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ON THE EQUALITY OF PERIODS OF KONTSEVICH-ZAGIER

JACKY CRESSON AND JUAN VIU-SOS

Abstract. Effective periods are defined by Kontsevich and Zagier as complex numbers whose real and imaginary parts are values of absolutely convergent integrals of $\mathbb{Q}$-rational functions over $\mathbb{Q}$-semi-algebraic domains in $\mathbb{R}^d$. The Kontsevich-Zagier period conjecture affirms that any two different integral expressions of a given period are related by a finite sequence of transformations only using three rules respecting the rationality of the functions and domains: additions of integrals by integrands or domains, change of variables and Stoke's formula.

In this paper, we discuss about possible geometric interpretations of this conjecture, viewed as a generalization of the Hilbert's third problem for compact semi-algebraic sets as well as for rational polyhedron equipped with piece-wise algebraic forms. Based on partial known results for analogous Hilbert's third problems, we study obstructions of possible geometric schemas to prove this conjecture.

Résumé. Les périodes effectives furent définies par Kontsevich et Zagier comme étant les nombres complexes dont les parties réelle et imaginaire sont valeurs d'intégrales absolument convergentes de fonctions $\mathbb{Q}$-rationnelles sur des domaines $\mathbb{Q}$-semi-algébriques dans $\mathbb{R}^d$. La conjecture des périodes de Kontsevich-Zagier affirme que si une période admet deux représentations intégrales, alors elles sont reliées par une suite finie d'opérations en utilisant uniquement trois règles respectant la rationalité des fonctions et domaines : sommes d'intégrales par intégrandes ou domaines, changement de variables et formule de Stokes.

Dans cet article, nous discutons des possibles interprétations géométriques de cette conjecture, vue comme une généralisation du troisième problème de Hilbert pour des ensembles semi-algébriques compacts et aussi comme pour des polyèdres rationnels munis d'une forme volume algébrique par morceaux. Basés sur des résultats partiels connus pour des problème de Hilbert analogues, nous étudions des obstructions de possibles schémas géométriques pour obtenir une preuve de la conjecture.

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1. Introduction

Introduced by M. Kontsevich and D. Zagier in their paper [KZ01] in 2001, *periods* are a class of numbers which contains most of the important constants in mathematics. They are strongly related to the study of transcendence in number theory [Wal06], Galois theory and motives ([And04], [And12], [Ayo15]) and differential equations [FR14]. We refer to [Wal15] and [MS14] for an overview of the subject.

Let $\overline{\mathbb{Q}}$ (resp. $\mathbb{R}_{\text{alg}}$) be the *field of complex* (resp. *real*) *algebraic numbers*. As described in its *affine* definition given in [KZ01], *a period of Kontsevich-Zagier* (also called *effective period*) is a complex number whose real and imaginary parts are values of absolutely convergent integrals of rational functions over domains in a real affine space given by polynomial inequalities both with coefficients in $\mathbb{R}_{\text{alg}}$, i.e. absolutely convergent integrals of the form

$$\mathcal{I}(S, P/Q) = \int_S \frac{P(x_1, \dots, x_d)}{Q(x_1, \dots, x_d)} \cdot \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_d \tag{1}$$

where $S \subset \mathbb{R}^d$ is a d-dimensional $\mathbb{R}_{\text{alg}}$-semi-algebraic set and $P, Q \in \mathbb{R}_{\text{alg}}[x_1, \dots, x_d]$ are coprime. We denote by $\mathcal{P}_{\text{KZ}}$ the set of periods of Kontsevich-Zagier and by $\mathcal{P}_{\text{KZ}}^{\mathbb{R}} = \mathcal{P}_{\text{KZ}} \cap \mathbb{R}$ the set of *real periods*. This numbers are *constructible*, in the sense that a period is directly associated with a set of integrands and domains of integrations given by polynomials of rational coefficients. The set $\mathcal{P}_{\text{KZ}}$ forms a constructible countable $\overline{\mathbb{Q}}$-algebra and contains many transcendental numbers such as π or the *multiple zeta values* (see [Wal00]).

Heuristically, the main point in the *periods philosophy* is the study of transcendence numbers and their relations, via integration, from those of functions and objects of algebraic/geometric nature. In particular, the restriction over $\mathbb{R}_{\text{alg}}$ on the coefficients plays a main role in this theory and is one of the main obstructions. In the following, any (semi-)algebraic object is assumed to be defined with coefficients in $\mathbb{R}_{\text{alg}}$.

The main conjecture for periods is called the *Kontsevich-Zagier period conjecture*, originally stated as follows.

Conjecture ([KZ01, Conjecture 1])**.** *If a real period admits two integral representations, then we can pass from one formulation to the other using only three operations: integral additions by domains or integrands, change of variables and the Stokes formula. Moreover, these operations should respect the class of the objects previously defined.*

The above three operations are called the *KZ-rules*. A common idea around the Kontsevich-Zagier period conjecture is that, in their own words, *"this problem looks completely intractable and may remain so for many years"* (see also [Wal06, And12, Ayo15]). This is due to distinct facts, some of them are the following:

— Although this conjecture is natural, its place among classical conjectures in Number Theory is still not well understood. This is not the case of the Diophantine conjecture, which is in the same spirit and deals with relations between multiple zeta values, see [Wal00, p. 587]. Nevertheless, there exists a motivic version of Kontsevich-Zagier's conjecture, which is better understood and related to the classical Grothendieck period conjecture, see [And12].

— Up to our knowledge, there is no strategy of proof for this conjecture.

