Resonance line polarization for arbitrary magnetic fields in optically thick media. 3: A generalization of the square root of epsilon law

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Astron. Astrophys. 284, 865–873 (1994)

ASTRONOMY AND ASTROPHYSICS

Resonance line polarization for arbitrary magnetic fields in optically thick media

III. A generalization of the $\sqrt{\varepsilon}$-law

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Received 3 September 1993 / Accepted 27 September 1993

Abstract. The well-known "$\sqrt{\varepsilon}$-law" – one of the few exact analytical results in the theory of radiative transfer – is generalized to the case of an isothermal, plane-parallel atmosphere in the presence of a magnetic field vector of arbitrary intensity and direction and in the presence of depolarizing collisions. A compact expression is found, relating the squares of the surface value of the upper level components of the atomic density matrix in the representation of the irreducible spherical tensors to the value of the constant Planck function. By taking the appropriate limits, the usual $\sqrt{\varepsilon}$-law for unpolarized radiation, and its generalization to resonance scattering polarization in a non-magnetic atmosphere (Ivanov 1990), are recovered as particular cases.

Key words: lines: formation – magnetic field – polarization – radiative transfer – Sun: magnetic fields

1. Introduction

In the previous papers of this series, we have tackled the problem of resonance polarization for a two-level atom in a three-dimensional medium embedded in a magnetic field of arbitrary strength and direction and irradiated by an arbitrary radiation field. In a first paper (Landi Degl'Innocenti et al. 1991a, hereafter referred to as Paper I), we presented the general physical background and established the relevant equations, while in a second paper (Landi Degl'Innocenti et al. 1991b, hereafter referred to as Paper II) we applied the formalism previously developed to the case of a plane-parallel atmosphere, neglecting however Zeeman coherences betweeen different magnetic sublevels.

In the present paper, we will first generalize the results of Paper II, taking properly into account the influence of Zeeman coherences. This will lead to a set of equations capable of describing the physics of resonance scattering polarization in a plane-parallel atmosphere embedded in an arbitrary (even if constant) magnetic field. The results thus obtained will fill the gap within those obtained in Paper II (valid for the so-called *strong field* regime of the Hanle effect, where Zeeman coherences can be neglected), and those obtained in two previous papers (Landi Degl'Innocenti et al. 1990; Bommier et al. 1991), valid for the so-called *Hanle effect* regime, where the splitting induced by the magnetic field has negligible effect on the line profile. In terms of physical quantities, in the present paper, the Larmor frequency ν_L is no longer restrained to the condition $\nu_L \gg A$ (strong field regime of the Hanle effect – A is the Einstein coefficient for spontaneous de-excitation), nor to the condition $\nu_L \ll \Delta\nu_D$ (Hanle effect regime – $\Delta\nu_D$ is the Doppler broadening in frequency units), so that also the intermediate regime between Hanle and Zeeman effects can be considered.

Analogously to what was done in the papers previously quoted, we will show how the physics of resonance scattering polarization in a magnetic atmosphere reduces to a set of coupled integral equations connecting, through suitable kernels, each statistical tensor of the atomic upper level at optical depth τ, $\rho^K_Q(\tau)$, with the statistical tensors at different optical depths τ'. The situation here is quite similar to the standard two-level atom radiative transfer problem, with the difference that the scalar source function is substituted by a set of source functions – the statistical tensors $\rho^K_Q(\tau)$.

Although suitable numerical methods can be devised to solve the system of coupled integral equations, we will derive in this paper an exact analytical result concerning the values of the surface statistical tensors (in other words of the surface source functions) for an isothermal atmosphere. For the case of a scalar source function it is known since a long time that in an isothermal atmosphere with constant parameters (rate of photon destruction, absorption profile), the source function at the surface is simply related to the source function at infinity. This is the so-called "$\sqrt{\varepsilon}$-

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law", that is discussed in classical papers or textbooks (see for instance Avrett & Hummer 1965; Mihalas 1970), and that can be proved by means of several different methods (Chandrasekhar 1947; Ivanov 1973; Frisch & Frisch 1975; Rybicki 1977; Landi Degl'Innocenti 1979).

Ivanov (1990) has recently generalized the $\sqrt{\varepsilon}$-law to the basic problem in the theory of transfer for polarized radiation, namely to the case of resonance scattering in a non-magnetic, plane-parallel atmosphere. The aim of the present paper is to provide a further generalization to the magnetic case.

Obviously, the results of the present paper can be very useful as a test for polarized radiative transfer numerical codes.

2. Coupled integral equations for the statistical tensors in a plane-parallel atmosphere

In Paper I, we have obtained a general equation describing the coupling of the statistical tensor $\rho^K_Q(\boldsymbol{x})$ at point $\boldsymbol{x}$ with the statistical tensors $\rho^{K'}_{Q'}(\boldsymbol{x}')$ at a different point $\boldsymbol{x}'$ in a medium of arbitrary shape [see Eq. (55 I)]. By particularizing this equation to the case of a plane-parallel atmosphere, and following the same procedure as in Paper II, one is led to the following system of Wiener–Hopf type equations:

$$(1+\varepsilon+\delta^{(K)}+\mathrm{i}\Gamma Q)\rho^K_Q(\tau) = \sum_{K'Q'}\int_0^\infty \tilde{K}_{KQ,K'Q'}(\tau,\tau')\rho^{K'}_{Q'}(\tau')\,\mathrm{d}\tau' + \varepsilon\beta_0\delta_{K,0}\delta_{Q,0}\,. \tag{1}$$

1. $\rho^K_Q(\tau)$ is the statistical tensor relative to the upper level involved in the atomic transition at optical depth τ. It has to be remarked here that, for Eq. (1) to hold, the statistical tensor has to be defined in a reference system having its z-axis directed along the magnetic field. The definition of ρ^K_Q is given in Eq. (31 I).

2. $\varepsilon = C_{J'J}/A$ is the ratio of the collisional de-excitation rate to the Einstein coefficient. Often, in the specialized literature concerning radiative transfer, the same symbol is employed to denote a different quantity, that we will rather call ε', and that is defined by

$$\varepsilon' = \frac{C_{J'J}}{C_{J'J}+A}\,.$$

Obviously ε and ε' are related through the equations:

$$\varepsilon' = \frac{\varepsilon}{1+\varepsilon}\,; \qquad \varepsilon = \frac{\varepsilon'}{1-\varepsilon'}\,,$$

and, in terms of ε', Eq. (1) will read:

$$[1+(\delta^{(K)}+\mathrm{i}\Gamma Q)(1-\varepsilon')]\rho^K_Q(\tau) = (1-\varepsilon')\sum_{K'Q'}\int_0^\infty \tilde{K}_{KQ,K'Q'}(\tau,\tau')\rho^{K'}_{Q'}(\tau')\,\mathrm{d}\tau' + \varepsilon'\beta_0\delta_{K,0}\delta_{Q,0}\,. \tag{2}$$

3. $\delta^{(K)} = D^{(K)}/A$ is the ratio of the relaxation rate of the statistical tensors of rank K to the Einstein coefficient. Through the quantities $\delta^{(K)}$, it is possible to account for depolarizing elastic collisions. In particular, one always has $\delta^{(0)} = 0$, as depolarizing collisions cannot act as sources or sinks on the total population of the upper level (proportional to ρ^0_0).

