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ANOSOV REPRESENTATIONS WITH LIPSCHITZ LIMIT SET

BEATRICE POZZETTI, ANDRÉS SAMBARINO, AND ANNA WIENHARD

Abstract. We study Anosov representation for which the image of the boundary map is the graph of a Lipschitz function, and show that the orbit growth rate with respect to an explicit linear function, the unstable Jacobian, is integral. Several applications to the orbit growth rate in the symmetric space are provided. We further study regularity of the limit curves of θ-positive representations as introduced by Guichard-W. [27].

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1. Introduction

Let $\Gamma \subset \mathsf{PGL}_d(\mathbb{R})$ be a discrete subgroup. Following Guivarc'h, Benoist [4] has shown that if Γ contains a proximal element and acts irreducibly on $\mathbb{R}^d$ then its action on projective space $\mathbb{P}(\mathbb{R}^d)$ has a smallest closed invariant set. This is usually called *Benoist's limit set* or simply *the limit set* of Γ on $\mathbb{P}(\mathbb{R}^d)$ and denoted by L_Γ.

In contrast with the negatively curved situation, the limit set of a subgroup Γ whose Zariski closure has rank $\geqslant 2$ need not be a fractal object. Examples of this phenomena are provided, for example, by Benoist [5] in his work on strictly convex divisible sets, showing that these groups have a C^1-sphere as limit set, or

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by Labourie [33] where he proved that the limit set of a Hitchin representation is a C^1-curve. We refer the reader to P.-S.-W [36] and to Zhang-Zimmer [45] for new examples of infinite co-volume Zariski-dense groups whose limit set is a proper C^1 submanifold.

Intermediate phenomena can also ocur. The easiest example could be by considering two different hyperbolizations $\rho, \eta : \pi_1 S \to \mathsf{PSL}_2(\mathbb{R})$ of a closed surface S with genus $\geqslant 2$. The limit set of the representation $(\rho, \eta) : \pi_1 S \to \mathsf{PSL}_2(\mathbb{R}) \times \mathsf{PSL}_2(\mathbb{R})$ in $\partial\mathbb{H}^2 \times \partial\mathbb{H}^2$ is a Lipschitz circle that is never C^1 . This Lipschitz property can be easily deduced from the fact that the Hölder map $\xi : \partial\mathbb{H}^2 \to \partial\mathbb{H}^2$ conjugating the action ρ with η is order preserving.

This actually fits in the broader framework of what is now known as *maximal representations*, introduced in Burger-Iozzi-W. [10, Theorem 8]. These are a class of representations of $\pi_1 S$ into an Hermitian Lie group, that have *maximal Toledo invariant*, a notion that generalizes the characterization, proved by Goldman [23], of the Teichmüller space of S as those representation with maximal Euler number.

It is proved by Burger-Iozzi-Labourie-W. [9] that maximal representations have Lipschitz limit set in the Shilov boundary of the target group (see Section 9 for further detail).

The main object of this paper are discrete groups whose limit set is a Lipschitz manifold, i.e. it is locally the graph of a Lipschitz map. The groups we will consider verify a stronger form of "quasi-isometrically embedded", called *projective Anosov* which we now define.

Let τ be an inner product on $\mathbb{R}^d$ and for $g \in \mathsf{GL}_d(\mathbb{K})$ denote by

$$\sigma_1(g) \geqslant \cdots \geqslant \sigma_d(g)$$

the singular values of g associated to τ, that is, the square roots of the eigenvalues of gg^*, where g^* is the adjoint operator of g. Given $g \in \mathsf{PGL}_d(\mathbb{K})$ one can consider a lift $\tilde{g} \in \mathsf{GL}_d(\mathbb{K})$ with $\det \tilde{g} \in \{-1, 1\}$, we define then $\sigma_i(g) = \sigma_i(\tilde{g})$.

Let Γ be a finitely generated group, fix a finite symmetric generating set and denote by $|\,|$ the associated word length on Γ. Let $\rho : \Gamma \to \mathsf{PGL}_d(\mathbb{K})$ be a homomorphism, then the following are equivalent:

i) There exist positive constants c, μ such that for all $\gamma \in \Gamma$ one has

$$\frac{\sigma_2}{\sigma_1}(\rho(\gamma)) \leqslant c e^{-\mu|\gamma|},$$

ii) The group Γ is hyperbolic and there exist equivariant maps

$$(\xi_\rho, \xi_\rho^*) : \partial\Gamma \to \mathbb{P}(\mathbb{R}^d) \times \mathbb{P}((\mathbb{R}^d)^*)$$

such that for every $x \neq y \in \partial\Gamma$ one has $\ker \xi^*(y) \oplus \xi(x) = \mathbb{R}^d$; and the bundle over $\mathsf{U}\Gamma$ whose fiber is (the induced on the quotient of) $\hom(\xi(x), \ker \xi^*(y))$ is contracting for the associated canonical flat bundle automorphism.

If either condition is satisfied we will say that ρ is a *projective Anosov representation.*

Remark 1.1. The implication ii)⇒i) comes from Labourie [33] and Guichard-W. [26]. The implication i)⇒ii) is due to Kapovich-Leeb-Porti [31], see also Guéritaud-Guichard-Kassel-W. [24] and Bochi-Potrie-S. [6] for different approaches.

Recall that if $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}_+$ is a sequence of positive numbers then the *critical exponent* of the Dirichlet series

$$s \mapsto \sum_{n\in\mathbb{N}} a_n^s$$

is defined by

$$\inf\{s : \sum_{n\in\mathbb{N}} a_n^s < \infty\} = \sup\{s : \sum_{n\in\mathbb{N}} a_n^s = \infty\}.$$

Inspired by Potrie-S. [35, Theorem B], the main purpose of this work is to prove the following result, whose scope of application is considerable broader.

