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► **To cite this version:**

Abraham Sylla. Influence of a slow moving vehicle on traffic: Well-posedness and approximation for a mildly non-local model. 2019. hal-02391844v1

**HAL Id: hal-02391844**

**<https://hal.science/hal-02391844v1>**

Preprint submitted on 3 Dec 2019 (v1), last revised 27 Jul 2021 (v4)

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# Influence of a slow moving vehicle on traffic: Well-posedness and approximation for a mildly non-local model

Abraham Sylla\*

## Abstract

In this paper, we propose a macroscopic model that describes the influence of a slow moving large vehicle on road traffic. The model consists of a scalar conservation law with a non-local constraint on the flux. The constraint level depends on the trajectory of the slower vehicle which is given by an ODE depending on the downstream traffic density. After proving well-posedness, we first build a finite volume scheme and prove its convergence, and then investigate numerically this model by performing a series of tests. In particular, the link with the limit local problem of [M.L. Delle Monache and P. Goatin, *J. Differ. Equ.* **257**(11), 4015–4029 (2014)] is explored numerically.

## Introduction

Delle Monache and Goatin developed in [20] a macroscopic model aiming at describing the situation in which a slow moving large vehicle – a bus for instance – reduces the road capacity and thus generates a moving bottleneck for the surrounding traffic flow. Their model is given by a Cauchy problem for Lightwell-Whitham-Richards scalar conservation law in one space dimension with local point constraint. The constraint is prescribed along the slow vehicle trajectory  $(y(t), t)$ , the unknown  $y$  being coupled to the unknown  $\rho$  of the constrained LWR equation. Point constraints were introduced in [19, 17] to account for localized in space phenomena that may occur at exits and which act as obstacles. The constraint in the model of [20] depends upon the slow vehicle speed  $\dot{y}$ , where its position  $y$  verifies the following ODE

$$\dot{y}(t) = \omega(\rho(y(t)^+, t)). \quad (\text{A})$$

Above,  $\rho = \rho(x, t) \in [0, R]$  is the traffic density and  $\omega : [0, R] \rightarrow \mathbb{R}^+$  is a non-increasing Lipschitz continuous function which links the traffic density to the slow vehicle velocity. Delle Monache and Goatin proved an existence result for their model in [20] with a wave-front tracking approach in the BV framework. Adjustments to the result were recently brought by Liard and Piccoli in [27]. Despite the step forward made in [21], the uniqueness issue remained open for a time. Indeed, the appearance of the trace  $\rho(y(t)^+, t)$  makes it fairly difficult to get a Lipschitz continuous dependency of the trajectory  $y = y(t)$  from the solution  $\rho = \rho(x, t)$ . Nonetheless, a highly nontrivial uniqueness result was achieved by Liard and Piccoli in [26].

In the present paper, we consider a modified model where the point constraint becomes non-local, making the velocity of the slow vehicle depend on the mean density evaluated in a small vicinity ahead the driver. More precisely, instead of (A), we consider the relation

$$\dot{y}(t) = \omega\left(\int_{\mathbb{R}} \rho(x + y(t), t)\mu(x)dx\right), \quad (\text{B})$$

where  $\mu \in \text{BV}(\mathbb{R}; \mathbb{R}^+)$  is a weight function used to average the density. From the mathematical point of view, this choice makes the study of the new model easier. Indeed, the authors of [3, 4, 5] put forward techniques for full well-posedness analysis of similar models with non-local point constraints. From the modeling point of view, considering (B) makes sense for several reasons outlined in Section 2.5.

The paper is organized as follows. Sections 1-2 are devoted to the proof of the well-posedness of the model.

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In Section 3 we introduce the numerical finite volume scheme and prove its convergence. An important step of the reasoning is to prove a BV regularity for the approximate solutions. It serves both in the existence proof and it is central in the uniqueness argument. In that optic, the appendix is essential. Indeed, it is devoted to the proof of a BV regularity for entropy solutions to a large class of limited flux models. Let us stress that, in passing, we highlight the interest of the  $BV_{\text{loc}}$  discrete compactness technique of Towers [31] in the context of general discontinuous-flux problems. In the numerical section 4, first we perform numerical simulations to validate our model. Then we investigate both qualitatively and quantitatively the proximity between our model – in which we considered (B) – as  $\delta \rightarrow \mu_{0+}$  and the model of [20] in which the authors considered (A).

## 1 Model, notion of solution and uniqueness

### 1.1 Model in the bus frame

Note that we find it convenient to study the problem in the bus frame, which means setting  $X = x - y(t)$  in the model of Delle Monache and Goatin in [20]. Keeping in mind what we said above about the non-local constraint, the problem we consider takes the form:

$$\begin{cases} \partial_t \rho + \partial_x F(\dot{y}(t), \rho) = 0 & \mathbb{R} \times (0, T) \\ \rho(x, 0) = \rho_0(x + y_0) & x \in \mathbb{R} \\ F(\dot{y}(t), \rho)|_{x=0} \leq Q(\dot{y}(t)) & t \in (0, T) \\ \dot{y}(t) = \omega \left( \int_{\mathbb{R}} \rho(x, t) \mu(x) dx \right) \\ y(0) = y_0. \end{cases} \quad (1.1)$$

Above,  $\rho = \rho(x, t)$  is the traffic density, of which maximum attainable value is  $R$  and

$$F(\dot{y}(t), \rho) = f(\rho) - \dot{y}(t)\rho$$

denotes the normal flux through the curve  $x = y(t)$ . We assume that the flux function  $f : [0, R] \rightarrow \mathbb{R}$  is Lipschitz continuous and bell-shaped, which are commonly used assumptions in traffic dynamics:

$$f(\rho) \geq 0, f(0) = f(R) = 0, \exists! \bar{\rho} \in (0, R), f'(\rho)(\bar{\rho} - \rho) > 0 \text{ for a.e. } \rho \in (0, R). \quad (1.2)$$

In [20], the authors chose the function  $Q(s) = \frac{\alpha}{4}(1 - s)^2$  with  $\alpha \in (0, 1)$  to prescribe the maximal flow allowed through a bottleneck located at  $x = 0$ . We can allow for more general choices. Specifically,

$$Q : [0, \|\omega\|_{L^\infty}] \rightarrow \mathbb{R}^+$$

can be any Lipschitz continuous function. It is a well known fact that in general, the total variation of an entropy solution to a constraint Cauchy problem may increase (see [17, Section 2] for an example). However, this increase can be controlled if the constraint level does not reach the maximum level. A mild assumption on  $Q$  – see Assumption (2.6) below – will guarantee availability of BV bounds, provided we suppose that

$$\rho_0 \in L^1(\mathbb{R}; [0, R]) \cap BV(\mathbb{R}).$$

### 1.2 Notion of solution

Throughout the paper, we denote by

$$\Phi_{\dot{y}(t)}(a, b) = \text{sign}(a - b)(F(\dot{y}(t), a) - F(\dot{y}(t), b)) = \Phi(a, b) - \dot{y}(t)|a - b|$$

the entropy flux associated with the Kruřkov entropy  $\rho \mapsto |\rho - \kappa|$ , for all  $\kappa \in [0, R]$ , see [25]. Following [20, 17, 6, 14], we give the following definition of solution for Problem (1.1).

**Definition 1.1.** A couple  $(\rho, y)$  with  $\rho \in L^\infty(\mathbb{R} \times [0, T])$  and  $y \in W^{1,\infty}((0, T))$  is called an admissible weak solution to Problem (1.1) if

(i) the following regularity is fulfilled:

$$\rho \in C([0, T]; L^1_{\text{loc}}(\mathbb{R})); \quad (1.3)$$

(ii) for all non-negative test function  $\varphi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^+)$  and  $\kappa \in [0, R]$ , the following entropy inequalities are verified for all  $0 \leq \tau < \tau' \leq T$ :

$$\begin{aligned} & \int_\tau^{\tau'} \int_{\mathbb{R}} |\rho - \kappa| \partial_t \varphi + \Phi_{\dot{y}(t)}(\rho, \kappa) \partial_x \varphi dx dt + \int_{\mathbb{R}} |\rho(x, \tau) - \kappa| \varphi(x, \tau) dx \\ & - \int_{\mathbb{R}} |\rho(x, \tau') - \kappa| \varphi(x, \tau') dx + 2 \int_\tau^{\tau'} \mathcal{R}_{\dot{y}(t)}(\kappa, q(t)) \varphi(0, t) dt \geq 0, \end{aligned} \quad (1.4)$$

where

$$\mathcal{R}_{\dot{y}(t)}(\kappa, q(t)) = F(\dot{y}(t), \kappa) - \min \{F(\dot{y}(t), \kappa), q(t)\} \text{ and } q(t) = Q(\dot{y}(t));$$

(iii) for all non-negative test function  $\psi \in C^\infty([0, T])$  and some given  $\varphi \in C_c^\infty(\mathbb{R})$  which verifies  $\varphi(0) = 1$ , the following weak constraint inequalities are verified for all  $0 \leq \tau < \tau' \leq T$ :

$$\begin{aligned} & - \int_\tau^{\tau'} \int_{\mathbb{R}^+} \rho \partial_t (\varphi \psi) + F(\dot{y}(t), \rho) \partial_x (\varphi \psi) dx dt - \int_{\mathbb{R}^+} \rho(x, \tau) \varphi(x) \psi(\tau) dx \\ & + \int_{\mathbb{R}^+} \rho(x, \tau') \varphi(x) \psi(\tau') dx \leq \int_\tau^{\tau'} q(t) \psi(t) dt; \end{aligned} \quad (1.5)$$

(iv) the following weak ODE formulation is verified for all  $t \in [0, T]$ :

$$y(t) = y_0 + \int_0^t \omega \left( \int_{\mathbb{R}} \rho(x, s) \mu(x) dx \right) ds. \quad (1.6)$$

**Definition 1.2.** We will call BV-regular solution any admissible weak solution  $(\rho, y)$  to Problem (1.1) which verifies

$$\rho \in L^\infty([0, T]; \text{BV}(\mathbb{R})).$$

*Remark 1.1.* It is more usual to formulate (1.4) with  $\tau = 0, \tau' = T$  and  $\varphi \in C_c^\infty(\mathbb{R} \times [0, T])$ . The equivalence between the two formulations is due to (1.3).

*Remark 1.2.* As it happens, the time-continuity regularity (1.3) is actually a consequence of inequalities (1.4). Indeed, we will use the result [11, Theorem 1.2] which states that if  $\Omega$  is an open subset of  $\mathbb{R}$  and if for all non-negative test function  $\varphi \in C_c^\infty(\Omega \times [0, T])$  and  $\kappa \in [0, R]$ ,  $\rho$  satisfies the following entropy inequalities:

$$\int_0^T \int_{\Omega} |\rho - \kappa| \partial_t \varphi + \Phi_{\dot{y}(t)}(\rho, \kappa) \partial_x \varphi dx dt + \int_{\Omega} |\rho_0(x) - \kappa| \varphi(x, 0) dx \geq 0,$$

then

$$\rho \in C([0, T]; L^1_{\text{loc}}(\Omega)).$$

Moreover, since  $\rho$  is bounded and  $\bar{\Omega} \setminus \Omega$  has a Lebesgue measure 0,  $\rho \in C([0, T]; L^1_{\text{loc}}(\bar{\Omega}))$ . We will use this remark several times in the sequel of the paper, with  $\Omega = \mathbb{R}^*$ .

*Remark 1.3.* Any admissible weak solution  $(\rho, y)$  to Problem (1.1) is also a distributional solution to the conservation law in (1.1). Therefore, inequalities (1.5) imply the following ones for all  $0 \leq \tau < \tau' \leq T$ :

$$\begin{aligned} & \int_\tau^{\tau'} \int_{\mathbb{R}^-} \rho \partial_t (\varphi \psi) + F(\dot{y}(t), \rho) \partial_x (\varphi \psi) dx dt + \int_{\mathbb{R}^-} \rho(x, \tau) \varphi(x) \psi(\tau) dx \\ & - \int_{\mathbb{R}^-} \rho(x, \tau') \varphi(x) \psi(\tau') dx \leq \int_\tau^{\tau'} q(t) \psi(t) dt, \end{aligned}$$

where  $\varphi$  and  $\psi$  are such as described in Definition 1.1 (iii).

The interest of weak formulations (1.5)-(1.6) for the flux constraint and for the ODE governing the slow vehicle lies in their stability with respect to  $\rho$ . Formulation (1.4)-(1.5)-(1.6) is well suited for passage to the limit of a.e. convergent sequences of exact or approximate solutions.

### 1.3 Uniqueness of the BV-regular solution

In this section, we prove stability with respect to the initial data and uniqueness for BV-regular solutions to Problem (1.1). We start with the

**Lemma 1.3.** *Fix  $(\rho, y)$  an admissible weak solution to Problem (1.1). Then  $y \in W^{1,\infty}((0, T))$ . In particular,  $y$  has bounded variation on  $[0, T]$ .*

*Proof.* Note for all  $t \in [0, T]$ ,

$$s(t) = \omega \left( \int_{\mathbb{R}} \rho(x, t) \mu(x) dx \right).$$

Since  $\mu \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$  and  $\rho \in C([0, T]; L^1_{\text{loc}}(\mathbb{R}))$ ,  $s \in C([0, T]; \mathbb{R})$ . By definition,  $y$  satisfies the weak ODE formulation (1.6). Consequently for a.e.  $t \in (0, T)$ ,  $\dot{y}(t) = s(t)$ . We are going to prove that  $s$  is Lipschitz continuous on  $[0, T]$ , which will ensure that  $y \in W^{1,\infty}((0, T))$ .

Since  $\mu \in \text{BV}(\mathbb{R})$ , there exists a sequence of functions  $(\mu_n)_{n \in \mathbb{N}}$  such that:

$$\begin{cases} \forall n \in \mathbb{N}, \mu_n \in \text{BV}(\mathbb{R}) \cap C^\infty(\mathbb{R}) \\ \mu_n \xrightarrow{n \rightarrow \infty} \mu \text{ in } L^1(\mathbb{R}) \\ \text{TV}(\mu_n) \xrightarrow{n \rightarrow \infty} \text{TV}(\mu). \end{cases}$$

Introduce for all  $n \in \mathbb{N}$  and  $t \in [0, T]$ , the function

$$\xi_n(t) = \int_{\mathbb{R}} \rho(x, t) \mu_n(x) dx.$$

Fix  $\psi \in C_c^\infty((0, T))$ . Since  $\rho$  is a distributional solution to the conservation law in (1.1), we have for all  $n \in \mathbb{N}$ ,

$$\begin{aligned} \int_0^T \xi_n(t) \dot{\psi}(t) dt &= \int_0^T \int_{\mathbb{R}} \rho \partial_t (\psi \mu_n) dx dt \\ &= - \int_0^T \int_{\mathbb{R}} F(\dot{y}(t), \rho) \partial_x (\psi \mu_n) dx dt \\ &= - \int_0^T \left( \int_{\mathbb{R}} F(\dot{y}(t), \rho) \mu_n'(x) dx \right) \psi(t) dt, \end{aligned}$$

which means that for all  $n \in \mathbb{N}$ ,  $\xi_n$  is differentiable in the weak sense, and that for a.e.  $t \in (0, T)$ ,

$$\dot{\xi}_n(t) = \int_{\mathbb{R}} F(\dot{y}(t), \rho) \mu_n'(x) dx.$$

In particular, since both the sequences  $(\|\mu_n\|_{L^1})_n$  and  $(\text{TV}(\mu_n))_n$  are bounded – say by  $C > 0$  – we also have for all  $n \in \mathbb{N}$ ,

$$\|\xi_n\|_{L^\infty} \leq RC, \quad \|\dot{\xi}_n\|_{L^\infty} \leq C(\|f\|_{L^\infty} + \|\omega\|_{L^\infty} R).$$

Therefore, the sequence  $(\xi_n)_n$  is bounded in  $W^{1,\infty}((0, T))$ . Now, for all  $t, \tau \in [0, T]$  and  $n \in \mathbb{N}$ ,

$$\begin{aligned} |s(t) - s(\tau)| &= \left| \omega \left( \int_{\mathbb{R}} \rho(x, t) \mu(x) dx \right) - \omega \left( \int_{\mathbb{R}} \rho(x, t) \mu_n(x) dx \right) + \omega \left( \int_{\mathbb{R}} \rho(x, t) \mu_n(x) dx \right) - \omega \left( \int_{\mathbb{R}} \rho(x, \tau) \mu_n(x) dx \right) \right. \\ &\quad \left. + \omega \left( \int_{\mathbb{R}} \rho(x, \tau) \mu_n(x) dx \right) - \omega \left( \int_{\mathbb{R}} \rho(x, \tau) \mu(x) dx \right) \right| \\ &\leq 2\|\omega'\|_{L^\infty} R \|\mu_n - \mu\|_{L^1} + \|\omega'\|_{L^\infty} \left| \int_{\mathbb{R}} \rho(x, t) \mu_n(x) dx - \int_{\mathbb{R}} \rho(x, \tau) \mu_n(x) dx \right| \\ &= 2\|\omega'\|_{L^\infty} R \|\mu_n - \mu\|_{L^1} + \|\omega'\|_{L^\infty} |\xi_n(t) - \xi_n(\tau)| \\ &\leq 2\|\omega'\|_{L^\infty} R \|\mu_n - \mu\|_{L^1} + \underbrace{C\|\omega'\|_{L^\infty} (\|f\|_{L^\infty} + \|\omega\|_{L^\infty} R)}_K |t - \tau|. \end{aligned}$$

Letting  $n \rightarrow \infty$ , we get that for all  $t, \tau \in [0, T]$ ,

$$|s(t) - s(\tau)| \leq K|t - \tau|.$$

This proves that  $s$  is Lipschitz continuous on  $[0, T]$ . This concludes the proof of the statement.  $\square$

Before stating the uniqueness result, we make the following additional assumption:

$$\forall s \in [0, \|\omega\|_{L^\infty}], \operatorname{argmax}_{\rho \in [0, R]} F(s, \rho) > 0. \quad (1.7)$$

This ensures that for all  $s \in [0, \|\omega\|_{L^\infty}]$ , the function  $F(s, \cdot)$  verifies the bell-shaped assumptions (A.2). For example, when considering the flux  $f(\rho) = \rho(R - \rho)$ , (1.7) reduces to  $\|\omega\|_{L^\infty} < R$ , which only means that the maximum velocity of the slow vehicle is lesser than the maximum velocity of the cars.

