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► **To cite this version:**

Antoine Benoit. Weak well-posedness of hyperbolic boundary value problems in a strip: when instabilities do not meet the geometry. 2019. hal-02391814

HAL Id: hal-02391814

<https://hal.science/hal-02391814>

Preprint submitted on 3 Dec 2019

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Weak well-posedness of hyperbolic boundary value problems in a strip: when instabilities do not meet the geometry

Antoine BENOIT

Université du Littoral Côte d'Opale, LMPA
50 rue Ferdinand Buisson CS 80699 62228 Calais, France

Email: `antoine.benoit@univ-littoral.fr`

Phone: +333 21463651

December 2, 2019

Abstract

In this article we investigate the possible losses of regularity of the solution for hyperbolic boundary value problems defined in the strip $\mathbb{R}^{d-1} \times]0, 1[$.

This question has already been widely studied in the half-space geometry in which a complete characterization is almost completed (see [ST88]-[Cou02]-[Cou05] and the references in the core of the text). In this setting it is known that several behaviours are possible, for example, a loss of a derivative on the boundary only or a loss of a derivative on the boundary combined with one or a half loss in the interior.

So crudely speaking the question addressed here is "can several boundaries make the situation becomes worse?".

Here we focus our attention to one special case of loss (namely the elliptic degeneracy of [ST88]) and we show that (in terms of losses of regularity) the situation is exactly the same as the one described in the half-space, meaning that the instability does not meet the geometry. This result has to be compared with the one of [Benc] in which the geometry has a real impact on the behaviour of the solution.

1 Introduction

The aim of this paper is to investigate the optimal loss of regularity for the following hyperbolic boundary value problem in the strip $\mathbb{R}^{d-1} \times]0, 1[$:

$$\begin{cases} L(\partial)u := \partial_t u + \sum_{j=1}^d A_j \partial_j u = f & (t, x', x_d) \in \mathbb{R}_t \times \mathbb{R}_{x'}^{d-1} \times]0, 1[:= \Omega, \\ B_0 u|_{x_d=0} = g_0 & (t, x') \in \mathbb{R}_t \times \mathbb{R}_{x'}^{d-1} := \omega, \\ B_1 u|_{x_d=1} = g_1 & (t, x') \in \omega, \\ u|_{t \leq 0} = 0 & (x', x_d) \in \mathbb{R}_{x'}^{d-1} \times [0, 1] := \Gamma, \end{cases} \quad (1)$$

where the coefficients $A_j \in \mathbf{M}_{N \times N}(\mathbb{R})$, $B_0 \in \mathbf{M}_{p \times N}(\mathbb{R})$ and $B_1 \in \mathbf{M}_{(N-p) \times N}(\mathbb{R})$ (we refer to Assumption 2.2 for the precise value of p).

By loss of regularity we mean that the (unique) solution u to (1) is not as regular (in terms of Sobolev spaces) as the source terms of (1), namely f , g_0 and g_1 . Such problems are referred as weakly well-posed problems, in contrast with problems for which the unique solution to (1) is as regular as the data of the system (referred as strongly well-posed problems).

Before to turn to a description of the results of this article let us recall what is known in the more classical geometry of the half-space.

The main result in the study of the strong well-posedness of the analogous of (1) in the half-space $\mathbb{R}^{d-1} \times \mathbb{R}_+$ is due to Kreiss in [Kre70] where the author gives a full characterization of the boundary conditions leading to strong well-posedness (see also [Sar65]).

This characterization, the so-called uniform Kreiss-Lopatinskii condition, relies on the normal mode analysis of the boundary value problem in the half-space. Without enter into technical details (we refer to [BGS07]-[CP81] for a complete study and to Section 2 for more informations) to the time/space half-space problem we associate its so-called resolvent frequency form (obtained after Laplace in time and Fourier in tangential space transforms):

$$\begin{cases} \frac{d}{dx_d} \widehat{u}(x_d; \zeta) = \mathcal{A}(\zeta) \widehat{u}(x_d; \zeta) + A_d^{-1} \widehat{f}(x_d; \zeta) & x_d > 0, \\ B_0 \widehat{u}|_{x_d=0}(\zeta) = \widehat{g}_0(\zeta), \end{cases} \quad (2)$$

where ζ stands for the frequency parameter. The behaviour of the solution to the ordinary differential equation (2) strongly depends on the matrix \mathcal{A} and [Kre70] is based on a very precise analysis of the possible structures of this matrix.

There are four different kinds of frequencies depending on the generalized eigenvalues of $\mathcal{A}(\zeta)$, namely: elliptic, mixed, hyperbolic and glancing.

The problem (2) satisfies the uniform Kreiss-Lopatinskii condition and thus is strongly well-posed if this condition (see Assumption 2.3 for more details) holds for all possible kinds of frequency. However it has also been observed on many examples (see for example the ones of [BGRSZ02]) that generically boundary conditions do not satisfy the uniform Kreiss-Lopatinskii condition.

A natural question is thus to determine what is the behaviour of the solution when the uniform Kreiss-Lopatinskii condition degenerates ?

The first possibility is that the problem generates a Hadamard instability and thus is ill-posed. However when the uniform Kreiss-Lopatinskii condition breaks down on one of the four kinds of frequency introduced above we may still have existence and uniqueness of the solution up to some losses of regularity.

Historically, this question has first been studied in [ST88] (in view of its applications in elastodynamics) in the case of a degeneracy of the uniform Kreiss-Lopatinskii condition for an elliptic frequency and a loss of one derivative on the boundary is shown.

Then the same question has been addressed in [Cou02] and [Cou05] for a degeneracy on mixed and on hyperbolic frequencies (in this context the main physical applications are phase transition and shock wave stability). For a mixed (resp. hyperbolic) degeneracy a loss of a half (resp. one) derivative in the interior appears in addition of the loss of one derivative on the boundary.