Theorem A. *Let Γ be a word-hyperbolic group such that $\partial\Gamma$ is topologically a sphere of dimension p. Let $\rho : \Gamma \to \mathsf{PSL}_d(\mathbb{R})$ be a strongly irreducible projective Anosov representation such that $\xi_\rho(\partial\Gamma)$ is locally the graph of a Lipschitz map. Then the Dirichlet series*

$$s \mapsto \sum_{\gamma\in\Gamma} \left(\frac{\sigma_1 \cdots \sigma_{p+1}}{\sigma_1^{p+1}} \big(\rho(\gamma)\big) \right)^s$$

has critical exponent equal to 1. Furthermore, if $p = 1$ the same holds replacing strong irreducibility with weak irreducibility.

We say that a projective Anosov representation ρ is *weakly irreducible* if the vector space span $\big(\xi_\rho(\partial\Gamma)\big)$ is $\mathbb{R}^d$. Let us introduce some more standard notation before explaining some consequences of such equality.

Let $\mathsf{E} = \{(a_1, \dots, a_d) \in \mathbb{R}^d : \sum_i a_i = 0\}$ and fix $p \in [\![1, d]\!]$. Denote by $\varepsilon_p \in \mathsf{E}^*$ the coordinate function $\varepsilon_p(a) = a_p$ and the *pth-root* $\mathsf{a}_p = \varepsilon_p - \varepsilon_{p+1}$. Consider the set of *simple roots* defined by $\Pi = \{\mathsf{a}_k : k \in [\![1, d-1]\!]\}$ and

$$\mathsf{E}^+ = \{a \in \mathsf{E} : a_1 \geqslant \cdots \geqslant a_d\}$$

the associated Weyl chamber. The *Cartan projection* $a : \mathsf{PGL}_d(\mathbb{R}) \to \mathsf{E}^+$ is defined by

$$a(g) = \big(\log \sigma_1(g), \dots, \log \sigma_d(g)\big),$$

note that this map depends on the choice of τ.

Consider a representation $\rho : \Gamma \to \mathsf{PGL}_d(\mathbb{R})$, then the *entropy* of a given linear form $\varphi \in \mathsf{E}^*$ is denoted by $h_\rho(\varphi)$ and is defined as the critical exponent of the Dirichlet series

$$s \mapsto \sum_{\gamma\in\Gamma} e^{-s\varphi\left(a\left(\rho(\gamma)\right)\right)}.$$

For each simple root a_p, we denote the associated *fundamental weight* by

$$\omega_p(a) = \sum_1^p a_i.$$

We introduce the *pth-unstable Jacobian* $\mathcal{J}_p^u \in \mathsf{E}^*$ defined by

$$\mathcal{J}_p^u = (p+1)\omega_{\mathsf{a}_1} - \omega_{\mathsf{a}_{p+1}},$$

so that Theorem A states that if ρ is projective Anosov and $\xi_\rho(\partial\Gamma)$ is a Lipschitz manifold of dimension p, then $h_\rho(\mathcal{J}_p^u) = 1$ (provided the corresponding irreducibility assumptions hold).

We proceed to explain some applications of Theorem A before explaining the main ideas of its proof.

Maximal representations. Let $\mathsf{G}_{\mathbb{R}}$ be a Hermitian Lie group. By definition, its symmetric space $X_{\mathsf{G}_{\mathbb{R}}}$ carries a $\mathsf{G}_{\mathbb{R}}$-invariant symplectic form Ω. Given a closed genus $g \geqslant 2$ surface and a representation $\rho : \pi_1 S \to \mathsf{G}_{\mathbb{R}}$, its *Toledo invariant* is defined by

$$\mathrm{T}(\rho) = \int_S f^*\Omega,$$

for a(ny) ρ-equivariant map $\tilde{f} : \tilde{S} \to X_{\mathsf{G}_{\mathbb{R}}}$. It is proved by Burger-Iozzi-W. [10] that $|\,\mathrm{T}(\rho)| \leqslant (2g-2)\,\mathrm{rank}(\mathsf{G}_{\mathbb{R}})$. One says that ρ is *maximal* if its Toledo invariant is maximal in modulus.

We obtain the following entropy one result for maximal representations.

Theorem 1.2. *Let $\mathsf{G}_{\mathbb{R}}$ be a classical simple Hermitian Lie group of tube type. Let $\rho : \Gamma \to \mathsf{G}_{\mathbb{R}}$ be a maximal representation, and let $\check{\mathsf{a}}$ denote the root associated to the stabilizer of a point in the Shilov boundary of $\mathsf{G}_{\mathbb{R}}$. Then $h_\rho(\check{\mathsf{a}}) = 1$.*

We give the precise linear form on E^*, where we denote by $\varepsilon_k \in \mathsf{E}^*$ the k-th coordinate function.

Target group	$\rho : \pi_1 S \to \mathsf{G}_{\mathbb{K}}$ maximal
$\mathsf{Sp}(2p, \mathbb{R})$	$h_\rho(2\varepsilon_p) = 1$
$\mathsf{SU}(p,p)$	$h_\rho(2\varepsilon_p) = 1$
$\mathsf{SO}^*(4p)$	$h_\rho(2\varepsilon_p) = 1$
$\mathsf{SO}^0(2,p)$	$h_\rho(\varepsilon_1 - \varepsilon_2) = 1$

Theorem 1.2 also holds for the exceptional Hermitian Lie group of tube type if the respresentation is Zariski-dense, and we expect it to hold unconditionally. We refer the reader to §9 for a slightly more general statement, further explanations and consequences, in particular concerning a sharp upper bound on the exponential orbit growth rate for the action on the symmetric space (see Proposition 9.9).

