$ Å, $g_{J'} = 1$, we obtain:

$$\frac{\Gamma}{\gamma} = 2.5\,10^2. \quad (6)$$

The high value of this ratio justifies to distinguish between two different regimes for the intensity of the magnetic field vector. The first regime ($\Gamma \approx 1$, $\gamma \ll 1$) is the so-called Hanle effect regime that has already been considered in two previous papers of this series (Landi Degl'Innocenti et al. 1990a; Bommier et al. 1990). The second regime ($\Gamma \gg 1$, $\gamma \approx 1$) is the one that we are going to discuss in the present paper, and is characterized by the fact that the magnetic field is sufficiently strong to completely remove coherences between different Zeeman sublevels. Translated in practical terms on the particular case considered above, this regime is characterized by values of magnetic fields larger than a critical value, B_c, that is of the order of 10 Gauss. The present theory can then be safely applied to the interpretation of chromospheric lines formed in typical magnetic regions of a stellar atmosphere.

The approximation of neglecting coherences between different Zeeman sublevels considerably simplifies Eq. (1) that is now reduced to an integral equation connecting only the diagonal terms of the form ρ^K_Q with $Q=0$. Introducing a more compact notation:

$$\rho^K_0 \equiv \rho_K; \quad \tilde{G}_{K0,K'0}(\boldsymbol{x},\boldsymbol{x}') \equiv \tilde{G}_{K,K'}(\boldsymbol{x},\boldsymbol{x}'); \quad E^K_0 \equiv E_K, \quad (7)$$

we have:

$$(1+\varepsilon+\delta^{(K)})\rho_K(\boldsymbol{x}) = \sum_{K'}\frac{1}{4\pi}\int_V \eta^{(0)}(\boldsymbol{x}')\frac{\tilde{G}_{K,K'}(\boldsymbol{x},\boldsymbol{x}')}{(\boldsymbol{x}-\boldsymbol{x}')^2}\rho_{K'}(\boldsymbol{x}')\mathrm{d}^3\boldsymbol{x}' + E_K(\boldsymbol{x}) + \varepsilon\beta_0\delta_{K,0}. \quad (8)$$

This is an integral equation which involves, in general, $2J'+1$ unknown functions of $\boldsymbol{x}$, namely the density matrix elements ρ_K. However, as we have already pointed out in Paper I, the coupling coefficients connecting multipole tensors of different parity are identically zero. Moreover, the tensors of odd rank do not have any source term, either when the boundary conditions Stokes parameters illuminating the medium from outside are identically zero (like in the case of a semi-infinite atmosphere), or when they satisfy certain symmetry properties (with respect to frequency inversion) that are specified in Appendix B of Paper I. In these cases, the tensors of odd rank are identically zero, and the unknowns in Eq. (8) are only the tensors of even rank. This reduces the number of unknown functions from $2J'+1$ to $N_{J'}$, where:

$$N_{J'} = \begin{cases} J'+1 & (J' \text{ integer}) \\ J'+1/2 & (J' \text{ half-integer}). \end{cases} \quad (9)$$

3. The case of a plane-parallel atmosphere

For a plane-parallel atmosphere, or, more in general, for a slab, like the one sketched in Fig. 1, the unknown functions $\rho_K(\boldsymbol{x})$ and the other parameters entering Eq. (8) ($\eta^{(0)}$, ε, $\delta^{(K)}$, β_0) depend only on the spatial coordinate z, and not on x and y. The radiation field that is illuminating the slab has obviously to fulfil the same requirement of being independent of x and y, so that also the external rate E_K is only a function of z.

Supposing that the point P is located on the z-axis ($x=y=0$), the integral appearing in Eq. (8) can be expressed in the following form:

$$\int_V \ldots \mathrm{d}^3\boldsymbol{x}' = \int_{z_a}^{z_b}\mathrm{d}z'\eta^{(0)}(z')\rho_{K'}(z') \times \int_{-\infty}^{\infty}\mathrm{d}x'\int_{-\infty}^{\infty}\mathrm{d}y'\frac{\tilde{G}_{K,K'}(\boldsymbol{x},\boldsymbol{x}')}{(z-z')^2+x'^2+y'^2}. \quad (10)$$

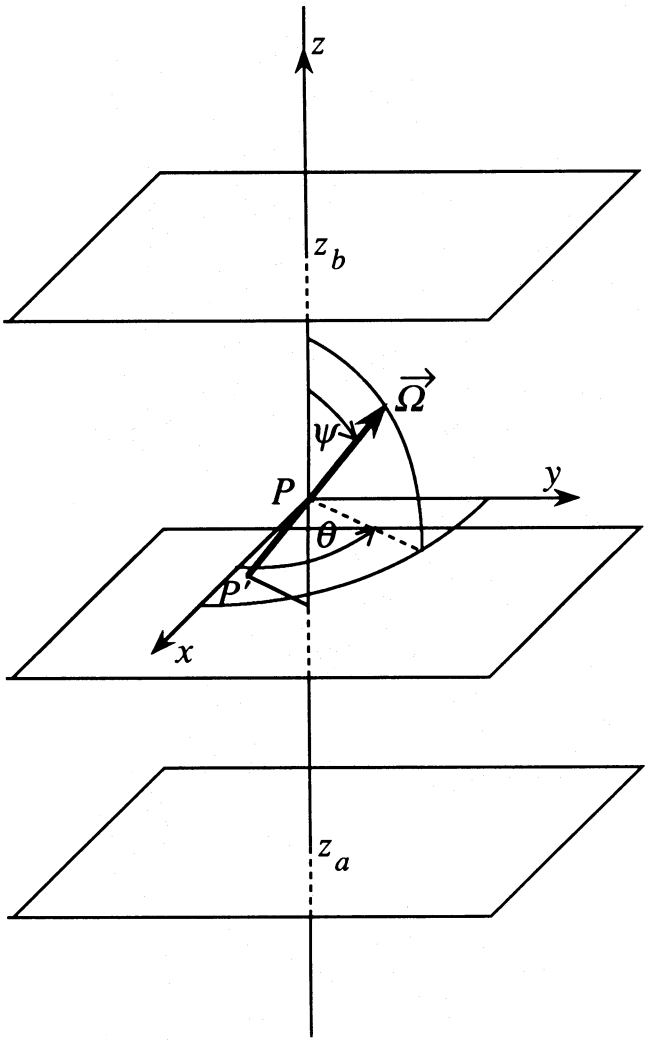

Fig. 1. Any direction Ω through the point P is specified by the polar angles ψ and θ