The aim of this paper is to discuss about possible reformulations of this conjecture which suggest a potential scheme of proof. Also, we give some hints about the role that each of the three KZ-rules plays in the conjecture and related problems. The main ingredient is the following algorithmic result, obtained in [VS15] by the second author:

Theorem (Semi-canonical reduction, [VS15, Thm. 1.1])**.** *Let p be a non-zero real period given in a certain integral form $\mathcal{I}(S, P/Q)$ in $\mathbb{R}^d$ as in (1). There exists an effective algorithm satisfying the KZ-rules such that $\mathcal{I}(S, P/Q)$ can be rewritten as*

$$\mathcal{I}(S, P/Q) = \operatorname{sgn}(p) \cdot \operatorname{vol}_m(K),$$

where K is a compact top-dimensional semi-algebraic set and $\operatorname{vol}_m(\cdot)$ is the canonical volume in $\mathbb{R}^m$, for some $0 < m \leq d+1$.

As a consequence, any real period can be written up to sign as the volume of a compact semi-algebraic set. The above theorem can be extended for the whole set of periods $\mathcal{P}_{\text{KZ}} \subset \mathbb{C}$ considering representations of the real and imaginary part respectively. Such a representation for a period p is called *a geometric semi-canonical representation of p*.

The semi-canonical reduction suggests a more geometrical point of view, in which we understand the KZ-rules as operations of geometrical nature between volumes. Doing so, we can interpret the Kontsevich-Zagier period conjecture as a kind of generalized Hilbert's third problem. This problem has a long history and we refer to [Car86] for a classical overview of the subject. Following the existing strategies dealing with analogues of Hilbert's third problem, we propose a scheme of proof for the *geometric Kontsevich-Zagier problem*, as well as some ways to obtain obstructions.

Concretely, we introduce the following problem:

Problem A (Geometric Kontsevich-Zagier's problem for periods, Problem 2.1)**.** *Let K_1, K_2 be two compact top-dimensional semi-algebraic sets in $\mathbb{R}^d$ such that $\operatorname{vol}_d(K_1) = \operatorname{vol}_d(K_2)$. Can we transform K_1 into K_2 only using the geometric operations: semi-algebraic scissors congruences, algebraic volume-preserving maps and product relations? Equivalently, are $[K_1]$ and $[K_2]$ equal in $\mathcal{K}_0(\mathcal{CSA}_{\mathbb{R}_{alg}})$, i.e. is* vol *an injective morphism?*

The ring $\mathcal{K}_0(\mathcal{CSA}_{\mathbb{R}_{\text{alg}}})$ is a Grothendieck ring of top-dimensional compact semi-algebraic sets modulo volume-preserving transformations and $\operatorname{vol} : \mathcal{K}_0(\mathcal{CSA}_{\mathbb{R}_{\text{alg}}}) \to \mathcal{P}_{\text{KZ}}^{\mathbb{R}}$ associates the class $[K]$ to its volume, see Section 2.2 for more details.

Analogously to this problem, we establish another one based on PL-geometry:

Problem B (Generalized Hilbert 3rd problem for rational polyhedra, Problem 3.1)**.** *Let P_1 and P_2 be two rational polyhedra in $\mathbb{R}^d$ equipped respectively by two piecewise algebraic volume forms ω_1 and ω_2 such that $\int_{P_1} \omega_1 = \int_{P_2} \omega_2$. Can we pass from an integral to the other one only by rational polyhedra scissors-congruence and rational polyhedra transformations?*

The plan of the paper is as follows: We introduce in Section 2 a first discussion about the nature of the Stokes formula in the original formulation of the Kontsevich-Zagier period conjecture, as well as its possible replacement by the *Fubini* and *Newton-Leibniz* formula as allowed operations in the conjecture. Then we present a geometric problem asking for a minimal set of relations between volumes of compact semi-algebraic sets, which follows the spirit of the Kontsevich-Zagier period conjecture (Problem A).

Based on the fact that any compact semi-algebraic set is triangulable, we are able to give in Section 3 an analogue to the previous problem in piecewise linear (PL) geometry: the generalized Hilbert's third problem for rational polyhedra (Problem B). In this case, the rules of transformation between rational polyhedra are of more combinatorial nature. Some known partial results of the generalized Hilbert third problem are given in Section 4, specially in the case of different polyhedra possessing the same volume with respect to the canonical volume form. In particular, we emphasize here the case of convex polyhedra.

Finally, we discuss in Section 5 how the previously introduced results can give a first schema to prove the Kontsevich-Zagier conjecture and also find some obstructions.

2. A reformulation of the Kontsevich-Zagier period conjecture

Our aim is to present a modified form of the Kontsevich-Zagier period conjecture from the point of view of volume-preserving operations between compact semi-algebraic sets. We are motivated by the following:

— Any operation should be simple and natural from the point of view of integral calculus.
— Easy to implement and manipulate.

2.1. Discussion about Stokes formula versus Fubini's theorem.

Sums by integrand functions or domains and change of variables are natural and induce explicit formulas between integrals. In the other hand, computation using Stokes's formula requires an exhaustive analysis in order to determine partitions and parameterizations of the boundary. Our main concern is then to find an alternative to use Stokes's formula, which leads to more tractable manipulations.

How such an alternative to Stokes formula could be determined? An idea is to recover basic operations of integral calculus satisfying our previous constraints and which suffices to prove Stokes formula. In the classical case, the *Fubini theorem* as well as the *Newton-Leibniz formula* (i.e. *Fundamental theorem of calculus*) are important ingredients of the proof [Spi79, p. 253-254].