θ-positive representations. In [27] Guichard-W. introduced the notion of *θ-positivity* for a subset θ of the simple roots of a real semisimple Lie group $\mathsf{G}_{\mathbb{R}}$, and used such notion to define θ-positive representations, a class of representations that encompasses Hitchin representations, maximal representations and includes a new class of representations into the group $\mathsf{SO}(p,q)$ of elements preserving a symmetric bilinear form Q of signature (p,q) in $\mathbb{R}^{p+q}$. The definition of positivity is rather involved and we postpone it to §10. In that Section we prove the following result:

Theorem 1.3. *Let $\rho : \Gamma \to \mathsf{SO}(p,q)$ be a θ-Anosov representation that is θ-positive. Then the images of the boundary maps $\xi^k : \partial\Gamma \to \mathsf{Is}_k(\mathbb{R}^{p,q})$ are C^1 submanifolds for each $1 \leqslant k < p-1$, and the image $\xi^{p-1}(\partial\Gamma)$ is Lipschitz.*

We will prove the two parts of Theorem 1.3 separately, respectively in Corollary 10.2 and Proposition 10.3. Theorem A implies then the following:

Corollary 1.4. *Let $\rho : \Gamma \to \mathsf{SO}(p,q)$ be a θ-Anosov representation that is θ-positive and weakly irreducible, then $h_\rho(\mathsf{a}_k) = 1$ for every $k \leqslant p-1$.*

Applying ideas from Potrie-S. [35], this corollary gives the following rigid upper bound for the critical exponent of the action of a positive representation in the symmetric space $X_{p,q}$ of $\mathsf{SO}(p,q)$ (see Theorem 10.5). Endow $X_{p,q}$ with a $\mathsf{SO}(p,q)$-invariant Riemannian metric such that the totally geodesic copy of $\mathbb{H}^2$ induced by the representation $\Lambda : \mathsf{SL}_2(\mathbb{R}) \to \mathsf{SO}(p,q)$ that stabilizes a subspace of $\mathbb{R}^d$ of

signature $(p, p-1)$, has constant curvature -1. Denote by d the induced distance in $X_{p,q}$.

For a representation $\rho : \Gamma \to \mathsf{SO}(p,q)$ and $x_0 \in X_{p,q}$ denote by $h_\rho^{X_{p,q}}$ the critical exponent of the Dirichlet series

$$s \mapsto \sum_{\gamma \in \Gamma} e^{-sd(x_0, \rho(\gamma)x_0)}.$$

Theorem 1.5. *Let Γ be the fundamental group of a surface and let $\rho : \Gamma \to \mathsf{SO}(p,q)$ be θ-positive. Then*

$$h_\rho^{X_{p,q}} \leqslant 1.$$

Furthermore if equality is achieved at a totally reducible representation η then η splits as $W \oplus V$ with W having signature $(p, p-1)$ and $\eta|W$ has Zariski closure the irreducible $\mathsf{PO}(2,1)$ in $\mathsf{PO}(p, p-1)$ and $\eta|V$ lies in a compact group.

Observe that the argument from [35] does not apply directly since the Anosov-Levi space of a θ-positive representation has codimension one (instead of 0, which is the case treated in [35]), see §10.

$\mathbb{H}^{p,q}$-convex-cocompact representations. Generalizing work of Mess [34] and Barbot-Mérigot [2], Danciger–Guéritaud–Kassel [16] introduced a class of representations called *$\mathbb{H}^{p,q}$-convex cocompact*.

The subspace of $\mathbb{P}(\mathbb{R}^d)$ consisting of negative definite lines for the form Q is called the *pseudo-Riemannian hyperbolic space* and denoted by

$$\mathbb{H}^{p,q-1} = \{\ell \in \mathbb{P}(\mathbb{R}^d) : Q|_{\ell - \{0\}} < 0\}.$$

The cone of isotropic lines is usually denoted by $\partial \mathbb{H}^{p,q-1}$.

Instead of the original definition of convex-cocompactness, we recall the characterization given by [16, Theorem 1.11].

Definition 1.6. A projective Anosov representation $\rho : \Gamma \to \mathsf{PO}(p,q)$ is *$\mathbb{H}^{p,q-1}$-convex cocompact* if for every pairwise distinct triple of points $x, y, z \in \partial\Gamma$, the restriction $Q|_{\xi(x) \oplus \xi(y) \oplus \xi(z)}$ has signature $(2,1)$.

Before stating the main consequence in this case let us introduce some more notation. Consider a representation $\Lambda : \mathsf{PO}(p,1) \to \mathsf{PO}(p,q)$ such that its image stabilizes a $p+1$-dimensional subspace V of $\mathbb{R}^d$ where $Q|V$ has signature $(p,1)$. Endow the symmetric space $X_{p,q}$ with a $\mathsf{PO}(p,q)$-invariant Riemannian metric such that the totally geodesic copy of $\mathbb{H}^p$ in $X_{p,q}$ induced be Λ has constant curvature -1. Then one has the following upper bound.

Proposition 1.7. *Assume that $\partial\Gamma$ is homeomorphic to a $p-1$-dimensional sphere and let $\rho : \Gamma \to \mathsf{PO}(p,q)$ be strongly irreducible and $\mathbb{H}^{p,q-1}$-convex-cocompact, then*

$$h_\rho^{X_{p,q}} \leqslant p-1.$$

One expects this upper bound to be rigid in the following sense: if the upper bound is attained then Γ is necessarily a co-compact lattice in $\mathsf{PO}(p,1)$ and the given representation preserves a totally geodesic copy of $\mathbb{H}^p$ of the type induced by Λ. However, only the case $p = 2$ is known due to Collier-Tholozan-Toulisse [14], this actually fits in the framework of the maximal representations.

Section 8 contains more information on $\mathbb{H}^{p,q}$-convex cocompact representations, in particular the relation with recent work by Glorieux-Monclair [20].

C^1-dichotomy for closed surface groups. In the case of fundamental groups of surfaces, or more generally of lattices in $\mathsf{PSL}_2(\mathbb{R})$, we deduce from our main result an interesting dichotomy for the regularity of the image of the boundary map.

Recall that an element $g \in \mathsf{PGL}_d(\mathbb{R})$ is *proximal* if the generalized eigenspace associated to its greatest eigenvalue (in modulus) has dimension 1. A representation $\Lambda : G \to \mathsf{PGL}_d(\mathbb{R})$ of a given group G is *proximal* if its image contains a proximal element.