Once these estimations for the loss of regularity of the solution are stated an interesting remaining question is the one of their sharpness. This question has been widely studied in the litterature and we refer to [Mar10] for the elliptic case to [Ben14] for the mixed case and to [CG10] for the hyperbolic case (see also the formal analysis of [MR83] and [Maj88]) and the sharpness of all the estimates described so far is now established.

So that at present time, the characterization of weakly well-posed boundary value problems in the half-space is achieved (up to the somewhat specific case of a degeneracy on a glancing frequency).

The common point of all these proofs is that they use WKB expansions (that is an approximate solution to (1) in the asymptotic of highly oscillating source terms) to saturate the known energy estimates in order to show their sharpness.

Concerning the strip geometry we can show that the uniform Kreiss-Lopatinskii condition on each side of the strip is sufficient to ensure strong well-posedness (up to pay some exponential growth of the solution with respect to time (see [Bena] in which this growth is studied)). However, in the author's knowledge, the classification of the possible weakly well-posed problems in the strip has not been investigated at present time.

In this article we restrict our attention to a degeneracy of the uniform Kreiss-Lopatinskii condition on an elliptic frequency, while in [Benc] the degeneracy takes place on a hyperbolic frequency.

The main results of this article, namely Theorems 2.1 and 2.2, are the following:

- Theorem 2.1 establishes the weak well-posedness of (1) when one of the boundary conditions is responsible of a failure of the uniform Kreiss-Lopatinskii on an elliptic frequency¹.
- Theorem 2.2 shows that the energy estimate of Theorem 2.1, with a loss of one derivative on the boundary where the uniform Kreiss-Lopatinskii condition fails but any loss in the interior, is sharp.

Consequently in the elliptic degeneracy framework the situation in the strip geometry is not worse than the one in the half-space geometry. This result is not surprising and it is linked to the fact that, contrary to hyperbolic modes, elliptic modes can not propagate information/singularities for one side of the strip to the other. However, we believe that this result is interesting for several reasons.

Firstly because it gives a class of hyperbolic boundary value problems in the strip for which the loss of derivatives is at most one² and in this framework one can expect to use iterative methods in order to treat non-linear problems.

Secondly, and in the author's opinion it is the most interesting point, because this non-amplification of singularities has to be compared with the result obtained in [Benc]. Indeed for a degeneracy of the uniform Kreiss-Lopatinskii condition on a hyperbolic frequency, [Benc] shows that the loss of derivatives of the solution has to be increasing with the final time of resolution of the system.

So the results of this paper combined with the one of [Benc] show that the behaviour of the solution can be totally different from one kind of degeneracy to the other (this is not really the case for weakly well-posed boundary value problems in the half-space for which the number of losses remain finite, no matter the time scale is). This is an example of the wide variety of possible phenomena for hyperbolic boundary value problems with several boundaries.

This paper is organized as follows: in Section 2 we describe the main assumptions on (1) and state the main results, namely Theorem 2.1 and Theorem 2.2. Then Section 3 gives the main steps in the proof of Theorem 2.1 while Section 4 exposes the ones of Theorem 2.2.

2 Notations and statements

2.1 Some general notations

In the core of the text we use $\cdot \lesssim \cdot$ as a short hand notation for: "there exists $C > 0$ such that $\cdot \leq C \cdot$ ". When C depends on some (relevant) parameter λ we specify it by using the notation \lesssim_λ . For $a, b \in \mathbb{R}$ we write $\llbracket a, b \rrbracket := [a, b] \cap \mathbb{Z}$.

For $T > 0$, we introduce:

$$\Omega_T :=]-\infty, T] \times \mathbb{R}_{x'}^{d-1} \times]0, 1[\quad \text{and} \quad \omega_T :=]-\infty, T] \times \mathbb{R}_{x'}^{d-1}.$$

The frequency space used in this article is given by:

$$\Xi := \{ \zeta := (\gamma + i\tau, \eta) \in \mathbb{C} \times \mathbb{R}^{d-1} \setminus \gamma \geq 0 \} \setminus \{ (0, 0) \},$$

and for convenience we use the notation $\Xi_0 := \Xi \cap \{ \gamma = 0 \}$.

For $s \in \mathbb{R}$, $\gamma > 0$ and $X \subset \mathbb{R}_t \times \mathbb{R}_{x'}^{d-1} \times]0, 1[$ we define the following weighted (in time) Sobolev spaces by

$$H_\gamma^s(X) := \{ u \in \mathcal{D}'(X) \setminus e^{-\gamma t} u \in H^s(X) \},$$

and we set $L_\gamma^2(X) := H_\gamma^0(X)$.

¹All the results of the paper hold *mutatis mutandis* if each boundary condition is responsible of a degeneracy of the uniform Kreiss-Lopatinskii condition on an elliptic frequency.

²It is in fact zero when the boundary conditions responsible of the failure of the uniform Kreiss-Lopatinskii condition are homogeneous.

2.2 Main notations and assumptions

The first assumption on the strip problem (1) is an hyperbolicity assumption for $L(\partial)$

Assumption 2.1 [Constant hyperbolicity] *There exist $M \in \llbracket 1, N \rrbracket$ and $\lambda_1, \dots, \lambda_M$ analytic real valued functions on $\mathbb{R}^d \setminus \{0\}$ and finally $\mu_1, \dots, \mu_M \geq 1$ such that*

$$\forall \xi \in \mathbb{S}^{d-1}, \det \left(\tau + \sum_{j=1}^d \xi_j A_j \right) = \prod_{k=1}^M (\tau + \lambda_k(\xi))^{\mu_k},$$

with $\lambda_1 < \dots < \lambda_M$ and the eigenvalues $\lambda_k(\xi)$ are semi-simple.