4. $\Gamma = 2\pi g_{J'}\nu_L/A$ is proportional to the ratio of the Zeeman splitting – induced by the magnetic field in the upper level – to the Einstein coefficient. In this formula $g_{J'}$ is the Landé factor of the upper level, while ν_L is the Larmor frequency:

$$\nu_L = \frac{e_0 B}{4\pi mc}\,.$$

Numerically one has $\Gamma = 0.88 g_{J'}B/A$, with B measured in Gauss and A in units of $10^7\ \mathrm{s}^{-1}$.

5. β_0 is the source term that is related to the local temperature of the medium. Due to the definition of the statistical tensors [see Eq. (31 I)] and to the fact that the density matrix is normalized to the overall population of the ground level [see Eq. (15 I)], one has:

$$\beta_0 = \frac{\sqrt{2J'+1}}{2J+1}\mathrm{e}^{-h\nu_0/kT}$$

where J and J' are the angular momentum quantum numbers of the lower and upper level, respectively, and where ν_0 is the frequency of the line. It is noteworthy that, in the standard scalar case, the source term β_0 is generally written as the local Planck function. It is possible to cast Eq. (1) in a form that bears more similarity to that of the scalar case, by simply multiplying it by the factor:

$$\frac{2h\nu_0^3}{c^2}\frac{2J+1}{\sqrt{2J'+1}}\,.$$

Defining then the source function tensors through the equation:

$$S^K_Q = \frac{2h\nu_0^3}{c^2}\frac{2J+1}{\sqrt{2J'+1}}\rho^K_Q\,, \tag{3}$$

Eq. (1) is cast in the form:

$$(1+\varepsilon+\delta^{(K)}+\mathrm{i}\Gamma Q)S^K_Q(\tau) = \sum_{K'Q'}\int_0^\infty \tilde{K}_{KQ,K'Q'}(\tau,\tau') \times S^{K'}_{Q'}(\tau')\,\mathrm{d}\tau' + \varepsilon B_0\delta_{K,0}\delta_{Q,0}\,, \tag{4}$$

where

$$B_0 = \frac{2h\nu_0^3}{c^2}\mathrm{e}^{-h\nu_0/kT}$$

is the Planck function in the Wien limit (consistently with the fact that stimulated emission has been neglected in our derivation). The tensorial source function defined in Eq. (3) is the direct generalization of the scalar source function to polarized radiative transfer. In particular, for the source function

tensor of rank 0, one obtains, with easy transformations:

$$S_0^0 = \frac{2h\nu_0^3}{c^2}\frac{N_{J'}/(2J'+1)}{N_J/(2J+1)},$$

where $N_{J'}$ and N_J are the overall atomic populations in the upper and lower level, respectively. This formula shows that, in the Wien limit, the source function tensor of rank 0 coincides with the scalar source function.

6. Finally, the kernels $\tilde{K}_{KQ,K'Q'}(\tau,\tau')$ appearing in Eqs. (1), (2) and (4) are given by an involved expression that can be found in the Appendix, where the main properties of the same kernels are also proved. Among these properties, it is relevant here to stress the following ones:

(a) the kernels are invariant under the interchange of the two arguments τ and τ', depending only on $|\tau-\tau'|$:

$$\tilde{K}_{KQ,K'Q'}(\tau,\tau') = \tilde{K}_{KQ,K'Q'}(\tau',\tau) = \tilde{K}_{KQ,K'Q'}(|\tau-\tau'|); \quad (5)$$

(b) for some of the kernels, an integral property can be proved:

$$\int_0^\infty \tilde{K}_{00,K'Q'}(t)\,\mathrm{d}t = \frac{1}{2}\delta_{K',0}\delta_{Q',0},$$

$$\int_0^\infty \tilde{K}_{KQ,00}(t)\,\mathrm{d}t = \frac{1}{2}\delta_{K,0}\delta_{Q,0}; \quad (6)$$

(c) all the kernels connecting statistical tensors of different parity are identically zero. This property is particularly important because it allows to decouple the system of equations (1) into two different systems, one for the statistical tensors of even rank, and another for the statistical tensors of odd rank. Since in this second set of equations there is no source term, all the ρ^K_Q statistical tensors with K odd are identically zero.

(d) for a particular choice of the reference system where the statistical tensors are defined, namely in a reference system having the z-axis along the magnetic field vector and the x-axis in the plane defined by the magnetic field vector and the vertical to the atmosphere, the kernels are symmetric relatively to the interchange of their arguments:

$$\tilde{K}_{KQ,K'Q'}(t) = \tilde{K}_{K'Q',KQ}(t). \quad (7)$$

This particular reference system, Σ_0, is shown in Fig. 1.

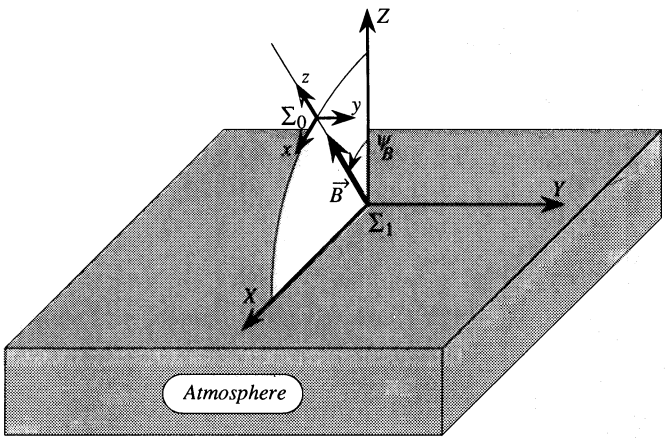

Fig. 1. The reference system $\Sigma_0 \equiv (x, y, z)$ is defined in the following way: the z-axis is aligned along the magnetic field vector, while the x-axis is contained in the plane defined by the magnetic field vector and the vertical to the atmosphere. The y-axis is defined accordingly. The figure also shows the reference system $\Sigma_1 \equiv (X, Y, Z)$ introduced in Sect. 5, and which is defined in the following way: the Z-axis is the vertical to the atmosphere, while the X-axis lays along the projection of the magnetic field vector onto the surface. The Y-axis is defined accordingly