Corollary 1.8. *Let $\Lambda : \mathsf{PSL}_2(\mathbb{R}) \to \mathsf{PSL}_d(\mathbb{R})$ be a (possibly reducible) proximal representation such that $\wedge^2\Lambda$ is also proximal. Let S be a closed surface of genus $\geqslant 2$ and let $\rho_0 : \pi_1 S \to \mathsf{PSL}_2(\mathbb{R})$ be discrete and faithful. Then we have the folloqing dichotomy:*

i) If the top two weights spaces of Λ belong to the same irreducible factor, then for every small deformation $\pi_1 S \to \mathsf{PSL}_d(\mathbb{R})$ of $\Lambda\rho_0$ the image of the boundary map to projective space is C^1.

ii) Otherwise, for every weakly irreducible small deformations $\pi_1 S \to \mathsf{PSL}_d(\mathbb{R})$ of $\Lambda\rho_0$ the image of the boundary map to projective space is not Lipschitz.

The regular case, item i) in Corollary 1.8, is inspired by Labourie [33], who treated the case (of arbitrary deformations) of the irreducible representations, and was proven in P.-S.-W [36, Proposition 9.4]. The novelty of this paper is item ii), inspired by Barbot [1] who proved it for $d = 3$. We believe both items placed together give a clearer picture.

Observe that it is easy to obtain similar results for other group G by considering suitable linear representations. On the other hand the double proximality assumption is necessary: the composition of a maximal representation not in the Hitchin component and the irreducible linear representation of $\mathsf{Sp}(2n, \mathbb{R})$ of highest weight w_n is proximal but its second exterior power is not proximal; it is possible to check that no small Zariski dense deformation satisfies either (i) or (ii).

We refer the reader to §7 for further explanations and consequences for hyperconvex representations.

1.1. **Two main ingredients of the proof of Theorem A.** The proof of Theorem A goes by proving indepently both inequalities. One inequality (Corollary 1.9) follows from a general result on Hausdorff dimension of limit sets (for projective Anosov representations) which we now explain. The other inequality follows from an improvement on a result by Quint [40, Théorème 8.1] concerning the existence of $(\rho(\Gamma), \varphi)$-Patterson-Sullivan measures. Its discussion is postponed to Subsection 1.1.2.

1.1.1. *The affinity exponent and the unstable Jacobian.* Recall that for a metric space (Λ, d) and for $s > 0$ one defines

$$\mathcal{H}^s(\Lambda) = \inf_{\varepsilon}\{\sum_{U\in\mathcal{U}} \operatorname{diam} U^s : \mathcal{U} \text{ is a covering of } \Lambda \text{ with } \sup_{U\in\mathcal{U}} \operatorname{diam} U < \varepsilon\}$$

and that the Hausdorff dimension of Λ is defined by

$$\dim_{\mathrm{Hff}}(\Lambda) = \inf\{s : \mathcal{H}^s(\Lambda) = 0\} = \sup\{s : \mathcal{H}^s(\Lambda) = \infty\}. \tag{1.1}$$

Given a representation $\rho : \Gamma \to \mathsf{PGL}_d(\mathbb{R})$, we consider, in analogy with Falconer's work [18], the *piecewise* Dirichlet series

$$\Phi_\rho^{\mathsf{Aff}}(s) = \sum_{\gamma\in\Gamma} \left(\frac{\sigma_2}{\sigma_1}\big(\rho(\gamma)\big)\cdots\frac{\sigma_{p-1}}{\sigma_1}\big(\rho(\gamma)\big)\right)\left(\frac{\sigma_p}{\sigma_1}\big(\rho(\gamma)\big)\right)^{s-(p-2)} : s \in [p-2, p-1].$$

As in Falconer [18], one directly observes that

$$h_\rho^{\mathsf{Aff}} = \inf\{s : \Phi_\rho^{\mathsf{Aff}}(s) < \infty\} = \sup\{s : \Phi_\rho^{\mathsf{Aff}}(s) = \infty\} \in (0, \infty],$$

this critical exponent is usually called the *affinity exponent* (or affinity dimension), probably because the target group in Falconer's work was the affine group $\mathsf{Aff}(d) = \mathsf{GL}_d(\mathbb{R}) \rtimes \mathbb{R}^d$.

Let $\rho : \Gamma \to \mathsf{PGL}_d(\mathbb{R})$ be a projective Anosov representation and denote by

$$\dim_{\mathrm{Hff}}(\xi(\partial\Gamma))$$

the Hausdorff dimension of $\xi(\partial\Gamma) \subset \mathbb{P}(\mathbb{R}^d)$ associated to a $\mathsf{PO}(d)$-invariant metric. This is independent of the choice of chosen inner product. Observe also that in this case, the terms in the sum of $\Phi_\rho^{\mathsf{Aff}}(s)$ can be bounded above by $ce^{-\mu s|\gamma|}$ and thus $h_\rho^{\mathsf{Aff}} < \infty$.

The second main result of this paper is the following (see §3 for a statement for arbitrary local fields):

Theorem B. *Let $\rho : \Gamma \to \mathsf{PGL}_d(\mathbb{R})$ be a projective Anosov representation, then*

$$\dim_{\mathrm{Hff}}\big(\xi(\partial\Gamma)\big) \leqslant h_\rho^{\mathsf{Aff}}.$$

Theorem B gives relations between the Hausdorff dimension of the limit set of a projective Anosov representation and the orbit growth rate with respect to explicit linear functionals on the Weyl chamber. Recall that we denote by $h_\rho(\mathcal{J}_{p-1}^u)$ the critical exponent of the Dirichlet series

$$s \mapsto \sum_{\gamma\in\Gamma}\left(\frac{\sigma_1\cdots\sigma_p}{\sigma_1^p}(\rho(\gamma))\right)^s$$

appearing in Theorem A. Theorem B provides directly the following corollary:

Corollary 1.9. *Let $\rho : \Gamma \to \mathsf{PGL}_d(\mathbb{R})$ be projective Anosov and assume furthermore that $\dim_{\mathrm{Hff}}(\xi(\partial\Gamma)) \geqslant p-1$. Then*

$$\dim_{\mathrm{Hff}}(\xi(\partial\Gamma)) \leqslant p h_\rho(\mathcal{J}_p^u).$$

Observe that $\mathcal{J}_1^u = \mathsf{a}_1$, and thus, since $\dim_{\mathrm{Hff}}(\xi(\partial\Gamma)) \geqslant 0$ we obtain as consequence Glorieux-Monclair-Tholozan [22, Theorem 4.1] and P.-S.-W. [36, Proposition 4.1].