Then a simplifying assumption is to restrict the study to non-characteristic boundary conditions.

Assumption 2.2 [Non-characteristic boundary] *The matrix A_d is invertible. Moreover p is the number of strictly positive eigenvalues of A_d .*

With Assumptions (2.1) and 2.2 in hand we define the resolvent matrix of (1) by:

$$\forall \zeta \in \Xi, \mathcal{A}(\zeta) := -A_d^{-1} \left((\gamma + i\tau)I + i \sum_{j=1}^{d-1} \eta_j A_j \right). \quad (3)$$

The result of [Her63] implies that for all $\zeta \in \Xi \setminus \Xi_0$, the stable (resp. unstable) subspace $E^s(\zeta)$ (resp. $E^u(\zeta)$), that is the space generated by the eigenmodes associated to the generalized eigenvalues with strictly negative (resp. positive) real part, is well-defined and of constant dimension equals to p (resp. $N - p$).

Moreover the results of [Kre70] and [Mét00] show that the stable subspace $E^s(\zeta)$ can be extended by continuity up to Ξ_0 . Of course in this extension some of the real parts may vanish. If such a situation occurs the associated frequency are called: mixed, hyperbolic or glancing (essentially depending on the number of eigenvalues that degenerate, we refer to [Mét00] for more details).

In this paper we are, however, interested in the case where all the eigenvalues keep a signed real part³ in this extension process, these frequencies are referred as elliptic frequencies. With more details we give the following definition:

Definition 2.1 (Elliptic area) *Under Assumptions 2.1 and 2.2 the elliptic area \mathcal{E} is the set of $\zeta \in \Xi_0$ such that there exist a neighbourhood \mathcal{V} of ζ in Ξ and a matrix $T \in \mathbf{GL}_N(\mathbb{C})$ regular on \mathcal{V} such that:*

$$\forall \zeta \in \mathcal{V}, \mathcal{A}(\zeta) = T^{-1}(\zeta) \text{diag}(\mathcal{A}^s(\zeta), \mathcal{A}^u(\zeta))T(\zeta),$$

where $\mathcal{A}^s, \mathcal{A}^u \in \mathbf{M}_{p \times p}(\mathbb{C})$ are regular on \mathcal{V} and satisfy that for all $\zeta \in \mathcal{V}$, $\text{Re}(\text{sp}(\mathcal{A}^s(\zeta))) \subset]-\infty, 0[$ and $\text{Re}(\text{sp}(\mathcal{A}^u(\zeta))) \subset]0, +\infty[$.

In particular, for $\zeta \in \mathcal{E}$ the decomposition

$$\mathbb{C}^N = E^s(\zeta) \oplus E^u(\zeta), \quad (4)$$

of [Her63] still holds and we denote by $\Pi^s(\zeta)$ (resp. $\Pi^u(\zeta)$) the projection on $E^s(\zeta)$ (resp. $E^u(\zeta)$) with respect to (4).

Our last main assumption deals with the boundary conditions B_0 and B_1 :

Assumption 2.3 *The boundary conditions B_0 and B_1 satisfy the following*

1. *The boundary condition on $\{x_d = 1\}$ satisfies the uniform Kreiss-Lopatinskii condition that is to say that*

$$\forall \zeta \in \Xi, \ker B_1 \cap E^u(\zeta) = \{0\}.$$

³Note that the existence of such a configuration implies that $N = 2p$.

2. The boundary condition on $\{x_d = 0\}$ violates the uniform Kreiss-Lopatinskii condition at order one in the elliptic area⁴. More precisely if we denote by $\Upsilon := \{\zeta \in \Xi \setminus \ker B_0 \cap E^s(\zeta) \neq \{0\}\}$ then

(a) the weak Kreiss-Lopatinskii condition is satisfied that is

$$\forall \zeta \in \Xi \setminus \Xi_0, \ker B_0 \cap E^s(\zeta) = \{0\},$$

and consequently $\Upsilon \subset \Xi_0$.

(b) We have $\Upsilon \neq \emptyset$ and $\Upsilon \subset \mathcal{E}$.

(c) Let $\underline{\zeta} \in \Upsilon$ then there exist a neighbourhood \mathcal{V} of $\underline{\zeta} \in \Xi$, a basis $(E_1, \dots, E_p)(\zeta)$ of $E^s(\zeta)$ regular on \mathcal{V} , a matrix $P = P(\zeta) \in \mathbf{GL}_p(\mathbb{C})$ regular on \mathcal{V} and a real valued function θ regular on \mathcal{V} such that

$$\forall \zeta \in \mathcal{V}, B_0 [E_1, \dots, E_p](\zeta) = P(\zeta) \text{diag}(\gamma + i\theta(\zeta), 1, \dots, 1).$$

2.3 Main results

The main results of the article are the following⁵:

Theorem 2.1 Under Assumptions 2.1, 2.2 and 2.3 there exists $\underline{\gamma} > 0$ such that for all $\gamma \geq \underline{\gamma}$ and for all $f \in L^2_\gamma(\Omega)$, $g_0 \in H^{1/2}_\gamma(\omega)$, $g_1 \in L^2_\gamma(\omega)$, the strip problem (1) admits a unique solution $u \in L^2_\gamma(\Omega)$ with traces $u|_{x_d=0} \in H^{-1/2}_\gamma(\omega)$ and $u|_{x_d=1} \in L^2_\gamma(\omega)$. Moreover u satisfies the energy estimate: for all $\gamma \geq \underline{\gamma}$ we have

$$\gamma \|u\|_{L^2_\gamma(\Omega)}^2 + \gamma \|u|_{x_d=0}\|_{H^{-1/2}_\gamma(\omega)}^2 + \|u|_{x_d=1}\|_{L^2_\gamma(\omega)}^2 \lesssim \frac{1}{\gamma} \|f\|_{L^2_\gamma(\Omega)}^2 + \frac{1}{\gamma} \|g_0\|_{H^{1/2}_\gamma(\omega)}^2 + \|g_1\|_{L^2_\gamma(\omega)}^2. \quad (5)$$