We now perform the surface integral contained in the r.h.s. by suitably changing the integration variables. We first characterize any direction Ω passing through the point P by means of the angles ψ and θ defined as in Fig. 1. Any direction crosses the plane $z=z'$ in a point P' having coordinates:

$$\begin{cases} x'=(z'-z)\,\mathrm{tg}\,\psi\cos\theta \\ y'=(z'-z)\,\mathrm{tg}\,\psi\sin\theta, \end{cases} \tag{11}$$

so that we can write, evaluating the Jacobian of this transformation

$$\mathrm{d}x'\mathrm{d}y'=(z'-z)^2\frac{|\mathrm{tg}\,\psi|}{\cos^2\psi}\,\mathrm{d}\psi\,\mathrm{d}\theta, \tag{12}$$

and we get:

$$\frac{\mathrm{d}x'\mathrm{d}y'}{(z-z')^2+x'^2+y'^2}=|\mathrm{tg}\,\psi|\,\mathrm{d}\psi\,\mathrm{d}\theta. \tag{13}$$

Substituting this expression in Eq. (10), we obtain:

$$\int_V \ldots \mathrm{d}^3x'=\int_{z_a}^{z_b}\mathrm{d}z'\eta^{(0)}(z')\,\rho_{K'}(z')\int_{\psi_1}^{\psi_2}\mathrm{d}\psi\,|\mathrm{tg}\,\psi| \times\int_0^{2\pi}\mathrm{d}\theta\,\tilde{G}_{K,K'}(\boldsymbol{x},\boldsymbol{x}'), \tag{14}$$

where:

$$\begin{cases}\psi_1=0\\ \psi_2=\pi/2\end{cases}\ \text{if } z>z';\qquad \begin{cases}\psi_1=\pi/2\\ \psi_2=\pi\end{cases}\ \text{if } z<z'. \tag{15}$$

Introducing finally the optical depth $t(z)$ measured from the top of the slab along the z-direction:

$$t(z)=\int_z^{z_b}\eta^{(0)}(z')\mathrm{d}z', \tag{16}$$

we obtain:

$$\int_V \ldots \mathrm{d}^3x'=\int_0^{t_a}\mathrm{d}t\,\rho_{K'}(t)\int_{\psi_1}^{\psi_2}\mathrm{d}\psi\,|\mathrm{tg}\,\psi|\int_0^{2\pi}\mathrm{d}\theta\,\tilde{G}_{K,K'}(\boldsymbol{x},\boldsymbol{x}'), \tag{17}$$

where $t_a=t(z_a)$ is the optical depth corresponding to the bottom of the slab.

Substituting this result in Eq. (2), we obtain the following integral equation:

$$\rho_K(t)=\frac{1}{1+\varepsilon+\delta^{(K)}}\sum_{K'}\int_0^{t_a}\tilde{K}_{KK'}(t,t')\rho_{K'}(t')\mathrm{d}t' +\frac{1}{1+\varepsilon+\delta^{(K)}}E_K+\frac{\varepsilon}{1+\varepsilon+\delta^{(K)}}\beta_0\delta_{K,0}, \tag{18}$$

where the kernels $\tilde{K}_{KK'}(t,t')$ are given by:

$$\tilde{K}_{KK'}(t,t')=\frac{1}{4\pi}\int_{\psi_1}^{\psi_2}\mathrm{d}\psi\,|\mathrm{tg}\,\psi|\int_0^{2\pi}\mathrm{d}\theta\,\tilde{G}_{K,K'}(\boldsymbol{x},\boldsymbol{x}'), \tag{19}$$

with:

$$\begin{cases}\psi_1=0\\ \psi_2=\pi/2\end{cases}\ \text{if } t<t';\qquad \begin{cases}\psi_1=\pi/2\\ \psi_2=\pi\end{cases}\ \text{if } t>t'.$$

The evaluation of the kernels requires a double integration, over the angles ψ and θ, of the multipole coupling coefficients. The explicit expression of these coefficients has been given in Eq. (53.I); we have:

$$\tilde{G}_{K,K'}(\boldsymbol{x},\boldsymbol{x}')=\sum_{i,j=0}^{3}\sum_{LL'}\int_0^{\infty}\mathrm{d}\nu\,\Phi_0^{K,L}(J,J';\nu)\Phi_0^{K',L'}(J,J';\nu) \times\mathscr{T}_0^{L}(i,\Omega)O_{ij}(\boldsymbol{x},\boldsymbol{x}';\nu)\mathscr{T}_0^{L'}(j,\Omega), \tag{20}$$

where $O_{ij}(\boldsymbol{x},\boldsymbol{x}';\nu)$ is given by (see Eq. (47.I)):

$$O_{ij}(\boldsymbol{x},\boldsymbol{x}';\nu)=[\exp\{-\tau(\boldsymbol{x},\boldsymbol{x}')\boldsymbol{H}(\nu,\Omega)\}]_{ij}. \tag{21}$$