3. The generalization of the $\sqrt{\varepsilon}$-law

Starting from Eq. (4), we introduce more shorthand notations by arranging all the elements of even rank of the source function tensor in a formal vector $S_i (i = 1, \ldots, N)$, starting with S_0^0 (apart from this restriction, the ordering is irrelevant). N, the total number of such elements, is given by:

$$N = \begin{cases} (J'+1)(2J'+1) & \text{for } J' \text{ integer}, \\ J'(2J'+1) & \text{for } J' \text{ half-integer}. \end{cases}$$

Accordingly, we arrange all the kernels $\tilde{K}_{KQ,K'Q'}$ in the matrix $K_{ij}(i, j = 1, \ldots, N)$ and all the quantities $(1 + \varepsilon + \delta^{(K)} + \mathrm{i}\Gamma Q)$ in the vector a_i $(i = 1, \ldots, N)$.

With these notations, Eq. (4) takes the form:

$$a_i S_i(\tau) = \sum_j \int_0^\infty K_{ij}(|\tau-\tau'|) S_j(\tau')\,\mathrm{d}\tau' + \varepsilon B_0 \delta_{i,1} \quad (i = 1, \ldots, N), \quad (8)$$

where, due to the properties of the kernels $\tilde{K}_{KQ,K'Q'}$:

$$K_{ij}(t) = K_{ji}(t). \quad (9)$$

$$\int_0^\infty K_{i1}(t)\,\mathrm{d}t = \int_0^\infty K_{1i}(t)\,\mathrm{d}t = \frac{1}{2}\delta_{i,1}, \quad (10)$$

where

$$a_1 = 1 + \varepsilon. \quad (11)$$

We then suppose that in Eq. (8) the quantities ε, B_0 and a_i $(i = 1, \ldots, N)$ are constant with τ (which implies that $\delta^{(K)}$ and Γ are also constant). With these notations and assumptions, the $\sqrt{\varepsilon}$-law is easily generalized.

We start by considering the limiting value of $S_i(\tau)$ for $\tau \to \infty$. Taking into account that all the kernels are integrable functions of their argument, the source functions $S_i(\tau)$ converge towards a fixed limit, $S_i(\infty)$, for $\tau \to \infty$. This limit is found through the set of equations:

$$a_i S_i(\infty) = \sum_j \left[\int_{-\infty}^{+\infty} K_{ij}(t)\,\mathrm{d}t\right] S_j(\infty) + \varepsilon B_0 \delta_{i,1} \quad (i = 1, \ldots, N),$$

which, taking into account Eqs. (10) and (11), decouples into

$$(1+\varepsilon)S_1(\infty) = S_1(\infty) + \varepsilon B_0\,,$$

$$a_i S_i(\infty) = \sum_{j=2}^{N}\left[\int_{-\infty}^{+\infty} K_{ij}(t)\,\mathrm{d}t\right] S_j(\infty) \quad (i=2,\ \ldots,\ N)\,.$$

This brings to the obvious solution:

$$S_1(\infty) = B_0\,,$$

$$S_2(\infty) = \cdots = S_N(\infty) = 0\,. \tag{12}$$

Taking the τ-derivative of Eq. (8), one gets

$$a_i\frac{\mathrm{d}S_i(\tau)}{\mathrm{d}\tau} = \sum_j \frac{\mathrm{d}}{\mathrm{d}\tau}\left[\int_0^\infty K_{ij}(|\tau-\tau'|)S_j(\tau')\,\mathrm{d}\tau'\right].$$

The r.h.s. can be evaluated by means of the derivative of the Wiener–Hopf Λ-operator (see e.g. Kourganoff 1963). This leads to

$$a_i\frac{\mathrm{d}S_i(\tau)}{\mathrm{d}\tau} = \sum_j K_{ij}(\tau)S_j(0) + \sum_j\int_0^\infty K_{ij}(|\tau-\tau'|)\frac{\mathrm{d}S_j(\tau')}{\mathrm{d}\tau'}\,\mathrm{d}\tau'\,.$$

Following Frisch and Frisch (1975), we now multiply this equation by $S_i(\tau)$, integrate in $\mathrm{d}\tau$ and sum over i to obtain

$$\sum_i a_i\int_0^\infty S_i(\tau)\frac{\mathrm{d}S_i(\tau)}{\mathrm{d}(\tau)}\,\mathrm{d}\tau = \sum_{ij} S_j(0)\int_0^\infty K_{ij}(\tau)S_i(\tau)\,\mathrm{d}\tau + \sum_{ij}\int_0^\infty \mathrm{d}\tau S_i(\tau)\int_0^\infty K_{ij}(|\tau-\tau'|)\frac{\mathrm{d}S_j(\tau')}{\mathrm{d}\tau'}\,\mathrm{d}\tau'\,. \tag{13}$$

For the first term in the r.h.s., we observe that, due to Eqs. (9) and (8), one has

$$\sum_i\int_0^\infty K_{ij}(\tau)S_i(\tau)\,\mathrm{d}\tau = \sum_i\int_0^\infty K_{ji}(\tau)S_i(\tau)\,\mathrm{d}\tau = a_jS_j(0) - \varepsilon B_0\delta_{j,1}\,, \tag{14}$$

while for the second term, interchanging the integrations we obtain, again through Eqs. (9) and (8)

$$\sum_{ij}\int_0^\infty \mathrm{d}\tau S_i(\tau)\int_0^\infty K_{ij}(|\tau-\tau'|)\frac{\mathrm{d}S_j(\tau')}{\mathrm{d}\tau'}\,\mathrm{d}\tau' = \sum_{ij}\int_0^\infty \mathrm{d}\tau'\frac{\mathrm{d}S_j(\tau')}{\mathrm{d}\tau'}\int_0^\infty K_{ji}(|\tau-\tau'|)S_i(\tau)\,\mathrm{d}\tau = \sum_j\int_0^\infty \mathrm{d}\tau'\frac{\mathrm{d}S_j(\tau')}{\mathrm{d}\tau'}[a_jS_j(\tau') - \varepsilon B_0\delta_{j,1}]\,. \tag{15}$$