Remark • If the side $\{x_d = 1\}$ also violates the uniform Kreiss-Lopatinskii condition in the elliptic area (see 2. of Assumption 2.3) then Theorem 2.1 still holds by assuming that $g_1 \in H^{1/2}_\gamma(\omega)$ and with the energy estimate: for all $\gamma \geq \underline{\gamma}$,

$$\|u\|_{L^2_\gamma(\Omega)}^2 + \|u|_{x_d=0}\|_{H^{-1/2}_\gamma(\omega)}^2 + \|u|_{x_d=1}\|_{H^{-1/2}_\gamma(\omega)}^2 \lesssim \frac{1}{\gamma^2} \left(\|f\|_{L^2_\gamma(\Omega)}^2 + \|g_0\|_{H^{1/2}_\gamma(\omega)}^2 + \|g_1\|_{H^{1/2}_\gamma(\omega)}^2 \right). \quad (6)$$

- If the boundary condition(s) on which the Kreiss-Lopatinskii condition degenerates in the elliptic area is (are) homogeneous then a direct consequence of (5) ((6)) is that we have the energy estimate without loss of derivative

$$\forall \gamma \geq \underline{\gamma}, \|u\|_{L^2(\Omega)}^2 \lesssim \frac{1}{\gamma^2} \|f\|_{L^2(\Omega)}^2.$$

In such a situation we then have examples of hyperbolic boundary value problems in the strip without loss of derivatives even if the uniform Kreiss-Lopatinskii condition is not satisfied.

The following theorem shows that (5) is sharp. More precisely

Theorem 2.2 Under Assumptions 2.1, 2.2 and 2.3, let $s > 0$, we assume that for all $f \in L^2(\Omega_T)$, $g_0 \in H^s(\omega_T)$, $g_1 \in L^2(\omega_T)$ the strip problem (1) admits a solution u such that

$$\|u\|_{L^2(\Omega_T)}^2 \lesssim_T \|f\|_{L^2(\Omega_T)}^2 + \|g_0\|_{H^s(\omega_T)}^2 + \|g_1\|_{L^2(\omega_T)}^2, \quad (7)$$

then necessarily $s \geq 1/2$.

⁴The definition given here differs a little in its formulation (but is equivalent) from the one given in [[ST88] condition (LC)]. The advantage of this formulation will be made precise in Section 4.

⁵Theorem 2.1 and its proof can, under the suitable assumptions, probably be extended to time/space depending coefficients matrices A_j , B_0 and B_1 .

3 Weak well-posedness of (1), proof of Theorem 2.1

The proof of Theorem 2.1 follows the classical proof to show the strong well-posedness of hyperbolic boundary value problems in the half-space already used in [Kre70]. It is a three steps proof:

1. we first show an *a priori* energy estimate for (1).
2. Then we show the existence of a weak solution.
3. At last we conclude to the uniqueness of the constructed weak solution, by showing that it satisfies the *a priori* energy estimate, thanks to a "weak=strong" lemma.

The fact that the same sketch of proof (even if, of course it involves many technical difficulties) also operates for weakly well-posed problems (see [ST88]-[Cou02] and [Cou05]) is quite interesting in its own and shows the robustness of the whole method.

In Paragraph 3.1 we give a simple proof of the *a priori* energy estimate (that is (5)) based on localization methods. The proof exposed here is (likely to be) the simplest possible one because we are not interested in the minimal exponential growth in time rate in the estimates.

Then Paragraph 3.2 contains some elements of proof concerning the existence and the "weak=strong" lemma.

3.1 A priori energy estimate for (1)

Let u be a smooth solution to (1). We introduce two cut-off functions $\chi_0, \chi_1 \in \mathcal{D}([-2, 2[, [0, 1])$ that satisfy

$$\chi_0(x_d) = \begin{cases} 1 & \text{if } x_d \in [0, \frac{1}{3}], \\ 0 & \text{if } x_d \geq \frac{2}{3}, \end{cases} \quad \text{and} \quad \chi_1(x_d) = \begin{cases} 0 & \text{if } x_d \leq \frac{1}{3}, \\ 1 & \text{if } x_d \in [\frac{2}{3}, 1]. \end{cases}$$

Define $\chi_{1/2} := 1 - \chi_0 - \chi_1$ and for all $j \in \llbracket 0, 2 \rrbracket$, $u_{j/2} := \chi_{j/2}u$.

By construction $u_{1/2}$ solves the Cauchy problem for the hyperbolic operator $L(\partial)$ so that we have the standard energy estimate: there exists $\gamma_{1/2} > 0$ such that for all $\gamma \geq \gamma_{1/2}$

$$\gamma \|u_{1/2}\|_{L_\gamma^2(\Omega)}^2 \lesssim \frac{1}{\gamma} \|L(\partial)u_{1/2}\|_{L_\gamma^2(\Omega)}^2. \quad (8)$$