The explicit dependence of $\tilde{G}_{K,K'}$ on the angles ψ and θ is contained in the irreducible tensors $\mathscr{T}_0^{K}(i,\Omega)$, in the matrix $\boldsymbol{H}(\nu,\Omega)$ and in the factor $\tau(\boldsymbol{x},\boldsymbol{x}')$. This factor can be simply expressed in the form:

$$\tau(\boldsymbol{x},\boldsymbol{x}')=\frac{|t-t'|}{\cos\psi}. \tag{22}$$

For what the matrix $\boldsymbol{H}(\nu,\Omega)$ is concerned, remembering Eqs. (19.I), (39.I), (40.I), and (46.I), it is readily seen that its explicit

dependence on the angles ψ and θ is again contained in the irreducible tensors $\mathscr{T}^K_0(i, \Omega)$. On the other hand, directly from the definition [see Landi Degl'Innocenti 1984, Eq. (A6)], we have:

$$\mathscr{T}^K_0(i, \Omega)=\sum_P t^K_P(i)\mathscr{D}^K_{P0}(R), \tag{23}$$

where $t^K_P(i)$ are the symbols defined in the same paper and where R is the rotation bringing the coordinate system of the polarization unit vectors $[e_a(\Omega), e_b(\Omega), \Omega]$ into the system having its z-axis directed along the magnetic field. If the reference direction $e_a(\Omega)$ for defining the Stokes parameters is the one of Fig. 2, denoting with θ_B and ψ_B the polar angles of the magnetic field, we have:

$$R\equiv(0, -\psi, -\theta)\otimes(\theta_B, \psi_B, 0). \tag{24}$$

Various properties of the kernels are discussed in Appendix A. In particular, the property described by Eq. (A1) allows to write the kernels in the form:

$$\tilde{K}_{KK'}(y)=\frac{1}{4\pi}\int_0^{\pi/2} d\psi \operatorname{tg}\psi \int_0^{2\pi} d\theta \sum_{i,j=0}^{3}\sum_{LL'}\int_0^{\infty} d\nu \Phi^{K,L}_0(J, J'; \nu) \times \Phi^{K',L'}_0(J, J'; \nu)\mathscr{T}^L_0(i, \Omega) \times\left[\exp\left\{-\frac{y}{\cos\psi}\boldsymbol{H}\right\}\right]_{ij}\mathscr{T}^{L'}_0(j, \Omega), \tag{25}$$

where $y=|t-t'|$.

Once these kernels have been evaluated, they can be substituted in Eq. (18). The same equation can then be solved to get the behavior of the multipole moments $\rho_K(t)$ as a function of optical depth.

For the case of a semi-infinite atmosphere not illuminated from above (the case to which we restrict in the following), Eq. (18) can be simplified by putting $E_K=0$ and letting $t_a\to\infty$. Moreover, introducing standard notations that are commonly employed in non-LTE transfer problems, and remembering that $\delta^{(0)}=0$, the same equation can be written in the form:

$$\rho_K(t)=(1-\varepsilon''_K)\sum_{K'}\int_0^{\infty}\tilde{K}_{KK'}(|t-t'|)\rho_{K'}(t')dt'+\varepsilon'\beta_0\delta_{K,0}, \tag{26}$$

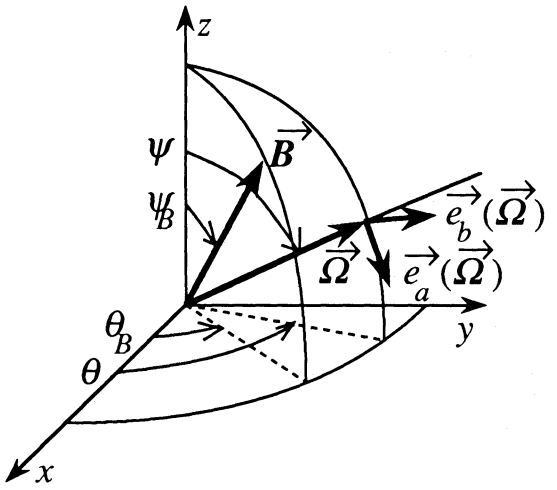

Fig. 2. Polar angles for reckoning the direction Ω and the magnetic field in the reference system having its z-axis perpendicular to the atmosphere. $e_a(\Omega)$ is the reference direction for defining the Q Stokes parameter propagating along Ω

where:

$$\varepsilon'=\frac{\varepsilon}{1+\varepsilon};\quad \varepsilon''_K=\begin{cases}\varepsilon' & \text{for } K=0\\ \dfrac{\varepsilon+\delta^{(K)}}{1+\varepsilon+\delta^{(K)}} & \text{for } K\neq 0.\end{cases} \tag{27}$$

An interesting particular case of this equation is obtained by considering the limiting case of complete depolarization of the upper level. This limit is obtained by letting $\delta^{(K)}\to\infty$ in Eq. (27), which results in $\varepsilon''_K=1$ for $K\neq 0$. In this case, the system of coupled integral equations reduces to a single integral equation for the density matrix element $\rho_0(t)$, a quantity proportional to the overall population of the upper level:

$$\rho_0(t)=(1-\varepsilon')\int_0^{\infty}\tilde{K}_{00}(|t-t'|)\rho_0(t')dt'+\varepsilon'\beta_0. \tag{28}$$

This is the basic equation that has been solved by Rees (1968) in his analysis of the problem of non-LTE line formation in a magnetic field. Rees considered the particular case of a normal Zeeman triplet ($J=0$, $J'=1$, according to our formalism), and neglected magneto-optical effects in the transfer of polarized radiation; he was able to show that, for a realistic atmospheric model, the presence of a magnetic field has a weak influence on the function $\rho_0(t)$ that is found by solving Eq. (28). In other words, the function $\rho_0(t)$ can be derived (within an accuracy of the order of 10%) by simply neglecting the influence of the magnetic field on the kernel $\tilde{K}_{00}(|t-t'|)$ and simply substituting it with the "zero-field" limit (see Eq. (A11)):

$$\lim_{\gamma\to 0}\tilde{K}_{00}(y)=\frac{1}{2}\int_{-\infty}^{\infty}dx\phi^2(x)E_1[y\phi(x)]. \tag{29}$$

The results obtained by Rees were later confirmed by Domke & Staude (1973), who included magneto-optical effect in their analysis and by Auer et al. (1977), who performed some numerical calculations on the Ca II K line under the same approximation of complete depolarization of the upper level.

