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Abstract: We introduce a new model describing multiple resonances in Kerr optical cavities. It agrees quantitatively with the Ikeda map and predicts complex phenomena such as super cavity solitons and coexistence of multiple nonlinear states.

OCIS codes: Nonlinear optics, fibers (190.4370); Pulse propagation and temporal solitons (190.5530); Resonators (230.5750).

Optical resonators featuring Kerr media display a wealth of phenomena, encompassing frequency combs, cavity solitons and instabilities [1–3], which are being intensively studied in view of their high impact applications. Given the complexity and the diversity of the physical phenomena, deriving simple, accurate and efficient models is of paramount importance. The workhorse for the description of nonlinear cavity dynamics is the celebrated Lugiato-Lefever equation (LLE) [4], which allows for deep theoretical insight and fast and accurate numerical modelling. Despite the fact that the LLE holds valid well beyond the mean-field approximation under which it has been historically derived, it is not capable of capturing certain physical phenomena of great interest. Indeed, LLE can model the evolution of only one Fabry-Pérot mode, related to a *single resonance*. Phenomena not captured by LLE range from 'super cavity solitons' (SCSs) [5] to the coexistence of stable modulational instability (MI) patterns and solitons, observed in Ref. [6]. The complete and exact dynamical scenario can be reproduced by the famous Ikeda map [7], but this model does not give any reasonable physical insight owing to its complex mathematical structure. A model sharing the accuracy of the Ikeda map and the simplicity of LLE will be of paramount importance in the design of the next-generation of resonators [9]. We describe here a multi-resonant LLE system, which agrees *quantitatively* with the Ikeda map. Each Fabry-Pérot resonance is described by an LLE-type equation, which is coupled to the others in a nontrivial way. An arbitrary number of resonances can be treated, making the model highly flexible and scalable to any situation of experimental interest [8]. For definiteness, we consider a fiber ring cavity, but the method can be of course straightforwardly applied to micro-resonators. We start from the Ikeda map in dimensional units:

$$E^{(n+1)}(Z=0,T) = \theta E_{in} + \rho e^{i\phi_0} E^{(n)}(Z=L,T), \qquad i\frac{\partial E^{(n)}}{\partial Z} - \frac{\beta_2}{2}\frac{\partial^2 E^{(n)}}{\partial T^2} + \gamma |E^{(n)}|^2 E^{(n)} = 0, \tag{1}$$

$E^{(n)}$ is the electric field envelope at the n-th round-trip (measured in $\sqrt{W}$), $P_{in} = |E_{in}|^2$ is the input pump power, ρ^2, θ^2 are respectively the power reflection and transmission coefficients of the coupler, and $\phi_0 = \beta_0 L$ is the linear cavity round-trip phase shift. For simplicity we lump all the losses in the boundary condition, with $1-\rho^2$ describing the total power lost per round-trip. Z measures the propagation distance inside the fiber of length L, and T is time in a reference frame traveling at the group velocity of the pulse. We note that the map above can be replaced – without loss of generality – with a single equation where the boundary conditions are explicitly incorporated in a type of Nonlinear Schrödinger Equation (NLSE) describing an "unfolded cavity". Using the Dirac delta comb to model the periodic application of the boundary conditions, and using the identity $\sum_n \delta(Z-nL) = \frac{1}{L}\sum_n e^{inkZ}$ (where $k = 2\pi/L$) and letting $Z \in [0,+\infty)$ ("unfolded" cavity), we arrive at the equation:

$$i\frac{\partial E}{\partial Z} - \frac{\beta_2}{2}\frac{\partial^2 E}{\partial T^2} + \gamma |E|^2 E = \frac{i\theta}{L} E_{in} \sum_n e^{i(nk-\beta_0)Z} + i\frac{\rho-1}{L} E \sum_n e^{inkZ}. \tag{2}$$

Equation (2) is the NLSE forced by two combs with equal wave-number spacing k and a relative shift β_0. The solution of (2) can be written as a sum of slowly-varying envelopes, which modulate the longitudinal Fabry-Pérot modes of the cavity. We assume that $N = N_R + N_L + 1$ Fabry-Pérot resonances are efficiently excited: $E(Z,T) = \sum_{n=-N_R}^{N_L} E_n(Z,T) e^{iknZ}$, where N_L (or N_R), are the number of modes corresponding to a resonance to the left towards smaller detuning (or to the right towards bigger detuning), of the central resonance denoted $n = 0$. By collecting exponentials oscillating with the same wave-number, we arrive at the following compact and general expression, consisting of N coupled LLEs (CLLEs):