Then u_0 (resp. u_1) solves the boundary value problem in the half-space $\mathbb{R}^{d-1} \times]0, \infty[$ (resp. $\mathbb{R}^{d-1} \times]-\infty, 1[$) so that from [ST88] (resp. [Kre70]) we have the energy estimate: there exists $\gamma_0 > 0$ (resp. $\gamma_1 > 0$) such that for all $\gamma \geq \gamma_0$ (resp. $\gamma \geq \gamma_1$):

$$\|u_0\|_{L_\gamma^2(\Omega)}^2 + \|u_{0|x_d=0}\|_{H_\gamma^{-1/2}(\omega)}^2 \lesssim \frac{1}{\gamma^2} \left(\|L(\partial)u_0\|_{L_\gamma^2(\Omega)}^2 + \|B_0u_{0|x_d=0}\|_{H_\gamma^{1/2}(\omega)}^2 \right), \quad (9)$$

$$\left(\text{resp. } \gamma \|u_1\|_{L_\gamma^2(\Omega)}^2 + \|u_{1|x_d=1}\|_{L_\gamma^2(\omega)}^2 \lesssim \frac{1}{\gamma} \|L(\partial)u_1\|_{L_\gamma^2(\Omega)}^2 + \|B_1u_{1|x_d=1}\|_{L_\gamma^2(\omega)}^2 \right). \quad (10)$$

However for all $j \in \llbracket 0, 2 \rrbracket$ we have $\|L(\partial)u_{j/2}\|_{L_\gamma^2(\Omega)} \lesssim_{A_d} \|f\|_{L_\gamma^2(\Omega)} + \|u\|_{L_\gamma^2(\Omega)}$. So that from the triangle inequality, the trace equalities $u_{|x_d=j} = u_{j|x_d=j}$ (for $j \in \{0, 1\}$) combined with (8), (9) and (10) it turns out that for all $\gamma \geq \max(\gamma_0, \gamma_{1/2}, \gamma_1)$:

$$\gamma \|u\|_{L_\gamma^2(\Omega)}^2 + \gamma \|u_{|x_d=0}\|_{H_\gamma^{-1/2}(\omega)}^2 + \|u_{|x_d=1}\|_{L_\gamma^2(\omega)}^2 \lesssim \frac{1}{\gamma} \|f\|_{L_\gamma^2(\Omega)}^2 + \frac{1}{\gamma} \|g_0\|_{H_\gamma^{1/2}(\omega)}^2 + \|g_1\|_{L_\gamma^2(\omega)}^2 + \frac{1}{\gamma} \|u\|_{L_\gamma^2(\Omega)}^2,$$

so that up to choose γ large enough we can absorb the term $\frac{1}{\gamma} \|u\|_{L_\gamma^2(\Omega)}^2$ from the right hand side into the left hand side and (5) follows for all regular solution to (1).

3.2 Existence and uniqueness

Once the *a priori* energy estimate for (1) is established, the existence and the uniqueness of the solution to (1) is proved exactly as exposed in [ST88] (in the simplified framework of constant coefficients in (1)).

A way to show the existence of a solution is to introduce a so-called dual problem of (1). More precisely we define:

$$\begin{cases} L^*(\partial)v := -\left(\partial_t + \sum_{j=1}^d A_j^T \partial_j\right)v = \tilde{f} & \text{in } \Omega, \\ C_0 v|_{x_d=0} = \tilde{g}_0 & \text{for } (t, x') \in \omega, \\ C_1 v|_{x_d=1} = \tilde{g}_1 & \text{for } (t, x') \in \omega, \\ v|_{t \leq 0} = 0 & \text{on } \Gamma, \end{cases} \quad (11)$$

where the boundary matrices are chosen in such a way that the following decompositions hold:

$$\text{for } \ell \in \{0, 1\}, A_d = C_\ell^T N_\ell + M_\ell^T B_\ell,$$

where $M_\ell, N_\ell, C_\ell \in \mathbf{M}_{p \times N}(\mathbb{R})$ in order to have the Green formula (for all u, v regular enough):

$$\langle L(\partial)u, v \rangle - \langle u, L^*(\partial)v \rangle = \sum_{\ell=0}^1 (-1)^{\ell+1} \left(\langle N_\ell u|_{x_d=\ell}, C_\ell v|_{x_d=\ell} \rangle + \langle B_\ell u|_{x_d=\ell}, M_\ell v|_{x_d=\ell} \rangle \right), \quad (12)$$

where

The end of the proof consists in showing that the dual problem (11) satisfies Assumptions 2.1, 2.2 and 2.3 in order to show that it satisfies the *a priori* energy estimate so that, essentially the Green formula (12) combined with Riesz representation theorem permits to show the existence of a weak solution to (1)⁶.

The fact that the dual problem (11) satisfies Assumptions 2.1 and 2.2 is readable from the definition of $L^*(\partial)$. The fact that (11) satisfies Assumption 2.3 is a consequence of the definition of the Lopatinskii determinant for the dual problem (we refer to Theorem 4.4 of [BGS07]).

At last the "weak=strong" lemma does not require a lot of comments because in the strip geometry there is only one normal variable, namely x_d . So that the classical regularization by a mollifier in the tangential variables applies exactly as in the half-space geometry (see [BGS07] Section 4.5).

This ends the proof of Theorem 2.1.

4 Saturation of (5), proof of Theorem 2.2

In this section we give a proof of Theorem 2.2. The sketch of proof that we follow here relies on the construction of a geometric optics expansion for (1) and consequently it is probably not the straightest possible proof⁷.

However we believe that this sketch of proof is interesting for several reasons. The first one is that giving an approximate solution to (1) is an interesting result in its own. The second is that the construction in itself explains why no new loss of derivative occurs.

And finally, because the construction of the geometric optics expansions for an elliptic frequency (on which the uniform Kreiss-Lopatinskii degenerates) is simpler than the one for a hyperbolic frequency (see [Benc]).

⁶We do not expose here the precise functional or the spaces used. However they are exactly the same as the ones used in [[ST88]-Section 1.3] up to the fact that the extra boundary condition on $\{x_d = 1\}$ has to be included. However this boundary condition is harmless because it satisfies the uniform Kreiss-Lopatinskii condition so the standard duality method applies directly.