It has to be remarked, however, that the approximation of complete depolarization is not valid, in general, for the higher layers of a stellar atmosphere, so that the Rees' approximation has to be applied with extreme care for the interpretation of Stokes parameters profiles of chromospheric lines.

4. Emerging Stokes parameters

The expression for the Stokes parameters of the radiation emerging, along any given direction, from the plane-parallel atmosphere can be derived from Eqs. (41.I) and (45.I). Characterizing a particular line of sight (Ω_l) through the angles θ_l and ψ_l defined as in Fig. 2, and assuming as reference direction the one specified in the same figure (positive Q meaning linear polarization directed perpendicularly to the atmospheric layers), we get:

$$S_i(\nu, \Omega_l)=\frac{A}{B}\sqrt{2J'+1}\sum_{KK'}\sum_{j=0}^{3}[P_K(\nu)]_{ij}\mathscr{T}^{K'}_0(j, \Omega_l)\Phi^{K,K'}_0(J, J'; \nu), \tag{30}$$

where:

$$[P_K(\nu)]_{ij}=\int_0^{\infty}\rho_K(\tau)\left[\exp\left\{-\frac{\tau}{\cos\psi_l}\boldsymbol{H}\right\}\right]_{ij}\frac{d\tau}{\cos\psi_l}, \tag{31}$$

and where the rotation R appearing in the tensor $\mathscr{T}^K_0(j, \Omega_l)$ is

given by:

$$R \equiv (0, -\psi_l, -\theta_l) \otimes (\theta_B, \psi_B, 0). \tag{32}$$

From the knowledge of the functions $\rho_K(\tau)$, the τ-integral leading to $[P_K(\nu)]_{ij}$ can be evaluated numerically; the substitution of these quantities into Eq. (30) yields the final results for the emerging Stokes parameters.

5. Limit for weak fields

In this section, we are going to deduce some results for the emerging Stokes parameters under the limit of weak fields ($\gamma \ll 1$). These results are particularly important as they concern a relationship between the profiles of the Stokes parameters $I(\nu, \Omega_l)$ and $V(\nu, \Omega_l)$ (intensity and circular polarization), which is often used for measuring the component of the magnetic field vector along the line of sight.

In Eq. (30), we consider a Taylor expansion, extended up to first order in ν_L, of the various quantities there contained (with the exception of $\rho_K(\tau)$ that we consider as fixed). For the generalized profiles, we have:

$$\Phi_0^{K,K'}(J, J'; \nu) = \delta_{K,K'} w_{J'J}^{(K)} \phi_\nu + g(K, K') \nu_L \phi'_\nu, \tag{33}$$

where ϕ_ν is the absorption profile relative to line center ($\phi_\nu = \phi(\nu_0 - \nu)$), ϕ'_ν is its derivative with respect to frequency ($\phi'_\nu = \partial\phi(\nu_0 - \nu)/\partial\nu$), $w_{J'J}^{(K)}$ are the symbols defined in Landi Degl'Innocenti (1983), and, finally, the quantities $g(K, K')$ are generalized, effective Landé factors whose explicit expression can be found in Appendix B.

We now take into account that, in Eq. (30), the sum over K is extended only to even values of K ($K = 0, 2, 4, \ldots$), while the sum over K' is extended only to $K' = 0, 1, 2$. We then have, for the only non-vanishing values of the generalized profiles (to first order in ν_L):

$$\begin{aligned}
&\Phi_0^{0,0}(J, J'; \nu) = \phi_\nu, \\
&\Phi_0^{0,1}(J, J'; \nu) = \nu_L g(0, 1) \phi'_\nu, \\
&\Phi_0^{2,1}(J, J'; \nu) = \nu_L g(2, 1) \phi'_\nu, \\
&\Phi_0^{2,2}(J, J'; \nu) = w_{J'J}^{(2)} \phi_\nu.
\end{aligned} \tag{34}$$

Remembering the expression for the propagation matrix $\boldsymbol{H}$ (see Eqs. (19.I), (39.I), and (40.I)), considering that for the generalized anomalous dispersion profiles $\Psi_0^{K,K'}$ we have expansions strictly analogous to Eqs. (34), and remembering that:

$$\mathscr{T}_0^1(i, \Omega_l) = \delta_{i,3} \sqrt{\frac{3}{2}} \mathscr{D}_{00}^1(R),$$

we obtain:

$$\boldsymbol{H} = \phi_\nu \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \nu_L \sqrt{\frac{3}{2}} \cos\Psi g(0, 1) \times \begin{pmatrix} 0 & 0 & 0 & \phi'_\nu \\ 0 & 0 & \psi'_\nu & 0 \\ 0 & -\psi'_\nu & 0 & 0 \\ \phi'_\nu & 0 & 0 & 0 \end{pmatrix}, \tag{35}$$

where:

$$\cos\Psi = \cos\psi_l \cos\psi_B + \sin\psi_l \sin\psi_B \cos(\theta_l - \theta_B)$$

is the cosine of the angle between the line of sight and the magnetic field, and where:

$$\psi'_\nu = \frac{\partial}{\partial\nu} \psi(\nu_0 - \nu).$$

From this expression, we obtain (to first order in ν_L):