We have the following result.

**Theorem 1.4.** *Suppose that  $f$  satisfies (1.2)-(1.7). Fix  $\rho_0^1, \rho_0^2 \in L^1(\mathbb{R}; [0, R]) \cap \text{BV}(\mathbb{R})$  and  $y_0^1, y_0^2 \in \mathbb{R}$ . We note  $(\rho^1, y^1)$  a BV-regular solution to Problem (1.1) corresponding to initial data  $(\rho_0^1, y_0^1)$ , and  $(\rho^2, y^2)$  an admissible weak solution with initial data  $(\rho_0^2, y_0^2)$ , respectively. Then there exist constants  $\alpha, \beta, \gamma > 0$  such that*

$$\text{for a.e. } t \in (0, T), \|\rho^1(t) - \rho^2(t)\|_{L^1} \leq (|y_0^1 - y_0^2| \text{TV}(\rho_0^1) + \|\rho_0^1 - \rho_0^2\|_{L^1}) e^{\alpha t} \quad (1.8)$$

and

$$\text{for every } t \in [0, T], |y^1(t) - y^2(t)| \leq |y_0^1 - y_0^2| + (\beta|y_0^1 - y_0^2| + \gamma\|\rho_0^1 - \rho_0^2\|_{L^1})(e^{\alpha t} - 1). \quad (1.9)$$

In particular, Problem (1.1) admits at most one BV-regular solution.

*Proof.* Since  $(\rho^1, y^1)$  is a BV-regular solution to Problem (1.1), there exists  $C \geq 0$  such that

$$\forall t \in [0, T], \text{TV}(\rho^1(t)) \leq C.$$

Lemma 1.3 ensures that  $\dot{y}^1, \dot{y}^2 \in \text{BV}([0, T]; \mathbb{R}^+)$ . We can use result (A.5) to obtain that for a.e.  $t \in (0, T)$ ,

$$\|\rho^1(t) - \rho^2(t)\|_{L^1} \leq |y_0^1 - y_0^2| \text{TV}(\rho_0^1) + \|\rho_0^1 - \rho_0^2\|_{L^1} + (2\|Q'\|_{L^\infty} + 2R + C) \int_0^t |\dot{y}^1(s) - \dot{y}^2(s)| ds. \quad (1.10)$$

Moreover, since for a.e.  $t \in (0, T)$ ,

$$|\dot{y}^1(t) - \dot{y}^2(t)| \leq \|\omega'\|_{L^\infty} \|\mu\|_{L^\infty} \|\rho^1(t) - \rho^2(t)\|_{L^1},$$

Gronwall's lemma yields (1.8) with  $\alpha = (2\|Q'\|_{L^\infty} + 2R + C) \|\omega'\|_{L^\infty} \|\mu\|_{L^\infty}$ . Then for every  $t \in [0, T]$ ,

$$\begin{aligned} |y^1(t) - y^2(t)| &\leq |y_0^1 - y_0^2| + \int_0^t |\dot{y}^1(s) - \dot{y}^2(s)| ds \\ &\leq |y_0^1 - y_0^2| + \|\omega'\|_{L^\infty} \|\mu\|_{L^\infty} \int_0^t \|\rho^1(s) - \rho^2(s)\|_{L^1} ds \\ &\leq |y_0^1 - y_0^2| + (\beta|y_0^1 - y_0^2| + \gamma\|\rho_0^1 - \rho_0^2\|_{L^1})(e^{\alpha t} - 1), \end{aligned}$$

where

$$\beta = \frac{\text{TV}(\rho_0^1)}{2\|Q'\|_{L^\infty} + 2R + C}, \quad \gamma = \frac{1}{2\|Q'\|_{L^\infty} + 2R + C}.$$

The uniqueness of a BV-regular solution is then clear.  $\square$

*Remark 1.4.* Up to inequality (1.10), our proof was very much following the one of [21, Theorem 2.1]. However, the authors of [21] faced an issue to derive a Lipschitz stability estimate between the car densities and the slow vehicle velocities starting from

$$|\omega(\rho^1(0+, t)) - \omega(\rho^2(0+, t))|.$$

For us, due to the non-locality of our problem, it was straightforward to obtain the bound

$$\left| \omega \left( \int_{\mathbb{R}} \rho^1(x, t) \mu(x) dx \right) - \omega \left( \int_{\mathbb{R}} \rho^2(x, t) \mu(x) dx \right) \right| \leq \|\omega'\|_{L^\infty} \|\mu\|_{L^\infty} \|\rho^1(t) - \rho^2(t)\|_{L^1}.$$

*Remark 1.5.* A noteworthy consequence of Theorem 1.4 is that the existence of a BV-regular solution will ensure the uniqueness of an admissible weak one.

## 2 Two existence results

### 2.1 Time-splitting technique

In [20], to prove existence for their problem, the authors took a wave-front tracking approach. We choose here to use a time-splitting technique. The main advantage of this technique is that it relies on a ready-to-use theory. More precisely, at each time step, we will deal with exact solutions to a problem – already solved – we are familiar with and which is relatively close to the one we want to solve.

Fix  $\rho_0 \in L^1(\mathbb{R}; [0, R])$  and  $y_0 \in \mathbb{R}$ . Let  $\delta > 0$  be a time step,  $N \in \mathbb{N}$  such that  $T \in [N\delta, (N+1)\delta)$  and note for all  $n \in \{0, \dots, N+1\}$ ,  $t^n = n\delta$ . We initialize with

$$\begin{aligned} \forall t \in \mathbb{R}, \quad \rho^0(t) &= \rho_0(\cdot + y_0). \\ \forall t \in [0, T], \quad y^0(t) &= y_0. \end{aligned}$$

Fix  $n \in \{1, \dots, N+1\}$ . First, we define for all  $t \in (t^{n-1}, t^n]$ ,

$$\sigma^n(t) = \omega \left( \int_{\mathbb{R}} \rho^{n-1}(x, t - \delta) \mu(x) dx \right), \quad s^n = \sigma^n(t^n) \text{ and } q^n = Q(s^n).$$

Using [6, Theorem 2.11] and taking into account Remark 1.1, we can define  $\rho^n \in L^\infty(\mathbb{R} \times [t^{n-1}, t^n])$  as the unique admissible weak solution to

$$\begin{cases} \partial_t \rho + \partial_x F(s^n, \rho) = 0 & \mathbb{R} \times (t^{n-1}, t^n) \\ \rho(x, t^{n-1}) = \rho^{n-1}(x, t^{n-1}) & x \in \mathbb{R} \\ F(s^n, \rho)|_{x=0} \leq q^n & t \in (t^{n-1}, t^n), \end{cases}$$

which means that  $\rho^n \in C([t^{n-1}, t^n]; L^1_{\text{loc}}(\mathbb{R}))$  and satisfies entropy/constraint inequalities analogous to (1.4)-(1.5) with suitable flux/constraint function and initial data, see Definition A.1. We then define the following functions:

$$\begin{aligned} \bullet \quad \rho_\delta(t) &= \rho^0 \mathbf{1}_{\mathbb{R}^-}(t) + \sum_{n=1}^{N+1} \rho^n(t) \mathbf{1}_{(t^{n-1}, t^n]}(t) \\ \bullet \quad \sigma_\delta(t), q_\delta(t), s_\delta(t) &= \sigma^n(t), q^n, s^n \text{ if } t \in (t^{n-1}, t^n] \\ \bullet \quad y_\delta(t) &= y_0 + \int_0^t \sigma_\delta(u) du \end{aligned}$$

First, let us prove that  $(\rho_\delta, y_\delta)$  solves an approximate version of Problem (1.1).

**Proposition 2.1.** *The couple  $(\rho_\delta, y_\delta)$  is an admissible weak solution to the problem*

$$\begin{cases} \partial_t \rho_\delta + \partial_x F(s_\delta(t), \rho_\delta) = 0 & \mathbb{R} \times (0, T) \\ \rho_\delta(x, 0) = \rho_0(x + y_0) & x \in \mathbb{R} \\ F(s_\delta(t), \rho_\delta)|_{x=0} \leq q_\delta(t) & t \in (0, T) \\ \dot{y}_\delta(t) = \omega \left( \int_{\mathbb{R}} \rho_\delta(x, t - \delta) \mu(x) dx \right) \\ y_\delta(0) = y_0. \end{cases} \quad (2.1)$$

*Proof.* (i) By construction, for all  $n \in \{1, \dots, N+1\}$ ,  $\rho^n \in C([t^{n-1}, t^n]; L^1_{\text{loc}}(\mathbb{R}))$ . Combining this with the “stop-and-restart” conditions  $\rho^n(\cdot, t^{n-1}) = \rho^{n-1}(\cdot, t^{n-1})$ , one ensures that  $\rho_\delta \in C([0, T]; L^1_{\text{loc}}(\mathbb{R}))$ .

(iv) Fix  $t \in [0, T]$  and  $n \in \{1, \dots, N+1\}$  such that  $t \in [t^{n-1}, t^n)$ . Then,

$$\begin{aligned} y_\delta(t) &= y_0 + \int_0^t \sigma_\delta(u) du = y_0 + \sum_{k=1}^n \int_{t^{k-1}}^{t^k} \sigma^k(u) du + \int_{t^n}^t \sigma^{n+1}(u) du \\ &= y_0 + \sum_{k=1}^n \int_{t^{k-1}}^{t^k} \omega \left( \int_{\mathbb{R}} \underbrace{\rho^{k-1}(x, u-\delta)}_{\rho_\delta(x, u-\delta)} \mu(x) dx \right) du + \int_{t^n}^t \omega \left( \int_{\mathbb{R}} \underbrace{\rho^n(x, u-\delta)}_{\rho_\delta(x, u-\delta)} \mu(x) dx \right) du \\ &= y_0 + \int_0^t \omega \left( \int_{\mathbb{R}} \rho_\delta(x, u-\delta) \mu(x) dx \right) du. \end{aligned} \quad (2.2)$$

(ii)-(iii) Fix  $\varphi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^+; \mathbb{R}^+)$  and  $\kappa \in [0, R]$ . By construction of the sequence  $((\rho^k, y^k))_k$ , we have for all  $n, m \in \{0, \dots, N+1\}$ ,

$$\begin{aligned} &\int_{t^n}^{t^m} \int_{\mathbb{R}} |\rho_\delta - \kappa| \partial_t \varphi + \Phi_{s_\delta(t)}(\rho_\delta, \kappa) \partial_x \varphi dx dt = \sum_{k=n+1}^m \int_{t^{k-1}}^{t^k} \int_{\mathbb{R}} |\rho^k - \kappa| \partial_t \varphi + \Phi_{s^k}(\rho^k, \kappa) \partial_x \varphi dx dt \\ &\geq \sum_{k=n+1}^m \left\{ \int_{\mathbb{R}} |\rho^k(x, t^k) - \kappa| \varphi(x, t^k) dx - \int_{\mathbb{R}} \underbrace{|\rho^k(x, t^{k-1}) - \kappa|}_{\rho^{k-1}(x, t^{k-1})} \varphi(x, t^{k-1}) dx - 2 \int_{t^{k-1}}^{t^k} \mathcal{R}_{s^k}(\kappa, q^k) \varphi(0, t) dt \right\} \\ &= \int_{\mathbb{R}} |\rho_\delta(x, t^m) - \kappa| \varphi(x, t^m) dx - \int_{\mathbb{R}} |\rho_\delta(x, t^n) - \kappa| \varphi(x, t^n) dx - 2 \int_{t^n}^{t^m} \mathcal{R}_{s_\delta(t)}(\kappa, q_\delta(t)) \varphi(0, t) dt. \end{aligned}$$

It is then straightforward to prove that for all  $0 \leq \tau < \tau' \leq T$ ,

$$\begin{aligned} &\int_\tau^{\tau'} \int_{\mathbb{R}} |\rho_\delta - \kappa| \partial_t \varphi + \Phi_{s_\delta(t)}(\rho_\delta, \kappa) \partial_x \varphi dx dt + \int_{\mathbb{R}} |\rho_\delta(x, \tau) - \kappa| \varphi(x, \tau) dx \\ &\quad - \int_{\mathbb{R}} |\rho_\delta(x, \tau') - \kappa| \varphi(x, \tau') dx + 2 \int_\tau^{\tau'} \mathcal{R}_{s_\delta(t)}(\kappa, q_\delta(t)) \varphi(0, t) dt \geq 0. \end{aligned} \quad (2.3)$$

The proof of (iii) is very similar so we omit the details. One simply has to start from

$$- \int_\tau^{\tau'} \int_{\mathbb{R}^+} \rho_\delta \partial_t(\varphi \psi) + F(s_\delta(t), \rho_\delta) \partial_x(\varphi \psi) dx dt$$

and make use once again of the construction of the sequence  $((\rho^k, y^k))_k$  to obtain

$$\begin{aligned} &- \int_\tau^{\tau'} \int_{\mathbb{R}^+} \rho_\delta \partial_t(\varphi \psi) + F(s_\delta(t), \rho_\delta) \partial_x(\varphi \psi) dx dt - \int_{\mathbb{R}^+} \rho_\delta(x, \tau) \varphi(x) \psi(\tau) dx \\ &\quad + \int_{\mathbb{R}^+} \rho_\delta(x, \tau') \varphi(x) \psi(\tau') dx \leq \int_\tau^{\tau'} q_\delta(t) \psi(t) dt. \end{aligned} \quad (2.4)$$

This concludes the proof. □

*Remark 2.1.* Note that we have for all  $\delta > 0$ ,

$$\|\sigma_\delta\|_{L^\infty} \leq \|\omega\|_{L^\infty} \text{ and } \|y_\delta\|_{L^\infty} \leq |y_0| + T\|\omega\|_{L^\infty}.$$

This means that the sequence  $(y_\delta)_\delta$  is bounded in  $W^{1,\infty}((0, T))$ . Using the compact embedding of  $W^{1,\infty}((0, T))$  in  $C([0, T])$ , we get the existence of  $y \in C([0, T])$  such that up to the extraction of subsequence,  $(y_\delta)_\delta$  converges uniformly to  $y$  on  $[0, T]$ .

At this point, we propose two ways to obtain compactness for the sequence  $(\rho_\delta)_\delta$ , which will lead to two existence results.



## 2.2 The case of a non-degenerately nonlinear flux

**Theorem 2.2.** Fix  $\rho_0 \in L^1(\mathbb{R}; [0; R])$  and  $y_0 \in \mathbb{R}$ . Suppose that  $f$  is Lipschitz continuous and satisfies (1.2)-(1.7). Suppose also that  $f$  verifies the following non-degeneracy assumption

$$\text{for a.e. } s \in (0, \|\omega\|_{L^\infty}), \text{ mes}\{\rho \in [0, R] \mid f'(\rho) - s = 0\} = 0. \quad (2.5)$$

Then Problem (1.1) admits at least one admissible weak solution.

*Proof.* Condition (2.5) combined with the obvious uniform  $L^\infty$  bound

$$\forall \delta > 0, \forall (x, t) \in \mathbb{R} \times [0, T], \rho_\delta(x, t) \in [0, R],$$

and the results proved by Panov in [28, 29] ensure the existence of a subsequence – which we do not relabel – that converges in  $L^1_{\text{loc}}(\mathbb{R}^* \times [0, T])$  to some  $\rho \in L^1_{\text{loc}}(\mathbb{R} \times [0, T])$ ; and a further extraction yields the almost everywhere convergence on  $\mathbb{R} \times [0, T]$  and also the fact that  $\rho \in L^\infty(\mathbb{R} \times [0, T]; [0, R])$ .

We show that the couple  $(\rho, y)$  constructed above is an admissible weak solution to the problem.

(iv) For all  $\delta > 0$  and  $t \in [0, T]$ ,

$$\begin{aligned} y_\delta(t) &= y_0 + \int_0^t \omega \left( \int_{\mathbb{R}} \rho_\delta(x, u - \delta) \mu(x) dx \right) du \\ &= y_0 + \int_{-\delta}^{t-\delta} \omega \left( \int_{\mathbb{R}} \rho_\delta(x, u) \mu(x) dx \right) du \\ &= y_0 + \int_0^t \omega \left( \int_{\mathbb{R}} \rho_\delta(x, u) \mu(x) dx \right) du + \left( \int_{-\delta}^0 - \int_{t-\delta}^t \right) \omega \left( \int_{\mathbb{R}} \rho_\delta(x, u) \mu(x) dx \right) du. \end{aligned}$$

The last term vanishes as  $\delta \rightarrow 0$  since  $\omega$  is bounded. Then, Lebesgue theorem combined with the continuity of  $\omega$  gives for all  $t \in [0, T]$

$$y_\delta(t) \xrightarrow{\delta \rightarrow 0} y_0 + \int_0^t \omega \left( \int_{\mathbb{R}} \rho(x, u) \mu(x) dx \right) du.$$

This last quantity is also equal to  $y(t)$  due to the uniform convergence of  $(y_\delta)_\delta$  to  $y$ . This proves that  $y$  verifies the weak ODE formulation (1.6).