⁷Indeed the fact that the estimate of [ST88] is sharp in the half-space geometry combined with finite speed of propagation and localization arguments make the sharpness of (5) almost readable.

4.1 Construction of a geometric optics expansion

Let $\underline{\zeta} := (i\underline{\tau}, \underline{\eta}) \in \Upsilon$ in all the following we consider the phase function $\psi := \psi(t, x') = \underline{\tau}t + \underline{\eta} \cdot x'$ and we consider the following highly oscillating boundary value problem: for $0 < \varepsilon \ll 1$

$$\begin{cases} L(\partial)u^\varepsilon = f^\varepsilon, & \text{in } \Omega, \\ B_0 u^\varepsilon|_{x_d=0} = g_0^\varepsilon & \text{for } (t, x') \in \omega, \\ B_1 u^\varepsilon|_{x_d=1} = g_1^\varepsilon & \text{for } (t, x') \in \omega, \\ u^\varepsilon|_{t \leq 0} = 0 & \text{on } \Gamma, \end{cases} \quad (13)$$

where the source terms in (13) are given by:

$$\begin{aligned} f^\varepsilon(t, x) &:= e^{\frac{i}{\varepsilon}\psi} \left(f_{ev} \left(t, x; \frac{x_d}{\varepsilon} \right) + f_{ex} \left(t, x; \frac{x_d - 1}{\varepsilon} \right) \right), \\ g_0^\varepsilon(t, x') &:= \varepsilon e^{\frac{i}{\varepsilon}\psi} g_0(t, x') \text{ and } g_1^\varepsilon(t, x') := e^{\frac{i}{\varepsilon}\psi} g_1(t, x'). \end{aligned}$$

the boundary amplitudes $g_0, g_1 \in H^\infty(\omega)$ vanish for negative times while the interior amplitudes $f_{ev} \in \mathcal{P}_{ev}(\Omega)$, $f_{ex} \in \mathcal{P}_{ex}(\Omega)$; the profile spaces defined as follows:

Definition 4.1 (Profile space) We denote by $\mathcal{P}_{ev}(\Omega)$ (resp. $\mathcal{P}_{ex}(\Omega)$) the set of evanescent (resp. explosive) profiles.

More precisely, $\mathcal{P}_{ev}(\Omega)$ (resp. $\mathcal{P}_{ex}(\Omega)$) is the set of functions $U = U(t, x; X) \in H^\infty(\Omega \times \mathbb{R}_+)$ (resp. $H^\infty(\Omega \times \mathbb{R}_-)$) vanishing on negative times and for which there exists $\delta > 0$ such that $e^{\delta X} U(t, x; X) \in H^\infty(\Omega \times \mathbb{R}_+)$ (resp. $H^\infty(\Omega \times \mathbb{R}_-)$).

For later use we also introduce the following vectors:

Definition 4.2 Under Assumptions 2.1, 2.2 and 2.3, let $\underline{\zeta} \in \Upsilon$ we can define:

- a vector $e := e(\underline{\zeta}) \in \mathbb{C}^N \setminus \{0\}$ such that $\ker B_0 \cap E^s(\underline{\zeta}) = \text{vect}\{e\}$.
- A vector $b := b(\underline{\zeta}) \in \mathbb{C}^p \setminus \{0\}$ such that for all $v \in E^s(\underline{\zeta})$ we have $b \cdot B_0 v = 0$.
- Finally a supplementary space $\check{E}^s := \check{E}^s(\underline{\zeta})$ to $\text{vect}\{e\}$ in $E^s(\underline{\zeta})$ such that B is an isomorphism from \check{E}^s to b^\perp .

We postulate for ansatz:

$$u^\varepsilon(t, x) \sim \sum_{n \geq 0} \varepsilon^n e^{\frac{i}{\varepsilon}\psi} \left(U_{ev,n} \left(t, x; \frac{x_d}{\varepsilon} \right) + U_{ex,n} \left(t, x; \frac{x_d - 1}{\varepsilon} \right) \right), \quad (14)$$

where for all $n \geq 0$, $U_{n,ev} \in \mathcal{P}_{ev}$ and $U_{n,ex} \in \mathcal{P}_{ex}$.

Injecting the ansatz (14) in the evolution equation of (13) gives (because of the definition of the profile spaces) the cascade of equations:

$$\begin{cases} L(\partial_X)U_{0,ev} = 0 & \text{on } \Omega \times \mathbb{R}_+, \\ L(\partial_X)U_{0,ex} = 0 & \text{on } \Omega \times \mathbb{R}_-, \\ L(\partial_X)U_{n+1,ev} + L(\partial)U_{n,ev} = \delta_{n,0} f_{ev} & \text{on } \Omega \times \mathbb{R}_+ \text{ and } \forall n \geq 0, \\ L(\partial_X)U_{n+1,ex} + L(\partial)U_{n,ex} = \delta_{n,0} f_{ex} & \text{on } \Omega \times \mathbb{R}_- \text{ and } \forall n \geq 0, \end{cases} \quad (15)$$

where $\delta_{\cdot, \cdot}$ stands for the Kronecker symbol and where the operator of derivation in the fast variable $L(\partial_X)$ is given by

$$L(\partial_X) := A_d (\partial_X - \mathcal{A}(\underline{\zeta})),$$

in which we recall that $\mathcal{A}(\underline{\zeta})$ stands for the resolvent matrix defined in (3).

