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Spectral analysis of a nonhomogeneous rotating Timoshenko beam: Riesz basis property

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Abstract

We consider a spatially nonhomogeneous Timoshenko beam mounted on the periphery of a rigid root of radius R rotating about its axis at a constant angular speed Ω . The junction between the beam and the root is assumed to be elastically restrained and damped. It is shown that under some hypotheses on the physical properties of the beam, one can find a sequence of root vectors of the unbounded operator associated to the problem which form a Riesz basis [1]. The outline of the proof is as follows. First we write a variational formulation of the problem which is transformed into a first order evolution equation. The unbounded operator of this evolution equation is the generator of a contraction semigroup so that the problem is well-posed. This operator has a compact resolvent but it is non-self-adjoint. The spectrum of this operator is studied by making a change of variable in order to reduce the problem to a first-order system asymptotically linear in the spectral parameter. By applying some results of [2], we obtain the asymptotics of the spectrum of this operator and we show that its eigenvalues are asymptotically algebraically simple. Then we show that the system of root vectors of the operator is complete and, applying a theorem from [3], that they form a Riesz basis. As a consequence of this result, one obtains an asymptotic expansion of the solution of the initial problem.

Keywords: Timoshenko beam, non-self-adjoint, Riesz basis

1 Introduction

In what follows, l is the length of the beam, A its cross-sectional area, E its Young modulus, ρ its mass density, I the moment of inertia of its cross-section, G its shear modulus, α_s its shear correction factor, w its infinitesimal transverse displacement, φ the infinitesimal rotation of its cross-section, k_w , k_φ , c_w and c_φ the corresponding spring and damping con-

stants at the junction. All the coefficients and their inverses are assumed to be in $W^{1,\infty}(0, l)$. If \mathbf{u}_0 , \mathbf{v}_0 are given in subspaces of $[H^1(0, l)]^2$ and $[L^2(0, l)]^2$, the corresponding variational problem reads as follows: find $\mathbf{u} = {}^t(w, \varphi) \in \mathcal{C}^1([0, +\infty[; [H^1(0, l)]^2) \cap \mathcal{C}^2([0, +\infty[; [L^2(0, l)]^2)$ such that $\forall \delta \mathbf{u} \in [H^1(0, l)]^2$,

$$m(\ddot{\mathbf{u}}, \delta \mathbf{u}) + b(\dot{\mathbf{u}}, \delta \mathbf{u}) + a(\mathbf{u}, \delta \mathbf{u}) = 0$$

and such that $\mathbf{u}(0) = \mathbf{u}_0$; $\dot{\mathbf{u}}(0) = \mathbf{v}_0$ (Problem (P)), where $\forall \mathbf{u}, \delta \mathbf{u} \in [H^1(0, l)]^2$,

$$a(\mathbf{u}, \delta \mathbf{u}) = \int_0^l [Tw' \overline{\delta w'} + EI \varphi' \overline{\delta \varphi'}] dx + \int_0^l \alpha_s GA \times [(w' - \varphi)(\overline{\delta w'} - \overline{\delta \varphi})] dx + k_w w(0) \overline{\delta w(0)} +$$

$$k_\varphi \varphi(0) \overline{\delta \varphi(0)} - \Omega^2 \int_0^l \rho I \varphi \overline{\delta \varphi} dx,$$

$$b(\mathbf{u}, \delta \mathbf{u}) = c_w w(0) \overline{\delta w(0)} + c_\varphi \varphi(0) \overline{\delta \varphi(0)},$$

and $\forall \mathbf{u}, \delta \mathbf{u} \in [L^2(0, l)]^2$,

$$m(\mathbf{u}, \delta \mathbf{u}) = \int_0^l \rho I \varphi \overline{\delta \varphi} dx + \int_0^l \rho A w \overline{\delta w} dx.$$