Thus, a first natural tentative would be to replace the Stokes formula by Fubini theorem and the Newton-Leibniz formula. This choice is also motivated by geometric considerations related to the semi-canonical reduction for periods. Indeed, Fubini theorem is a convenient tool to bring down the dimension of the semi-canonical reduction, which plays an important role in the discussions about complexity of periods (see [CVS]). However, this strategy can not be achieved for at least two reasons. First, primitives of algebraic functions are in general transcendental functions. As a consequence, the integral representations obtained by Fubini theorem are out of the algebraic class. This is illustrated in the following expression:

$$\int_{P\times Q} f(x,y)\ \mathrm{d}x\mathrm{d}y = \int_P \left[\int_Q f(x,y)\ \mathrm{d}x\right] \mathrm{d}y \tag{2}$$

For example:

$$\begin{aligned}\int_{\left\{\substack{0<x<1\\1<y<x+1}\right\}} \frac{x}{y}\ \mathrm{d}x\mathrm{d}y &= \int_0^1 \left(\int_1^{1+x} \frac{x}{y}\ \mathrm{d}y\right) \mathrm{d}x\\ &= \int_0^1 x\log(x+1)\ \mathrm{d}x\\ &= \frac{1}{4}.\end{aligned}$$

Remark 2.1. In [KZ01, p. 5], Kontsevich and Zagier says that an integral of a transcendental function can be a period *"by accident"*, giving as example

$$\int_0^1 \frac{x}{\log\frac{1}{1-x}}\ \mathrm{d}x = \log 2.$$

However, the previous discussion shows that many periods will be possible to write as an integral of a transcendental function by using Fubini's theorem.

Secondly, in the case where Fubini's theorem gives an integral in the algebraic class, for example when $f(x,y) = g(x)h(y)$ in (2), this induces in fact quadratic relations between periods.

2.2. **A geometric Kontsevich-Zagier's problem for periods.** Our idea is to take advantage of the geometric representation obtained in the semi-canonical reduction theorem in order to formulate a related geometric problem in terms of volumes of compact semi-algebraic sets. It is worth noticing that the rules appearing in the Kontsevich-Zagier conjecture do not respect the representation of periods as volumes.

Let $\mathcal{CSA}^d_{\mathbb{R}_{\text{alg}}}$ be the collection of compact d-dimensional semi-algebraic sets in $\mathbb{R}^d$, containing the empty set and closed under finite unions. Define:

$$\mathcal{CSA}_{\mathbb{R}_{\text{alg}}} = \bigcup_{d\geq 1} \mathcal{CSA}^d_{\mathbb{R}_{\text{alg}}} \tag{3}$$

We study the different relations in $\mathcal{CSA}_{\mathbb{R}_{\text{alg}}}$ which are possible to obtain with the geometric operations described as follows. Let $(\mathcal{K}_0(\mathcal{CSA}_{\mathbb{R}_{\text{alg}}}),+)$ be the free abelian group generated by the classes $[K]$, with $K \in \mathcal{CSA}_{\mathbb{R}_{\text{alg}}}$, modulo the relations:

— (**Semi-algebraic scissors congruences**) If $K, K' \in \mathcal{CSA}^d_{\mathbb{R}_{\text{alg}}}$ such that $\text{codim}(K \cap K') \geq 1$, then
$$[K \cup K'] = [K] + [K'].$$

— (**Algebraic volume-preserving maps**) Let K. If there exist $U, V \subset \mathbb{R}^d$ open semi-algebraic sets and $f : U \to V$ an algebraic[1] diffeomorphism such that $K \subset U$ and $\det(\text{Jac}(f)) = 1$. Then
$$[K] = [f(K)].$$

We denote by $K \simeq_{\text{vol}} K'$ when such a volume-preserving map f exists and $K' = f(K)$. A natural connection between compact semi-algebraic sets in $\mathcal{CSA}_{\mathbb{R}_{\text{alg}}}$ comes from the direct product and Fubini's theorem, which induces a ring structure in $\mathcal{K}_0(\mathcal{CSA}_{\mathbb{R}_{\text{alg}}})$:

— (**Product relation**) For any $K \in \mathcal{CSA}^d_{\mathbb{R}_{\text{alg}}}$ and $K' \in \mathcal{CSA}^{d'}_{\mathbb{R}_{\text{alg}}}$, then
$$[K \times K'] = [K] \cdot [K'].$$

Remark 2.2.

(1) In our setting, it is particularly interesting to study when a compact semi-algebraic set can be expressed as a cylinder by the unit interval $I = [0,1]$, i.e. consider $K \in \mathcal{CSA}^d_{\mathbb{R}_{\text{alg}}}$ and $\widetilde{K} \in \mathcal{CSA}^{d-n}_{\mathbb{R}_{\text{alg}}}$ verifying that $K = \widetilde{K} \times I^n$. Then, we obtain the *flattening relation*:
$$[K] = [\widetilde{K} \times I^n] = [\widetilde{K}].$$

(2) Notice that the class of I is the neutral element of the product in $\mathcal{K}_0(\mathcal{CSA}_{\mathbb{R}_{\text{alg}}})$, and we denote $1 = [I]$. Analogously, the class of the empty set is the neutral element of the sum, denoted $0 = [\emptyset]$.