Then plugging the ansatz (14) in the boundary conditions of (13) leads to:

$$\forall n \geq 0 \begin{cases} B_0 U_{ev,n}(t, x', 0; 0) + B_0 U_{ex,n}(t, x', 0; -\frac{1}{\varepsilon}) = \delta_{n,1} g_0 & \text{on } \omega, \\ B_1 U_{ev,n}(t, x', 1; \frac{1}{\varepsilon}) + B_1 U_{ex,n}(t, x', 1; 0) = \delta_{n,0} g_1 & \text{on } \omega. \end{cases}$$

However by definition of the profile spaces the terms $U_{ex,n}(t, x', 0; -\frac{1}{\varepsilon})$ and $U_{ev,n}(t, x', 1; \frac{1}{\varepsilon})$ are $O(\varepsilon^\infty)$ so that they can be neglected in the previous boundary conditions.

Consequently in the particular setting where the frequency is elliptic we obtain a total decoupling of the cascade of equations. It turns out that the cascades of equations to solve are:

$$\begin{cases} L(\partial_X)U_{0,ev} = 0 & \text{on } \Omega \times \mathbb{R}_+, \\ L(\partial_X)U_{n+1,ev} + L(\partial)U_{n,ev} = \delta_{n,0} f_{ev} & \text{on } \Omega \times \mathbb{R}_+ \text{ and } \forall n \geq 0, \\ B_0 U_{ev,n}(t, x', 0; 0) = \delta_{n,1} g_0 & \text{on } \omega \text{ and } \forall n \geq 0, \\ U_{ev,n}|_{t \leq 0} = 0 & \text{on } \Gamma \text{ and } \forall n \geq 0, \end{cases} \quad (16)$$

and

$$\begin{cases} L(\partial_X)U_{0,ex} = 0 & \text{on } \Omega \times \mathbb{R}_-, \\ L(\partial_X)U_{n+1,ex} + L(\partial)U_{n,ex} = \delta_{n,0} f_{ex} & \text{on } \Omega \times \mathbb{R}_- \text{ and } \forall n \geq 0, \\ B_1 U_{ex,n}(t, x', 1; 0) = \delta_{n,0} g_1 & \text{on } \omega \text{ and } \forall n \geq 0, \\ U_{ex,n}|_{t \leq 0} = 0 & \text{on } \Gamma \text{ and } \forall n \geq 0. \end{cases} \quad (17)$$

In the following the construction of a sequence $(U_{ex,n})_{n \in \mathbb{N}}$ satisfying (17) will not be described because it is a slight adaptation of the works of [Mar10]-[Les07] (see also [Wil96]-[Wil00] for a construction in which the elliptic modes are treated mode by mode, and not by a one block argument)⁸.

The construction of a sequence $(U_{ev,n})_{n \in \mathbb{N}}$ satisfying (16) is based on the work of [Mar10] (up to the little modification induced by a non-zero source term on the boundary). However, because this mode carries the loss of regularity of the boundary value problem (1) we sketch in the following the main steps in the construction of $U_{0,ev}$.

The key ingredient in the construction of $U_{0,ev}$ is the following lemma (a consequence of Duhamel formula)):

Lemma 4.1 *[[Les07]] For $X \in \mathbb{R}_+$ and $U \in \mathcal{P}_{ev}$ define*

$$(\mathbf{P}_{ev}U)(t, x; X) := e^{X\mathcal{A}(\zeta)} \Pi^s(\zeta) U(t, x; 0)$$

and

$$(\mathbf{Q}_{ev}U)(t, x; X) := \int_0^X e^{(X-y)\mathcal{A}(\zeta)} \Pi^s(\zeta) A_d^{-1} U(t, x; y) dy - \int_X^\infty e^{(X-y)\mathcal{A}(\zeta)} \Pi^u(\zeta) A_d^{-1} U(t, x; y) dy.$$

Then for all $F \in \mathcal{P}_{ev}$ the equation $L(\partial_X)U = F$ on $\Omega \times \mathbb{R}_+$ admits a solution $U \in \mathcal{P}_{ev}$ written under the form $U = \mathbf{P}_{ev}U + \mathbf{Q}_{ev}F$.

From Lemma 4.1 we obtain the "polarization" relation on the leading order amplitude that is $U_{0,ev} = \mathbf{P}_{ev}U_{0,ev}$. Consequently from the definition of \mathbf{P}_{ev} we deduce that to determine $U_{0,ev}$ it is sufficient to determine $U_{0,ev}(t, x; 0)$ and that $U_{0,ev} \in E^s(\zeta)$.

To determine $U_{0,ev}(t, x; 0)$, we first determine the "double" trace $U_{0,ev}(t, x', 0; 0)$ and then to extend this trace to $x_d > 0$ we consider it as a boundary layer.

The boundary condition of (17) written for $n = 0$ implies that $U_{0,ev}(t, x', 0; 0) \in \ker B_0$ so that there exists a scalar function ϱ_0 defined on ω such that $U_{0,ev}(t, x', 0; 0) = \varrho_0(t, x')e$ and the construction of $U_{0,ev}$ only requires the determination of ϱ_0 .

⁸Moreover this construction is more or less contained (in a simpler setting) in the construction of the $(U_{ev,n})_{n \in \mathbb{N}}$ below.