Now, we aim at passing to the limit in (2.3)-(2.4). With this in mind, we prove the a.e. convergence of the sequence  $(\sigma_\delta)_\delta$  towards  $\dot{y}$ . Since  $\mu \in \text{BV}(\mathbb{R})$ , there exists a sequence of functions  $(\mu_n)_{n \in \mathbb{N}}$  such that:

$$\begin{cases} \forall n \in \mathbb{N}, \mu_n \in \text{BV}(\mathbb{R}) \cap C^\infty(\mathbb{R}) \\ \mu_n \xrightarrow{n \rightarrow \infty} \mu \text{ in } L^1(\mathbb{R}) \\ \text{TV}(\mu_n) \xrightarrow{n \rightarrow \infty} \text{TV}(\mu). \end{cases}$$

Introduce for every  $\delta > 0$  and  $n \in \mathbb{N}$ , the function

$$\xi_\delta^n(t) = \int_{\mathbb{R}} \rho_\delta(x, t) \mu_n(x) dx.$$

Since for all  $\delta > 0$ ,  $\rho_\delta$  is a distributional solution to the conservation law in (2.1), one can show – following the proof of Lemma 1.3 for instance – that for every  $n \in \mathbb{N}$ ,  $\xi_\delta^n \in W^{1,\infty}((0, T))$ , and that for a.e.  $t \in (0, T)$ ,

$$\dot{\xi}_\delta^n(t) = \int_{\mathbb{R}} F(s_\delta(t), \rho_\delta) \mu_n'(x) dx.$$

Moreover, since both the sequences  $(\|\mu_n\|_{L^1})_n$  and  $(\text{TV}(\mu_n))_n$  are bounded, it is clear that  $(\xi_\delta^n)_{\delta,n}$  is uniformly bounded in  $W^{1,\infty}((0, T))$ , therefore so is  $(\omega(\xi_\delta^n))_{\delta,n}$ . Consequently, for all  $n \in \mathbb{N}$ ,  $\delta > 0$  and almost every

$t \in (0, T)$ ,

$$\begin{aligned}
 & \left| \sigma_\delta(t) - \omega \left( \int_{\mathbb{R}} \rho(x, t) \mu(x) dx \right) \right| = \left| \omega \left( \int_{\mathbb{R}} \rho_\delta(x, t - \delta) \mu(x) dx \right) - \omega \left( \int_{\mathbb{R}} \rho_\delta(x, t - \delta) \mu_n(x) dx \right) \right. \\
 & + \omega \left( \int_{\mathbb{R}} \rho_\delta(x, t - \delta) \mu_n(x) dx \right) - \omega \left( \int_{\mathbb{R}} \rho_\delta(x, t) \mu_n(x) dx \right) + \omega \left( \int_{\mathbb{R}} \rho_\delta(x, t) \mu_n(x) dx \right) - \omega \left( \int_{\mathbb{R}} \rho_\delta(x, t) \mu(x) dx \right) \\
 & \left. + \omega \left( \int_{\mathbb{R}} \rho_\delta(x, t) \mu(x) dx \right) - \omega \left( \int_{\mathbb{R}} \rho(x, t) \mu(x) dx \right) \right| \\
 & \leq 2 \|\omega'\|_{L^\infty} R \|\mu_n - \mu\|_{L^1} + \delta \sup_{n \in \mathbb{N}} \|\omega(\xi_\delta^n)\|_{W^{1,\infty}} + \|\omega'\|_{L^\infty} \left| \int_{\mathbb{R}} \rho_\delta(x, t) \mu(x) dx - \int_{\mathbb{R}} \rho(x, t) \mu(x) dx \right| \xrightarrow[\delta \rightarrow 0]{n \rightarrow \infty} 0,
 \end{aligned}$$

which proves that  $(\sigma_\delta)_\delta$  converges a.e. on  $(0, T)$  to  $\dot{y}$ .

(i) To prove the time-continuity regularity, we first apply inequality (2.3) with  $\tau = 0$ ,  $\tau' = T$  (which is licit since  $\rho_\delta$  is continuous in time),  $\varphi \in C_c^\infty(\mathbb{R}^* \times [0, T]; \mathbb{R}^+)$  and  $\kappa \in [0, R]$ :

$$\int_0^T \int_{\mathbb{R}} |\rho_\delta - \kappa| \partial_t \varphi + \Phi_{\sigma_\delta(t)}(\rho_\delta, \kappa) \partial_x \varphi dx dt + \int_{\mathbb{R}} |\rho_0(x + y_0) - \kappa| \varphi(x, 0) dx \geq 0.$$

Then, we let  $\delta \rightarrow 0$  to get

$$\int_0^T \int_{\mathbb{R}} |\rho - \kappa| \partial_t \varphi + \Phi_{\dot{y}(t)}(\rho, \kappa) \partial_x \varphi dx dt + \int_{\mathbb{R}} |\rho_0(x + y_0) - \kappa| \varphi(x, 0) dx \geq 0.$$

Consequently,  $\rho \in C([0, T]; L^1_{\text{loc}}(\mathbb{R}))$ , see Remark 1.2.

(ii)-(iii) The a.e. convergences of  $(\sigma_\delta)_\delta$  to  $\dot{y}$  and of  $(\rho_\delta)$  to  $\rho$  are enough to pass to the limit in (2.3). This ensures that for all non-negative test function  $\varphi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^+)$  and  $\kappa \in [0, R]$ , the following inequalities hold for a.e.  $0 \leq \tau < \tau' \leq T$ :

$$\begin{aligned}
 & \int_\tau^{\tau'} \int_{\mathbb{R}} |\rho - \kappa| \partial_t \varphi + \Phi_{\dot{y}(t)}(\rho, \kappa) \partial_x \varphi dx dt + \int_{\mathbb{R}} |\rho(x, \tau) - \kappa| \varphi(x, \tau) dx \\
 & - \int_{\mathbb{R}} |\rho(x, \tau') - \kappa| \varphi(x, \tau') dx + 2 \int_\tau^{\tau'} \mathcal{R}_{\dot{y}(t)}(\kappa, q(t)) \varphi(0, t) dt \geq 0.
 \end{aligned}$$

To conclude, observe that the expression in the left-hand side of the previous inequality is a continuous function of  $(\tau, \tau')$  which is almost everywhere greater than the continuous function 0. By continuity, this expression is everywhere greater than 0, which proves that  $\rho$  satisfies the entropy inequalities (1.4). Using similar arguments, we also show that  $\rho$  satisfies the constraint inequalities (1.5). This shows that the couple  $(\rho, y)$  is an admissible weak solution to Problem (1.1), and this concludes the proof.  $\square$

In this section, we prove an existence result for  $L^\infty$  initial data, but we have no guarantee of uniqueness since a priori we have no information regarding the BV regularity of such solutions.

Assumption (2.5) ensures the compactness for sequence of entropy solutions to conservation laws with flux function  $F$ . However, it prevents us from using flux functions with linear parts – which corresponds to constant traffic velocity for small densities – whereas such fundamental diagrams are often used in traffic modeling. The results of the next section will extend to this interesting case, under the extra BV assumption on the data.

### 2.3 Well-posedness for BV data

To obtain compactness for  $(\rho_\delta)_\delta$ , an alternative to the setting of Section 2.2 is to derive uniform BV bounds.

**Theorem 2.3.** Fix  $\rho_0 \in L^1(\mathbb{R}; [0, R]) \cap BV(\mathbb{R})$  and  $y_0 \in \mathbb{R}$ . Suppose that  $f$  satisfies (1.2)-(1.7). Suppose also that

$$\forall s \in [0, \|\omega\|_{L^\infty}], F(s, \cdot) \in C^1([0, R] \setminus \{\bar{\rho}_s\}),$$

where  $\bar{\rho}_s = \operatorname{argmax}_{\rho \in [0, R]} F(s, \rho)$ . Finally assume that  $Q$  satisfies the condition

$$\exists \varepsilon > 0, \forall s \in [0, \|\omega\|_{L^\infty}], Q(s) \leq \max_{\rho \in [0, R]} F(s, \rho) - \varepsilon. \quad (2.6)$$

Then Problem (1.1) admits a unique admissible weak solution, which is also BV-regular.

*Proof.* Fix  $\delta > 0$ . Recall that  $(\rho_\delta, y_\delta)$  is an admissible weak solution to

$$\begin{cases} \partial_t \rho_\delta + \partial_x F(s_\delta(t), \rho_\delta) = 0 & \mathbb{R} \times (0, T) \\ \rho_\delta(x, 0) = \rho_0(x + y_0) & x \in \mathbb{R} \\ F(s_\delta(t), \rho_\delta)|_{x=0} \leq q_\delta(t) & t \in (0, T) \\ \dot{y}_\delta(t) = \omega \left( \int_{\mathbb{R}} \rho_\delta(x, t - \delta) \mu(x) dx \right) \\ y_\delta(0) = y_0. \end{cases}$$

It is clear from the splitting construction that for a.e.  $t \in (0, T)$ ,

$$\sigma_\delta(t) = \omega \left( \int_{\mathbb{R}} \rho_\delta(x, t - \delta) \mu(x) dx \right).$$

Following the steps of the proof of Lemma 1.3, we can show that for all  $\delta > 0$ ,  $\sigma_\delta \in BV([0, T]; \mathbb{R}^+)$ . Even more than that, by doing so we show that the sequence  $(TV(\sigma_\delta))_\delta$  is bounded. Therefore, so is the sequence  $(TV(s_\delta))_\delta$ . Moreover, since  $Q$  verifies (2.6), all the hypotheses of Corollary A.7 are fulfilled. Consequently, there exists a constant  $C^\varepsilon = C^\varepsilon(\|\partial_s F\|_{L^\infty})$  such that for all  $t \in [0, T]$ ,

$$\begin{aligned} TV(\rho_\delta(t)) &\leq TV(\rho_0) + 4R + C^\varepsilon (TV(q_\delta) + TV(s_\delta)) \\ &\leq TV(\rho_0) + 4R + C^\varepsilon (1 + \|Q'\|_{L^\infty}) TV(s_\delta). \end{aligned} \quad (2.7)$$

Consequently for all  $t \in [0, T]$ , the sequence  $(\rho_\delta(t))_\delta$  is bounded in  $BV(\mathbb{R})$ . A classical analysis argument – see [24, Theorem A.8] – ensures the existence of  $\rho \in C([0, T]; L^1_{\text{loc}}(\mathbb{R}))$  such that

$$\forall t \in [0, T], \rho^\Delta(t) \xrightarrow{\delta \rightarrow 0} \rho(t) \text{ in } L^1_{\text{loc}}(\mathbb{R}).$$

With this convergence, we can follow the proof of Theorem 2.2 to show that  $(\rho, y)$  is an admissible weak solution to the problem. While by passing to the limit in (2.7), the lower semi-continuity of the BV semi-norm ensures that  $(\rho, y)$  is also BV-regular. By Remark 1.5, it ensures uniqueness and concludes the proof.  $\square$

## 2.4 Stability with respect to $\mu$

To end this section, we now study the stability of Problem (1.1) with respect to the weight function  $\mu$ . More precisely, let  $(\mu^\ell)_\ell \subset BV(\mathbb{R}; \mathbb{R}^+)$  be a sequence of weight functions that converges to  $\mu$  in the weak  $L^1$  sense:

$$\forall g \in L^\infty(\mathbb{R}), \int_{\mathbb{R}} g(x) \mu^\ell(x) dx \xrightarrow{\ell \rightarrow \infty} \int_{\mathbb{R}} g(x) \mu(x) dx. \quad (2.8)$$

Let  $(y_0^\ell)_\ell \subset \mathbb{R}$  be a sequence of real numbers that converges to some  $y_0$  and let  $(\rho_0^\ell)_\ell \subset L^1(\mathbb{R}; [0, R])$  be a sequence of initial data that converges to  $\rho_0$  in the strong  $L^1$  sense. We suppose that the flux function  $f$

satisfies Assumptions (1.2)-(1.7)-(2.5). Theorem 2.2 allows us to define for all  $\ell \in \mathbb{N}$ , the couple  $(\rho^\ell, y^\ell)$  as an admissible weak solution to the problem

$$\begin{cases} \partial_t \rho^\ell + \partial_x F(y^\ell(t), \rho^\ell) = 0 & \mathbb{R} \times (0, T) \\ \rho^\ell(x, 0) = \rho_0^\ell(x + y_0^\ell) & x \in \mathbb{R} \\ F(y^\ell(t), \rho^\ell)|_{x=0} \leq Q(y^\ell(t)) & t \in (0, T) \\ y^\ell(t) = \omega \left( \int_{\mathbb{R}} \rho^\ell(x, t) \mu^\ell(x) dx \right) \\ y^\ell(0) = y_0^\ell. \end{cases} \quad (2.9)$$

*Remark 2.2.* Using the same arguments as in Remark 2.1 and as in the proof of Theorem 2.2, we get that up to the extraction of a subsequence,  $(y^\ell)_\ell$  converges uniformly on  $[0, T]$  to some  $y \in C([0, T])$  and  $(\rho^\ell)_\ell$  converges a.e. on  $\mathbb{R} \times [0, T]$  to some  $\rho \in L^\infty(\mathbb{R} \times [0, T])$ .

We have the following result.

**Theorem 2.4.** *The couple  $(\rho, y)$  constructed above is an admissible weak solution to Problem (1.1).*

*Proof.* (iv) The sequence  $(\mu^\ell)_\ell$  converges in the weak  $L^1$  sense and is bounded in  $L^1(\mathbb{R})$ . Then by the Dunford-Pettis theorem, this sequence is equi-integrable:

$$\forall \varepsilon > 0, \exists \alpha > 0, \forall A \in \mathcal{B}(\mathbb{R}), \text{mes}(A) < \alpha \implies \forall \ell \in \mathbb{N}, \int_A \mu^\ell(x) dx \leq \varepsilon \quad (2.10)$$

and

$$\forall \varepsilon > 0, \exists X > 0, \forall \ell \in \mathbb{N}, \int_{|x| \geq X} \mu^\ell(x) dx \leq \varepsilon. \quad (2.11)$$

Fix  $t \in (0, T)$  and  $\varepsilon > 0$ . Fix  $\alpha, X > 0$  given by (2.10)-(2.11). Egoroff theorem yields the existence of a measurable subset  $E_t \subset [-X, X]$  such that

$$\text{mes}([-X, X] \setminus E_t) < \alpha \text{ and } \rho^\ell(\cdot, t) \longrightarrow \rho(\cdot, t) \text{ uniformly on } E_t. \quad (2.12)$$

For a sufficiently large  $\ell \in \mathbb{N}$ ,

$$\begin{aligned} & \left| \int_{\mathbb{R}} \rho^\ell(x, t) \mu^\ell(x) dx - \int_{\mathbb{R}} \rho(x, t) \mu(x) dx \right| \\ & \leq \int_{|x| \geq X} |\rho^\ell - \rho| \mu^\ell dx + \left| \int_{E_t} (\rho^\ell - \rho) \mu^\ell dx \right| + \left| \int_{[-X, X] \setminus E_t} (\rho^\ell - \rho) \mu^\ell dx \right| + \left| \int_{\mathbb{R}} \rho \mu^\ell dx - \int_{\mathbb{R}} \rho \mu dx \right| \\ & \leq R\varepsilon + \|\rho^\ell - \rho\|_{L^\infty(E_t)} \int_{E_t} \mu^\ell(x) dx + R \int_{[-X, X] \setminus E_t} \mu^\ell(x) dx + \varepsilon \\ & \leq 2(R+1)\varepsilon, \end{aligned}$$

which proves that for a.e.  $t \in (0, T)$ ,

$$\int_{\mathbb{R}} \rho^\ell(x, t) \mu^\ell(x) dx \xrightarrow{\ell \rightarrow \infty} \int_{\mathbb{R}} \rho(x, t) \mu(x) dx. \quad (2.13)$$

We get that  $y$  verifies the weak ODE formulation (1.6) by passing to the limit in

$$y^\ell(t) = y_0^\ell + \int_0^t \omega \left( \int_{\mathbb{R}} \rho^\ell(x, s) \mu^\ell(x) dx \right) ds.$$

(i) By definition, for all  $\ell \in \mathbb{N}$ , the couple  $(\rho^\ell, y^\ell)$  satisfies the analogue of entropy/constraint inequalities (1.4)-(1.5) with suitable flux/constraint functions. Applying these inequalities with  $\tau = 0$ ,  $\tau' = T$ ,  $\varphi \in C_c^\infty(\mathbb{R}^* \times [0, T]; \mathbb{R}^+)$  and  $\kappa \in [0, R]$ , we get

$$\int_0^T \int_{\mathbb{R}} |\rho^\ell - \kappa| \partial_t \varphi + \Phi_{y^\ell(t)}(\rho^\ell, \kappa) \partial_x \varphi dx dt + \int_{\mathbb{R}} |\rho_0^\ell(x + y_0^\ell) - \kappa| \varphi(x, 0) dx \geq 0.$$

The continuity of  $\omega$  and the convergence (2.13) ensure that  $(\dot{y}^\ell)_\ell$  converges a.e. to  $\dot{y}$ . This combined with the a.e. convergence of  $(\rho^\ell)_\ell$  to  $\rho$  and Riesz-Frechet-Kolmogorov theorem –  $(\rho_0^\ell)_\ell$  being strongly compact in  $L^1(\mathbb{R})$  – is enough to show that when letting  $\ell \rightarrow \infty$  in the inequality above, we get up to the extraction of a subsequence:

$$\int_0^T \int_{\mathbb{R}} |\rho - \kappa| \partial_t \varphi + \Phi_{\dot{y}(t)}(\rho, \kappa) \partial_x \varphi dx dt + \int_{\mathbb{R}} |\rho_0(x + y_0) - \kappa| \varphi(x, 0) dx \geq 0.$$

Thus  $\rho$  is a Kruřkov entropy solution away from the interface. Consequently  $\rho \in C([0, T]; L^1_{\text{loc}}(\mathbb{R}))$ , see Remark 1.2.