(3) It is worth noticing that the point $\mathbb{R}^0 = \{\text{pt}\}$ is not considered as a part of $\mathcal{CSA}_{\mathbb{R}_{\text{alg}}}$ in (3). Beyond the fact that points have not intrinsic euclidean volume, one can see that any class of a point would define a zero divisor in $\mathcal{K}_0(\mathcal{CSA}_{\mathbb{R}_{\text{alg}}})$, since $\{\text{pt}\} \times I \subset \mathbb{R}^2$ has codimension 1.

1. We said that f is *algebraic* if its graph Γ_f is an algebraic set in $U \times V$.

Example 2.1. Let $a, b \in \mathbb{R}_{\text{alg}}$ be positive. Define the compact semi-algebraic set

$$K_{a,b} = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq a, 0 \leq y + \sqrt{x+1} \leq b\}$$

Then, $K_{a,b} \simeq_{\text{vol}} C_{ab}$, where C_{ab} is the closed square of segments with lengths 1 and ab respectively, via the composition of a linear isometric maps of $\mathbb{R}^2_{\text{alg}}$ and the algebraic map $y \mapsto y - \sqrt{x+1}$ defined in the positive semi-plane. In particular,

$$[K_{a,b}] = [C_{ab}] = [I_{ab}],$$

via the flattening relation, where I_{ab} is the closed interval $[0, ab]$ in $\mathbb{R}^2$.

The product on $\mathcal{K}_0(\mathcal{CSA}_{\mathbb{R}_{\text{alg}}})$ extends to a particular class of fibrations. Let $E, B, F \in \mathcal{CSA}_{\mathbb{R}_{\text{alg}}}$ and $\rho : E \to B$ be a *proper map*, i.e. verifying that $\rho^{-1}(C)$ is compact for any compact subset C of B. We say that $\rho : E \to B$ is a *piecewise trivial proper fibration with fiber* F if there exist a finite partition $B = \bigcup_i C_i$ such that $C_i \in \mathcal{CSA}_{\mathbb{R}_{\text{alg}}}$, $\text{codim}(C_i \cap C_j) \geq 1$ and $\rho^{-1}(C_i) \simeq_{\text{vol}} C_i \times F$, for any i, j.

Considering the above notion, the following holds directly.

Proposition 2.1. *If $\rho : E \to B$ is a piecewise trivial proper fibration with fiber F, then $[E] = [F] \cdot [B]$.*

We extend naturally the volume of semi-algebraic sets into a well-defined ring morphism $\text{vol} : \mathcal{K}_0(\mathcal{CSA}_{\mathbb{R}_{\text{alg}}}) \to \mathbb{R}$ by defining $\text{vol}([K]) = \text{vol}_d(K)$, where $d \geq 1$ is such that $K \in \mathcal{CSA}^d_{\mathbb{R}_{\text{alg}}}$. By the semi-canonical reduction theorem, the latter restrains into a surjective morphism

$$\mathcal{K}_0(\mathcal{CSA}_{\mathbb{R}_{\text{alg}}}) \xrightarrow{\text{vol}} \mathcal{P}^{\mathbb{R}}_{\text{KZ}}.$$

Problem 2.1 (Problem A)**.** *Let K_1, K_2 be two compact top-dimensional semi-algebraic sets in $\mathbb{R}^d$ such that $\text{vol}_d(K_1) = \text{vol}_d(K_2)$. Can we transform K_1 into K_2 only using the geometric operations: semi-algebraic scissors congruences, algebraic volume-preserving maps and product relations? Equivalently, are $[K_1]$ and $[K_2]$ equal in $\mathcal{K}_0(\mathcal{CSA}_{\mathbb{R}_{alg}})$, i.e. is* vol *an injective morphism?*

This problem is reminiscent of two classical problems about *cutting-and-pasting* in geometry: Hilbert's third problem (see [Car86]) and Tarski's circle-squaring problem (see [Lac90]).

The Hilbert's third problem asks for scissor-congruence of polyhedra of the same volume. The 3-dimensional case was solved by Dehn [Deh01, Syd65] in 1900 introducing its *Dehn invariant* to distinguish scissor-congruent polyhedra. The main difference with our problem in terms of *cutting-and-pasting* operations is that we work in semi-algebraic class (with coefficients in $\mathbb{R}_{\text{alg}}$) which is a weaker scissor-congruence constrain than in original Hilbert's problem, where one must stay in the polyhedral class. Also, the class of transformations allowed for each piece is larger than classical isometries on the affine space.

The Tarski's circle-squaring problem is the equivalent question about scissor-congruence between a square and a disk of same area, but without restriction on the class of decompositions. This problem was solved by M. Laczkovich [Lac90] in 1990, proving that there exists a decomposition of the circle by non-measurable sets covering the square only using translations. In [CVS], we prove that this problem has no solution in our setting, and this is due to the fact that the semi-algebraic class has strong measurable properties, as well as constraints of arithmetic nature.

Here, we present our problem in a more modern language, where the construction of $\mathcal{K}_0(\mathcal{CSA}_{\mathbb{R}_{\text{alg}}})$ presents the geometrical relations in the same way as algebraic K-theory and Grothendieck rings (see [Car86, p. 268] and [Zak16]).

There exist a recent cutting-and-pasting problem in the setting of the Grothendieck ring of complex varieties $\mathcal{K}_0(\mathrm{Var}_{\mathbb{C}})$, which is a ring generated by classes $[X]$ of complex varieties modulo isomorphisms, subtractions $[X \setminus F] = [X] - [F]$ by closed subsets $F \subset X$ and the product relation. This problem is known as the *Larsen-Lunts Conjecture* [LL14], which asks whether an equality of classes $[X] = [Y]$ implies that X and Y admit a decomposition into isomorphic locally closed subvarieties. This was proved to be true for $\dim X \leq 1$ [LS10], but false in general [Bor18] as a consequence of the existence of zero divisors in $\mathcal{K}_0(\mathrm{Var}_{\mathbb{C}})$.