$$\begin{aligned}
\exp\left\{-\frac{\tau}{\cos\psi_l} \boldsymbol{H}\right\} = \exp\left\{-\frac{\tau}{\cos\psi_l} \phi_\nu\right\} \\
\times \left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \frac{\tau}{\cos\psi_l} \nu_L \sqrt{\frac{3}{2}} \cos\Psi g(0, 1) \right. \\
\left. \times \begin{pmatrix} 0 & 0 & 0 & \phi'_\nu \\ 0 & 0 & \psi'_\nu & 0 \\ 0 & -\psi'_\nu & 0 & 0 \\ \phi'_\nu & 0 & 0 & 0 \end{pmatrix} \right],
\end{aligned} \tag{36}$$

and, substituting into Eq. (30), we get for I and V:

$$\begin{aligned}
I(\nu, \Omega_l) = \frac{A}{B} \sqrt{2J'+1}\, \phi_\nu \left\{ \int_0^\infty \rho_0(\tau) e^{-(\tau/\cos\psi_\ell)\phi_\nu} \frac{d\tau}{\cos\psi_l} \right. \\
\left. + w_{J'J}^{(2)} \frac{1}{\sqrt{2}} \mathscr{D}_{00}^2(R) \int_0^\infty \rho_2(\tau) e^{-(\tau/\cos\psi_\ell)\phi_\nu} \frac{d\tau}{\cos\psi_l} \right\},
\end{aligned} \tag{37}$$

where:

$$\mathscr{D}_{00}^2(R) = \frac{1}{2}(3\cos^2\Psi - 1); \tag{38}$$

$$\begin{aligned}
V(\nu, \Omega_l) = \frac{A}{B} \sqrt{2J'+1}\, \nu_L \sqrt{\frac{3}{2}} \cos\Psi \\
\times \left\{ g(0, 1) \phi'_\nu \int_0^\infty \rho_0(\tau) e^{-(\tau/\cos\psi_\ell)\phi_\nu} \frac{d\tau}{\cos\psi_l} + \right. \\
+ g(2, 1) \phi'_\nu \int_0^\infty \rho_2(\tau) e^{-(\tau/\cos\psi_\ell)\phi_\nu} \frac{d\tau}{\cos\psi_l} \\
- g(0, 1) \phi_\nu \phi'_\nu \int_0^\infty \rho_0(\tau) e^{-(\tau/\cos\psi_\ell)\phi_\nu} \frac{\tau}{\cos\psi_l} \frac{d\tau}{\cos\psi_l} \\
- g(0, 1) w_{J'J}^{(2)} \frac{1}{\sqrt{2}} \mathscr{D}_{00}^2 \phi_\nu \phi'_\nu \\
\left. \times \int_0^\infty \rho_2(\tau) e^{-(\tau/\cos\psi_\ell)\phi_\nu} \frac{\tau}{\cos\psi_l} \frac{d\tau}{\cos\psi_l} \right\}.
\end{aligned} \tag{39}$$

From these equations, we obtain, after some algebra:

$$\begin{aligned}
V(\nu, \Omega_l) = \nu_L \sqrt{\frac{3}{2}} \cos\Psi g(0, 1) \frac{dI(\nu, \Omega_l)}{d\nu} \\
+ \frac{A}{B} \sqrt{2J'+1}\, \nu_L \sqrt{\frac{3}{2}} \cos\Psi \phi'_\nu \left[g(2, 1) - g(0, 1) \right. \\
\left. \times w_{J'J}^{(2)} \frac{1}{\sqrt{2}} \mathscr{D}_{00}^2(R) \right] \int_0^\infty \rho_2(\tau) e^{-(\tau/\cos\psi_\ell)\phi_\nu} \frac{d\tau}{\cos\psi_l}.
\end{aligned} \tag{40}$$

Taking into account that the quantity $\sqrt{\frac{3}{2}}g(0, 1)$ is nothing but $\bar{g}$, or the usual effective Landé factor of the spectral line, we see from this equation that the relationship

$$V(v, \Omega_l)=v_L \bar{g} \cos\Psi \frac{dI(v, \Omega_l)}{dv} \tag{41}$$

that is commonly used to find the line of sight component of the magnetic field vector (in the limiting case of weak fields), is valid only when the atom is devoided of atomic polarization ($\rho_2(\tau)=0$). For $\rho_2(\tau)\neq 0$, the same relationship is no longer true, but it has to be substituted with the more general Eq. (40).

Finally, substituting Eqs. (34) and (36) in Eq. (25), it is possible to show that the various kernels $\tilde{K}_{KK'}(y)$ have a quadratic behavior, as a function of v_L, around $v_L=0$. In other words:

$$\tilde{K}_{KK'}(y)=[\tilde{K}_{KK'}(y)]_{v_L=0}+\mathcal{O}(v_L^2). \tag{42}$$

6. Conclusion

In this paper, we have attacked the general problem of resonance scattering for a two-level atom in a magnetized plane-parallel atmosphere. By particularizing a more general formalism presented in a previous paper (Paper I), and assuming that the magnetic field is sufficiently strong to remove coherences between different Zeeman sublevels ($B\gg 10$ Gauss), we have found a system of coupled integral equations relating the density matrix elements ρ^K_Q of even rank at different depths in the atmosphere. The coupling of the various density matrix elements is described by suitable kernels whose relevant properties are thoroughly discussed in the appendices. The solution of the system of integral equations leads, in principle, to the determination of the run with optical depth of the various density matrix elements, from which the emerging Stokes parameters can be derived for any assigned direction of the line of sight.