1.1.2. *On the existence of Patterson-Sullivan measures.* Let $\{u_1, \dots, u_d\}$ be a τ-orthonormal basis of $\mathbb{R}^d$ and denote by $A = \exp \operatorname{diag} \mathsf{E}$ the group of determinant one matrices, diagonal in this ordered basis. Denote also by N the group of unipotent upper triangular matrices and recall that the *Iwasawa decomposition* of $\mathsf{PGL}_d(\mathbb{R})$ states that every $g \in \mathsf{PGL}_d(\mathbb{R})$ can be uniquely written as a (non-commutative) product

$$g = k_g a_g n_g,$$

where $k_g \in K = \mathsf{PO}(\tau)$, $a_g \in A$ and $n_g \in N$.

Recall that a *full flag* of $\mathbb{R}^d$ is a collection of subspaces $(V_i)_{i=0}^d$ with $V_i \subset V_{i+1}$ and $\dim V_i = i$. The space of full flags of $\mathbb{R}^d$ is denoted by $\mathcal{F}$, it is a K-homogeneus space and the stabilizer in K of the flag $e = (u_1 \oplus \cdots \oplus u_i)_{i=1}^d$ is denoted by M. Given a flag $x \in \mathcal{F}$, we will denote by x^p the associated p-dimensional subspace.

The *Iwasawa cocycle* introduced by Quint [40] is the cocycle $\mathrm{b} : \mathsf{PGL}_d(\mathbb{R}) \times \mathcal{F} \to \mathsf{E}$ defined such that, if $g \in \mathsf{PGL}_d(\mathbb{R})$ and $x = k_x e \in \mathcal{F}$ then

$$gk_x = k\big(\exp \operatorname{diag} \mathrm{b}(g,x)\big)n,$$

where the right hand side is the Iwasawa decomposition of gk_x. It is the higher rank analog of the more studied *Busemann cocycle* in negative curature (see for example Quint [40, Lemma 6.6]).

A more hands-on definition is the following (see Lemma 4.9). For $p \in \Pi = [\![1,d]\!]$ one has

$$\omega_p\big(\mathrm{b}(g,x)\big) = \log \frac{\|gv_1 \wedge \cdots \wedge gv_p\|}{\|v_1 \wedge \cdots \wedge v_p\|} \tag{1.2}$$

where $\{v_1, \dots, v_p\}$ is any basis of the p-dimensional space x^p of x and $\| \, \|$ is the norm on $\wedge^p \mathbb{R}^d$ induced by τ.

Notice that $\omega_p\big(\mathrm{b}(g,x)\big)$, the sum of the first p coordinates of $\mathrm{b}(g,x)$, only depends on x^p, so it is actually more natural to consider the Iwasawa cocycle defined on *partial flags*. Given $\theta \subset \Pi$ denote by $\mathcal{F}_\theta$ the space of partial flags with dimension jumps only in θ. Consider also the subspace E_θ of E defined by

$$\mathsf{E}_\theta = \bigcap_{p \notin \theta} \ker \mathsf{a}_p.$$

The fundamental weights $\{\omega_p|\mathsf{E}_\theta : p \in \theta\}$ span its dual $(\mathsf{E}_\theta)^*$ so Riesz Theorem gives an Iwasawa cocycle

$$\mathrm{b}_\theta : \mathsf{PGL}_d(\mathbb{R}) \times \mathcal{F}_\theta \to \mathsf{E}_\theta.$$

We recall the following definition from [40].

Definition 1.10. Given a discrete subgroup $\Delta < \mathsf{PGL}_d(\mathbb{R})$ and $\varphi \in (\mathsf{E}_\theta)^*$ a (Δ, φ)-*Patterson-Sullivan measure* on $\mathcal{F}_\theta$ is a finite Radon measure μ such that for every $g \in \Delta$ one has

$$\frac{\mathsf{d} g_* \mu}{\mathsf{d}\mu}(x) = e^{-\varphi\left(\mathrm{b}_\theta(g^{-1},x)\right)}.$$

The core of the second inequality in Theorem A is the following result (see also Theorem 5.14). Given $\theta \subset \Pi$ denote by $\mathrm{i}\,\theta = \{d - p : p \in \theta\}$. We say that $(x,y) \in \mathcal{F}_\theta \times \mathcal{F}_{\mathrm{i}\,\theta}$, are *transverse* if for every $p \in \theta$ one has that $x^p \cap y^{d-p}$. A *complementary hyperplane of* $\mathcal{F}_\theta$ is a subset of $\mathcal{F}_\theta$ of the form

$$\{x \in \mathcal{F}_\theta : x \text{ is not transverse to } y_0\}$$

for a given $y_0 \in \mathcal{F}_{\mathrm{i}\,\theta}$.

Theorem C. *Let $\rho : \Gamma \to \mathsf{PGL}_d(\mathbb{R})$ be a proyective Anosov representation and consider $\theta \subset \Pi$ such that $\mathsf{a}_1 \in \theta$. Let $\varphi \in (\mathsf{E}_\theta)^*$. If there exists a $(\rho(\Gamma), \varphi)$-Patterson-Sullivan measure on $\mathcal{F}_\theta$ whose support is not contained on a complementary subspace, then*

$$h_\rho(\varphi) \leqslant 1.$$

A similar result has been earlier proved by Quint [40, Théorème 8.1], inspired on a classical result by Sullivan [43]. However, Quint's theorem does not require the representation to be Anosov and the obtained inequality is not enough for what we need.