Remark 2.3.

(1) Following Cartier's presentation of Dehn's invariant, it would be interesting to look for a similar invariant based on $\mathcal{K}_0(\mathcal{CSA}_{\mathbb{R}_{\mathrm{alg}}})$. Also, it is natural to ask if $\mathcal{K}_0(\mathcal{CSA}_{\mathbb{R}_{\mathrm{alg}}})$ is a domain, i.e. if it contains zero divisors as $\mathcal{K}_0(\mathrm{Var}_{\mathbb{C}})$, even if intuitively this should not be the case.

(2) Giving an affirmative answer to the geometric Kontsevich-Zagier problem implies in fact to prove the Kontsevich-Zagier period conjecture. A negative answer will be also interesting, in order to determine possible obstructions.

(3) The study of the *flattering relation* and trivializations by fibration is central in the characterization of "minimal" representations of compact semi-algebraic sets in $\mathcal{K}_0(\mathcal{CSA}_{\mathbb{R}_{\mathrm{alg}}})$. This problem will play a fundamental role in future works concerning *degree theory and transcendence of periods.*

(4) It is worth noticing that the general problem of classifying $\mathbb{R}$-semi-algebraic sets up to $\mathbb{R}$-semi-algebraic volume-preserving bijections modulo lower-dimensional sets was already considered by A. Blass and S. Schanuel in [BS, p. 3] as an intractable problem.

In [Ayo15, Conj. 1.1], Ayoub consider a different approach of the Kontsevich-Zagier conjecture, consisting in concentrate all the complexity of a period's representation in the differential form, instead of putting all on the domain. More precisely, he identify periods as integrals over a real unit hypercube

$$\int_{[0,1]^d} f(z_1, \ldots, z_d) \mathrm{d}z_1 \cdots \mathrm{d}z_d, \tag{4}$$

where f is a holomorphic function over the closed poly-disk $\mathbb{D}^d = \{|z_i| \leq 1\} \subset \mathbb{C}^d$ and which is algebraic over $\mathbb{R}_{\mathrm{alg}}(z_1, \ldots, z_d)$, see [Ayo14, §2.2]. Using this representation, he formulates a *compact from* of the Kontsevitch-Zagier conjecture: any relation between periods (4) is consequence of linear combinations of expressions

$$\frac{\partial f}{\partial z_i} - f_{|z_i=1} + f_{|z_i=0}. \tag{5}$$

Note that we recover the Stokes formula from the above. Moreover, these expressions are supposed to contain the change of variables relation [Ayo15, Rem. 1.5]. Nevertheless, at this moment it is not known by the authors if there exist an algorithmic way to pass from an expression (1) to another of type (4) corresponding to the same effective period, as in the case of the semi-canonical representation.

Using this representation, a *relative version* of the Kontsevich-Zagier conjecture is proved [Ayo15, Thm. 4.25], showing explicitly the "geometric" relations between Laurent series where periods appears as coefficients in the form (4). Roughly speaking, those relations are linearly generated by expressions (5) adapted for such Laurent series, and the submodule generated by functions which have zero-valuated integrals (4).

Ayoub's point of view follows the same idea realized differently in the following section: to simplify the expression of periods by concentrating the complexity in the differential form. In our case, we are going to keep some informations by allowing integrals over rational polyhedra, instead of a cube. In fact, both approaches could be studied using *Moser's theorem* [Mos65] about volume-preserving maps for manifolds, which is cited in Section 4. It is worth noticing that one of the results in Moser's article could potentially give a solution of the conjecture following Ayoub's point of view: it is proved in [Mos65, Lemma 2] that there exists a change of variables relating two integrals of differentiable functions over the unit hypercube giving the same value. Moreover, such a change of variables is explicitly constructed in the proof, but it turns to be *transcendental* in most of the cases, since the construction is made by using integration inductively over each variable.

3. From Semi-algebraic to Piecewise Linear geometry

We consider an analogous problem to the previous one in the *Piecewise Linear* (PL) class. The interest of this new problem is twofold. First, we can make a connection between the PL and semi-algebraic cases. Second, we can take advantages of know results in the PL geometry.

Up to now, we have not consider all the good properties of semi-algebraic sets. In particular, it is well-know that compact semi-algebraic sets admit "good" triangulations, which allows us to obtain a representation in the PL category. Using this result, called *Semi-algebraic Triangulation Theorem*, we make a connection between the geometric Kontsevich-Zagier problem and a PL version.

3.1. Semi-algebraic triangulations.

Let $a_0, \dots, a_k$ be $k+1$ points affinely independent in $\mathbb{R}^d$. Recall that a *k-simplex* $[a_0, \dots, a_k]$ is the set of points $p \in \mathbb{R}^d$ such that there exist non-negative $\lambda_0, \dots, \lambda_k \in \mathbb{R}$ verifying $\sum_{i=1}^k \lambda_i = 1$ and $p = \sum_{i=1}^k \lambda_i a_i$. In this case, the numbers $(\lambda_0, \dots, \lambda_k)$ are called the *barycentric coordinates* of p. For any non-empty subset $\{a_{i_0}, \dots, a_{i_\ell}\} \subset \{a_0, \dots, a_k\}$, the ℓ-simplex $[a_{i_0}, \dots, a_{i_\ell}]$ is called a *ℓ-face* of $[a_0, \dots, a_k]$. If σ is a simplex, we denote by $\mathring{\sigma}$ the *open simplex*, composed bu the points of σ whose barycentric coordinates are all positive.