(ii)-(iii) The combined a.e. convergences of  $(\dot{y}^\ell)_\ell$  to  $\dot{y}$  and of  $(\rho^\ell)_\ell$  to  $\rho$  guarantee that for all  $0 \leq \tau < \tau' \leq T$ ,

$$\begin{aligned} \int_\tau^{\tau'} \int_{\mathbb{R}} |\rho^\ell - \kappa| \partial_t \varphi + \Phi_{\dot{y}^\ell(t)}(\rho^\ell, \kappa) \partial_x \varphi dx dt &\xrightarrow{\ell \rightarrow \infty} \int_\tau^{\tau'} \int_{\mathbb{R}} |\rho - \kappa| \partial_t \varphi + \Phi_{\dot{y}(t)}(\rho, \kappa) \partial_x \varphi dx dt, \\ \int_\tau^{\tau'} \int_{\mathbb{R}^+} \rho^\ell \partial_t(\varphi \psi) + F(\dot{y}^\ell(t), \rho^\ell) \partial_x(\varphi \psi) dx dt &\xrightarrow{\ell \rightarrow \infty} \int_\tau^{\tau'} \int_{\mathbb{R}^+} \rho \partial_t(\varphi \psi) + F(\dot{y}(t), \rho) \partial_x(\varphi \psi) dx dt \end{aligned}$$

and that for a.e.  $0 \leq \tau < \tau' \leq T$ ,

$$\int_\tau^{\tau'} \mathcal{R}_{\dot{y}^\ell(t)}(\kappa, Q(\dot{y}^\ell(t))) \varphi(0, t) dt, \int_\tau^{\tau'} Q(\dot{y}^\ell(t)) \psi(t) dt \xrightarrow{\ell \rightarrow \infty} \int_\tau^{\tau'} \mathcal{R}_{\dot{y}(t)}(\kappa, Q(\dot{y}(t))) \varphi(0, t) dt, \int_\tau^{\tau'} Q(\dot{y}(t)) \psi(t) dt.$$

Consequently, the couple  $(\rho, y)$  verifies inequalities (1.4)-(1.5) for almost every  $0 \leq \tau < \tau' \leq T$ . The same continuity argument we used in the proof Theorem 2.2 holds here to ensure that  $(\rho, y)$  actually satisfies the inequalities for all  $0 \leq \tau < \tau' \leq T$ . This concludes the proof of our stability claim.  $\square$

## 2.5 Discussion

The last section concludes the theoretical analysis of Problem (1.1). The non-locality in space of the constraint delivers an easy proof of stability with respect to the initial data in the BV framework. Although a proof of existence using a fixed point theorem was possible (*c.f.* [5]), we chose to propose a proof based on a time-splitting technique. The stability with respect to  $\mu$  is a noteworthy feature, which shows a certain sturdiness of the model. However, the case we had in mind – namely  $\mu \rightarrow \delta_{0+}$  – is not reachable with the assumptions we used to prove the stability, especially (2.8). We will explore this singular limit numerically, after having built a robust convergent numerical scheme for Problem (1.1). Let us also underline that unlike in [26, 27] where the authors required a particular form for the function  $\omega$  to prove well-posedness for their model. Regarding  $\omega$ , our result holds as long as  $\omega$  is Lipschitz continuous.

As evoked earlier, the non-locality in space of the constraint makes the mathematical study of the model easier. But in the modeling point of view, this choice also makes sense for several reasons. First of all, one can think that the velocity  $\dot{y}$  of the slow moving vehicle – unlike its acceleration – is a rather continuous value. Even the driver of the slow vehicle suddenly apply the brakes, the vehicle will not decelerate instantaneously. Note that the LWR model allows for discontinuous averaged velocity of the agents, however while modeling the slow vehicle we are concerned with an individual agent and can model its behavior more precisely. Moreover, considering the mean value of the traffic density in a vicinity ahead of the driver could be seen as taking into account both the driver anticipation and a psychological effect. For example, if the driver sees – several dozens of meters ahead of him/her – a speed reduction on traffic, he/she will start to slow down. This observation can be related to the fact that, compared to the fluid mechanics models where the typical number of agents is governed by the Avogadro constant, in traffic models the number of agents is at least  $10^{20}$  times less. Therefore, a mild non-locality (evaluation of the downstream traffic flow via averaging over a handful of preceding cars) is a reasonable assumption in the macroscopic traffic models inspired by fluid mechanics. This point of view is exploited in the model of [15]. Note that it is feasible to substitute the basic LWR equation on  $\rho$  by the non-local LWR introduced in [15] in our non-local model for the slow vehicle. Such mildly non-local model remains close to the basic local model of [20]. It can be studied combining the techniques of [15] and the ones we developed in this section.

### 3 Numerical approximation of the model

In this section, we aim at constructing a finite volume scheme and at proving its convergence toward the BV-regular solution to

$$\begin{cases} \partial_t \rho + \partial_x F(\dot{y}(t), \rho) = 0 & \mathbb{R} \times (0, T) \\ \rho(x, 0) = \rho_0(x + y_0) & x \in \mathbb{R} \\ F(\dot{y}(t), \rho)|_{x=0} \leq Q(\dot{y}(t)) & t \in (0, T) \\ \dot{y}(t) = \omega \left( \int_{\mathbb{R}} \rho(x, t) \mu(x) dx \right) \\ y(0) = y_0. \end{cases} \quad (3.1)$$

From now on, we note

$$a \vee b = \max(a, b) \text{ and } a \wedge b = \min(a, b).$$

Fix  $\rho_0 \in L^1(\mathbb{R}; [0, R])$  and  $y_0 \in \mathbb{R}$ .

#### 3.1 Finite volume scheme in the bus frame

For a fixed spatial mesh size  $\Delta x$  and time mesh size  $\Delta t$ , let  $x_j = j\Delta x$ ,  $t^n = n\Delta t$ . We define the grid cells  $\mathcal{K}_{j+\frac{1}{2}} = (x_j, x_{j+1})$  and  $N \in \mathbb{N}$  such that  $T \in [N\Delta t, (N+1)\Delta t)$ . We write

$$\mathbb{R} \times [0, T] \subset \bigcup_{n=0}^N \bigcup_{j \in \mathbb{Z}} \mathcal{P}_{j+\frac{1}{2}}^n, \quad \mathcal{P}_{j+\frac{1}{2}}^n = \mathcal{K}_{j+\frac{1}{2}} \times [t^n, t^{n+1}).$$

We choose to discretize the initial data  $\rho_0(\cdot + y_0)$  and the weight function  $\mu$  with  $(\rho_{j+\frac{1}{2}}^0)_{j \in \mathbb{Z}}$  and  $(\mu_{j+\frac{1}{2}})_{j \in \mathbb{Z}}$  where for all  $j \in \mathbb{Z}$ ,  $\rho_{j+\frac{1}{2}}^0$  and  $\mu_{j+\frac{1}{2}}$  are their mean values on the cell  $\mathcal{K}_{j+\frac{1}{2}}$ .

*Remark 3.1.* Others choice could be made, for instance in the case  $\rho_0 \in C(\mathbb{R})$  such that  $\lim_{|x| \rightarrow \infty} \rho_0(x)$  exists (in which case, the limit is zero due to the integrability assumption), the values  $\rho_{j+\frac{1}{2}}^0 = \rho_0(x_{j+\frac{1}{2}} + y_0)$  can be used. The only requirements are

- $\forall j \in \mathbb{Z}, \rho_{j+\frac{1}{2}}^0 \in [0, R]$
- $\rho_0^\Delta = \sum_{j \in \mathbb{Z}} \rho_{j+\frac{1}{2}}^0 \mathbf{1}_{\mathcal{K}_{j+\frac{1}{2}}} \xrightarrow{L^1_{\text{loc}}} \rho_0(\cdot + y_0)$  as  $\Delta x \rightarrow 0$ .

Fix  $n \in \{0, \dots, N\}$ . At each time step we first define an approximate velocity of the slow vehicle  $s^{n+1}$  and a constraint level  $q^{n+1}$ :

$$s^{n+1} = \omega \left( \sum_{j \in \mathbb{Z}} \rho_{j+\frac{1}{2}}^n \mu_{j+\frac{1}{2}} \Delta x \right), \quad q^{n+1} = Q(s^{n+1}). \quad (3.2)$$

With these values, we update the approximate traffic density with the marching formula

$$\forall j \in \mathbb{Z}, \rho_{j+\frac{1}{2}}^{n+1} = \rho_{j+\frac{1}{2}}^n - \frac{\Delta t}{\Delta x} \left( \mathcal{F}_{j+1}^{n+1}(\rho_{j+\frac{1}{2}}^n, \rho_{j+\frac{3}{2}}^n) - \mathcal{F}_j^{n+1}(\rho_{j-\frac{1}{2}}^n, \rho_{j+\frac{1}{2}}^n) \right), \quad (3.3)$$

where, following the recipe of [6, 14],

$$\mathcal{F}_j^{n+1}(a, b) = \begin{cases} \mathcal{F}^{n+1}(a, b) & \text{if } j \neq 0 \\ \min \{ \mathcal{F}^{n+1}(a, b), q^{n+1} \} & \text{if } j = 0, \end{cases} \quad (3.4)$$

$\mathcal{F}^{n+1}$  being a monotone consistent and Lipschitz numerical flux associated to  $F(s^{n+1}, \cdot)$ . We will also use the notation

$$\rho_{j+\frac{1}{2}}^{n+1} = H_j^{n+1}(\rho_{j-\frac{1}{2}}^n, \rho_{j+\frac{1}{2}}^n, \rho_{j+\frac{3}{2}}^n), \quad (3.5)$$

where  $H_j^{n+1}$  is given by the expression in the right-hand side of (3.3). We then define the functions

- $\rho^\Delta(x, t) = \rho_{j+\frac{1}{2}}^n$  if  $(x, t) \in \mathcal{P}_{j+\frac{1}{2}}^n$
- $s^\Delta(t) = s^{n+1}$  if  $t \in [t^n, t^{n+1})$
- $q^\Delta(t) = q^{n+1}$  if  $t \in [t^n, t^{n+1})$
- $y^\Delta(t) = y_0 + \int_0^t s^\Delta(u) du.$

Let  $\Delta = (\Delta x, \Delta t)$ . For our convergence analysis, we will assume that  $\Delta \rightarrow 0$ , with  $\lambda = \Delta t / \Delta x$  verifying the CFL condition

$$\lambda \underbrace{\sup_{s \in [0, \|\omega\|_{L^\infty}]} \left( \left\| \frac{\partial \mathcal{F}^s}{\partial x} \right\|_{L^\infty} + \left\| \frac{\partial \mathcal{F}^s}{\partial y} \right\|_{L^\infty} \right)}_{L} \leq 1, \quad (3.6)$$

where  $\mathcal{F}^s = \mathcal{F}^s(x, y)$  is the numerical flux – associated to  $F(s, \cdot)$  – we use in the scheme (3.3).

*Remark 3.2.* When considering the Rusanov flux or the Godunov one, (3.6) is guaranteed when

$$2\lambda(\|f'\|_{L^\infty} + \|\omega\|_{L^\infty}) \leq 1.$$

### 3.2 Stability and discrete entropy inequalities

**Proposition 3.1** ( $L^\infty$  stability). *The scheme (3.5) is*

- (i) *monotone: for all  $n \in \{0, \dots, N\}$  and  $j \in \mathbb{Z}$ ,  $H_j^{n+1}$  is non-decreasing with respect to its three arguments;*
- (ii) *stable:*

$$\forall n \in \{0, \dots, N+1\}, \forall j \in \mathbb{Z}, \rho_{j+\frac{1}{2}}^n \in [0, R]. \quad (3.7)$$

*Proof.* (i) In the classical case  $-j \notin \{-1, 0\}$  – we simply differentiate the Lipschitz function  $H_j^{n+1}$  and make use of both the CFL condition (3.6) and the monotonicity of  $\mathcal{F}^{n+1}$ . For  $j \in \{-1, 0\}$ , note that the authors of [6] pointed out (in Proposition 4.2) that the modification done in the numerical flux (3.4) does not change the monotonicity of the scheme.

(ii) The  $L^\infty$  stability is a consequence of the monotonicity and also of the fact that 0 and  $R$  are stationary solutions of the scheme *i.e.* for all  $n \in \{0, \dots, N\}$  and  $j \in \mathbb{Z}$ ,

$$H_j^{n+1}(0, 0, 0) = 0, \quad H_j^{n+1}(R, R, R) = R,$$

see [6, Proposition 4.2]. □

In order to show that the limit of  $(\rho^\Delta)_\Delta$  – under the a.e. convergence up to a subsequence – is a solution of the conservation law in (3.1), we derive discrete entropy inequalities. These inequalities also contain terms that will help to pass to the limit in the constrained formulation of the conservation law, as soon as the sequence  $(q^\Delta)_\Delta$  of constraints is proved convergent as well.

**Proposition 3.2** (Discrete entropy inequalities). *The numerical scheme (3.5) fulfills the following inequalities for all  $n \in \{0, \dots, N\}$ ,  $j \in \mathbb{Z}$  and  $\kappa \in [0, R]$ :*

$$\begin{aligned} & \left( |\rho_{j+\frac{1}{2}}^{n+1} - \kappa| - |\rho_{j+\frac{1}{2}}^n - \kappa| \right) \Delta x + (\Phi_{j+1}^n - \Phi_j^n) \Delta t \\ & \leq \mathcal{R}_{s^{n+1}}(\kappa, q^{n+1}) \Delta t \delta_{j \in \{-1, 0\}} + \left( \Phi_0^n - \bar{\Phi}_0^n \right) \Delta t (\delta_{j=-1} - \delta_{j=0}), \end{aligned} \quad (3.8)$$

where

$$\begin{aligned} \Phi_j^n &= \mathcal{F}^{n+1}(\rho_{j-\frac{1}{2}}^n \vee \kappa, \rho_{j+\frac{1}{2}}^n \vee \kappa) - \mathcal{F}^{n+1}(\rho_{j-\frac{1}{2}}^n \wedge \kappa, \rho_{j+\frac{1}{2}}^n \wedge \kappa), \\ \bar{\Phi}_0^n &= \min\{\mathcal{F}^{n+1}(\rho_{-\frac{1}{2}}^n \vee \kappa, \rho_{\frac{1}{2}}^n \vee \kappa), q^{n+1}\} - \min\{\mathcal{F}^{n+1}(\rho_{-\frac{1}{2}}^n \wedge \kappa, \rho_{\frac{1}{2}}^n \wedge \kappa), q^{n+1}\} \end{aligned}$$

denote the approximate numerical fluxes and

$$\mathcal{R}_{s^{n+1}}(\kappa, q^{n+1}) = F(s^{n+1}, \kappa) - \min\{F(s^{n+1}, \kappa), q^{n+1}\}.$$

*Proof.* This result is a direct consequence of the scheme monotonicity. When the constraint does not enter the calculations *i.e.*  $j \notin \{-1, 0\}$ , the proof follows [23, Lemma 5.4]. The key point is not only the monotonicity, but also the fact that in the classical case, all the constants  $\kappa \in [0, R]$  are stationary solutions of the scheme. This observation does not hold when the constraint enters the calculations. For example if  $j = -1$ ,

$$H_{-1}^{n+1}(\kappa, \kappa, \kappa) = \kappa + \lambda \mathcal{R}_{s^{n+1}}(\kappa, q^{n+1}).$$

Consequently, we have both

$$\rho_{-\frac{1}{2}}^{n+1} \vee \kappa \leq H_{-1}^{n+1}(\rho_{-\frac{3}{2}}^n \vee \kappa, \rho_{-\frac{1}{2}}^n \vee \kappa, \rho_{\frac{1}{2}}^n \vee \kappa)$$

and

$$\rho_{-\frac{1}{2}}^{n+1} \wedge \kappa \geq H_{-1}^{n+1}(\rho_{-\frac{3}{2}}^n \wedge \kappa, \rho_{-\frac{1}{2}}^n \wedge \kappa, \rho_{\frac{1}{2}}^n \wedge \kappa) - \lambda \mathcal{R}_{s^{n+1}}(\kappa, q^{n+1}).$$

By subtracting these last two inequalities, we get

$$\begin{aligned} |\rho_{-\frac{1}{2}}^{n+1} - \kappa| &= \rho_{-\frac{1}{2}}^{n+1} \vee \kappa - \rho_{-\frac{1}{2}}^{n+1} \wedge \kappa \\ &\leq H_{-1}^{n+1}(\rho_{-\frac{3}{2}}^n \vee \kappa, \rho_{-\frac{1}{2}}^n \vee \kappa, \rho_{\frac{1}{2}}^n \vee \kappa) - H_{-1}^{n+1}(\rho_{-\frac{3}{2}}^n \wedge \kappa, \rho_{-\frac{1}{2}}^n \wedge \kappa, \rho_{\frac{1}{2}}^n \wedge \kappa) + \lambda \mathcal{R}_{s^{n+1}}(\kappa, q^{n+1}) \\ &= |\rho_{-\frac{1}{2}}^n - \kappa| - \lambda \left( \min\{\mathcal{F}^{n+1}(\rho_{-\frac{1}{2}}^n \vee \kappa, \rho_{\frac{1}{2}}^n \vee \kappa), q^{n+1}\} - \mathcal{F}^{n+1}(\rho_{-\frac{1}{2}}^n \vee \kappa, \rho_{\frac{1}{2}}^n \vee \kappa) \right) \\ &\quad + \lambda \left( \min\{\mathcal{F}^{n+1}(\rho_{-\frac{1}{2}}^n \wedge \kappa, \rho_{\frac{1}{2}}^n \wedge \kappa), q^{n+1}\} - \mathcal{F}^{n+1}(\rho_{-\frac{1}{2}}^n \wedge \kappa, \rho_{\frac{1}{2}}^n \wedge \kappa) \right) + \lambda \mathcal{R}_{s^{n+1}}(\kappa, q^{n+1}) \\ &= |\rho_{-\frac{1}{2}}^n - \kappa| - \lambda (\Phi_0^n - \Phi_{-1}^n) + \lambda (\Phi_0^n - \bar{\Phi}_0^n) + \lambda \mathcal{R}_{s^{n+1}}(\kappa, q^{n+1}), \end{aligned}$$

which is exactly (3.8) in the case  $j = -1$ . The case  $j = 0$  is similar so we omit the details of the proof for this case.  $\square$

Starting from (3.3) and (3.8), we can obtain approximate versions of (1.4) and (1.5). We start with the approximate entropy inequalities.

**Proposition 3.3** (Approximate entropy inequalities). *Fix  $\varphi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^+; \mathbb{R}^+)$ ,  $\kappa \in [0, R]$ . Then there exists a constant  $C_1^\varphi = C_1^\varphi(R, T, L)$ , non-decreasing with respect to its arguments, such that the following inequalities hold for all  $0 \leq \tau < \tau' \leq T$ :*

$$\begin{aligned} &\int_\tau^{\tau'} \int_{\mathbb{R}} |\rho^\Delta - \kappa| \partial_t \varphi + \Phi^\Delta(\rho^\Delta, \kappa) \partial_x \varphi dx dt + \int_{\mathbb{R}} |\rho^\Delta(x, \tau) - \kappa| \varphi(x, \tau) dx \\ &- \int_{\mathbb{R}} |\rho^\Delta(x, \tau') - \kappa| \varphi(x, \tau') dx + 2 \int_\tau^{\tau'} \mathcal{R}_{s^\Delta(t)}(\kappa, q^\Delta(t)) \varphi(0, t) dt \geq -C_1^\varphi(\Delta t + \Delta x), \end{aligned} \quad (3.9)$$

where

$$\Phi^\Delta(\rho^\Delta, \kappa) = \sum_{n=0}^N \sum_{j \in \mathbb{Z}} \Phi_j^n \mathbb{1}_{\mathcal{P}_{j+\frac{1}{2}}^n}(x, t).$$

*Proof.* Define

- $\forall n \in \mathbb{N}, \forall j \in \mathbb{Z}, \varphi_{j+\frac{1}{2}}^n = \frac{1}{\Delta x \Delta t} \iint_{\mathcal{P}_{j+\frac{1}{2}}^n} \varphi(x, t) dx dt$
- $k, m \in \mathbb{N}$  such that  $\tau \in [t^k, t^{k+1})$  and  $\tau' \in [t^m, t^{m+1})$ .