Substituting Eqs. (14) and (15) into Eq. (13), we obtain, through easy transformations:

$$\sum_j a_j[S_j(0)]^2 = \varepsilon B_0 S_1(\infty)\,,$$

or, by means of Eq. (12):

$$\sum_j a_j[S_j(0)]^2 = \varepsilon B_0^2\,.$$

Remembering the notations introduced at the beginning of this section, these equations can also be written in the form:

$$\sum_{KQ}(1+\varepsilon+\delta^{(K)}+\mathrm{i}\Gamma Q)[S_Q^K(0)]^2 = \varepsilon B_0 S_0^0(\infty) = \varepsilon B_0^2\,, \tag{16}$$

or, in terms of the statistical tensors:

$$\sum_{KQ}(1+\varepsilon+\delta^{(K)}+\mathrm{i}\Gamma Q)[\rho_Q^K(0)]^2 = \varepsilon\beta_0\rho_0^0(\infty) = \varepsilon\beta_0^2\,. \tag{17}$$

Note that the left-hand-sides of these equations are real quantities, as it can be easily verified from the conjugation property of the statistical tensors:

$$[\rho_Q^K]^* = (-1)^Q\rho_{-Q}^K\,.$$

It has to be remarked that the results contained in Eqs. (16) and (17) are totally independent of the detailed form of the kernels. The only constraints that are imposed to the kernels for Eqs. (16) and (17) to hold are those expressed by Eqs. (9)–(11). In particular, these results are valid for any transition, irrespectively of the J-values and Landé factors of the upper and lower levels and, consequently, irrespectively of the Zeeman pattern of the transition (the Landé factor of the upper level, $g_{J'}$, enters however the definition of Γ). Moreover, the same results are valid for any type of absorption profile (Lorentzian, Gaussian, Voigt, etc.).

4. Discussion and particular cases

The equations now derived are very general and they encompass, as limiting cases, both the $\sqrt{\varepsilon}$-law and its extension to resonance polarization. By particularizing Eq. (16) to different physical regimes, we are led to the following considerations:

4.1. Scalar case

The scalar case can be recovered in our formalism by simply taking the limit:

$$\delta^{(K)} \to \infty \quad \text{for } K \neq 0\,.$$

Under this limit, all the components of the source function tensor are identically zero, except for S_0^0, and a direct application of Eq. (16) gives:

$$S_0^0(0) = \sqrt{\frac{\varepsilon}{1+\varepsilon}}B_0 = \sqrt{\varepsilon'}B_0\,, \tag{18}$$

which is the classical $\sqrt{\varepsilon}$-law (obviously, in our formalism, it would be more convenient to call it the $\sqrt{\varepsilon'}$-law). Note that this equation is valid also in magnetic atmospheres and

for any value of the magnetic field vector provided that depolarizing collisions are capable of destroying any atomic polarization that might be induced by the anisotropy of the radiation field. Note also that Eq. (18) is valid in any reference system (and not only in the reference system Σ_0 defined in Fig. 1), because S^0_0 is an invariant. It has finally to be remarked that, if the angular momentum J' of the upper level is either 0 or 1/2, the only component S^K_Q different from 0 is just S^0_0, and the scalar case is recovered. The following cases, 4.2–4.4, refer only to $J' \geq 1$.

4.2. Resonance polarization, non-magnetic case

If the magnetic field is zero, Eq. (16) is valid in any reference system (provided, however, that the x-axis is contained in the plane defined by the arbitrarily chosen z-axis and the vertical to the atmosphere). If we consider in particular the reference system having its z-axis aligned with the vertical, the expression for the source function tensor becomes simpler because, for evident symmetry reasons, the only non-vanishing components are S^0_0 and S^2_0 (cf. Landi Degl'Innocenti et al. 1990). Equation (16) then becomes:

$$(1+\varepsilon)[S^0_0(0)]^2 + (1+\varepsilon+\delta^{(2)})[S^2_0(0)]^2 = \varepsilon B^2_0\,, \tag{19}$$

or, for a non-depolarizing atmosphere ($\delta^{(2)} = 0$):

$$\sqrt{[S^0_0(0)]^2 + [S^2_0(0)]^2} = \sqrt{\varepsilon'} B_0\,. \tag{20}$$

This is the result obtained by Ivanov (1990), who employs the notations $S^0_0 = S_i$ and $-S^2_0 = S_q$.

4.3. Resonance polarization, Hanle effect regime

In the Hanle effect regime ($\nu_L \approx A \ll \Delta\nu_D$), the only non-vanishing components of S^K_Q are S^0_0 and S^2_Q (see Landi Degl'Innocenti et al. 1990). We then have, in the reference system Σ_0:

$$(1+\varepsilon)[S^0_0(0)]^2 + \sum_Q (1+\varepsilon+\delta^{(2)}+\mathrm{i}\Gamma Q)[S^2_Q(0)]^2 = \varepsilon B^2_0\,. \tag{21}$$

4.4. Resonance polarization, strong field regime of the Hanle effect

In the strong field regime of the Hanle effect ($\nu_L \gg A$), the only non-vanishing components of the source function tensor are those of the form $S^K_0 (K = 0, 2, 4, \ldots, K_{\max}$; with $K_{\max} = 2J'$ for J' integer or $2J'-1$ for J' half-integer). Equation (16) now becomes (in Σ_0):

$$\sum_K (1+\varepsilon+\delta^{(K)})[S^K_0(0)]^2 = \varepsilon B^2_0\,, \tag{22}$$

or, for non-depolarizing media:

$$\sqrt{[S^0_0(0)]^2 + [S^2_0(0)]^2 + \cdots + [S^{K_{\max}}_0(0)]^2} = \sqrt{\varepsilon'} B_0\,. \tag{23}$$

5. The generalized $\sqrt{\varepsilon}$-law in a different reference system

Equations (16) and (17) are not invariant. Indeed, they are valid only in the reference system Σ_0 defined in Fig. 1, and, changing the reference system, they generally acquire a more complicated expression. A plane-parallel magnetic atmosphere defines however, in a quite natural way, another reference system, namely the system $\Sigma_1 \equiv (X, Y, Z)$, whose Z-axis coincides with the vertical to the atmosphere and X-axis lays along the projection of the magnetic field vector onto the surface (see also Fig. 1). Since, in some cases, it may be more advantageous to carry out detailed calculations in Σ_1 (instead of Σ_0) it is worth obtaining the relevant equations for the surface source function tensor in this system too. To avoid confusion, we denote by R^K_Q the source function tensors in Σ_1. Given our definition of the statistical tensors, we have the following relationship between the source function tensors in the two systems (cf. Landi Degl'Innocenti 1984):

$$S^K_Q = \sum_{Q'} R^K_{Q'} [\mathcal{D}^K_{Q'Q}(R)]^*\,,$$

where R is the rotation that brings the reference system Σ_1 into Σ_0, and where $\mathcal{D}$ are the ordinary rotation matrices. On the other hand, we have

$$R \equiv (0, \psi_B, 0)\,,$$

where ψ_B ($0 \leq \psi_B \leq \pi$) is the angle between the magnetic field and the vertical, and we can write

$$S^K_Q = \sum_{Q'} R^K_{Q'} d^K_{Q'Q}(\psi_B)\,,$$

where d are the reduced rotation matrices. In the reference system Σ_1, Eq. (16) can then be written in the form

$$\sum_{KQ} (1+\varepsilon+\delta^{(K)}+\mathrm{i}\Gamma Q) \times \sum_{Q'Q''} R^K_{Q'}(0) R^K_{Q''}(0) d^K_{Q'Q}(\psi_B) d^K_{Q''Q}(\psi_B) = \varepsilon B^2_0\,. \tag{24}$$