Setting $\mathbf{v} = {}^t(z, \psi)$, $\mathbf{U} = \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix}$, $\mathbf{U}_0 = \begin{pmatrix} \mathbf{u}_0 \\ \mathbf{v}_0 - \Omega \mathbf{u}_0 \end{pmatrix}$, after replacing $\mathbf{u}(t)$ by $e^{\Omega t} \mathbf{u}(t)$, (P) boils down to

$$\dot{\mathbf{U}} = \mathcal{A} \mathbf{U}, \mathbf{U}(0) = \mathbf{U}_0,$$

where \mathcal{A} is an unbounded operator of $\mathcal{H} = [H^1(0, l)]^2 \times [L^2(0, l)]^2$ with domain $D(\mathcal{A})$. \mathcal{A}^* is an unbounded operator of \mathcal{H} with a domain $D(\mathcal{A}^*) \neq D(\mathcal{A})$ so that \mathcal{A} is non-self-adjoint. On the other hand $0 \in \rho(\mathcal{A})$ and \mathcal{A}^{-1} is a compact operator of \mathcal{H} therefore the spectrum of \mathcal{A} consists of isolated eigenvalues of finite algebraic multiplicity. \mathcal{A} is maximal-dissipative hence \mathcal{A} is the generator of a contraction semigroup on \mathcal{H} . Returning to the initial problem (P) it follows that if $\mathbf{U}_0 \in D(\mathcal{A})$, there exists a unique solution to (P) such that $\mathbf{u} \in \mathcal{C}^0([0, +\infty); [H^2(0, l)]^2) \cap \mathcal{C}^1([0, +\infty); [H^1(0, l)]^2) \cap \mathcal{C}^2([0, +\infty); [L^2(0, l)]^2)$ and such that $\mathbf{U} \in \mathcal{C}^0([0, +\infty); D(\mathcal{A}))$.

2 Spectral theory of \mathcal{A}

Setting $\alpha_1 = \sqrt{\frac{\rho A}{T + \alpha_s G A}}$, $\alpha_2 = \sqrt{\frac{\rho}{E}}$, $\tilde{\Lambda}_1 = \text{diag}(\alpha_1, \alpha_2, -\alpha_1, -\alpha_2)$ there exist matrices T_0 , $\tilde{\Lambda}_0$, $\tilde{\Lambda}_{-1}$, \tilde{B}_1 , \tilde{B}_0 , \tilde{C}_1 , \tilde{C}_0 such that for $\lambda \in \mathbb{C}^*$, the conditions: $\mathbf{U} \in D(\mathcal{A})$, $\mathbf{U} \neq 0$ and $(\mathcal{A} - \lambda Id)\mathbf{U} = 0$ are equivalent to the conditions: $Z \in [H^1(0, l)]^4$, $Z \neq 0$ and $T(\lambda)Z = 0$ where $Z = T_0^{-1} \begin{pmatrix} \mathbf{u} \\ \mathbf{u}'/\lambda \end{pmatrix}$, $T(\lambda) = \begin{pmatrix} T^D(\lambda) \\ T^R(\lambda) \end{pmatrix}$,

$$T^D(\lambda)Z = Z' - (\lambda \tilde{\Lambda}_1 + \tilde{\Lambda}_0 + \frac{1}{\lambda} \tilde{\Lambda}_{-1})Z,$$

$$T^R(\lambda)Z = (\tilde{B}_1 + \frac{1}{\lambda} \tilde{B}_0)T_0 Z(0) + (\tilde{C}_1 + \frac{1}{\lambda} \tilde{C}_0)T_0 Z(l).$$

Assume that there exists a constant $C > 0$ such that $|\alpha_1 - \alpha_2| \geq C > 0$ on $[0, l]$. Then from [2], Theorem 2.8.2, there exists an asymptotic fundamental matrix function of $T^D(\lambda)Z = 0$ denoted $\tilde{E}(\cdot, \lambda)$ such that for $\lambda \in \mathbb{C}^*$, the conditions: $Z \in [H^1(0, l)]^4$, $T^D(\lambda)Z = 0$ are equivalent to the condition: $\exists c(\lambda) \in \mathbb{C}^4$ with $Z(\cdot, \lambda) = \tilde{E}(\cdot, \lambda)c(\lambda)$. It follows that λ is an eigenvalue of \mathcal{A} iff $\det M(\lambda) = 0$ where $M(\lambda) = T^R(\lambda)\tilde{E}(\cdot, \lambda)$. If $c_w \neq (\alpha_1(T + \alpha_s G A))(0)$ and $c_\varphi \neq (\alpha_2 E I)(0)$ the eigenvalues of \mathcal{A} are given asymptotically by the formulas $\lambda_n^1 = \mu^1 - \Omega + \nu_n^1$, $\lambda_n^2 = \mu^2 - \Omega + \nu_n^2$ where $\mu_1, \mu_2 \leq 0$, $\nu_n^1, \nu_n^2 \in i\mathbb{R}$, $|\nu_n^1|, |\nu_n^2| \rightarrow +\infty$ when $|n| \rightarrow +\infty$. If $\mu_1 \neq \mu_2$ the zeros of $\det M(\lambda)$ are asymptotically simple and $\inf_{n \neq m} |\lambda_n - \lambda_m| > 0$.