Multiplying the discrete entropy inequalities (3.8) by  $\varphi_{j+\frac{1}{2}}^n$ , then summing on  $n \in \{k, \dots, m-1\}$  and  $j \in \mathbb{Z}$ , one obtains after reorganization of the sums (using in particular the Abel/“summation-by-parts” procedure)

$$A + B + C + D + E \geq 0, \quad (3.10)$$



with

$$\begin{aligned}
 A &= \sum_{n=k+1}^{m-1} \sum_{j \in \mathbb{Z}} |\rho_{j+\frac{1}{2}}^n - \kappa| \left( \frac{\varphi_{j+\frac{1}{2}}^n - \varphi_{j+\frac{1}{2}}^{n-1}}{\Delta t} \right) \Delta x \Delta t \\
 B &= \sum_{n=k}^{m-1} \sum_{j \in \mathbb{Z}} \Phi_j^n \left( \frac{\varphi_{j+\frac{1}{2}}^n - \varphi_{j-\frac{1}{2}}^n}{\Delta x} \right) \Delta x \Delta t \\
 C &= \sum_{j \in \mathbb{Z}} |\rho_{j+\frac{1}{2}}^k - \kappa| \varphi_{j+\frac{1}{2}}^k \Delta x - \sum_{j \in \mathbb{Z}} |\rho_{j+\frac{1}{2}}^m - \kappa| \varphi_{j+\frac{1}{2}}^{m-1} \Delta x \\
 D &= \sum_{n=k}^{m-1} \mathcal{R}_{s^{n+1}}(\kappa, q^{n+1}) \left( \varphi_{-\frac{1}{2}}^n + \varphi_{\frac{1}{2}}^n \right) \Delta t, \quad E = \sum_{n=k}^{m-1} \left( \Phi_0^n - \tilde{\Phi}_0^n \right) \left( \varphi_{-\frac{1}{2}}^n - \varphi_{\frac{1}{2}}^n \right) \Delta t.
 \end{aligned}$$

Inequality (3.9) follows from (3.10) with

$$\begin{aligned}
 C_1^\varphi &= R \left( T \max_{t \in [0, T]} \|\partial_{tt}^2 \varphi(\cdot, t)\|_{L^1} + 4 \max_{t \in [0, T]} \|\partial_t \varphi(\cdot, t)\|_{L^1} \right) \\
 &\quad + RL \left( T \max_{t \in [0, T]} \|\partial_{xx}^2 \varphi(\cdot, t)\|_{L^1} + 2 \max_{t \in [0, T]} \|\partial_x \varphi(\cdot, t)\|_{L^1} + 4 \|\varphi\|_{L^\infty} + 2T \|\partial_x \varphi\|_{L^\infty} \right),
 \end{aligned}$$

making use of the bounds:

$$\begin{aligned}
 \left| A - \int_\tau^{\tau'} \int_{\mathbb{R}} |\rho^\Delta - \kappa| \partial_t \varphi dx dt \right| &\leq R \left( T \max_{t \in [0, T]} \|\partial_{tt}^2 \varphi(\cdot, t)\|_{L^1} + 2 \max_{t \in [0, T]} \|\partial_t \varphi(\cdot, t)\|_{L^1} \right) \Delta t \\
 \left| B - \int_\tau^{\tau'} \int_{\mathbb{R}} \Phi^\Delta(\rho^\Delta, \kappa) \partial_x \varphi dx dt \right| &\leq RL \left( T \max_{t \in [0, T]} \|\partial_{xx}^2 \varphi(\cdot, t)\|_{L^1} \Delta x + 2 \max_{t \in [0, T]} \|\partial_x \varphi(\cdot, t)\|_{L^1} \Delta t \right) \\
 \left| C - \int_{\mathbb{R}} |\rho^\Delta(x, \tau) - \kappa| \varphi(x, \tau) dx + \int_{\mathbb{R}} |\rho^\Delta(x, \tau') - \kappa| \varphi(x, \tau') dx \right| &\leq 2R \max_{t \in [0, T]} \|\partial_t \varphi(\cdot, t)\|_{L^1} \Delta t, \\
 \left| D - 2 \int_\tau^{\tau'} \mathcal{R}_{s^\Delta(t)}(\kappa, q^\Delta(t)) \varphi(0, t) dt \right| &\leq RL (4 \|\varphi\|_{L^\infty} \Delta t + T \|\partial_x \varphi\|_{L^\infty} \Delta x) \quad \text{and} \quad |E| \leq 2RTL \|\partial_x \varphi\|_{L^\infty} \Delta x.
 \end{aligned}$$

This concludes the proof.  $\square$

**Proposition 3.4** (Approximate constraint inequalities). *Fix  $\psi \in C^\infty([0, T]; \mathbb{R}^+)$  and  $\varphi \in C_c^\infty(\mathbb{R})$  such that  $\varphi(0) = 1$ . Then there exists a constant  $C_2^{\varphi, \psi} = C_2^{\varphi, \psi}(R, T, L, \|Q\|_{L^\infty})$ , non-decreasing with respect to its arguments, such that the following inequalities hold for all  $0 \leq \tau < \tau' \leq T$ :*

$$\begin{aligned}
 & - \int_{\mathbb{R}^+} \rho^\Delta(x, \tau) \varphi(x) \psi(\tau) dx - \int_\tau^{\tau'} \int_{\mathbb{R}^+} \rho^\Delta \partial_t(\varphi \psi) + \mathcal{F}^\Delta(s^\Delta(t), \rho^\Delta) \partial_x(\varphi \psi) dx dt \\
 & + \int_{\mathbb{R}^+} \rho^\Delta(x, \tau') \varphi(x) \psi(\tau') dx \leq \int_\tau^{\tau'} q^\Delta(t) \psi(t) dt + C_2^{\varphi, \psi} (\Delta x + \Delta t),
 \end{aligned} \tag{3.11}$$

where

$$\mathcal{F}^\Delta(s^\Delta(t), \rho^\Delta) = \sum_{n=0}^N \sum_{j \in \mathbb{Z}} \mathcal{F}^{n+1}(\rho_{j-\frac{1}{2}}^n, \rho_{j+\frac{1}{2}}^n) \mathbf{1}_{\mathcal{P}_{j+\frac{1}{2}}^n}(x, t).$$

*Proof.* In this case, the constant  $C_2^{\varphi, \psi}$  reads

$$\begin{aligned}
 C_2^{\varphi, \psi} &= R \|\varphi\|_{L^1} (T \|\psi''\|_{L^\infty} + 4 \|\psi'\|_{L^\infty}) + \|Q\|_{L^\infty} \|\psi\|_{L^\infty} (2 + T \|\varphi'\|_{L^\infty}) \\
 &\quad + RL \|\psi\|_{L^\infty} (2 \|\varphi'\|_{L^1} + T \|\varphi'\|_{L^1} + T \|\varphi''\|_{L^1}).
 \end{aligned}$$

Indeed, following the proof of (3.9), define

- $\forall n \in \mathbb{N}, \forall j \in \mathbb{Z}, \psi^n = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \psi(t) dt, \varphi_{j+\frac{1}{2}} = \frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} \varphi(x) dx$
- $k, m \in \mathbb{N}$  such that  $\tau \in [t^k, t^{k+1})$  and  $\tau' \in [t^m, t^{m+1})$ .

Multiplying the scheme (3.3) by  $\varphi_{j+\frac{1}{2}} \psi^n$ , then summing on  $n \in \{k, \dots, m-1\}$  and  $j \geq 0$ , one obtains after reorganization of the sums (still using the Abel transformation)

$$A + B + C + D = 0, \quad (3.12)$$

with

$$\begin{aligned} A &= \sum_{n=k+1}^{m-1} \sum_{j \geq 0} \rho_{j+\frac{1}{2}}^n \varphi_{j+\frac{1}{2}}^n \left( \frac{\psi^n - \psi^{n-1}}{\Delta t} \right) \Delta x \Delta t \\ B &= \sum_{n=k}^{m-1} \sum_{j \geq 1} \mathcal{F}^{n+1}(\rho_{j-\frac{1}{2}}^n, \rho_{j+\frac{1}{2}}^n) \left( \frac{\varphi_{j+\frac{1}{2}} - \varphi_{j-\frac{1}{2}}}{\Delta x} \right) \psi^n \Delta x \Delta t \\ C &= \sum_{j \geq 0} \rho_{j+\frac{1}{2}}^k \varphi_{j+\frac{1}{2}} \psi^k \Delta x - \sum_{j \geq 0} \rho_{j+\frac{1}{2}}^m \varphi_{j+\frac{1}{2}} \psi^m \Delta x \\ D &= \sum_{n=k}^{m-1} \min \left\{ \mathcal{F}^{n+1}(\rho_{-\frac{1}{2}}^n, \rho_{\frac{1}{2}}^n), q^{n+1} \right\} \varphi_{\frac{1}{2}} \psi^n \Delta t. \end{aligned}$$

Inequality (3.11) follows from (3.12) making use of the following bounds:

$$\begin{aligned} \left| A - \int_{\tau}^{\tau'} \int_{\mathbb{R}^+} \rho^\Delta \partial_t(\varphi \psi) dx dt \right| &\leq R \|\varphi\|_{L^1} (T \|\psi''\|_{L^\infty} + 2 \|\psi'\|_{L^\infty}) \Delta t, \\ \left| B - \int_{\tau}^{\tau'} \int_{\mathbb{R}^+} \mathcal{F}^\Delta(s^\Delta(t), \rho^\Delta) \partial_x(\varphi \psi) dx dt \right| &\leq RL \|\psi\|_{L^\infty} (2 \|\varphi'\|_{L^1} \Delta t + T \|\varphi'\|_{L^1} \Delta x + T \|\varphi''\|_{L^1} \Delta x) \\ \left| C - \int_{\mathbb{R}^+} \rho^\Delta(x, \tau) \varphi(x) \psi(\tau) dx + \int_{\mathbb{R}^+} \rho^\Delta(x, \tau') \varphi(x) \psi(\tau') dx \right| &\leq 2R \|\varphi\|_{L^1} \|\psi'\|_{L^\infty} \Delta t \\ \left| D - \int_{\tau}^{\tau'} \min \{ \mathcal{F}^\Delta(s^\Delta(t), \rho^\Delta), q^\Delta(t) \} \psi(t) dt \right| &\leq \|Q\|_{L^\infty} \|\psi\|_{L^\infty} (2 \Delta t + T \|\varphi'\|_{L^\infty} \Delta x). \end{aligned}$$

□

The final step is to obtain compactness for the sequences  $(\rho^\Delta)_\Delta$  and  $(y^\Delta)_\Delta$  in order to pass to the limit in (3.9)-(3.11). We start with  $(y^\Delta)_\Delta$ .

**Proposition 3.5.** *The sequence  $(y^\Delta)_\Delta$  verifies the following properties:*

(i) for all  $t \in [0, T]$ ,

$$y^\Delta(t) = y_0 + \int_0^t \omega \left( \int_{\mathbb{R}} \rho^\Delta(x, u) \mu(x) dx \right) du; \quad (3.13)$$

(ii) there exists  $y \in C([0, T])$  such that up to an extraction,  $(y^\Delta)_\Delta$  converges uniformly to  $y$  on  $[0, T]$ .

*Proof.* (i) For all  $t \in [0, T]$ , if  $t \in [t^n, t^{n+1})$  for some  $n \in \{0, \dots, N\}$ , we can write

$$\begin{aligned}
 y^\Delta(t) &= y_0 + \int_0^t s^\Delta(u) du = y_0 + \sum_{k=0}^{n-1} \int_{t^k}^{t^{k+1}} s^\Delta(u) du + \int_{t^n}^t s^\Delta(u) du \\
 &= y_0 + \sum_{k=0}^{n-1} \int_{t^k}^{t^{k+1}} s^{k+1} du + \int_{t^n}^t s^{n+1} du \\
 &= y_0 + \sum_{k=0}^{n-1} \int_{t^k}^{t^{k+1}} \omega \left( \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}} \rho_{j+\frac{1}{2}}^k \mu_{j+\frac{1}{2}} \Delta x \right) du + \int_{t^n}^t \omega \left( \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}} \rho_{j+\frac{1}{2}}^n \mu_{j+\frac{1}{2}} \Delta x \right) du \\
 &= y_0 + \int_0^t \omega \left( \int_{\mathbb{R}} \rho^\Delta(x, u) \mu(x) dx \right) du.
 \end{aligned}$$

Let us point out that from (3.2), we get that for all  $\Delta$  and almost every  $t \in (0, T)$ ,

$$s^\Delta(t) = \omega \left( \int_{\mathbb{R}} \rho^\Delta(x, t) \mu(x) dx \right). \quad (3.14)$$

(ii) A consequence of (3.13) and (3.14) is that for all  $\Delta$ ,

$$\| \dot{y}^\Delta \|_{L^\infty} = \| s^\Delta \|_{L^\infty} \leq \| \omega \|_{L^\infty} \text{ and } \| y^\Delta \|_{L^\infty} \leq |y_0| + T \| \omega \|_{L^\infty}.$$

The sequence  $(y^\Delta)_\Delta$  is therefore bounded in  $W^{1,\infty}((0, T))$ . Making use of the compact embedding of  $W^{1,\infty}((0, T))$  in  $C([0, T])$ , we get the existence of  $y \in C([0, T])$  such that up to the extraction of subsequence,  $(y^\Delta)_\Delta$  converges uniformly to  $y$  on  $[0, T]$ .  $\square$

The presence of a time dependent flux in the conservation law of (3.1) complicates the obtaining of compactness for  $(\rho^\Delta)_\Delta$ . In particular, the techniques used in [8, 9] to derive localized BV estimates don't apply here since our problem lacks time translation invariance. In our situation, it would be possible to derive weak BV estimates ([6, 23]) or to use a singular mapping technique ([2, 16, 30]). But, we take different options. Similarly to what we did in Section 2, we propose two ways to obtain compactness, which will lead to two convergence results.

### 3.3 Compactness via one-sided Lipschitz condition technique

First, we choose to adapt techniques and results put forward by Towers in [31]. With this in mind, we suppose in this section that  $f \in C^2([0, R])$  and satisfies the following uniform concavity assumption:

$$\exists \alpha > 0, \forall \rho \in [0, R], f''(\rho) \leq -\alpha. \quad (3.15)$$

Though this assumption is stronger than the non-degeneracy one (2.5), since  $f$  is bell-shaped, these two assumptions are similar in their spirit. We will also assume, following [31], that

$$\text{the numerical flux chosen in (3.3) is either the Engquist-Osher one or the Godunov one.} \quad (3.16)$$

Actually, the choice made for the numerical flux at the interface – *i.e.* when  $j = 0$  in (3.4) – does not play any role. What is important is that away from the interface, one chooses either the Engquist-Osher flux or the Godunov one. We denote for all  $n \in \{0, \dots, N + 1\}$  and  $j \in \mathbb{Z}$ ,

$$D_j^n = \max \left\{ \rho_{j-\frac{1}{2}}^n - \rho_{j+\frac{1}{2}}^n, 0 \right\}.$$

We will also use the notation

$$\hat{\mathbb{Z}} = \mathbb{Z} \setminus \{-1, 0, 1\}.$$

In [31], the author dealt with a discontinuous in both time and space flux and the specific “vanishing viscosity” coupling at the interface. The discontinuity in space was localized along the curve  $\{x = 0\}$ . Here, we deal

with only a discontinuous in time flux, but we also have a flux constraint along the curve  $\{x = 0\}$  since we work in the bus frame. The applicability of the technique of [31] for our case with moving interface and flux-constrained interface coupling relies on the fact that one can derive a bound on  $D_j^n$  as long as the “interface” does not enter the calculations for  $D_j^n$  i.e.  $j \in \hat{\mathbb{Z}}$ . This is what the following lemma points out under Assumptions (3.15)-(3.16). For readers’ convenience and in order to highlight the generality of the technique of Towers [31], let us provide the key elements of the argumentation leading to compactness.

**Lemma 3.6.** *Let  $n \in \{0, \dots, N\}$  and  $j \in \hat{\mathbb{Z}}$ . Then,*

$$D_j^{n+1} \leq \max \{D_{j-1}^n, D_j^n, D_{j+1}^n\} - a (\max \{D_{j-1}^n, D_j^n, D_{j+1}^n\})^2 \quad (3.17)$$

and

$$D_j^{n+1} \leq \frac{1}{\min\{|j| - 1, n + 1\}a}, \quad (3.18)$$

where

$$a = \frac{\lambda\alpha}{4}.$$

*Proof (Sketched).* Inequality (3.18) is an immediate consequence of inequality (3.17), see [31, Lemma 4.3]. Obtaining inequality (3.17) however, is less immediate. Let us give some details of the proof. First, note that by introducing the function  $\psi : z \mapsto z - az^2$ , inequality (3.17) can be stated as:

$$D_j^{n+1} \leq \psi (\max \{D_{j-1}^n, D_j^n, D_{j+1}^n\}). \quad (3.19)$$

Then, one can show – only using the monotonicity of both the scheme and the function  $\psi$  – that under the assumption

$$\text{inequality (3.19) holds when } (\rho_{j+\frac{3}{2}}^n - \rho_{j+\frac{1}{2}}^n), (\rho_{j-\frac{1}{2}}^n - \rho_{j-\frac{3}{2}}^n) \leq 0, \quad (3.20)$$

it follows that inequality (3.19) holds for all cases. And finally in [31, Page 23], the author proves that if the flux considered is either the Engquist-Osher flux or the Godunov flux, then (3.20) holds.  $\square$

The following lemma is an immediate consequence of inequality (3.18).