Through the properties of the reduced rotation matrices, one can easily prove that

$$\sum_Q d^K_{Q'Q}(\psi_B) d^K_{Q''Q}(\psi_B) = \delta_{Q',Q''}\,,$$

$$\sum_Q Q d^K_{Q'Q}(\psi_B) d^K_{Q''Q}(\psi_B) = (-1)^{K-Q''} \sqrt{K(K+1)(2K+1)} \times \begin{pmatrix} K & K & 1 \\ Q' & -Q'' & P \end{pmatrix} d^1_{P0}(\psi_B)\,,$$

so that one has, in Σ_1:

$$\sum_{KQ} (1+\varepsilon+\delta^{(K)})[R^K_Q(0)]^2 + \mathrm{i}\Gamma \sum_{KQ'Q''} (-1)^{K-Q''} \times \sqrt{K(K+1)(2K+1)} \begin{pmatrix} K & K & 1 \\ Q' & -Q'' & P \end{pmatrix} \times d^1_{P0}(\psi_B) R^K_{Q'}(0) R^K_{Q''}(0) = \varepsilon B^2_0\,. \tag{25}$$

For the particular case of $K_{\max} = 2$, this formula yields, in terms of real and imaginary parts:

$$(1+\varepsilon)[R_0^0(0)]^2 + (1+\varepsilon+\delta^{(2)})\{[R_0^2(0)]^2 + 2[\Re R_1^2(0)]^2 - 2[\Im R_1^2(0)]^2 + 2[\Re R_2^2(0)]^2 - 2[\Im R_2^2(0)]^2\} - 4\Gamma\cos\psi_B\{[\Re R_1^2(0)][\Im R_1^2(0)] + 2[\Re R_2^2(0)]^2[\Im R_2^2(0)]\} - 2\Gamma\sin\psi_B\{2[\Re R_1^2(0)][\Im R_2^2(0)] + 2[\Re R_2^2(0)][\Im R_1^2(0)] + \sqrt{6}[R_0^2(0)][\Im R_1^2(0)]\} = \varepsilon B_0^2 . \quad (26)$$

6. Connection of the surface tensorial source functions with the Stokes parameters of the radiation emitted tangentially to the surface

Through this connection (that is particularly simple only in the weak field limit of the Hanle effect), the $\sqrt{\varepsilon}$-law can be expressed in terms of Stokes parameters rather than in terms of the tensorial source functions.

From Eq. (30) of Paper II, generalized to the "intermediate" regime, where all the coherences are accounted for, the Stokes parameters I_i ($i = 0, \ldots, 3$) of the radiation emitted at frequency ν in the direction Ω_l are given by

$$I_i(\nu,\Omega_l) = \frac{A}{B}\sqrt{2J'+1}\sum_{KK'Q}\sum_{j=0}^{3}[P_Q^K(\nu)]_{ij} \times \mathcal{T}_Q^{K'}(j,\Omega_l)\Phi_Q^{K,K'}(J,J';\nu) , \quad (27)$$

where

$$[P_Q^K(\nu)]_{ij} = \int_0^\infty \rho_Q^K(\tau)\left[\exp\left\{\frac{-\tau}{\cos\psi_l}\boldsymbol{H}\right\}\right]_{ij}\frac{\mathrm{d}\tau}{\cos\psi_l} .$$

Introducing the source function tensors S_Q^K in place of the ρ_Q^K, we have, from Eq. (3) and being:

$$\frac{A}{B} = \frac{2h\nu_0^3}{c^2}\frac{(2J+1)}{(2J'+1)}$$

$$I_i(\nu,\Omega_l) = \sum_{KK'Q}\sum_{j=0}^{3}\left\{\int_0^\infty S_Q^K(\tau)\left[\exp\left\{-\frac{\tau}{\cos\psi_l}\boldsymbol{H}\right\}\right]_{ij}\frac{d\tau}{\cos\psi_l}\right\} \times \mathcal{T}_Q^{K'}(j,\Omega_l)\Phi_Q^{K,K'}(J,J';\nu) . \quad (28)$$

If we consider a line-of-sight parallel to the surface of the atmosphere ($\psi_l \to \pi/2$), we have (such a direction is denoted as Ω_t):

$$I_i(\nu,\Omega_t) = \sum_{KK'Q}\sum_{j=0}^{3}S_Q^K(0)[\boldsymbol{H}^{-1}]_{ij} \times \mathcal{T}_Q^{K'}(j,\Omega_l)\Phi_Q^{K,K'}(J,J';\nu) \quad (29)$$

This equation gives, in general, a ν-dependent profile.

An important simplification occurs in the case of the *Hanle effect* regime ($\nu_L \approx A \ll \Delta\nu_D$), where we have:

$$\Phi_Q^{K,K'}(J,J';\nu) = \delta_{K,K'}w_{J'J}^{(K)}\phi(\nu_0-\nu) ,$$

$$H_{ij} = \phi(\nu_0-\nu)\delta_{ij} ,$$

where $w_{J'J}^{(K)}$ is a symbol that has been defined in Eq. (38) of Landi Degl'Innocenti (1984), and that only depends on K and on the quantum numbers J' and J of upper and lower levels respectively (one has $w_{10}^{(2)} = 1$ for $K = 2$ and for a normal Zeeman triplet line, namely $J' = 1$ and $J = 0$; a series of values for $K = 2$ and for other J' and J quantum numbers can be found in Table 1 of Landi Degl'Innocenti (1984)), and $\phi(\nu_0-\nu)$ is the absorption profile. Then one has:

$$I_i(\nu,\Omega_t) = \sum_{KQ}S_Q^K(0)\mathcal{T}_Q^K(i,\Omega_t) \times w_{J'J}^{(K)} \quad (\nu-\text{independent}) . \quad (30)$$