If Ω is an open nonempty subset of \mathbb{C} , E and F are Banach spaces, $\mu \in \Omega$, $S \in H(\Omega, L(E, F))$ and $x \in H(\Omega, E)$ are holomorphic families, x is called a root function of S at μ if $x(\mu) \neq 0$ and $(Sx)(\mu) = 0$. The order of the zero of Sx at μ is called the multiplicity of x with respect to S at μ . One can also define the geometric and algebraic multiplicities of S at μ [2], Chapter I. By comparing the multiplicities of the root functions and the geometric multiplicities of the three holomorphic families $\lambda \in \mathbb{C}^* \mapsto T(\lambda) \in L([H^1(0, l)]^4, [L^2(0, l)]^4 \times \mathbb{C}^4)$, $\lambda \in \mathbb{C}^* \mapsto A(\lambda) = \mathcal{A} - \lambda Id \in L(D(\mathcal{A}), \mathcal{H})$ and $\lambda \in \mathbb{C}^* \mapsto M(\lambda) \in L(\mathbb{C}^4, \mathbb{C}^4)$ at an eigenvalue of \mathcal{A} , it can be shown that if $\mu_1 \neq \mu_2$, the eigenvalues of \mathcal{A} are asymptotically algebraically simple (in the "classical" sense). By estimating the resolvent of \mathcal{A} first on the negative real axis (comparing \mathcal{A} with the self-adjoint operator obtained by taking $c_w = c_\varphi = 0$) and then on the

right half-plane (using the Hille-Yosida Theorem) and by applying the Phragmen-Lindelöf Theorem to a certain function of exponential type, it is shown that the system of root vectors of \mathcal{A} is complete in \mathcal{H} .

Recall that a sequence $(h_n)_{n \in \mathbb{N}}$ of a Hilbert space V is a Riesz basis of V iff there exists an isomorphism U of V and a Hilbert basis $(e_n)_{n \in \mathbb{N}}$ of V such that $\forall n \in \mathbb{N}$, $h_n = Ue_n$. Applying Theorem 1.1 of [3], it follows that there exists a sequence of root vectors of \mathcal{A} which forms a Riesz basis of \mathcal{H} . The following theorem gives an expansion of the solution of (P) .

Theorem 1 Assume that $\mathbf{U}_0 \in D(\mathcal{A})$. Let $\{\lambda_k, k \in \mathbb{N}\}$ be the set of eigenvalues of \mathcal{A} . For all $k \in \mathbb{N}$, let r_k be the geometric multiplicity of λ_k , $\mathbf{U}_{k,i,j}$, $i = 1, \dots, r_k$, $j = 0, \dots, m_{k,i} - 1$ a family of root vectors corresponding to λ_k such that $(\mathcal{A} - \lambda_k Id)\mathbf{U}_{k,i,j} = \mathbf{U}_{k,i,j-1}$, $i = 1, \dots, r_k$, $j = 0, \dots, m_{k,i} - 1$ (with $\mathbf{U}_{k,i,-1} = 0$) and set $\mathbf{U}_{k,i,j} = \begin{pmatrix} \mathbf{u}_{k,i,j} \\ \mathbf{v}_{k,i,j} \end{pmatrix}$.

Then there exists a family of polynomial functions $P_{k,i,j}$ of degree $\leq m_{k,i} - 1 - j$, $k \in \mathbb{N}$, $i = 1, \dots, r_k$, $j = 0, \dots, m_{k,i} - 1$ such that if for all $n \in \mathbb{N}$ and $t \in \mathbb{R}$ we set

$$\mathbf{u}^n(t) = \sum_{k=0}^n \sum_{i=1}^{r_k} \sum_{j=0}^{m_{k,i}-1} a_{k,i,j}(t) \mathbf{u}_{k,i,j}$$

where $a_{k,i,j}(t) = e^{(\Omega + \lambda_k)t} P_{k,i,j}(t)$, then for all $T > 0$, $\mathbf{u}^n \rightarrow \mathbf{u}$ in $C^0([0, T]; [H^2(0, l)]^2) \cap C^1([0, T]; [H^1(0, l)]^2) \cap C^2([0, T]; [L^2(0, l)]^2)$.

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