In order to determine ϱ_0 we consider the amplitude of order one. Because of Lemma 4.1 applied to the second equation of (16) (for $n = 1$) we have $(I - \mathbf{P}_{ev})U_{1,ev} = \mathbf{Q}_{ev}(f_{ev} - L(\partial)U_{ev,0})$. Injecting this expression in the boundary condition of (16) (for $n = 1$) then gives

$$\underbrace{b \cdot B_0(\mathbf{P}_{ev}U_{1,ev})|_{x_d=X=0}}_{=0} - b \cdot B_0 \int_0^\infty e^{-s\mathcal{A}(\underline{\zeta})} \Pi^u(\underline{\zeta}) A_d^{-1} (f_{ev} - L(\partial)U_{0,ev})(t, x', 0; s) ds = b \cdot g_0.$$

Expanding the left hand side of the previous equation shows that ϱ_0 satisfies the transport equation:

$$\begin{cases} \left(b \cdot B_0 I_0 \partial_t + \sum_{j=1}^{d-1} b \cdot B_0 I_j \partial_j \right) \varrho_0 = \tilde{g}_0 & \text{in } \omega, \\ \varrho_0|_{t \leq 0} = 0, \end{cases} \quad (18)$$

where $\tilde{g}_0 := b \cdot \left(g_0 + B_0 \int_0^\infty e^{-s\mathcal{A}(\underline{\zeta})} \Pi^u(\underline{\zeta}) A_d^{-1} f_{ev} ds \right)$ and for all $j \in \llbracket 0, d-1 \rrbracket$,

$$I_j := \int_0^\infty e^{-s\mathcal{A}(\underline{\zeta})} \Pi^u(\underline{\zeta}) A_d^{-1} A_j e^{s\mathcal{A}(\underline{\zeta})} e ds \quad (\text{with the convention } A_0 = I).$$

Reiterating the same proof as the one of [Mar10], one can show that for all $j \in \llbracket 0, d-1 \rrbracket$, $I_j = \Pi^u(\underline{\zeta}) \partial_{\eta_j} \Pi^s(\underline{\zeta}) e = \partial_{\eta_j} e(\underline{\zeta})$ (with the convention $\eta_0 = \tau$). A direct consequence of Assumption 2.3 is that the I_j can be expressed in terms of the (real valued) function θ . More precisely, we obtain that (18) is equivalent to the following transport equation (with real coefficients)

$$\begin{cases} b \cdot \beta(\underline{\zeta}) (\partial_\tau \theta(\underline{\zeta}) \partial_t + \nabla_\eta \theta(\underline{\zeta}) \cdot \nabla_{x'}) \varrho_0 = -i \tilde{g}_0 & \text{in } \omega, \\ \varrho_0|_{t \leq 0} = 0, \end{cases} \quad (19)$$

where $\beta(\underline{\zeta}) \in \mathbb{C}^p$ the first column of $P(\underline{\zeta})$ (see Assumption 2.3) satisfies $b \cdot \beta(\underline{\zeta}) \neq 0$.

A simple integration along the characteristics then determines the value of ϱ_0 and the value of the "double" trace $U_{0,ev}(t, x', 0; 0)$. To complete the determination of $U_{0,ev}$ it is then sufficient to extend this "double" trace in a single one by setting, for example,

$$U_{0,ev}(t, x; X) := \chi(x_d) \varrho_0(t, x') e^{X\mathcal{A}(\underline{\zeta})} e, \quad (20)$$

where $\chi \in \mathcal{D}([-1/2, 1/2[; \mathbb{R})$ satisfies $\chi(0) = 1$.

Reiterating essentially the same kind of computations, we can easily construct the higher order amplitudes, namely the $(U_{n,ev})_{n \geq 1}$ and the $(U_{n,ex})_{n \geq 1}$ and show that the truncated ansatz (14) is a good approximation of the exact solution u^ε to (13). More precisely we have the following result:

Theorem 4.1 *Under Assumptions 2.1-2.2 and 2.3, there exists $(U_{n,ev})_{n \geq 0} \in \mathcal{P}_{ev}^{\mathbb{N}}$ (resp. $(U_{n,ex})_{n \geq 0} \in \mathcal{P}_{ex}^{\mathbb{N}}$) satisfying the cascade of equations (16) (resp. (17)). Moreover let u^ε be the exact solution to (16) and for $N_0 \in \mathbb{N}$ let*

$$u_{N_0}^\varepsilon := \sum_{n=0}^{N_0} \varepsilon^n e^{\frac{i}{\varepsilon} \psi} \left[U_{n,ev} \left(t, x; \frac{x_d}{\varepsilon} \right) + U_{n,ex} \left(t, x; \frac{x_d - 1}{\varepsilon} \right) \right], \quad (21)$$

then for all $T > 0$, $\alpha \in \mathbb{N}^{d+1}$ and $\varepsilon \in]0, 1]$ we have the estimate $\varepsilon^{|\alpha|} \|u^\varepsilon - u_{N_0}^\varepsilon\|_{L^2(\Omega_T)} \lesssim_\alpha \varepsilon^{N_0+3/2}$.

4.2 Proof of Theorem 2.2

With Theorem 4.1 in hand, the proof of Theorem 2.2 is quite immediate.

By contradiction we assume that $s = \frac{1}{2} - \delta$ for some $\delta > 0$. In (13) let the source terms be given by $f^\varepsilon \equiv 0$, $g_1^\varepsilon \equiv 0$ and

$$g_0^\varepsilon(t, x') := \varepsilon e^{\frac{i}{\varepsilon} \psi(t, x')} \kappa(t, x') b,$$

where $\kappa \in \mathcal{D}(\omega; \mathbb{R})$. In the following, u^ε stands for the exact solution to (13).

By standard interpolation it turns out g_0^ε is $O(\varepsilon^{1/2+\delta})$ in $H^s(\omega_T)$. So the energy estimate (7) implies that u^ε is $O(\varepsilon^{1/2+\delta})$ in $L^2(\Omega_T)$. Theorem 4.1 applied to $N_0 = 1$ combined with the triangle inequality show that u_0^ε (see (21)) is also $O(\varepsilon^{1/2+\delta})$.

Using a technical lemma from [Ben14] (more precisely Lemma 4.1) it implies that $U_{0,ev}$ is zero in $L^2(\Omega_T)$. This is a contradiction with the explicit formula (20) in which ϱ_0 can not be zero because it is the solution to the transport equation (19) with interior source term $\tilde{g}_0 = \kappa|b|^2 \neq 0$.

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