When the tensorial source functions S_Q^K are defined in the reference system Σ_1 (see Fig. 1 for its definition), where they are denoted as R_Q^K according to the particular case studied in Sect. 5, and for a line-of-sight specified by the angle θ (see Fig. 2), we have:

$$I_t(\theta) = R_0^0(0) + w_{J'J}^{(2)}\left[-\frac{1}{2\sqrt{2}}R_0^2(0) + \frac{\sqrt{3}}{2}(\cos 2\theta\,\Re R_2^2(0) - \sin 2\theta\,\Im R_2^2(0))\right] ,$$

$$Q_t(\theta) = w_{J'J}^{(2)}\left[-\frac{3}{2\sqrt{2}}R_0^2(0) - \frac{\sqrt{3}}{2}(\cos 2\theta\,\Re R_2^2(0) - \sin 2\theta\,\Im R_2^2(0))\right] ,$$

$$U_t(\theta) = w_{J'J}^{(2)}\sqrt{3}[\sin\theta\,\Re R_1^2(0) + \cos\theta\,\Im R_1^2(0)] , \quad (31)$$

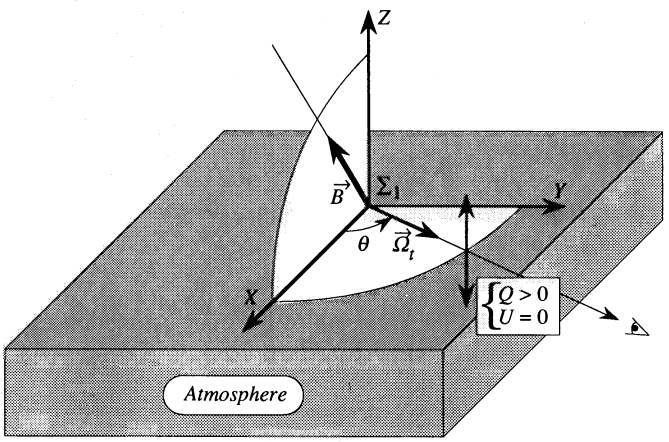

Fig. 2. In addition to the reference system $\Sigma_1 \equiv (X, Y, Z)$ defined in Fig. 1, the figure shows the reference direction for the definition of the Stokes parameters – namely the direction of linear polarization for the positive Q and zero U – for a ray propagating along the surface of the atmosphere in the direction Ω_t referred by the angle θ. This reference direction coincides with the vertical to the atmosphere

where the index t reminds that we are considering a ray propagating along the surface of the atmosphere. Figure 2 shows the definition of the reference direction for the Stokes parameters – namely the direction of linear polarization for positive Q and zero U – that coincides with the vertical to the atmosphere. These formulae – that incidentally show that the quantity $[I_t(\theta)+Q_t(\theta)]$ is θ-independent – allow to connect the various components of the source function tensor to the Stokes parameters propagating along different θ-directions. In terms of the Stokes parameters relative $\theta = 0°$, $45°$, $90°$, and $135°$, we have:

$$R_0^0(0) = \frac{1}{6}[3I_t(0°) + 3I_t(90°) - Q_t(0°) - Q_t(90°)],$$

$$w^{(2)}_{J'J} R_0^2(0) = -\frac{\sqrt{2}}{3}[Q_t(0°) + Q_t(90°)],$$

$$w^{(2)}_{J'J} \Re R_1^2(0) = \frac{1}{\sqrt{3}} U_t(90°),$$

$$w^{(2)}_{J'J} \Im R_1^2(0) = \frac{1}{\sqrt{3}} U_t(0°),$$

$$w^{(2)}_{J'J} \Re R_2^2(0) = \frac{1}{\sqrt{3}}[I_t(0°) - I_t(90°)]$$

$$= -\frac{1}{\sqrt{3}}[Q_t(0°) - Q_t(90°)],$$

$$w^{(2)}_{J'J} \Im R_2^2(0) = \frac{1}{\sqrt{3}}[I_t(45°) - I_t(135°)]$$

$$= \frac{1}{\sqrt{3}}[Q_t(45°) - Q_t(135°)]. \quad (32)$$

In the particular case of resonance polarization in a non-magnetic atmosphere, Ivanov (1990) provides analogous formulae (with, in his notations: $R_0^0 = S_i$ and $-R_0^2 = S_q$), which can be recovered by the previous ones by noting that, in zero magnetic field, the Stokes parameters of a ray propagating along the surface are θ-independent, for evident symmetry reasons.

In the *Hanle effect* regime ($\nu_L \approx A \ll \Delta\nu_D$), the $\sqrt{\varepsilon}$-law, expressed in terms of some of the Stokes parameters of the radiation propagating along the surface, can be derived by combining Eqs. (32) and (26).

7. Conclusions

In this paper, we have generalized the well known $\sqrt{\varepsilon}$-law to the case of an isothermal, plane-parallel atmosphere in the presence of a (constant) magnetic field vector having arbitrary intensity and direction, and in the presence of depolarizing collisions. The final formula that we have derived [Eq. (16) or (17)] relates the value of the components of the statistical tensor (or, equivalently, of the source function tensor) at the surface of the atmosphere to the value of the Planck function. By taking the appropriate limits, Eq. (16) yields to well-known results in the theory of radiative transfer, including the $\sqrt{\varepsilon}$-law for unpolarized radiation, and its generalization to resonance scattering polarization in a non-magnetic atmosphere (Ivanov 1990). Such formulae, as Eqs. (16) and (17), and their expressions in the particular cases given in Sect. 4, can be used as a test for polarized radiative transfer numerical codes.

The particular simplicity of the equation derived gives a clear demonstration of the power of the formalism of the irreducible spherical tensors for dealing with the physical phenomena involved in radiative transfer for polarized radiation.

Appendix: properties of the kernels $\tilde{K}_{KQ,K'Q'}(t,t')$

The expression of the kernels can be obtained through a procedure that has been outlined in Paper II and need not to be repeated here. The result is the following [cf. Eq. (19 II)]; (see also Fig. 1 of Paper II for the definition of the angles ψ and θ):

$$\tilde{K}_{KQ,K'Q'}(t,t') = \frac{1}{4\pi}\int_{\psi_1}^{\psi_2} d\psi \,|\tan\psi|\, \times \int_0^{2\pi} d\theta \tilde{G}_{KQ,K'Q'}(\boldsymbol{x},\boldsymbol{x}'), \quad (A1)$$

where

$$\begin{cases} \psi_1 = 0 \\ \psi_2 = \pi/2 \end{cases} \text{ if } t < t'; \qquad \begin{cases} \psi_1 = \pi/2 \\ \psi_2 = 0 \end{cases} \text{ if } t > t',$$

and where the multipole coupling coefficients $\tilde{G}_{KQ,K'Q'} \times (\boldsymbol{x},\boldsymbol{x}')$

$$\tilde{G}_{KQ,K'Q'}(\boldsymbol{x},\boldsymbol{x}') = (-1)^Q \sum_{ij=0}^{3} \sum_{LL'} \int_0^\infty d\nu \Phi_Q^{K,L}(J,J';\nu) \times \Phi_{Q'}^{K',L'}(J,J';\nu) \mathcal{T}^L_{-Q}(i,\Omega) \times O_{ij}(\boldsymbol{x},\boldsymbol{x}';\nu) \mathcal{T}^{L'}_{Q'}(j,\Omega). \quad (A2)$$

All the quantities entering this expression have been defined in the previous papers of this series; the properties of the generalized profiles $\Phi_Q^{K,K'}(J,J';\nu)$ and those of the multipole coupling coefficients $\tilde{G}_{KQ,K'Q'},(\boldsymbol{x},\boldsymbol{x}')$ are summarized in Appendices A and B of Paper I, respectively.