Particularly interesting are the results for the case of weak fields ($v_L \ll \Delta\nu_D$) that have been derived in Sect. 5. These results show that the presence of atomic polarization avoids to relate the circular polarization Stokes parameter to the derivative with respect to frequency of the intensity profile. This fact has to be kept in mind when trying to deduce the magnetic field from magnetograph-type observation of chromospheric lines.

Appendix A: properties of the kernels $\tilde{K}_{KK'}(t, t')$

a) Inversion of the arguments t and t'

$$\tilde{K}_{KK'}(t, t')=\tilde{K}_{KK'}(t', t)=\tilde{K}_{KK'}(|t-t'|). \tag{A1}$$

b) Reality

$$[\tilde{K}_{KK'}(y)]^*=\tilde{K}_{KK'}(y). \tag{A2}$$

c) Inversion of the indices

$$\tilde{K}_{KK'}(y)=\tilde{K}_{K'K}(y). \tag{A3}$$

d) Vanishing values

$$\tilde{K}_{KK'}(y)=(-1)^{K+K'}\tilde{K}_{KK'}(y). \tag{A4}$$

All these properties descend directly from analogous properties of the multipole coupling coefficients that have been proved in Paper I. In particular, property *a*) descends from Eq. (B18.I), and, analogously, property *b*) from Eq. (B11.I), property *c*) from Eq. (B27.I), and property *d*) from Eq. (B17.I). Moreover, we derive the further properties given in the following.

e) Integral properties

$$\int_0^\infty \tilde{K}_{0K}(y)dy=\frac{1}{2}\delta_{0,K}. \tag{A5}$$

To prove this property, we start from Eq. (25), and integrate in dy. We have:

$$\int_0^\infty dy\left[\exp\left\{-\frac{y}{\cos\psi}\boldsymbol{H}\right\}\right]_{ij}=\cos\psi(\boldsymbol{H}^{-1})_{ij}, \tag{A6}$$

where $\boldsymbol{H}^{-1}$ is the inverse of $\boldsymbol{H}$.

We next observe that (see Eqs. (19.I) and (39.I)):

$$\sum_L \mathcal{T}^L_0(i, \Omega)\Phi^{0,L}_0(J, J'; \nu)=H_{0i}, \tag{A7}$$

so that we obtain, summing over the indices i and j:

$$\int_0^\infty \tilde{K}_{0K}(y)dy=\frac{1}{4\pi}\int_0^{\pi/2} d\psi \sin\psi \int_0^{2\pi} d\theta \sum_{L'}\int_0^\infty d\nu \Phi^{K,L'}_0(J, J'; \nu) \times \mathcal{T}^{L'}_0(0, \Omega). \tag{A8}$$

We now integrate over θ and ψ and take into account the Weil theorem which gives:

$$\frac{1}{4\pi}\int_0^{\pi/2} d\psi \sin\psi \int_0^{2\pi} d\theta \mathcal{T}^{L'}_0(0, \Omega)=\frac{1}{2}\delta_{0,L'};$$

we thus obtain:

$$\int_0^\infty \tilde{K}_{0K}(y)dy=\frac{1}{2}\int_0^\infty d\nu \Phi^{K,0}_0(J, J'; \nu), \tag{A9}$$

and, taking finally into account Eq. (A3.I), we get the proof of Eq. (A5).

f) Limit for weak fields

In the limit of weak fields ($\Gamma\gg 1$, $\gamma\ll 1$), we have (see Eq. (B5) in Landi Degl'Innocenti et al. (1990a)):

$$\lim_{\gamma\to 0}\tilde{G}_{K,K'}(\boldsymbol{x}, \boldsymbol{x}')=G_{K0,K'0}(\Omega)\phi^2(\nu_0-\nu)\exp[-\tau(\boldsymbol{x}, \boldsymbol{x}')\phi(\nu_0-\nu)], \tag{A10}$$

where $G_{K0,K'0}(\Omega)$ is defined in the same paper. The integration over the solid angle can now be performed according to the transformations described in Sect. 3 of the same paper. The results for the only non-vanishing values of the kernel $\tilde{K}_{KK'}$ are the following:

$$\lim_{\gamma\to 0}\tilde{K}_{00}(y)=\frac{1}{2}\int_{-\infty}^\infty dx\phi^2(x)E_1[y\phi(x)],$$

$$\lim_{\gamma\to 0}\tilde{K}_{20}(y)=\lim_{\gamma\to 0}\tilde{K}_{02}(y)=\frac{1}{8\sqrt{2}}a_2(3\cos^2\psi_B-1) \times\int_{-\infty}^\infty dx\phi^2(x)\{3E_3[y\phi(x)]-E_1[y\phi(x)]\},$$

$$\lim_{\gamma\to 0}\tilde{K}_{22}(y)=\frac{1}{64}a_3(19-6\cos^2\psi_B+27\cos^4\psi_B)\times\int_{-\infty}^{\infty}dx\phi^2(x)E_1[y\phi(x)]-\frac{3}{32}a_3(1-30\cos^2\psi_B+45\cos^4\psi_B)\times\int_{-\infty}^{\infty}dx\phi^2(x)E_3[y\phi(x)]+\frac{9}{64}a_3(3-30\cos^2\psi_B+35\cos^4\psi_B)\times\int_{-\infty}^{\infty}dx\phi^2(x)E_5[y\phi(x)],$$

$$\lim_{\gamma\to 0}\tilde{K}_{11}(y)=\frac{3}{8}a_1\sin^2\psi_B\int_{-\infty}^{\infty}dx\phi^2(x)E_1[y\phi(x)]+\frac{3}{8}a_1(3\cos^2\psi_B-1)\int_{-\infty}^{\infty}dx\phi^2(x)E_3[y\phi(x)], \quad (A11)$$

where a_1, a_2, a_3 are the symbols defined in Landi Degl'Innocenti et al. (1990a), and where $E_n(x)$ is the usual exponential integral.