We refer the reader to §5 for a version of Theorem C where the target group is an arbitrary semi-simple group over a local field.

Recently Dey–Kapovich developed Patterson-Sullivan theory for Anosov representations [DK19]. There are important differences between their approach and ours: while we consider Hausdorff dimension with respect to the Riemannian distance on the flag manifold and measures quasi-invariant with respect to a functional that doesn't induce a norm on the Weyl chamber, they consider distances on the flag varieties which are induced from Finsler metrics on the symmetric space. In particular our results seem complementary.

1.2. **Plan of the paper.** In §2 we introduce some required preliminaries, and recall some needed results from Bochi-Potrie-S. [6] and P.-S.-W. [36]. Section 3 deals with the affinity exponent and Hausdorff dimension for Anosov representations, in it we prove Theorem B for any local field.

Section §4 is basically a reminder on (more or less) standard definitions on semi-simple algebraic groups over a local field.

In §5 we recall objets from higher rank Patterson-Sullivan Theory and in subsection 5.3 we prove Theorem 5.14 (a broader version of Theorem C).

Section §6 glues the pieces to complete the proof of Theorem A. The remaining sections deal with applications of this result, as explained earlier in this introduction.

Acknowledgements. We would like to thank J.-F. Quint for pointing us to Falconer's work and suggesting to consider the affinity exponent.

2. Preliminaries

We recall in this section the notions we will need concerning Anosov representations and cone types. We refer the reader to [36] and the references therein for more details.

Throughout the paper $\mathbb{K}$ will denote a local field with absolute value $|\cdot| : \mathbb{K} \to \mathbb{R}^+$. If $\mathbb{K}$ is non-Archimedean, we require that $|\omega| = \frac{1}{q}$ where ω denotes the *uniformizing element*, namely a generator of the maximal ideal of the valuation ring $\mathcal{O}$, and q is the cardinality of the residue field $\mathcal{O}/\omega\mathcal{O}$ (this is finite because $\mathbb{K}$ is, by assumption, local). This guarantees that the Hausdorff dimension of $\mathbb{P}^1(\mathbb{K}) = 1$.

2.1. **Singular values and Anosov representations into $\mathsf{PGL}_d(V_{\mathbb{K}})$.** A $\mathbb{K}$-norm $\| \, \|$ on a $\mathbb{K}$ vector space $V_{\mathbb{K}}$ induces a norm on every exterior power of V; the angle between two vectors $\measuredangle(v, w)$ is the unique number in $[0, \pi]$ such that

$$\sin \measuredangle(v, w) := \frac{\|v \wedge w\|}{\|v\|\|w\|}$$

Given two points $[v], [w] \in \mathbb{P}V$, we define their distance as

$$d([v], [w]) := \sin \measuredangle(v, w),$$

and given any two subspaces $P, Q < V$ we define their minimal angle as

$$\measuredangle(P,Q) = \min_{v\in P\setminus\{0\}} \min_{w\in Q\setminus\{0\}} \measuredangle(v,w).$$

An element $a \in \mathsf{GL}(V_{\mathbb{K}})$ is a *semi-homothecy* (for a norm $\|\cdot\|$) if there exists a a-invariant $\mathbb{K}$-orthogonal[1] decomposition $V = V_1 \oplus \cdots \oplus V_k$ and $\sigma_1, \cdots, \sigma_k \in \mathbb{R}_+$ such that for every $i \in [\![1,k]\!]$ and every $v_i \in V_i$ one has

$$\|av_i\| = \sigma_i \|v_i\|.$$

The numbers σ_i are called the ratios of the semi-homothecy a.

Following Quint [38, Théorème 6.1], we fix a maximal abelian subgroup of diagonalizable matrices $A \subset \mathsf{GL}(V_{\mathbb{K}})$, a compact subgroup $K \subset \mathsf{GL}(V_{\mathbb{K}})$ such that, if N is the normalizer of A in $\mathsf{GL}(V_{\mathbb{K}})$, then $N = (N \cap K)A$, and a $\mathbb{K}$-norm $\| \|$ on V preserved by K, and such that A acts on V by semi-homothecies. Let $e_1 \oplus \cdots \oplus e_d$ be the eigenlines of A (here $d = \dim V$) and choose the Weyl chamber A^+ consistsing of those elements $a \in A$ whose corresponding semi-homothecy ratios verify $\sigma_1(a) \geqslant \cdots \geqslant \sigma_d(a)$.

For every $g \in \mathsf{GL}(V_{\mathbb{K}})$ we choose a Cartan decomposition $g = k_g a_g l_g$ with a_g in A^+, $k_g, l_g \in K$, and denote by

$$\sigma_1(g) \geqslant \sigma_2(g) \geqslant \cdots \geqslant \sigma_d(g)$$

the semi-homotecy ratios of the Cartan projection $a_g \in A^+$ (these do not depend on the choice of the Cartan decomposition once K and $\|\cdot\|$ are fixed). In order to simplify notation we will often write $\frac{\sigma_i}{\sigma_j}(g) = \frac{\sigma_i(g)}{\sigma_j(g)}$.

We define, for $p \in [\![1, d-1]\!]$,

$$u_p(g) = k_g \cdot e_p \in V.$$

The set $\{u_p(g) : p \in [\![1, d-1]\!]\}$ is an *arbitrary* orthogonal choice of the axes (ordered in decreasing length) of the ellipsoid $\{Av : \|v\| = 1\}$, and, by construction, for every $v \in g^{-1}u_p(g)$ one has $\|gv\| = \sigma_p(g)\|v\|$. Let

$$U_p(g) = u_1(g) \oplus \cdots \oplus u_p(g) = k_g \cdot (e_1 \oplus \cdots \oplus e_p).$$

If g is such that $\sigma_p(g) > \sigma_{p+1}(g)$, then we say that g *has a gap of index* p. In that case the decomposition

$$U_{d-p}(g^{-1}) \oplus g^{-1}(U_p(g))$$

is orthogonal (cfr. [36, Remark 2.4]) and, if $\mathbb{K}$ is Archimedean, the p-dimensional space $U_p(g)$ is independent of the Cartan decomposition of g.