The kernels satisfy the properties given in the following. These properties generalize (to values $Q, Q' \neq 0$) similar properties obtained in Paper II for $\tilde{K}_{K0,K'0}(t,t')$, see Appendix A of Paper II:

(a) Inversion of the arguments τ and τ' [see Eq. (5)]

$$\tilde{K}_{KQ,K'Q'}(\tau,\tau') = \tilde{K}_{KQ,K'Q'}(\tau',\tau) = \tilde{K}_{KQ,K'Q'}(|\tau-\tau'|). \quad (A3)$$

(b) Conjugation property

$$[\tilde{K}_{KQ,K'Q'}(y)]^* = (-1)^{Q'-Q}\tilde{K}_{K-Q,K'-Q'}(y). \quad (A4)$$

(c) Vanishing values

$$\widetilde{K}_{KQ,K'Q'}(y) = (-1)^{K+K'}\widetilde{K}_{KQ,K'Q'}(y)\,, \tag{A5}$$

which implies that

$$\widetilde{K}_{KQ,K'Q'}(y) = 0$$

if K and K' are of different parity.

(d) Integral properties

$$\int_0^\infty \widetilde{K}_{00,KQ}(y)\,\mathrm{d}y = \int_0^\infty \widetilde{K}_{KQ,00}(y)\,\mathrm{d}y = \frac{1}{2}\delta_{K,0}\delta_{Q,0}\,. \tag{A6}$$

Properties (a), (b) and (c), descend from analogous properties of the multipole coupling coefficients that are given in Appendix B of Paper I. In particular, (a) descends from Eq. (B18 I), (b) from Eq. (B11 I), and (c) from Eq. (B17 I). The proof of property (d) is similar to the one given in Paper II for the special case $Q = 0$ (see Eq. (A5 II)) and is not repeated here. A further property is the following:

(e) Symmetry about the indices (valid only in the reference system Σ_0)

$$\widetilde{K}_{KQ,K'Q'}(y) = \widetilde{K}_{K'Q',KQ}(y)\,. \tag{A7}$$

To prove this property, we write down the expression of the kernel by combining Eqs. (A1) and (A2); recalling that

$$O_{ij}(\boldsymbol{x},\boldsymbol{x}';\nu) = \left[\exp\left\{-\frac{y}{\cos\psi}\boldsymbol{H}\right\}\right]_{ij}\,,$$

where $\boldsymbol{H}$ is the absorption matrix defined in Paper I [see Eqs. (19 I) and (46 I)], one has

$$\begin{aligned}\widetilde{K}_{KQ,K'Q'}(y) = \frac{1}{4\pi}\int_0^{\pi/2}\mathrm{d}\psi\tan\psi\int_0^{2\pi}\mathrm{d}\theta(-1)^Q \\ \times\sum_{ij=0}^{3}\sum_{LL'}\int_0^\infty \mathrm{d}\nu\Phi_Q^{K,L}(J,J';\nu) \\ \times\Phi_{Q'}^{K',L'}(J,J';\nu)\mathcal{T}_{-Q}^{L}(i,\Omega) \\ \times\left[\exp\left\{-\frac{y}{\cos\psi}\boldsymbol{H}\right\}\right]_{ij}\mathcal{T}_{Q'}^{L'}(j,\Omega)\,.\end{aligned} \tag{A8}$$

We next consider the inner quantity:

$$\begin{aligned}\mathcal{C}_{LQ,L'Q'}(\Omega) = (-1)^Q\sum_{ij=0}^{3}\mathcal{T}_{-Q}^{L}(i,\Omega)\left[\exp\left\{-\frac{y}{\cos\psi}\boldsymbol{H}\right\}\right]_{ij} \\ \times\mathcal{T}_{Q'}^{L'}(j,\Omega)\,,\end{aligned} \tag{A9}$$

and observe that both the tensor $\mathcal{T}_Q^L$ and the attenuation operator O_{ij} depend on the particular choice of the reference direction $\vec{e}_a(\Omega)$ that defines the Stokes parameters relatively to the direction Ω. However, it can be directly proved that this quantity is invariant relatively to the choice of the reference direction, a result that is also physically intuitive because the kernel cannot depend on the (arbitrary) definition of the linear polarization Stokes parameters. Given this invariance property, we can choose a particular reference direction to define the Stokes parameters relative to the direction Ω, in such a way to obtain the maximum simplification. This is done by aligning $\boldsymbol{e}_a(\Omega)$ in the plane containing the direction Ω and the magnetic field vector (see Fig. 3). With this choice, the absorption matrix $\boldsymbol{H}$ acquires the simplified form

$$\boldsymbol{H} = \begin{pmatrix} h_I & h_Q & 0 & h_V \\ h_Q & h_I & r_V & 0 \\ 0 & -r_V & h_I & r_Q \\ h_V & 0 & -r_Q & h_I \end{pmatrix}\,,$$

so that one can write

$$H_{ij} = \tau_i\tau_j H_{ji}\,, \tag{A10}$$

where τ_i $(i = 0,\ldots,3)$ is the formal vector defined by $\tau_i \equiv (1,1,-1,1)$. On the other hand, as property (A10) is conserved for any power of the matrix $\boldsymbol{H}$, one also has

$$\left[\exp\left\{-\frac{y}{\cos\psi}\boldsymbol{H}\right\}\right]_{ij} = \tau_i\tau_j\left[\exp\left\{-\frac{y}{\cos\psi}\boldsymbol{H}\right\}\right]_{ji}\,,$$

or

$$O_{ij} = \tau_i\tau_j O_{ji}\,.$$

Substituting this result in Eq. (A9), and recalling the definition of the irreducible tensor $\mathcal{T}_Q^L$, one gets:

$$\begin{aligned}\mathcal{C}_{LQ,L'Q'}(\Omega) = (-1)^Q\sum_{ij=0}^{3}\sum_{PP'}t_P^L(i)\mathcal{D}_{P-Q}^{L}(R)\tau_i\tau_j O_{ji} \\ \times t_{P'}^{L'}(j)\mathcal{D}_{P'Q'}^{L'}(R)\,,\end{aligned}$$