By integrating these expressions with respect to y, Eq. (A5) (which is valid in general and not only under the limit $\gamma\to 0$) is easily recovered. Moreover we obtain (independently of the angle ψ_B):

$$\lim_{\gamma\to 0}\int_0^{\infty}\tilde{K}_{11}(y)dy=\frac{1}{4}a_1,$$
$$\lim_{\gamma\to 0}\int_0^{\infty}\tilde{K}_{22}(y)dy=\frac{7}{20}a_3. \quad (A12)$$

Equations (A12) and (A5) can also be condensed in a unique formula that can be obtained through some Racah algebra. By means of the identity:

$$\sum_{i=0}^{3}\sum_{P}t^K_P(i)t^K_{-P}(i)=(2K+1)\left[\frac{2}{3}+(-1)^K\begin{Bmatrix}1&1&K\\1&1&2\end{Bmatrix}\right], \quad (A13)$$

it is possible to prove that:

$$\lim_{\gamma\to 0}\int_0^{\infty}\tilde{K}_{KK'}(y)dy=\delta_{K,K'}(2J'+1)\begin{Bmatrix}1&1&K\\J'&J'&J\end{Bmatrix}^2\left[1+\frac{3}{2}(-1)^K\begin{Bmatrix}1&1&K\\1&1&2\end{Bmatrix}\right]. \quad (A14)$$

g) Limit for large fields

$$\lim_{\gamma\to 0}\int_0^{\infty}\tilde{K}_{KK'}(y)dy=\frac{1}{2}\delta_{K,K'}. \quad (A15)$$

To prove this identity, we observe that, when $\gamma\to\infty$, all the profiles $\Phi(\nu_{J'M,JN}-\nu)$ relative to the different Zeeman components are well separated in frequency so that they do not overlap. From the direct expression of the generalized profiles (see Eq. (36.I)), we then get, for values of ν neighbouring the Zeeman component due to the transition between M and N:

$$\Phi_0^{K,K'}(J,J';\nu\sim\nu_{J'M,JN})=(-1)^{J'-M}\sqrt{(2K+1)(2J'+1)}\begin{pmatrix}J'&K&J'\\-M&0&M\end{pmatrix}\times\Phi_0^{0,K'}(J,J';\nu\sim\nu_{J'M,JN}). \quad (A16)$$

We see from this equation that the two profiles $\Phi_0^{K,K'}$ and $\Phi_0^{0,K'}$ are proportional through a factor depending on the Zeeman component and on K. Substituting this result into Eq. (25), we can write:

$$\lim_{\gamma\to\infty}\tilde{K}_{KK'}(y)=\frac{1}{4\pi}\int_0^{\pi/2}d\psi\,\mathrm{tg}\,\psi\int_0^{2\pi}d\theta\sum_{i,j=0}^{3}\sum_{LL'}\sum_{MN}\int_{(M,N)}d\nu\times\Phi_0^{0,L}(J,J';\nu)\Phi_0^{0,L'}(J,J';\nu)\mathcal{T}^L_0(i,\Omega)\times\left[\exp\left\{-\frac{y}{\cos\psi}H\right\}\right]_{ij}\mathcal{T}^{L'}_0(j,\Omega)\times\sqrt{(2K+1)(2K'+1)}(2J'+1)\begin{pmatrix}J'&K&J'\\-M&0&M\end{pmatrix}\times\begin{pmatrix}J'&K'&J'\\-M&0&M\end{pmatrix}, \quad (A17)$$

where, with the symbol $\int_{(M,N)}d\nu$, we mean the integral extended over an interval in frequency fully covering the M, N component, but excluding the other components.

If we now integrate in dy and repeat the same arguments developed in subsection *e*), we get:

$$\lim_{\gamma\to\infty}\int_0^{\infty}\tilde{K}_{KK'}(y)dy=\frac{1}{2}\sum_{MN}\sqrt{(2K+1)(2K'+1)}(2J'+1)\begin{pmatrix}J'&K&J'\\-M&0&M\end{pmatrix}\times\begin{pmatrix}J'&K'&J'\\-M&0&M\end{pmatrix}\int_{(M,N)}d\nu\Phi_0^{0,0}(J,J';\nu). \quad (A18)$$

On the other hand, this last integral is given by:

$$\int_{(M,N)}d\nu\,\Phi_0^{0,0}(J,J';\nu)=\begin{pmatrix}J'&J&1\\-M&N&q\end{pmatrix}^2. \quad (A19)$$

Substituting this result in the former equation and performing the sum over M and N, we finally get:

$$\lim_{\gamma\to\infty}\int_0^{\infty}\tilde{K}_{KK'}(y)dy=\frac{1}{2}\sum_{MN}\sqrt{(2K+1)(2K'+1)}\times(2J'+1)\times\begin{pmatrix}J'&K&J'\\-M&0&M\end{pmatrix}\begin{pmatrix}J'&K'&J'\\-M&0&M\end{pmatrix}\times\begin{pmatrix}J'&J&1\\-M&N&q\end{pmatrix}^2=\frac{1}{2}\delta_{K,K'}. \quad (A20)$$

h) Further symmetry properties

The kernel $\tilde{K}_{KK'}(y)$ depends implicitly on the type of transition considered (specified by the quantum numbers J and J' and by the Landé factors g_J and $g_{J'}$), and on the amplitude and direction of the magnetic field (specified by the parameters γ, ψ_B, θ_B). For symmetry reasons, however, the dependence on the angle θ_B

disappears once the θ-integration is performed, as specified in the definition of the kernel given in Eq. (25). For what the dependence on the angle ψ_B is concerned, the inversion property specified in Eq. (A1) allows to state that the kernel is invariant under interchange of ψ_B with $\pi-\psi_B$.