**Lemma 3.7.** *Fix  $0 < \varepsilon < X$ . Note  $i, J \in \mathbb{N}^*$  such that  $\varepsilon \in \mathcal{K}_{i+\frac{1}{2}}$  and  $X \in \mathcal{K}_{J-\frac{1}{2}}$ . Then if  $\Delta x/\varepsilon$  is sufficiently small, there exists a constant  $B = B(R, X, \frac{1}{a}, \frac{1}{\varepsilon})$ , non-increasing with respect to its arguments, such that for all  $n \geq i - 1$ ,*

$$\sum_{j=i}^{J-1} |\rho_{j+\frac{1}{2}}^n - \rho_{j-\frac{1}{2}}^n|, \quad \sum_{j=-J+1}^{-i-1} |\rho_{j+\frac{1}{2}}^n - \rho_{j-\frac{1}{2}}^n| \leq B \quad (3.21)$$

and

$$\sum_{j=i}^{J-2} |\rho_{j+\frac{1}{2}}^{n+1} - \rho_{j+\frac{1}{2}}^n|, \quad \sum_{j=-J+1}^{-i-2} |\rho_{j+\frac{1}{2}}^{n+1} - \rho_{j+\frac{1}{2}}^n| \leq 2\lambda LB. \quad (3.22)$$

**Proposition 3.8.** *There exists  $\rho \in L^\infty(\mathbb{R} \times [0, T])$  such that up to the extraction of a subsequence,  $(\rho^\Delta)_\Delta$  converges almost everywhere to  $\rho$  in  $\mathbb{R} \times [0, T]$ .*

*Proof.* Fix  $0 < \varepsilon < X$  and  $t > \lambda\varepsilon$ . We note

$$\Omega(X, \varepsilon) = (-X, -\varepsilon) \cup (\varepsilon, X).$$

Introduce  $i, J, n \in \mathbb{N}$  such that  $\varepsilon \in \mathcal{K}_{i+\frac{1}{2}}$ ,  $X \in \mathcal{K}_{J-\frac{1}{2}}$  and  $t \in [t^n, t^{n+1})$ . Note that

$$(n+1)\Delta t > t > \lambda\varepsilon \geq \lambda(i \Delta x) = i\Delta t,$$

i.e.  $n \geq i - 1$ . Then, if we suppose that  $\Delta x/\varepsilon$  is sufficiently small, we can use Lemma 3.7. From (3.21), we get

$$\text{TV}(\rho^\Delta(t)|_{\Omega(X, \varepsilon)}) \leq 2B \quad (3.23)$$

and from (3.22), we deduce

$$\int_{\Omega(X,\varepsilon)} |\rho^\Delta(x, t + \Delta t) - \rho^\Delta(x, t)| dx \leq 4LB\Delta t. \quad (3.24)$$

Combining (3.23)-(3.24) and the  $L^\infty$  bound (3.7), a functional analysis result ([24, Theorem A.8]) ensures the existence of a subsequence which converges almost everywhere to some  $\rho$  on  $\Omega(X, \varepsilon) \times (\lambda\varepsilon, T)$ . By a standard diagonal process we can extract a further subsequence (which we do not relabel) such that  $(\rho^\Delta)_\Delta$  converges almost everywhere to  $\rho$  on  $\mathbb{R} \times (0, T)$ .  $\square$

**Theorem 3.9.** *Fix  $\rho_0 \in L^1(\mathbb{R}; [0, R])$  and  $y_0 \in \mathbb{R}$ . Suppose that  $f \in C^2$  satisfies Assumptions (1.2)-(3.15). Suppose that in (3.4), we use the Engquist-Osher flux or the Godunov one when  $j \neq 0$  and any other monotone consistent and Lipschitz numerical flux when  $j = 0$ . Then under the CFL condition (3.6), the scheme (3.2)-(3.3)-(3.4) converges to an admissible weak solution to Problem (3.1).*

*Proof.* We have shown that – up to the extraction of a subsequence –  $y^\Delta$  converges uniformly on  $[0, T]$  to some  $y \in C([0, T])$  and that  $\rho^\Delta$  converges a.e. on  $\mathbb{R} \times [0, T]$  to some  $\rho \in L^\infty(\mathbb{R} \times [0, T])$ . We now prove that this couple  $(\rho, y)$  is an admissible weak solution to Problem (3.1).

(iv) Recall that for all  $\Delta$  and  $t \in [0, T]$ ,

$$y^\Delta(t) = y_0 + \int_0^t \omega \left( \int_{\mathbb{R}} \rho^\Delta(x, u) \mu(x) dx \right) du.$$

Letting  $\Delta \rightarrow 0$  above, by dominated convergence, we obtain that for all  $t \in [0, T]$ ,

$$y(t) = y_0 + \int_0^t \omega \left( \int_{\mathbb{R}} \rho(x, u) \mu(x) dx \right) du.$$

(i) To prove the time-continuity regularity, we first apply inequality (3.9) with  $\tau = 0$ ,  $\tau' = T$ ,  $\varphi \in C_c^\infty(\mathbb{R}^* \times [0, T]; \mathbb{R}^+)$  and  $\kappa \in [0, R]$  to obtain

$$\int_0^T \int_{\mathbb{R}} |\rho^\Delta - \kappa| \partial_t \varphi + \Phi^\Delta(\rho^\Delta, \kappa) \partial_x \varphi dx dt + \int_{\mathbb{R}} |\rho_0^\Delta - \kappa| \varphi(x, 0) dx \geq -C_1^\varphi(\Delta x + \Delta t).$$

Then the a.e. convergence of  $(s^\Delta)_\Delta$  to  $\dot{y}$  – coming from (3.14) – and the a.e. convergence of  $(\rho^\Delta)_\Delta$  to  $\rho$  ensure that when letting  $\Delta \rightarrow 0$ , we get

$$\int_0^T \int_{\mathbb{R}} |\rho - \kappa| \partial_t \varphi + \Phi_{\dot{y}(t)}(\rho, \kappa) \partial_x \varphi dx dt + \int_{\mathbb{R}} |\rho_0(x + y_0) - \kappa| \varphi(x, 0) dx \geq 0,$$

and consequently  $\rho \in C([0, T]; L^1_{\text{loc}}(\mathbb{R}))$ , see Remark 1.2.

(ii)-(iii) Now, we pass to the limit in the approximate inequalities (3.9)-(3.11) using the a.e. convergence of  $(s^\Delta)_\Delta$  to  $\dot{y}$  and of  $(\rho^\Delta)_\Delta$  to  $\rho$  as well as the continuity of  $Q$  and  $\omega$ . Consequently, for all non-negative test function  $\varphi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^+)$  and  $\kappa \in [0, R]$ , the following inequalities hold for almost every  $0 \leq \tau < \tau' \leq T$ :

$$\begin{aligned} & \int_\tau^{\tau'} \int_{\mathbb{R}} |\rho - \kappa| \partial_t \varphi + \Phi_{\dot{y}(t)}(\rho, \kappa) \partial_x \varphi dx dt + \int_{\mathbb{R}} |\rho(x, \tau) - \kappa| \varphi(x, \tau) dx \\ & - \int_{\mathbb{R}} |\rho(x, \tau') - \kappa| \varphi(x, \tau') dx + 2 \int_\tau^{\tau'} \mathcal{R}_{\dot{y}(t)}(\kappa, q(t)) \varphi(0, t) dt \geq 0. \end{aligned}$$

To conclude, note that the expression in the left-hand side of the previous inequality is a continuous function of  $(\tau, \tau')$  which is almost everywhere greater than the continuous function 0. By continuity, this expression is everywhere greater than 0, which proves that  $\rho$  satisfies the entropy inequalities (1.4). Using similar arguments, one shows that  $\rho$  also satisfies the constraint inequalities (1.5).

This shows that the couple  $(\rho, y)$  is an admissible weak solution to Problem (3.1), and that concludes the proof of convergence.  $\square$

We proved than in the  $L^\infty$  framework, the scheme converges to an admissible weak solution, but note that there is no guarantee of uniqueness in this construction. Also stress that we cannot extend this result to general consistent monotone numerical fluxes beyond hypothesis (3.16).

### 3.4 Compactness via global BV bounds

The following result is the discrete version of Lemma 1.3 so it is consistent that the proof used the discrete analogou arguments of the ones we used in the proof of Lemma 1.3.

**Lemma 3.10.** *Introduce for all  $\Delta > 0$  the function  $\xi^\Delta$  defined for all  $t \in [0, T]$  by*

$$\xi^\Delta(t) = \int_{\mathbb{R}} \rho^\Delta(x, t) \mu(x) dx.$$

*Then  $\xi^\Delta$  has bounded variation and consequently, so does  $s^\Delta$ .*

*Proof.* Since  $\mu \in \text{BV}(\mathbb{R})$ , there exists a sequence of smooth functions  $(\mu_\ell)_{\ell \in \mathbb{N}} \subset \text{BV}(\mathbb{R}) \cap C^\infty(\mathbb{R})$  such that

$$\|\mu_\ell - \mu\|_{L^1} \xrightarrow{\ell \rightarrow \infty} 0 \quad \text{and} \quad \text{TV}(\mu_\ell) \xrightarrow{\ell \rightarrow \infty} \text{TV}(\mu).$$

Introduce for all  $\ell \in \mathbb{N}$  and  $t \in [0, T]$ , the function  $\xi_\ell^\Delta(t) = \int_{\mathbb{R}} \rho^\Delta(x, t) \mu_\ell(x) dx$  and let  $K > 0$  such that

$$\forall \ell \in \mathbb{N}, \quad \|\mu_\ell\|_{L^1}, \text{TV}(\mu_\ell) \leq K.$$

For all  $\ell \in \mathbb{N}$  and  $t, s \in [0, T]$ , if  $t \in [t^k, t^{k+1})$  and  $s \in [t^m, t^{m+1})$ , we have

$$\begin{aligned} |\xi_\ell^\Delta(t) - \xi_\ell^\Delta(s)| &= |\xi_\ell^\Delta(t^k) - \xi_\ell^\Delta(t^m)| = \left| \int_{\mathbb{R}} \rho^\Delta(x, t^k) \mu_\ell(x) dx - \int_{\mathbb{R}} \rho^\Delta(x, t^m) \mu_\ell(x) dx \right| \\ &= \left| \sum_{j \in \mathbb{Z}} (\rho_{j+\frac{1}{2}}^k - \rho_{j+\frac{1}{2}}^m) \mu_{j+\frac{1}{2}}^\ell \Delta x \right| = \left| \sum_{j \in \mathbb{Z}} \sum_{p=m}^{k-1} (\rho_{j+\frac{1}{2}}^{p+1} - \rho_{j+\frac{1}{2}}^p) \mu_{j+\frac{1}{2}}^\ell \Delta x \right| \\ &= \left| \sum_{p=m}^{k-1} \sum_{j \in \mathbb{Z}} \left( \mathcal{F}_j^{p+1}(\rho_{j-\frac{1}{2}}^p, \rho_{j+\frac{1}{2}}^p) - \mathcal{F}_{j+1}^{p+1}(\rho_{j+\frac{1}{2}}^p, \rho_{j+\frac{3}{2}}^p) \right) \mu_{j+\frac{1}{2}}^\ell \Delta t \right| \\ &= \left| \sum_{p=m}^{k-1} \sum_{j \in \mathbb{Z}} \mathcal{F}_{j+1}^{p+1}(\rho_{j+\frac{1}{2}}^p, \rho_{j+\frac{3}{2}}^p) (\mu_{j+\frac{3}{2}}^\ell - \mu_{j+\frac{1}{2}}^\ell) \Delta t \right| \leq RL \sum_{p=m}^{k-1} \text{TV}(\mu_\ell) \Delta t \leq RLK(|t-s| + 2\Delta t). \end{aligned}$$

Consequently, for all  $\ell \in \mathbb{N}$ ,  $\Delta > 0$  and  $t, \tau \in [0, T]$ , the triangle inequality yields:

$$|\xi^\Delta(t) - \xi^\Delta(\tau)| \leq 2R\|\mu - \mu_\ell\|_{L^1} + RLK(|t - \tau| + 2\Delta t).$$

Letting  $\ell \rightarrow \infty$ , we get that for all  $\Delta > 0$  and  $t, \tau \in [0, T]$ ,

$$|\xi^\Delta(t) - \xi^\Delta(\tau)| \leq RLK(|t - \tau| + 2\Delta t),$$

which leads to

$$\text{TV}(\xi^\Delta) = \sum_{k=0}^N |\xi^\Delta(t^{k+1}) - \xi^\Delta(t^k)| \leq 3RLK(T + \Delta t).$$

This proves that  $\xi^\Delta \in \text{BV}([0, T])$ . Since  $\omega$  is Lipschitz continuous,  $s^\Delta$  also has bounded variation.  $\square$

**Theorem 3.11.** *Fix  $\rho_0 \in L^1(\mathbb{R}; [0, R]) \cap \text{BV}(\mathbb{R})$  and  $y_0 \in \mathbb{R}$ . Suppose that  $f$  satisfies (1.2)-(1.7) and that*

$$\forall s \in [0, \|\omega\|_{L^\infty}], \quad F(s, \cdot) \in C^1([0, R] \setminus \{\bar{\rho}_s\}),$$

where  $\bar{\rho}_s = \operatorname{argmax}_{\rho \in [0, R]} F(s, \rho)$ . Suppose also that in (3.4), we use the Godunov flux when  $j = 0$  and any other monotone consistent and Lipschitz numerical flux when  $j \neq 0$ . Finally assume that  $Q$  satisfies the condition

$$\exists \varepsilon > 0, \forall s \in [0, \|\omega\|_{L^\infty}], Q(s) \leq \max_{\rho \in [0, R]} F(s, \rho) - \varepsilon. \quad (3.25)$$

Then under the CFL condition (3.6), the scheme (3.2)-(3.3)-(3.4) converges to the unique BV-regular solution to Problem (3.1).

*Proof.* All the hypotheses of Lemma A.4 are fulfilled. Consequently, there exists a constant  $C^\varepsilon$  such that for all  $n \in \{0, \dots, N\}$ ,

$$\begin{aligned} \operatorname{TV}(\rho^\Delta(t^{n+1})) &\leq \operatorname{TV}(\rho_0) + 4R + C^\varepsilon \left( \sum_{k=0}^n |q^{k+1} - q^k| + \sum_{k=0}^n |s^{k+1} - s^k| \right) \\ &\leq \operatorname{TV}(\rho_0) + 4R + C^\varepsilon (1 + \|Q'\|_{L^\infty}) \sum_{k=0}^n |s^{k+1} - s^k|. \end{aligned} \quad (3.26)$$

Making use of Lemma 3.10, we get that for all  $n \in \{0, \dots, N\}$ ,

$$\sum_{k=0}^n |s^{k+1} - s^k| = \sum_{k=0}^n |s^\Delta(t^{k+1}) - s^\Delta(t^k)| \leq \|\omega\|_{L^\infty} \sum_{k=0}^n |\xi^\Delta(t^{k+1}) - \xi^\Delta(t^k)| \leq 3RLK\|\omega\|_{L^\infty}(T + \Delta t).$$

where the constant  $K$  was introduced in the proof of Lemma 3.10. The two last inequalities lead to

$$\forall t \in [0, T], \operatorname{TV}(\rho^\Delta(t)) \leq \operatorname{TV}(\rho_0) + 4R + 3C^\varepsilon (1 + \|Q'\|_{L^\infty}) \|\omega\|_{L^\infty} RLK(T + \Delta t). \quad (3.27)$$

Therefore, the sequence  $(\rho^\Delta)_\Delta$  is uniformly in time bounded in  $\operatorname{BV}(\mathbb{R})$ . Using [22, Appendix], we get the existence of  $\rho \in C([0, T]; L^1_{\text{loc}}(\mathbb{R}))$  such that

$$\forall t \in [0, T], \rho^\Delta(t) \xrightarrow{\Delta \rightarrow 0} \rho(t) \text{ in } L^1_{\text{loc}}(\mathbb{R}).$$

Following the proof of Theorem 3.9, we show that  $(\rho, y)$  is an admissible weak solution. Then passing to the limit in (3.27), the lower semi-continuity of the BV semi-norm ensures that  $(\rho, y)$  is also BV-regular, and therefore it is the unique solution of the limit problem. This concludes the proof.  $\square$

*Remark 3.3.* Note the complementarity of the hypotheses made in the above Theorem with the ones of Theorem 3.9. Recall that in Theorem 3.9, where we needed the Godunov flux only away from the interface.

## 4 Numerical simulations

In this section we present some numerical tests performed with the scheme analyzed in Section 3. In all the simulations we take the uniformly concave flux  $f(\rho) = \rho(1 - \rho)$  (the maximal car velocity and the maximal density are assumed to be equal to one). Following the hypotheses of Theorem 3.11, we choose the Godunov flux at the interface, and the Rusanov one away from the interface. We will use weight functions of the kind

$$\mu_n(x) = 2^n \mathbf{1}_{[0; \frac{1}{2^n}]}(x),$$

for one (in Section 4.1) or several (in Section 4.2) values of  $n \in \mathbb{N}^*$ .

### 4.1 Validation of the scheme

In this section, to link the traffic density to the slow vehicle – say a bus – velocity, we consider the function

$$\omega(\rho) = \frac{1 - \rho}{\alpha(\rho + \beta)^{3/2}},$$

where  $\alpha, \beta$  are chosen such that  $\omega(0) = 0.8$  and  $\omega$  has a sufficiently large (non-positive) slope at point  $\rho = 0$ , as illustrated in Figure 1 (left). We choose this function guided by the idea that even a moderate traffic density disrupts the slow vehicle, and the disruption can be progressive. This explains the important decrease of the bus velocity at low density ( $0 \leq \rho \leq 0.4$ ). And when the bus is really slow, the high density ( $0.4 < \rho \leq 1$ ) does not really affect its velocity.

*Remark 4.1.* The function  $\omega$  we chose above is not of the form as required in [26, 27]. Once again, let us stress that the particular form  $\omega(\rho) = \min(V_{\text{bus}}, 1 - \rho)$ , where  $V_{\text{bus}}$  is the maximum bus velocity, is crucial for the well-posedness result of [26, 27] to hold. Indeed, it is essential in the analysis of [26, 27] that the velocity of the bus be constant (equal to  $V_{\text{bus}}$ ) across the non-classical shocks. Our non-local model is not bound to this restriction.