where R is the rotation bringing the reference system $(\boldsymbol{e}_a(\Omega), \boldsymbol{e}_b(\Omega), \Omega)$ into the reference system where the density matrix is defined. When this is the system Σ_0 defined in Fig. 1, one has

$$R \equiv (0,\alpha,\beta)\,,$$

where α and β are the angles defined in Fig. 3. We now take into account a property of the symbol t_P^L(i), namely:

$$\tau_i t_P^L(i) = t_{-P}^L(i)\,,$$

and we write down explicitly the expressions for the rotation matrices in terms of reduced rotation matrices. We get, changing also P in $-P$ and P' in $-P'$:

$$\begin{aligned}\mathcal{C}_{LQ,L'Q'}(\Omega) = (-1)^Q\sum_{ij=0}^{3}\sum_{PP'}t_P^L(i) \\ \times d_{-P-Q}^{L}(\alpha)\mathrm{e}^{\mathrm{i}Q\beta}O_{ji}t_{P'}^{L'}(j)d_{-P'Q'}^{L'}(\alpha)\mathrm{e}^{-\mathrm{i}Q'\beta}\,.\end{aligned}$$

We take into account a property of the reduced rotation matrices, namely:

$$d_{-P-Q}^{L}(\alpha) = (-1)^{-P+Q}d_{PQ}^{L}(\alpha)\,,$$

and observe that the symbole $t_P^L(i)$ is different from zero only

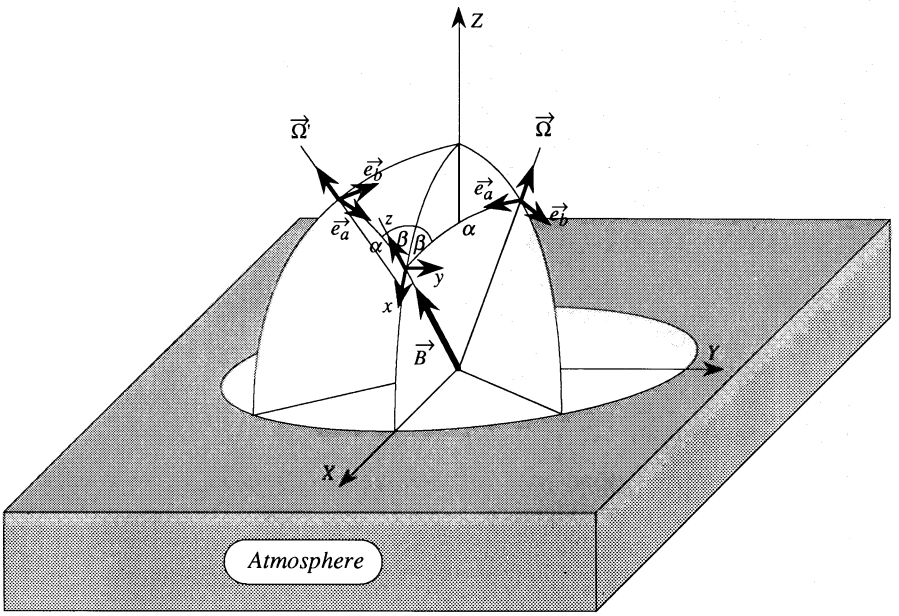

Fig. 3. The two directions Ω and Ω' are symmetric relatively to the plane defined by the magnetic field vector and the vertical. The reference direction $\boldsymbol{e}_a(\Omega)$ is contained in the plane defined by the magnetic field vector and the direction Ω. Similarly for the reference direction $\boldsymbol{e}_a(\Omega')$. α is the angle between the vectors $\boldsymbol{B}$ and Ω ($\boldsymbol{B}$ and Ω') while β is the angle between the plane ($\boldsymbol{B}$, Ω) and the plane ($\boldsymbol{B}$, Z) (or ($\boldsymbol{B}$, Ω') and ($\boldsymbol{B}'Z$)).(x, y, z) is the reference system Σ_0 defined in Fig. 1

if P is 0, 2, or -2. We obtain:

$$\mathcal{C}_{LQ,L'Q'}(\Omega) = (-1)^{Q'} \sum_{ij=0}^{3} \sum_{PP'} t_P^L(i) d_{PQ}^L(\alpha) \mathrm{e}^{iQ\beta} O_{ji} \times t_{P'}^{L'}(j) d_{P'-Q'}^{L'}(\alpha) \mathrm{e}^{-\mathrm{i}Q'\beta} .$$

On the other hand, defining R' as the rotation:

$$R' \equiv (0, \alpha, -\beta) ,$$

We have:

$$d_{PQ}^L(\alpha)\mathrm{e}^{\mathrm{i}Q\beta} = \mathcal{D}_{PQ}^L(R') ,$$

$$d_{P'-Q'}^{L'}(\alpha)\mathrm{e}^{-\mathrm{i}Q'\beta} = \mathcal{D}_{P'-Q'}^{L'}(R') ,$$

and we can write, rearranging the order of the various terms:

$$\mathcal{C}_{LQ,L'Q'}(\Omega) = (-1)^{Q'} \sum_{ij=0}^{3} \mathcal{T}_{-Q'}^{L'}(j, \Omega') \times \left[\exp\left\{ -\frac{y}{\cos\psi} \boldsymbol{H} \right\} \right]_{ji} \mathcal{T}_Q^L(i, \omega') ,$$

where Ω' is the mirror image of Ω with respect to the plane containing the vertical and the magnetic field vector (see Fig. 3). Taking into account that, to the direction Ω', it corresponds a matrix $\boldsymbol{H}'$ identical to $\boldsymbol{H}$, and an angle with the vertical $\psi' = \psi$, one obtains:

$$\mathcal{C}_{LQ,L'Q'}(\Omega) = \mathcal{C}_{L'Q',LQ}(\Omega') .$$

Substituting this result ino Eq. (A8), and taking into account that an integration has to be performed to all directions, Eq. (A7) is finally obtained.

Acknowledgements. The authors are grateful to V.V. Ivanov and H. Frisch for fruitful discussions, and for having pointed to their attention the method of the derivative of the Wiener-Hopf Λ-operator, generalized to matrix-type kernels (Ivanov, unpublished; Frisch and Frisch, 1975). This method has been employed in the present paper to give a proof of Eqs. (16) and (17) that is much simpler and more elegant than an alternative one, based on the Rybicki's quadratic integral, previously obtained by the authors (Landi Degl'Innocenti & Bommier, unpublished; Ivanov, 1990)

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