Moreover, a further symmetry property with respect to the plane containing the vertical to the atmosphere and the magnetic field allows to extend the θ-integration in Eq. (25) to the partial interval $[0, \pi]$. Taking into account also the inversion properties with respect to line center, and indicating with x the reduced frequency from line center, Eq. (25) can be written in the simpler form:

$$\tilde{K}_{KK'}(y)=\frac{1}{\pi}\int_0^{\pi/2} d\psi \,\mathrm{tg}\,\psi \int_0^{\pi} d\theta \sum_{i,j=0}^{3}\sum_{LL'}\int_0^{\infty} dx \Phi_0^{K,L}(J, J'; x) \times \Phi_0^{K',L'}(J, J'; x)\,\mathscr{T}_0^{L}(i, \Omega) \times \left[\exp\left\{-\frac{y}{\cos\psi}\boldsymbol{H}\right\}\right]_{ij} \mathscr{T}_0^{L'}(j, \Omega). \quad \text{(A21)}$$

Appendix B: Taylor expansion of the generalized profiles as a function of v_L

The generalized profile $\Phi_0^{K,K'}(J, J'; v)$ is given by the expression [see Eq. (36.I)]:

$$\Phi_0^{K,K'}(J, J'; v)=\sum_{MNq}(-1)^{J'-N-1}\sqrt{3(2K+1)(2K'+1)(2J'+1)} \times\begin{pmatrix} J' & K & J' \\ -M & 0 & M\end{pmatrix}\begin{pmatrix} J' & J & 1 \\ -M & N & q\end{pmatrix}^2 \times\begin{pmatrix} 1 & 1 & K' \\ -q & q & 0\end{pmatrix}\phi(v_{J'M,JN}-v). \quad \text{(B1)}$$

Observing that:

$$v_{J'M,JN}-v=v_0+v_L(g_{J'}M-g_JN)-v, \quad \text{(B2)}$$

we have, up to first order in v_L:

$$\phi(v_{J'M,JN}-v)=\phi_v-v_L(g_{J'}M-g_JN)\phi'_v, \quad \text{(B3)}$$

where:

$$\phi_v=\phi(v_0-v); \qquad \phi'_v=\frac{\partial}{\partial v}\phi(v_0-v). \quad \text{(B4)}$$

Substituting Eq. (B3) into Eq. (B4), remembering Eq. (A2.I), and observing that, with the help of some Racah algebra, one can prove the following identities:

$$\sum_{MNq}(-1)^{J'-N-1}M\begin{pmatrix} J' & K & J' \\ -M & 0 & M\end{pmatrix} \times\begin{pmatrix} J' & J & 1 \\ -M & N & q\end{pmatrix}^2\begin{pmatrix} 1 & 1 & K' \\ -q & q & 0\end{pmatrix} =(-1)^{J-J'+1}\sqrt{J'(J'+1)(2J'+1)}\begin{pmatrix} 1 & K & K' \\ 0 & 0 & 0\end{pmatrix} \times\begin{Bmatrix} 1 & K & K' \\ J' & J' & J'\end{Bmatrix}\begin{Bmatrix} 1 & 1 & K' \\ J' & J' & J\end{Bmatrix},$$

$$\sum_{MNq}(-1)^{J'-N-1}N\begin{pmatrix} J' & K & J' \\ -M & 0 & M\end{pmatrix} \times\begin{pmatrix} J' & J & 1 \\ -M & N & q\end{pmatrix}^2\begin{pmatrix} 1 & 1 & K' \\ -q & q & 0\end{pmatrix} =-\sqrt{J(J+1)(2J+1)}\begin{pmatrix} 1 & K & K' \\ 0 & 0 & 0\end{pmatrix}\begin{Bmatrix} 1 & K & K' \\ J & J' & 1 \\ J & J' & 1\end{Bmatrix}, \quad \text{(B5)}$$

one gets:

$$\Phi_0^{K,K'}(J, J'; v)=\delta_{K,K'}w_{J'J}^{(K)}\phi_v+g(K, K')v_L\phi'_v, \quad \text{(B6)}$$

where:

$$g(K, K')=-\sqrt{3(2K+1)(2K'+1)(2J'+1)}\begin{pmatrix} 1 & K & K' \\ 0 & 0 & 0\end{pmatrix} \times\Bigg[g_{J'}\sqrt{J'(J'+1)(2J'+1)} \times(-1)^{J-J'+1}\begin{Bmatrix} 1 & K & K' \\ J' & J' & J'\end{Bmatrix}\begin{Bmatrix} 1 & 1 & K' \\ J' & J' & J\end{Bmatrix} +g_J\sqrt{J(J+1)(2J+1)}\begin{Bmatrix} 1 & K & K' \\ J & J' & 1 \\ J & J' & 1\end{Bmatrix}\Bigg]. \quad \text{(B7)}$$

In particular, through the analytical expressions of the 3-j, 6-j, and 9-j symbols, we obtain, after some algebra:

$$g(0, 1)=\sqrt{\frac{2}{3}}\left\{\frac{1}{2}(g_{J'}+g_J)+\frac{1}{4}(g_{J'}-g_J)[J'(J'+1)-J(J+1)]\right\} =\sqrt{\frac{2}{3}}\bar{g}, \quad \text{(B8)}$$

where $\bar{g}$ is the so-called effective Landé factor (see, for instance, Landi Degl'Innocenti (1982));

$$g(2, 1)=\frac{1}{2}\frac{1}{\sqrt{30J'(J'+1)(2J'-1)(2J'+3)}} \times\{g_{J'}(2J'-1)(2J'+3)[J'(J'+1)-J(J+1)+2] -g_J(J+J'-1)(J+J'+3)[(J'-J)(2J+1)+4]\}. \quad \text{(B9)}$$

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