A *finite simplicial complex* of $\mathbb{R}^d$ is a collection of simplices $\mathcal{K} = (\sigma_i)_{i=1,\dots,m}$ verifying that the faces of every σ_i belongs to $\mathcal{K}$ and such that, for every $1 \leq i < j \leq m$, $\sigma_i \cap \sigma_j$ is either empty or a common face of σ_i and σ_j. The *realization* of the complex is $|\mathcal{K}| = \bigcup_{i=1}^m \sigma_i$. Note that the open simplices $\mathring{\sigma}_i$ form a partition of $|\mathcal{K}|$.

Following [BCR98, Sec. 9.2], we have:

Theorem 3.1 (Semi-algebraic Triangulation Theorem)**.** *Every compact semi-algebraic set K in $\mathbb{R}^d$ is* semi-algebraically triangulable*, i.e. there exists a finite simplicial complex $\mathcal{K} = (\sigma)_{i=1,\dots,p}$ and a semi-algebraic homeomorphism $\Phi : |\mathcal{K}| \to K$.*

It is important to notice that the triangulation procedure is algorithmic (see [BPR06, Sec. 5.7, p. 183]) and moreover, that any operation used in this procedure belong to the KZ-rules. In particular, every strata of the decomposition is obtained by a *Nash embedding* [BPR06, Remark 9.2.5]. In fact, this triangulation can be chosen such that the triangulation $\Phi : |\mathcal{K}| \to K$ is a C^1-map, see [OS17], but this construction is not defined algorithmically.

Lemma 3.1. *For any compact $K \in \mathcal{SA}^d_{\mathbb{R}_{alg}}$ in $\mathbb{R}^d$, there exist a rational polyhedron P in $\mathbb{R}^d$ and a piecewise algebraic volume form ω such that*

$$\mathrm{vol}_d(K) = \int_P \omega$$

Moreover, the passage between the two integrals respects the KZ-rules.

Proof. It follows form [OS17, Thm 1.1]. There exists a semi-algebraic triangulation $(\mathcal{K}, \Phi)$ of K with C^1 realization. Then, the pull-back Φ^* is well defined over K and taking $P = |\mathcal{K}|$, we have:

$$\mathrm{vol}_d(K) = \int_K \omega_0 = \int_P \Phi^*(\omega_0),$$

where ω_0 is the canonical volume form in $\mathbb{R}^d$. Taking $\omega = \Phi^*(\omega_0)$, the result follows. □

3.2. **A PL version of the geometric Kontsevich-Zagier problem.** The previous lemma leads us to formulate an analogous version of the geometric Kontsevich-Zagier problem in terms of PL-geometry.

Problem 3.1 (Problem B)**.** *Let P_1 and P_2 be two rational polyhedra in $\mathbb{R}^d$ equipped respectively by two piecewise algebraic volume forms ω_1 and ω_2 such that $\int_{P_1} \omega_1 = \int_{P_2} \omega_2$. Can we pass from an integral to the other one only by rational polyhedra scissors-congruence and rational polyhedra transformations?*

Due to the similarity between Problem A and Problem B, it seems that we have not gain any advantage from the new formulation. Moreover, it is worth noticing that the flatness condition does not appear naturally in this setting and we are fixing the dimension of our objects. Nevertheless, this problem is now a part of discrete and polyhedral geometry, for which many powerful algorithmic methods already exist to deal with scissor-congruences. We refer to the book of I. Pak [Pak15] for an overview on the state of art of discrete and polyhedral geometry with combinatorial methods.

In the next section, we give some hints about why the previous problem could have a positive answer. However, we are still far away from constructing a complete proof, and this is related to the following fact: we are using some classical results for which the current known proof is unsatisfying in our setting.

We show again in this other approach how essential is the decomposition-on-domains/scissors-congruence operation in the conjecture. The main idea to find a counter-example to the latter without the scissors-congruences is to construct two PL-manifolds in the arithmetic class which are homeomorphic but not PL-homeomorphic. This is non trivial and follows from Milnor, Kirby and Siebenmann's example [Mil61, KS69]. As a consequence, if a period admits a representation by two such polyhedra, we cannot find a global volume preserving PL-map between them. A natural question is then: even if a PL-map cannot be found, can we nevertheless construct a semi-algebraic map between them? The answer is again negative, and it follows from a non-trivial result due to M. Shiota and Yokoi: the *Hauptvermutung* for compact semi-algebraic sets [SY84, Cor 4.3].

4. About volume of rational polyhedra, scissor-congruences and mappings

In this section, we exhibit some known partial results in the problem of constructing transformations between two polyhedra equipped respective volume forms which have the same total volume.

4.1. **Canonical volume form.**

Before discussing the general case, a first simplification of Problem B is to consider the polyhedra equipped with the canonical volume form.

4.1.1. *A general result.* In that case, we have found an interesting result discussed by A. Henriques and I. Pak in [HP04] which gives a priori a positive answer to the problem. The strategy is first to look at this problem restricted to convex polytopes and then to reduce the general case using appropriate decompositions by convex parts.