$$i\frac{\partial U_n}{\partial Z} - \frac{\delta_n}{L}U_n - \frac{\beta_2}{2}\frac{\partial^2 U_n}{\partial T^2} + \gamma \sum_{p=-N_R}^{N_L} \sum_{q=q_{min}}^{q_{max}} U_p U_q U^*_{p-n+q} = i\frac{\theta}{L}E_{in} - i\frac{\alpha}{L}\sum_{p=-N_R}^{N_L} U_p, \quad (n = -N_R, \ldots, N_L) \tag{3}$$

where $\alpha = 1-\rho$, $U_n = E_n \exp[i\delta_0 Z/L]$, $q_{min} = \max\{-N_R, n-p-N_R\}$, $q_{max} = \min\{N_L, n-p+N_L\}$, $\delta_n = \delta_0 + 2\pi n$. The conditions on the integers $q_{min,max}$ select only the correct nonlinear couplings. The cavity detuning from the central mode is defined as $\delta_0 = mk - \beta_0$, $m = \arg\min_n |nk - \beta_0|$, entailing $-\pi \leq \delta_0 \leq \pi$, which is consistent with the 2π periodicity of the Ikeda map. If we assume that only one mode is excited in the cavity ($N_R = N_L = 0$) we recognize in (3) the standard, single-resonance LLE.

In order to test our model, we simulate the fiber ring resonator described in Ref. [6] (parameter listed in the caption of Fig. 1). We found that $N = 3$ resonances ($N_L = 2, N_R = 0$) describe perfectly the full range of detuning $-\pi \leq \delta_0 \leq \pi$ for $P_{in} = 1.5$ W. Figure 1a) shows the tilted cavity resonances obtained by solving the stationary ($\partial/\partial Z = \partial/\partial T = 0$) CLLEs Eq. (3) via a Newton-Raphson method (solid black curve). The agreement with the Ikeda map (dashed blue curve) is *perfect* over the full range $-\pi \leq \delta_0 \leq \pi$.

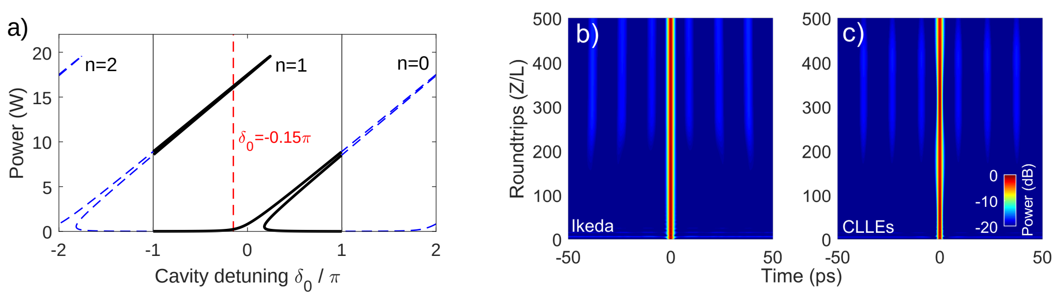

Fig. 1. a) Steady-state response of the cavity from Ikeda map (dashed blue) and CLLEs Eq. (3) (solid black) with $N = 3$ resonances ($N_L = 2, N_R = 0$). Parameters $L = 300$ m, $\alpha = 0.0619$, $\theta^2 = 0.05$, $\beta_2 = -22$ ps^2/km, $\gamma = 1.2$ W^{-1}km^{-1}, $P_{in} = |E_{in}|^2 = 1.5$ W. b-c) Coexistence between CS and MI pattern at $\delta_0 = -0.15\pi$ (vertical dashed red line in Fig. 1). Input for Ikeda map $|E^{(0)}(Z=0)|^2 = P_S \text{sech}(T/T_S)^2$; for CLLEs $|U_1(Z=0)|^2 = P_S \text{sech}(T/T_S)^2$, $U_{0,2}(Z=0) = 0$. Small white noise is added to the initial condition. Parameters: $P_S = 2(\delta_0 + 2\pi)/(\gamma L)$, $T_S^2 = |\beta_2|L/(2(\delta_0 + 2\pi))$ [5]

At a detuning $\delta_0 = -0.15\pi$ (dashed red vertical line in Fig. 1)a), we expect the coexistence of a stable MI pattern, given by the resonance $n = 0$, and a SCS supported by the resonance $n = 1$. This is confirmed by numerical solution of the Ikeda map in Fig. 1b), which matches remarkably well with the CLLEs [Fig. 1c)].

To conclude, we have derived a model based on coupled LLEs which allows for the accurate description of Kerr optical cavities when several Fabry-Pérot resonances interact. It is extremely accurate even when including a small number of modes, and will allow to acquire great physical insight thanks to its simplicity. We envision important applications in the modelling of new generations of nonlinear, large Q-factor microring resonators.

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