We will denote by $\Pi = \{\mathsf{a}_1, \ldots, \mathsf{a}_{d-1}\}$ the root system of $\mathsf{PGL}(V_{\mathbb{K}})$, and, given a subset $\theta \subset \Pi$, by $\mathcal{F}_\theta$ the associated partial flag manifold. Given $\theta \subset \Pi$ we also denote by $U^\theta(g)$ the partial flag $U^\theta(g) = \{U_p(g) : \mathsf{a}_p \in \theta\}$. The *$\theta$-basin of attraction of g*

$$B_{\theta,\alpha}(g) = \{x^\theta \in \mathcal{F}_\theta(\mathbb{K}^d) : \min_{\mathsf{a}_p\in\theta} \measuredangle\big(x^p, U_{d-p}(g^{-1})\big) > \alpha\} \tag{2.1}$$

is the complement of the α-neighborhood of $U^{\theta^c}(g^{-1})$. When θ consists of a single root a we will write $B_{\mathsf{a},\alpha}(g)$ instead of $B_{\{\mathsf{a}\},\alpha}(g)$

[1]Recall that for $\mathbb{K}$ non-Archimedean a decomposition $V = V_1 \oplus \cdots \oplus V_k$ is *ortogonal* if, for every $v_i \in V_i$, it holds $\|\sum v_i\| = \max_i \|v_i\|$.

Remark 2.1. If g has a gap of index p, then $U_{d-p}(g^{-1})$ is well defined if $\mathbb{K}$ is Archimedean, and any two possible choices have distance at most $\frac{\sigma_{p+1}}{\sigma_p}(g)$ if $\mathbb{K}$ is non-Archimedean. It follows that, also in the non-Archimedean case, $B_{\theta,\alpha}(g)$ only depends on K provided α is bigger than the minimal singular value gap.

We recall for later use the following lemma, which explains the choice of the term basin of attraction:

Lemma 2.2 (Bochi-Potrie-S. [6, Lemma A.6])**.** *For every $g \in \mathsf{PGL}_d(\mathbb{K})$, and $x \in B_{\mathsf{a}_1,\alpha}(g)$ it holds*

$$d(U_1(g), g \cdot x) \leqslant \frac{1}{\sin(\alpha)} \frac{\sigma_2}{\sigma_1}(g).$$

2.2. **Anosov representations.** Let Γ be a word-hyperbolic group with identity element e, fix a finite symmetric generating set S_Γ. For $\gamma \in \Gamma - \{e\}$ denote by $|\gamma|$ the least number of elements of S_Γ needed to write γ as a word on S, and define the induced distance $d_\Gamma(\gamma, \eta) = |\gamma^{-1}\eta|$. A geodesic segment on Γ is a sequence $\{\alpha_i\}_0^k$ of elements in Γ such that $d_\Gamma(\alpha_i, \alpha_j) = |i - j|$.

Definition 2.3. A representation $\rho : \Gamma \to \mathsf{PGL}_d(\mathbb{K})$ is a_p-*Anosov*[2] if there exist positive constants c, μ such that for all $\gamma \in \Gamma$ one has

$$\frac{\sigma_{p+1}}{\sigma_p}\big(\rho(\gamma)\big) \leqslant ce^{-\mu|\gamma|}. \tag{2.2}$$

An a_1-Anosov representation will be called *projective Anosov*.

The following result was proven in Bochi-Potrie-S. [6] for $\mathbb{K} = \mathbb{R}$, the same arguments also give the result for any local field:

Proposition 2.4 ([6, Lemma 2.5])**.** *Let $\rho : \Gamma \to \mathsf{PGL}_d(\mathbb{K})$ be a projective Anosov representation. Then there exists $\eta_\rho > 0$ and $L \in \mathbb{N}$ such that for every geodesic segment $\{\alpha_i\}_0^k$ in Γ through e with $|\alpha_0|, |\alpha_k| \geqslant L$ one has*

$$\measuredangle\Big(U_1\big(\rho(\alpha_k)\big), U_{d-1}\big(\rho(\alpha_0)\big)\Big) > \eta_\rho.$$

Proposition 2.4 is a key ingredient in the construction of boundary maps:

Proposition 2.5 ([6, Lemma 4.9])**.** *Let $\rho : \Gamma \to \mathsf{PGL}_d(\mathbb{K})$ be projective Anosov and $(\alpha_i)_0^\infty \subset \Gamma$ a geodesic ray based at the identity converging to $x \in \partial\Gamma$ then*

$$\xi_\rho^1(x) := \lim_{i\to\infty} U_1\big(\rho(\alpha_i)\big) \quad \xi_\rho^{d-1}(x) := \lim_{i\to\infty} U_{d-1}\big(\rho(\alpha_i)\big)$$

exist, do not depend on the ray and define continuous ρ-equivariant transverse maps $\xi_\rho^1 : \partial\Gamma \to \mathbb{P}(\mathbb{K}^d)$, $\xi_\rho^{d-1} : \partial\Gamma \to \mathbb{P}\big((\mathbb{K}^d)^*\big)$. Furthermore, there are positive constants C, μ depending only on ρ such that

$$d\Big(U_1\big(\rho(\alpha_k)\big), \xi_\rho^1(x)\Big) \leqslant Ce^{-\mu k}$$

The following Lemma from concerning properties of boundary maps will be precious in Section 3.1:

Lemma 2.6 (Bochi-Potrie-S. [6, Lemma 3.9])**.** *Let $\rho : \Gamma \to \mathsf{PGL}_d(\mathbb{K})$ be projective Anosov, then there exist constants $\nu \in (0, 1)$, $a_0 > 0$ and $a_1 > 0$ such that for every $\gamma, \eta \in \Gamma$ one has*

$$d_\Gamma(\gamma, \eta) \geqslant \nu(|\gamma| + |\eta|) - a_0 - a_1|\log d\big(U_1(\rho(\gamma)), U_1(\rho(\eta))\big)|.$$

[2]In the language of Bochi-Potrie-S. [6, Section 3.1] a a_p-Anosov representation is called *p-dominated*.