Theorem 4.1 ([Pak15, Example 18.3, p. 171])**.** *Let $P, Q \subset \mathbb{R}^d$ be two rational polyhedra of equal volume, i.e.* $\mathrm{vol}_d(P) = \mathrm{vol}_d(Q)$*. Then one can pass from P to Q only using rational polyhedra scissors-congruence and rational polyhedra transformations.*

It is worth noticing that this result only deals with periods coming from volumes of rational polyhedra, i.e. $\mathbb{Q} \oplus i\mathbb{Q} \subset \mathcal{P}_{\mathrm{KZ}}$. Nevertheless, the main difference here is that we allow an operation which does not appear in the original Hilbert third problem: rational polyhedra transformations. The importance of allowing such type of operations is clear, since the original Dehn invariant becomes an obstruction otherwise.

The main ingredient of the proof given for the latter is the following result:

Theorem 4.2 ([HP04, Theorem 2])**.** *Let $P, Q \subset \mathbb{R}^d$ be two convex rational polyhedra of equal volume, i.e.* $\mathrm{vol}_d(P) = \mathrm{vol}_d(Q)$*. Then there exists a one-to-one rational map $f : P \to Q$, which is continuous, piecewise linear and volume-preserving.*

Note that no decomposition of the polyhedra is used in the previous result, due to the restrictive hypothesis in this setting. However, the proof of the latter relies fundamentally on *Moser's theorem* [Mos65] about volume-preserving maps for manifolds, which is not constructive. In fact, to obtain a combinatorial proof of Moser's Theorem for convex polyhedra is still an open problem, see [HP04, Sec. 8.7, p. 17]. Pak refers to continuous, piecewise linear and volume-preserving maps as *Monge maps* (see [Pak15, Sec. 18.1, p. 170]).

We give an illustration of Theorem 4.1 for the case of convex plane polygons in the following. The main point is that in this case the Monge maps are explicitly constructed and moreover we see directly how these maps are rational. We then discuss how Theorem 4.1 (non-convex case) can be derived from Theorem 4.2 (convex case).

4.1.2. *Convex polygons.* Restricting to polygons in the real plane, Pak describes a simple way to obtain a Monge maps between two convex polygons of same area [Pak15, Ex. 18.2, p. 170]. The idea is to reduce any convex polygon into a triangle, by a recursively series of continuous, piecewise linear and volume-preserving transformations displacing vertex in order to reduce the number of faces of the polygon.

Let P a polygon of $n > 3$ faces and consider a vertex $v \in P$. Take the triangle (uvw) formed by u and its neighbors u and w, which is contained in P. Let z be the second neighboring vertex of w other than v. We go to transform the triangle (uvw) into another one $(uv'w)$ by shifting v along the parallel line to (uw) passing through v, such that the new vertex v' lies in the line (wz). This induces a global transformation keeping the rest of the polygon $P \setminus (uvw) = P \setminus (uv'w)$ unchanged (see Figure 1). Remark that we have obtained a polygon of $(n-1)$ faces and that this transformation is continuous and piecewise linear. In fact, the triangles (uvw) and $(uv'w)$ has the same area since we are not modifying neither

the base of the triangle (lying in (uw)) or the height (because v and v' lyes in the same parallel line to (uw)). Thus it is also a volume-preserving map.

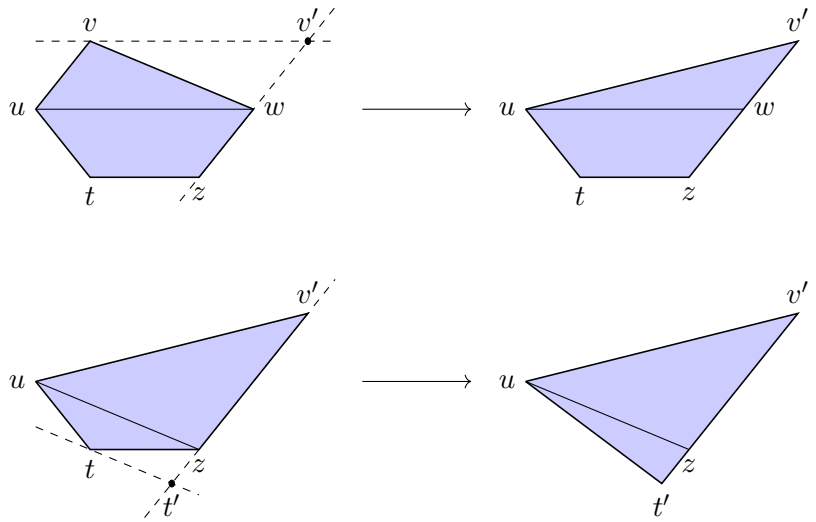

FIGURE 1. Illustration of the Monge map described by Pak.

It is worth noticing that the previously described transformation respects the rationality of the polygon. Repeating this procedure $n-3$ times, we can transform P on to a triangle $\triangle_1$ of same area by a Monge map $\zeta_1 : P \to \triangle_1$.

Let Q be a polygon with same area of P. Repeating the previous procedure on Q, we obtain a triangle $\triangle_2$ a Monge map $\zeta_2 : Q \to \triangle_2$. As $\triangle_1$ and $\triangle_2$ has the same area by hypothesis, there exist a volume-preserving linear map $\phi : \triangle_1 \to \triangle_2$. Thus, by composing the above functions we obtain a Monge map $\zeta = \zeta_2^{-1} \circ \phi \circ \zeta_1 : P \to Q$ between P and Q.

Remark 4.1. Note that the previous procedure is not unique and depends on many choices during the process.