The set-up of the experiment is the following. Consider a domain of computation  $[0, 6]$ , the weight function  $\mu_1$  and the initial data

$$\rho_0(x) = 0.4\mathbf{1}_{[0.2;0.7]}(x), \quad y_0 = 0.7. \tag{4.1}$$

We consider a constraint defined by  $Q(s) = 0.8 \times \left(\frac{1-s}{2}\right)^2$ .

As we can see in Figure 1 (right), at first the bus travels at near maximum velocity. Then as the cars behind him overtake it, the density  $\xi$  ahead of it increases which makes him go slower. Finally when all the cars have overtaken it, the bus can once again travel at maximum speed. The approximate solution  $\rho^\Delta$  is represented in Figure 2.

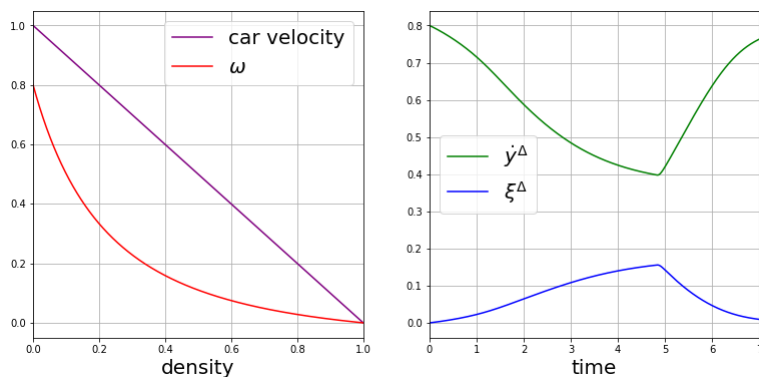


Figure 1: Evolution in time of the subjective density  $\xi$  and the bus velocity  $\dot{y}$  (640 cells).

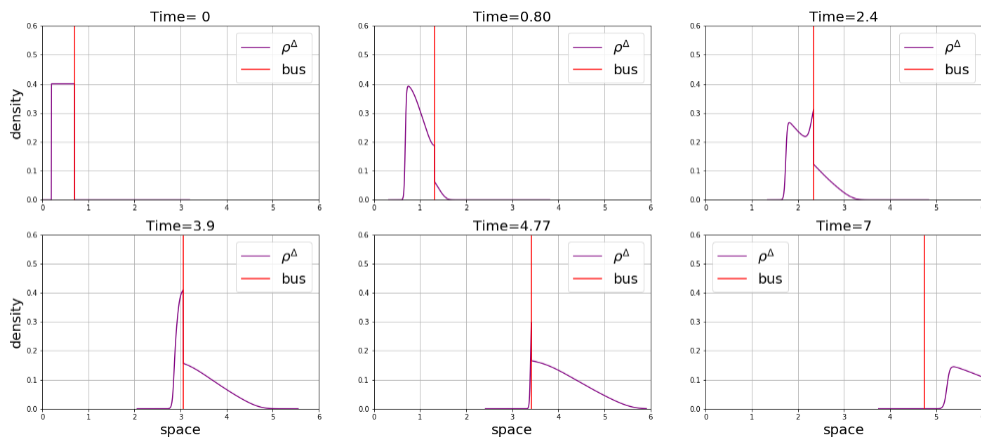


Figure 2: The numerically computed solution  $x \mapsto \rho^\Delta(x, t)$  at different fixed times  $t$  (640 cells).



A convergence analysis is also performed for this test. We introduce the relative errors

$$E_\rho^\Delta = \|\rho^\Delta - \rho^{\Delta/2}\|_{L^1((0,T);L^1(\mathbb{R}))} \quad \text{and} \quad E_y^\Delta = \|y^\Delta - y^{\Delta/2}\|_{L^\infty}.$$

In Table 1, we computed these errors for different number of space cells at the final time  $T = 7$ . We deduce that the order of convergence is approximately 0.80 for the car density and is approximately 0.95 for the slow vehicle position.

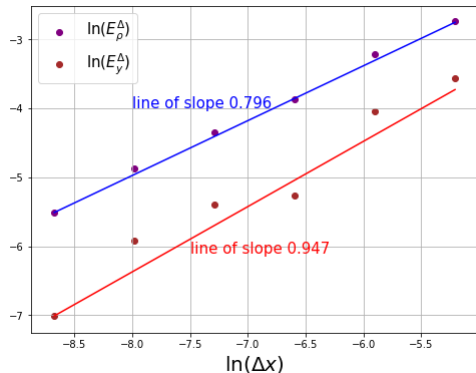


Figure 3: Rates of convergence.

Number of cells	$E_\rho^\Delta$	$E_y^\Delta$
640	$6.504 \times 10^{-2}$	$2.825 \times 10^{-2}$
1280	$4.006 \times 10^{-2}$	$1.763 \times 10^{-2}$
2560	$2.089 \times 10^{-2}$	$5.178 \times 10^{-3}$
5120	$1.301 \times 10^{-2}$	$4.525 \times 10^{-3}$
10240	$7.673 \times 10^{-3}$	$2.697 \times 10^{-3}$
20480	$4.055 \times 10^{-3}$	$9.047 \times 10^{-4}$

Table 1: Measured errors at time  $T = 7$ .

## 4.2 Comparisons with experiments on the local model

Now we confront the numerical tests performed with our model with the tests done by the authors in [13] approximating the original problem of [20]. We deal with a road of length 1 parametrized by the interval  $[0, 1]$  and choose the weight function  $\mu_1$ . Moreover,

$$\omega(\rho) = \min\{0.3 ; 1 - \rho\} \quad \text{and} \quad Q(s) = 0.6 \times \left(\frac{1-s}{2}\right)^2.$$

- Case 1. First, consider the initial datum

$$\rho_0(x) = \begin{cases} 0.4 & \text{if } x < 0.5 \\ 0.5 & \text{if } x > 0.5 \end{cases} \quad y_0 = 0.5. \quad (4.2)$$

The numerical solution is composed of two classical shocks separated by a non-classical discontinuity, as illustrated in Figure 4 (left).

- Case 2. Next, we choose

$$\rho_0(x) = \begin{cases} 0.8 & \text{if } x < 0.5 \\ 0.5 & \text{if } x > 0.5 \end{cases} \quad y_0 = 0.5. \quad (4.3)$$

The values of the initial condition create a rarefaction wave followed by a non-classical and classical shocks, as illustrated in Figure 4 (right).

- Case 3. Finally, still following [13], we consider

$$\rho_0(x) = \begin{cases} 0.8 & \text{if } x < 0.5 \\ 0.4 & \text{if } x > 0.5 \end{cases} \quad y_0 = 0.4. \quad (4.4)$$

Here the solution is composed of a rarefaction wave followed by non-classical and classical shocks on the density that are created when the slow vehicle approaches the rarefaction and initiates a moving bottleneck, as illustrated in Figure 5.

With these three tests, we can already see – in a qualitative way – the resemblance between the numerical approximations to the solutions to our model and the numerical approximations of [13]. One way to quantify their proximity is for example to evaluate the  $L^1$  error between the car densities and the  $L^\infty$  error between the bus positions. More precisely, note  $(\rho^\Delta, y^\Delta)$  the approximation of the BV-regular solution to (3.1) obtained

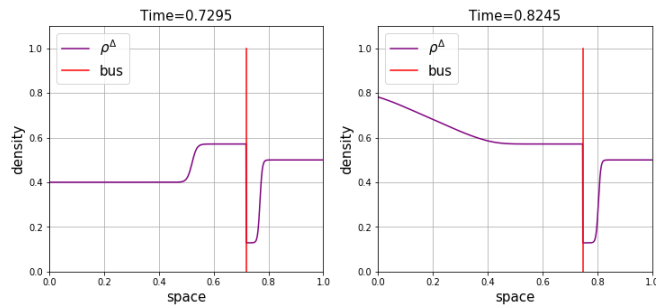


Figure 4: Evolution in time of the approximate density corresponding to initial data (4.2) (left) and (4.3) (right), with 640 cells.

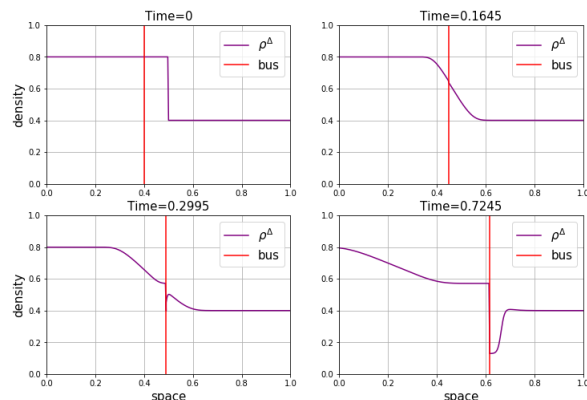


Figure 5: Evolution in time of the approximate density corresponding to initial data (4.4) (640 cells).

with the scheme (3.2)-(3.3)-(3.4), and note  $(\bar{\rho}^\Delta, \bar{y}^\Delta)$  the couple obtained with this same scheme but replacing the line

$$s^{n+1} = \omega \left( \sum_{j \in \mathbb{Z}} \rho_{j+\frac{1}{2}}^n \mu_{j+\frac{1}{2}} \Delta x \right) \quad \text{by} \quad s^{n+1} = \omega \left( \rho_{\frac{1}{2}}^n \right).$$

Let us precise that this is not the scheme the authors of [13] proposed. However, this scheme is consistent with the problem

$$\begin{cases} \partial_t \rho + \partial_x F(\dot{y}(t), \rho) = 0 & \mathbb{R} \times (0, T) \\ \rho(x, 0) = \rho_0(x + y_0) & x \in \mathbb{R} \\ F(\dot{y}(t), \rho)|_{x=0} \leq Q(\dot{y}(t)) & t \in (0, T) \\ \dot{y}(t) = \omega(\rho(0+, t)) \\ y(0) = y_0 \end{cases} \quad (4.5)$$

and behaves in a stable way in the calculations we performed. Therefore, the couple  $(\bar{\rho}^\Delta, \bar{y}^\Delta)$  is expected to give a reasonable approximation of the solution to (4.5). With this in mind, for the case (4.4) and with the weight function  $\mu_1$ , we computed in Table 2 the measured errors

$$E_{L^1}^\Delta = \|\rho^\Delta - \bar{\rho}^\Delta\|_{L^1((0,T);L^1(\mathbb{R}))} \quad \text{and} \quad E_{L^\infty}^\Delta = \|y^\Delta - \bar{y}^\Delta\|_{L^\infty}.$$

These calculations indicate that for a sufficiently large number of cells  $J \geq 81920$ ,

$$E_{L^1}^\Delta \sim 3.71 \times 10^{-3} \quad \text{and} \quad E_{L^\infty}^\Delta \sim 1.42 \times 10^{-2}.$$

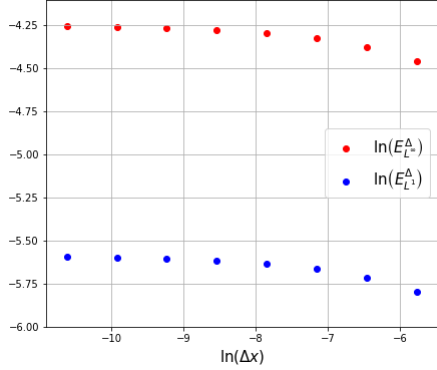


Figure 6: Computed errors between the approximate solutions of the local and non-local model.

This indicates the discrepancy between our non-local and the local model (4.5) of [20]. The idea is now to fix the number of cells  $J = 81920$  and to make the length of the weight function support go to zero. In Table 3, we have computed for different weight functions, the error between the approximations of the two models. This error corresponds, as in the above calculation, to the residual error observed starting from a sufficiently small  $\Delta x$ .

Number of cells	$E_{L1}^\Delta$	$E_{L\infty}^\Delta$
640	$3.028 \times 10^{-3}$	$1.158 \times 10^{-2}$
1280	$3.290 \times 10^{-3}$	$1.255 \times 10^{-2}$
2560	$3.463 \times 10^{-3}$	$1.322 \times 10^{-2}$
5120	$3.571 \times 10^{-3}$	$1.365 \times 10^{-2}$
10240	$3.638 \times 10^{-3}$	$1.391 \times 10^{-2}$
20480	$3.677 \times 10^{-3}$	$1.407 \times 10^{-2}$
40960	$3.699 \times 10^{-3}$	$1.416 \times 10^{-2}$
81920	$3.713 \times 10^{-3}$	$1.421 \times 10^{-2}$

Table 2: Measured errors at time  $T = 0.7245$ .

weight function	$E_{L1}^\Delta$	$E_{L\infty}^\Delta$
$\mu_1$	$3.713 \times 10^{-3}$	$1.421 \times 10^{-2}$
$\mu_3$	$3.676 \times 10^{-3}$	$1.409 \times 10^{-2}$
$\mu_5$	$6.056 \times 10^{-4}$	$2.539 \times 10^{-3}$
$\mu_7$	$1.883 \times 10^{-4}$	$7.845 \times 10^{-4}$

Table 3: Measured errors at time  $T = 0.7245$

Even if we are unable – at the moment – to rigorously link our problem (1.1) with  $\mu \rightarrow \delta_{0+}$  and the original problem (4.5) of the authors in [20], this last experiment corroborates the conjecture that our model (3.1) is the singular limit of the local model (4.5), in the case  $\omega$  is of the form  $\omega(\rho) = \min(V_{\text{bus}}, 1 - \rho)$ . The other interesting question is whether the local model is well posed beyond this particular choice of  $\omega$ .

*Acknowledgements.* The author is most grateful to Boris Andreianov for his constant support and many enlightening discussions.

## A On BV bounds for limited flux models

We focus on the study of the following class of models:

$$\begin{cases} \partial_t \rho + \partial_x F(s(t), \rho) = 0 & \mathbb{R} \times (0, T) \\ \rho(x, 0) = \rho_0(x) & x \in \mathbb{R} \\ F(s(t), \rho)|_{x=0} \leq q(t) & t \in (0, T), \end{cases} \quad (\text{A.1})$$

where  $s \in \text{BV}([0, T]; [0, \Sigma])$  for some  $\Sigma > 0$  and  $q \in \text{BV}([0, T]; \mathbb{R}^+)$ . We suppose that  $F$  is continuously differentiable on  $[0, \Sigma] \times [0, R]$  and that for all  $s \in [0, \Sigma]$ ,  $F(s, \cdot)$  is bell-shaped *i.e.*

$$\forall s \in [0, \Sigma], F(s, 0) = 0, F(s, R) \leq 0, \exists! \bar{\rho}_s \in (0, R), \partial_\rho F(s, \rho)(\bar{\rho}_s - \rho) > 0 \text{ for a.e. } \rho \in (0, R). \quad (\text{A.2})$$

This framework covers the particular case when  $F$  takes the form:

$$F(s(t), \rho) = f(\rho) - s(t)\rho,$$

with bell-shaped  $f$ , which our model (1.1) was based on. This class of models is well known, especially when the flux function is not time dependent, *c.f.* [17, 6]. In this appendix, we establish in passing the

well-posedness of Problem (A.1), but our main interest lies in BV in space regularity for the solutions of Problem (A.1). We aim at obtaining a bound on the total variation of the solutions to (A.1), using a finite volume approximation which allows for sharp control of the variation at the constraint. Note that alternative offered by wave-front tracking would be cumbersome because of the explicit time-dependency in (A.1). In the general case, entropy solutions to limited flux problems like (A.1) do not belong to  $L^\infty([0, T]; \text{BV}(\mathbb{R}))$ , see [1]. We will show that it is the case under a mild assumption on the constraint function  $q$  – see Assumption (A.10) below – and provided that

$$\rho_0 \in L^1(\mathbb{R}; [0, R]) \cap \text{BV}(\mathbb{R}).$$

Throughout the appendix, for all  $s \in [0, \Sigma]$  and  $a, b \in [0, R]$ , we denote by

$$\Phi_s(a, b) = \text{sign}(a - b)(F(s, a) - F(s, b))$$

the classical Kruřkov entropy flux associated with the Kruřkov entropy  $\rho \mapsto |\rho - \kappa|$ , for all  $\kappa \in [0, R]$ , see [25].

### A.1 Equivalent definitions of solution and uniqueness

Let us first recall the following definition.

**Definition A.1.** A function  $\rho \in L^\infty(\mathbb{R} \times [0, T])$  is called an admissible weak solution to Problem (A.1) if

(i) the following regularity is fulfilled:

$$\rho \in C([0, T]; L^1_{\text{loc}}(\mathbb{R}));$$

(ii) for all non-negative test function  $\varphi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^+)$  and  $\kappa \in [0, R]$ , the following entropy inequalities are verified for all  $0 \leq \tau < \tau' \leq T$ :

$$\begin{aligned} & \int_\tau^{\tau'} \int_{\mathbb{R}} |\rho - \kappa| \partial_t \varphi + \Phi_{s(t)}(\rho, \kappa) \partial_x \varphi dx dt + \int_{\mathbb{R}} |\rho(x, \tau) - \kappa| \varphi(x, \tau) dx \\ & - \int_{\mathbb{R}} |\rho(x, \tau') - \kappa| \varphi(x, \tau') dx + 2 \int_\tau^{\tau'} \mathcal{R}_{s(t)}(\kappa, q(t)) \varphi(0, t) dt \geq 0, \end{aligned} \quad (\text{A.3})$$

where

$$\mathcal{R}_{s(t)}(\kappa, q(t)) = F(s(t), \kappa) - \min \{F(s(t), \kappa), q(t)\};$$

(iii) for all non-negative test function  $\psi \in C^\infty([0, T])$  and some given  $\varphi \in C_c^\infty(\mathbb{R})$  which verifies  $\varphi(0) = 1$ , the following weak constraint inequalities are verified for all  $0 \leq \tau < \tau' \leq T$ :

$$\begin{aligned} & - \int_\tau^{\tau'} \int_{\mathbb{R}^+} \rho \partial_t (\varphi \psi) + F(s(t), \rho) \partial_x (\varphi \psi) dx dt - \int_{\mathbb{R}^+} \rho(x, \tau) \varphi(x) \psi(\tau) dx \\ & + \int_{\mathbb{R}^+} \rho(x, \tau') \varphi(x) \psi(\tau') dx \leq \int_\tau^{\tau'} q(t) \psi(t) dt. \end{aligned} \quad (\text{A.4})$$

**Definition A.2.** An admissible weak solution  $\rho$  will be called BV-regular if it verifies

$$\rho \in L^\infty([0, T]; \text{BV}(\mathbb{R})).$$

As we pointed out before, this notion of solution is well suited for passage to the limit of a.e. convergent sequences of exact or approximate solutions. However, it is not so well-adapted to prove uniqueness. An equivalent notion of solution, based on explicit treatment of traces of  $\rho$  at the constraint, was introduced by the authors of [7]. This notion of solution leads to the following stability estimate.

**Theorem A.3.** Fix  $s^1, s^2 \in \text{BV}([0, T]; [0, \Sigma])$ . Note  $\rho^1, \rho^2$  two BV-regular solutions to (A.1) respectively associated with initial data  $\rho_0^1, \rho_0^2 \in L^1(\mathbb{R}; [0, R]) \cap \text{BV}(\mathbb{R})$ , constraint functions  $q^1, q^2 \in \text{BV}([0, T]; \mathbb{R}^+)$  and flux functions  $(t, \rho) \mapsto F(s^1(t), \rho), F(s^2(t), \rho)$  satisfying (A.2). Then for a.e.  $t \in (0, T)$ ,

$$\begin{aligned} \|\rho^1(t) - \rho^2(t)\|_{L^1} & \leq \|\rho_0^1 - \rho_0^2\|_{L^1} + 2 \int_0^t |q^1(\tau) - q^2(\tau)| d\tau + 2 \int_0^t \|F(s^1(\tau), \cdot) - F(s^2(\tau), \cdot)\|_{L^\infty} d\tau \\ & + \int_0^t \|\partial_\rho F(s^1(\tau), \cdot) - \partial_\rho F(s^2(\tau), \cdot)\|_{L^\infty} \text{TV}(\rho^1(\tau)) d\tau. \end{aligned} \quad (\text{A.5})$$

In particular, Problem (A.1) admits at most one admissible weak solution.

*Proof.* Since our interest to details lies rather on the numerical approximation point of view, we do not fully prove this statement but we give the essential steps leading to this stability result.

- *Definition of solution.* First, the authors of [7] introduce a subset of  $\mathbb{R}^2$  called *germ*, which can be seen as the set of all the possible traces of a solution to (A.1). Then, they say that  $\rho$  is a solution to (A.1) if  $\rho$  satisfies entropy inequalities away from the interface – i.e. with  $\varphi \in C_c^\infty(\mathbb{R}^* \times \mathbb{R}^+)$  in (A.3) – and if the couple constituted of left-side and the right-side traces of  $\rho$  belongs to this so-called germ.

- *Equivalence of the two definitions.* The next step is to prove that this latter definition of solution is equivalent to Definition A.1. This part is done using good choices of test functions, see [7, Theorem 3.18] or [6, Proposition 2.5, Theorem 2.9].

- *First stability estimate.* One first shows that if  $s^1 = s^2$ , then

$$\|\rho^1(t) - \rho^2(t)\|_{L^1} \leq \|\rho_0^1 - \rho_0^2\|_{L^1} + 2 \int_0^t |q^1(\tau) - q^2(\tau)| d\tau. \quad (\text{A.6})$$

The proof starts with the classical doubling of variables method of Kružkov[25, Theorem 1] and then uses the germ structure, what the authors of [7] called  $L^1$ -dissipativity, see [7, Definition 3.1] and [6, Lemma 2.7]. Note that at this point, uniqueness for admissible weak solutions is proved.

- *Proof of estimate (A.5).* The proof is based upon estimate (A.6) and elements borrowed from [10, 18]. Most details can be found in the proof of [21, Theorem 2.1].  $\square$

## A.2 Existence of BV-regular solutions

We now turn to the proof of the existence of BV-regular solutions by the means of a finite volume scheme.

Fix  $\rho_0 \in L^1(\mathbb{R}; [0, R])$ . For a fixed spatial mesh size  $\Delta x$  and time mesh size  $\Delta t$ , let  $x_j = j\Delta x$ ,  $t^n = n\Delta t$ . We define the grid cells  $\mathcal{K}_{j+\frac{1}{2}} = (x_j, x_{j+1})$  and  $N \in \mathbb{N}^*$  such that  $T \in [N\Delta t, (N+1)\Delta t)$ . We write

$$\mathbb{R} \times [0, T] \subset \bigcup_{n=0}^N \bigcup_{j \in \mathbb{Z}} \mathcal{P}_{j+\frac{1}{2}}^n, \quad \mathcal{P}_{j+\frac{1}{2}}^n = \mathcal{K}_{j+\frac{1}{2}} \times [t^n, t^{n+1}).$$

We choose to discretize the initial data  $\rho_0$  and the functions  $s, q$  with  $(\rho_{j+\frac{1}{2}}^0)_j$ ,  $(s^n)_n$  and  $(q^n)_n$  where for all  $j \in \mathbb{Z}$  and  $n \in \{0, \dots, N\}$ ,  $\rho_{j+\frac{1}{2}}^0$ ,  $s^n$  and  $q^n$  are their mean values on each cell  $\mathcal{K}_{j+\frac{1}{2}}$  and  $[t^n, t^{n+1})$ . Following [6], the marching formula of the scheme takes the form:

$$\forall n \in \{0, \dots, N\}, \forall j \in \mathbb{Z}, \rho_{j+\frac{1}{2}}^{n+1} = \rho_{j+\frac{1}{2}}^n - \lambda \left( \mathcal{F}_{j+1}^n(\rho_{j+\frac{1}{2}}^n, \rho_{j+\frac{3}{2}}^n) - \mathcal{F}_j^n(\rho_{j-\frac{1}{2}}^n, \rho_{j+\frac{1}{2}}^n) \right), \quad (\text{A.7})$$

where

$$\mathcal{F}_j^n(a, b) = \begin{cases} \mathcal{F}^n(a, b) & \text{if } j \neq 0 \\ \min \{ \mathcal{F}^n(a, b), q^n \} & \text{if } j = 0, \end{cases} \quad (\text{A.8})$$

$\mathcal{F}^n$  being a monotone consistent and Lipschitz numerical flux associated to  $F(s^n, \cdot)$ . We then define

- $\rho^\Delta(x, t) = \rho_{j+\frac{1}{2}}^n$  if  $(x, t) \in \mathcal{P}_{j+\frac{1}{2}}^n$
- $s^\Delta(t) = s^n$  if  $t \in [t^n, t^{n+1})$
- $q^\Delta(t) = q^n$  if  $t \in [t^n, t^{n+1})$ .

Let  $\Delta = (\Delta x, \Delta t)$ . For the convergence analysis, we will assume that  $\Delta \rightarrow 0$ , with  $\lambda = \Delta/\Delta x$ , verifying the CFL condition

$$\lambda \underbrace{\sup_{s \in [0, \Sigma]} \left( \left\| \frac{\partial \mathcal{F}^s}{\partial x} \right\|_{L^\infty} + \left\| \frac{\partial \mathcal{F}^s}{\partial y} \right\|_{L^\infty} \right)}_L \leq 1, \quad (\text{A.9})$$

where  $\mathcal{F}^s = \mathcal{F}^s(x, y)$  is the numerical flux – associated to  $F(s, \cdot)$  – we use in the scheme (A.7). From now, the analysis of the scheme follows the same path as in Section 3. In that order, we prove that the scheme (A.7)-(A.8) is  $L^\infty$  stable, satisfies discrete entropy inequalities similar to (3.8) and approximate entropy/constraint inequalities similar to (3.9)-(3.11). Only the obtaining of compactness for  $(\rho^\Delta)_\Delta$  is left since the  $L^1_{\text{loc}}$  compactness for the sequence  $(s^\Delta)_\Delta$  and  $(q^\Delta)_\Delta$  is clear. One way to do so is to derive uniform BV bounds.

**Lemma A.4.** *We suppose that  $\rho_0 \in L^1(\mathbb{R}; [0, R]) \cap \text{BV}(\mathbb{R})$  and that  $q$  verifies the assumption*

$$\exists \varepsilon > 0, \forall t \in [0, T], \forall s \in [0, \Sigma], q(t) \leq \max_{\rho \in [0, R]} F(s, \rho) - \varepsilon = q_\varepsilon(s). \quad (\text{A.10})$$

*Then there exists a constant  $C^\varepsilon = C^\varepsilon(\|\partial_s F\|_{L^\infty})$  non-decreasing with respect to its argument such that for all  $n \in \{0, \dots, N\}$ ,*

$$\text{TV}(\rho^\Delta(t^{n+1})) \leq \text{TV}(\rho_0) + 4R + C^\varepsilon \left( \sum_{k=0}^n |q^{k+1} - q^k| + \sum_{k=0}^n |s^{k+1} - s^k| \right), \quad (\text{A.11})$$

where  $\rho^\Delta = \left( \rho_{j+\frac{1}{2}}^n \right)_{n,j}$  is the finite volume approximation constructed with the scheme (A.7)-(A.8), using the Godunov numerical flux when  $j = 0$  in (A.8).

*Proof.* With this set up we can follow the proofs of [12, Section 2] to obtain the following estimate:

$$\forall n \in \{0, \dots, N\}, \sum_{j \in \mathbb{Z}} |\rho_{j+\frac{1}{2}}^{n+1} - \rho_{j-\frac{1}{2}}^{n+1}| \leq \text{TV}(\rho_0) + 4R + 2 \sum_{k=0}^n |(\widehat{\rho}_{s^{k+1}}(q^{k+1}) - \widehat{\rho}_{s^k}(q^k)) - (\check{\rho}_{s^{k+1}}(q^{k+1}) - \check{\rho}_{s^k}(q^k))|,$$

where for all  $k \in \{0, \dots, N\}$ , the couple  $(\widehat{\rho}_{s^k}(q^k), \check{\rho}_{s^k}(q^k)) \in [0, R]^2$  is uniquely defined by the conditions

$$F(s^k, \widehat{\rho}_{s^k}(q^k)) = F(s^k, \check{\rho}_{s^k}(q^k)) = q^k, \widehat{\rho}_{s^k}(q^k) > \check{\rho}_{s^k}(q^k).$$

Note  $\Omega(\varepsilon)$  the open subset

$$\Omega(\varepsilon) = \bigcup_{s \in [0, \Sigma]} \Omega_s(\varepsilon)$$

where for all  $s \in [0, \Sigma]$ ,  $\Omega_s(\varepsilon) = (\check{\rho}_s(q_\varepsilon(s)), \widehat{\rho}_s(q_\varepsilon(s)))$ . By Assumption (A.10), the continuous function  $(s, \rho) \mapsto |\partial_\rho F(s, \rho)|$  is strictly non-negative on the compact subset  $[0, \Sigma] \times [0, R] \setminus \Omega(\varepsilon)$ . Hence, it attains its minimal value  $C_0 > 0$ . Consequently, for all  $s \in [0, \Sigma]$ , if one denotes by  $I_s : [0, \check{\rho}_s(q_\varepsilon(s))] \rightarrow [0, q_\varepsilon(s)]$  the non-decreasing part of  $F(s, \cdot)$ , this function carries out a  $C^1$ -diffeomorphism. Moreover,

$$\forall q \in [0, q_\varepsilon(s)], \left| (I_s^{-1})'(q) \right| \leq \frac{1}{C_0}.$$

Then, for all  $k \in \{0, \dots, N-1\}$ ,

$$\begin{aligned} & |\check{\rho}_{s^{k+1}}(q^{k+1}) - \check{\rho}_{s^k}(q^k)| = |(I_{s^{k+1}}^{-1})(q^{k+1}) - \check{\rho}_{s^k}(q^k)| \\ & \leq \frac{1}{C_0} |q^{k+1} - q^k| + |(I_{s^{k+1}}^{-1})(q^k) - \check{\rho}_{s^k}(q^k)| = \frac{1}{C_0} |q^{k+1} - q^k| + |(I_{s^{k+1}}^{-1})(q^k) - (I_{s^{k+1}}^{-1}) \circ I_{s^{k+1}}(\check{\rho}_{s^k}(q^k))| \\ & \leq \frac{1}{C_0} (|q^{k+1} - q^k| + |q^k - I_{s^{k+1}}(\check{\rho}_{s^k}(q^k))|) = \frac{1}{C_0} (|q^{k+1} - q^k| + |F(s^k, \check{\rho}_{s^k}(q^k)) - F(s^{k+1}, \check{\rho}_{s^k}(q^k))|) \\ & \leq \frac{1}{C_0} (|q^{k+1} - q^k| + \|\partial_s F\|_{L^\infty} |s^{k+1} - s^k|) \leq \frac{1 + \|\partial_s F\|_{L^\infty}}{C_0} (|q^{k+1} - q^k| + |s^{k+1} - s^k|). \end{aligned}$$

Using the same techniques, one can show that the same inequality holds when considering  $|\widehat{\rho}_{s^{k+1}}(q^{k+1}) - \widehat{\rho}_{s^k}(q^k)|$ . Therefore, inequality (A.11) follows with

$$C^\varepsilon = 4 \left( \frac{1 + \|\partial_s F\|_{L^\infty}}{C_0} \right).$$

□

*Remark A.1.* Recall we suppose that  $F : [0, \Sigma] \times [0, R]$  is continuously differentiable, but if we look in the details of the proof above, we actually need  $F = F(s, \rho)$  to be continuously differentiable with respect to  $s$  and

$$\forall s \in [0, \Sigma], F(s, \cdot) \in C^1([0, R] \setminus \{\bar{\rho}_s\}), \quad \bar{\rho}_s = \operatorname{argmax}_{\rho \in [0, R]} F(s, \rho).$$

**Corollary A.5.** Fix  $\rho_0 \in L^1(\mathbb{R}; [0, R]) \cap \text{BV}(\mathbb{R})$ ,  $s \in \text{BV}([0, T], [0, \Sigma])$  and  $q \in \text{BV}([0, T], \mathbb{R}^+)$ . Suppose that  $q$  verifies Assumption (A.10). We note  $\rho^\Delta = \left( \rho_{j+\frac{1}{2}}^n \right)_{n,j}$  the finite volume approximate solution constructed with the scheme (A.7)-(A.8), using the Godunov numerical flux when  $j = 0$  in (A.8), and any other monotone consistent and Lipschitz numerical flux when  $j \neq 0$ . Then there exists  $\rho \in C([0, T]; L^1_{\text{loc}}(\mathbb{R}))$  such that

$$\forall t \in [0, T], \rho^\Delta(t) \xrightarrow{\Delta \rightarrow 0} \rho(t) \text{ in } L^1_{\text{loc}}(\mathbb{R}).$$

*Proof.* Since  $s$  and  $q$  have bounded variation, inequality (A.11) leads to an uniform in time BV bound for the sequence  $(\rho^\Delta)_\Delta$ . Then the result from [22, Appendix] establish the compactness statement. □

**Theorem A.6.** Fix  $\rho_0 \in L^1(\mathbb{R}; [0, R]) \cap \text{BV}(\mathbb{R})$ ,  $s \in \text{BV}([0, T]; [0, \Sigma])$ ,  $F \in C^1([0, \Sigma] \times [0, R])$  verifying (A.2) and  $q \in \text{BV}([0, T]; \mathbb{R}^+)$ . Suppose that in (A.8), we use the Godunov flux when  $j = 0$  and any other monotone consistent and Lipschitz numerical flux when  $j \neq 0$ . Finally, suppose that  $q$  satisfies the condition

$$\exists \varepsilon > 0, \forall t \in [0, T], \forall s \in [0, \Sigma], q(t) \leq \max_{\rho \in [0, R]} F(s, \rho) - \varepsilon. \quad (\text{A.12})$$

Then under the CFL condition (A.9), the scheme (A.7)-(A.8) converges to an admissible weak solution to Problem (A.1), which is also BV-regular. More precisely, there exists a constant  $C^\varepsilon = C^\varepsilon(\|\partial_s F\|_{L^\infty})$  non-decreasing with respect to its argument such that

$$\forall t \in [0, T], \text{TV}(\rho(t)) \leq \text{TV}(\rho_0) + 4R + C^\varepsilon (\text{TV}(q) + \text{TV}(s)). \quad (\text{A.13})$$

*Proof.* From the scheme (A.7), one can derive approximate entropy/constraint inequalities analogous to (3.9)-(3.11) of Section 3. We note  $\rho$  the limit to the finite volume scheme, the compactness of  $(\rho^\Delta)_\Delta$  coming from the last corollary. We already know that  $\rho \in C([0, T]; L^1_{\text{loc}}(\mathbb{R}))$ . Moreover, by passing to the limit in the approximate entropy/constraint inequalities verified by  $(\rho^\Delta)_\Delta$  we get that  $\rho$  satisfies (A.3)-(A.4). This shows that  $\rho$  is an admissible weak solution to Problem (A.1). Finally, from (A.11), the lower semi-continuity of the BV semi-norm ensures that  $\rho \in L^\infty([0, T]; \text{BV}(\mathbb{R}))$  and verifies (A.13). This concludes the proof. □

**Corollary A.7.** Fix  $\rho_0 \in L^1(\mathbb{R}; [0, R]) \cap \text{BV}(\mathbb{R})$ ,  $s \in \text{BV}([0, T]; [0, \Sigma])$ ,  $F \in C^1([0, \Sigma] \times [0, R])$  verifying (A.2) and  $q \in \text{BV}([0, T]; \mathbb{R}^+)$ . Suppose that  $q$  satisfies Assumption (A.12).

Then the problem

$$\begin{cases} \partial_t \rho + \partial_x F(s(t), \rho) = 0 & \mathbb{R} \times (0, T) \\ \rho(x, 0) = \rho_0(x) & x \in \mathbb{R} \\ F(s(t), \rho)|_{x=0} \leq q(t) & t \in (0, T) \end{cases}$$

admits a unique admissible weak solution  $\rho$  which is also BV-regular. Moreover,  $\rho$  satisfies the bound (A.13).

*Proof.* Uniqueness comes from Theorem A.3, the existence and the BV bound comes from Theorem A.6. □

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