4.1.3. *Non-convex case.* In the non-convex case, the idea is first to decompose each polyhedron in the same number of convex pieces $P = P_1 \cup \cdots \cup P_n$ and $Q = Q_1, \cup \cdots \cup Q_n$ with respective volumes $\alpha_i = \mathrm{vol}_d(P_i)$ and $\beta_j = \mathrm{vol}_d(Q_j)$. Then, we can produce a partition with second subdivisions $P = \bigcup_{i,j=1}^n P_{i,j}$ and $Q = \bigcup_{i,j=1}^n Q_{i,j}$ such that $\mathrm{vol}_d(P_{i,j}) = \mathrm{vol}_d(Q_{i,j}) = \alpha_i\beta_j$, for any $i, j = 1, \ldots, n$. Such a subdivision is detailed in [HP04, Sec. 1, p. 2]. Moreover, such a decomposition can be made in order to preserve the rationality. Applying the previous theorem at each pair of convex pieces, the result holds. Note that this proof is constructed by making an essential use of two allowed transformation rules: scissor-congruence and decomposition.

It is worth noticing that the previous results are still non-constructive in the general case, and this fact comes entirely from the convex case in dimension strictly greater than two.

4.2. **The general case.** Now, let us come back to the original setting, where the polyhedra P and Q are equipped with two piecewise algebraic volume forms ω_1 and ω_2 verifying

$$\int_{P_1} \omega_1 = \int_{P_2} \omega_2.$$

Up to our knowledge, the following result due to G. Kuperberg in [Kup96, Thm. 3] is the most general result in the case where non-canonical volume forms are considered. This

result is formulated and proved in [HP04, Theorem 1] in a more suitable way with respect to our problem.

Theorem 4.3. *Let $M_1, M_2 \subset \mathbb{R}^d$ be two* PL*-manifolds, possibly with boundary, which are* PL*-homeomorphic and equipped with piecewise constant volume forms ω_1 and ω_2. Suppose that M_1 and M_2 have equal volume, i.e. $\int_{M_1} \omega_1 = \int_{M_2} \omega_2$. Then there exists a volume-preserving* PL*-homeomorphism $f : M_1 \to M_2$, in particular $f^*(\omega_2) = \omega_1$.*

Remark 4.2. In fact, Pak and Henriques only give a sketch of proof of the above theorem, saying that it follows more or less directly from the canonical volume form case. However, certain parts need to be detailed, in particular the one analogous to the construction named in Section 4.1.3, and dealing with two simplicial decompositions for P and Q by pieces $\{P_{i,j}\}_{i,j=1}^n$ and $\{Q_{i,j}\}_{i,j=1}^n$ such that $\int_{P_{i,j}} \omega_1 = \int_{Q_{i,j}} \omega_2$ for any $i = 1, \dots, n$.

From our point of view, the above equality between values of integrals express exactly a equality between periods! This is in fact one of the main points of the Kontsevich-Zagier conjecture and other related problems in periods theory: to compare periods between them from integral expressions [KZ01, Sec. 1.2].

Looking at the hypotheses of Theorem 4.3, we first remark that they are very strong. As long as we are looking for *global* continuous, piecewise-linear, volume-preserving maps, one can not avoid to impose the PL-homeomorphism condition. With respect to our problem, we can not ensure that this condition is satisfied in general. Moreover, the volume forms which appear from our construction are a priori non piecewise constant.

The first condition can be removed by restricting our attention on convex polyhedra. Indeed, we have the following:

Lemma 4.1 ([HP04, Lemma 1.1])**.** *For any two convex polyhedra $P, Q \subset \mathbb{R}^d$ there exist a* PL*-homeomorphism $f : P \to Q$.*

In this direction, we obtain a more satisfying result:

Theorem 4.4. *Let $P, Q \subset \mathbb{R}^d$ be two convex polyhedra equipped with piecewise constant volume forms ω_1 and ω_2. Suppose that P and Q have equal volume, i.e. $\int_P \omega_1 = \int_Q \omega_2$. Then there exists a volume-preserving* PL*-homeomorphism $f : P \to Q$, in particular $f^*(\omega_2) = \omega_1$.*

5. Conclusion

We have presented two reformulations of the Kontsevich-Zagier period conjecture which take place in different settings (Grothendieck rings, K-theory, PL-geometry, volumes and forms in convex polyhedra,...) where there already exists some similar tools to work in possible schemas of proof and obstructions.

The latter open different ways to attack this problem, generalizing some of the previous results in more general settings for constructing a complete proof. On the other hand, the considerable quantity of obstructions which exist in other similar scissor-congruence problems give some evidence to believe that the Kontsevitch-Zagier period conjecture could be, in fact, false.

Problem A presents an algebraic setting to work in the construction of obstructions and geometric invariants. This could lead to possible counter-examples, in the spirit of other cutting-and-pasting problems. The main goal here is to develop a Dehn-like invariant for general compact semi-algebraic sets.

Concerning Problem B, even if the type of result that Theorem 4.4 represents is exactly in the spirit of our problem, the gap between this result and a complete answer in the PL case is large due to the following reasons:

— We think that the more difficult problem is to extend the previous theorem for volume forms in the algebraic class.

— We would like to avoid the convexity assumption. In this case, the situation seems more tractable following the same strategy as for canonical volume forms. Indeed, the setting of our problem allows scissors-congruences between the polyhedra and not only global volume-preserving PL-homeomorphism: it seems reasonable that we could give decompositions of non-convex polyhedra in convex parts with a predefined volume and then "compare" these parts one by one.

Due to the connection between the Kontsevich-Zagier period conjecture, Problem B and Problem A, the above describes the main problems to be solved in order to advance in the knowledge of periods from both new settings.

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