3. Hausdorff dimension of the limit set and the affinitiy exponent

Generalizing the definition given in the introduction, we define the affinity exponent h_ρ^{Aff} of a projective Anosov representation $\rho : \Gamma \to \mathsf{PGL}(V_{\mathbb{K}})$ as the critical exponent of the broken Dirichlet series

$$\Phi_\rho^{\mathsf{Aff}}(s) = \sum_{\gamma\in\Gamma} \left(\frac{\sigma_2}{\sigma_1}\big(\rho(\gamma)\big)\cdots\frac{\sigma_{p-1}}{\sigma_1}\big(\rho(\gamma)\big)\right)^{d_{\mathbb{K}}} \left(\frac{\sigma_p}{\sigma_1}\big(\rho(\gamma)\big)\right)^{s-d_{\mathbb{K}}(p-2)} \quad s\in[d_{\mathbb{K}}(p-2), d_{\mathbb{K}}(p-1)]$$

where the dimension $d_{\mathbb{K}}$ of $\mathbb{P}^1(\mathbb{K})$ is 1 unless $\mathbb{K}=\mathbb{C}$ in which case $d_{\mathbb{C}}=2$.

The goal of the section is to prove the following result:

Theorem 3.1. *Let $\mathbb{K}$ be a local field. If $\rho : \Gamma \to \mathsf{PGL}(V_{\mathbb{K}})$ is a_1-Anosov then*

$$\dim_{\mathrm{Hff}}\left(\xi_\rho^1(\partial\Gamma)\right) \leqslant h_\rho^{\mathsf{Aff}}.$$

The proof of Theorem 3.1 is elementary and based on the construction of a good cover of the image of the limit map (explicitely constructed in § 3.1) which we show, in § 3.2 to be contained in ellipses of controlled axis.

3.1. Coarse Cone types.

In P.-S.-W. [36, Section 2.3.1] we used cone types at infinity to construct well behaved coverings of the boundary of the group. For the purposes of this paper a coarse version of these sets will be more useful, which we now introduce.

Recall that a sequence $(\alpha_i)_0^\infty$ is a (c_0,c_1)-quasigeodesic if for every pair i,j it holds

$$\frac{1}{c_0}|i-j| - c_1 \leqslant d_\Gamma(\alpha_i,\alpha_j) \leqslant c_0|i-j| + c_1.$$

We associate to every element γ a *coarse cone type at infinity*, consisting of endpoints at infinity of quasi geodesic rays based at γ^{-1} passing through the identity:

$$\mathcal{C}_\infty^{c_0,c_1}(\gamma) = \Big\{[(\alpha_i)_0^\infty]\in\partial\Gamma \,|\, (\alpha_i)_0^\infty \text{ is a } (c_0,c_1)\text{-quasigeodesic with } \alpha_0=\gamma^{-1}, e\in\{\alpha_i\}\Big\}.$$

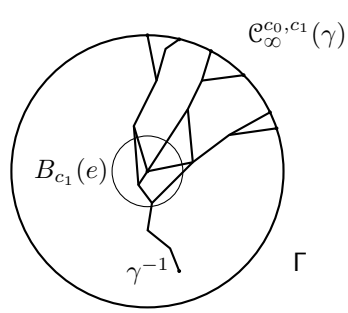

Figure 1. The coarse cone type at infinity, the black broken lines are (c_0,c_1)-quasigeodesics. All endpoints of geodesic rays from γ^{-1} intersecting the ball $B_{c_1}(e)$ clearly belong to $\mathcal{C}_\infty^{c_0,c_1}(\gamma)$

Hyperbolicity of Γ lets us understand the overlaps of coarse cone types; this will be crucial in Section 5.3 to guarantee bounded overlap of suitable covers of the limit set.

Proposition 3.2. *Let Γ be word-hyperbolic. For every c_0, c_1 there exists $C > 0$ such that if*

$$\gamma\mathcal{C}_\infty^{c_0,c_1}(\gamma) \cap \eta\mathcal{C}_\infty^{c_0,c_1}(\eta) \neq \varnothing$$

then

$$d_\Gamma(\gamma,\eta) \leqslant \big||\gamma| - |\eta|\big| + C.$$

Proof. Assume that $x \in \gamma\mathcal{C}_\infty^{c_0,c_1}(\gamma) \cap \eta\mathcal{C}_\infty^{c_0,c_1}(\eta)$. Since Γ is hyperbolic, by the Morse Lemma, there exists $K > 0$ (only depending on c_0, c_1 and the hyperbolicity constant of Γ) such that γ is at distance at most K from a geodesic ray from e to x. The same holds then for η, and, using the hyperbolicity of Γ again, we can assume, up to making the constant K worse (but still depending on c_0, c_1 only), that the two rays agree. This implies that there exist g_0 and g_1 on a geodesic ray from e to x such that $d(\gamma, g_0) \leqslant K$ and $d(\eta, g_1) \leqslant K$. Since g_0 and g_1 lie in a geodesic we have $d(g_0, g_1) \leqslant \big||g_0| - |g_1|\big|$ and thus

$$d(\gamma,\eta) \leqslant 4K + \big||\gamma| - |\eta|\big|.$$

□

Our next goal is to show that, for an Anosov representation, the intersections of Cartan's basins of attraction $B_{\theta,\alpha}(\rho(\gamma))$ with the image of the boundary map are contained in the image of a suitably big coarse cone type of γ. Let now $\theta \subset \Pi$ be a subset containing the first root a_1. We will denote by $\pi_{\theta,1} : \mathcal{F}_\theta(\mathbb{K}^d) \to \mathbb{P}(\mathbb{K}^d)$ the canonical projection. Recall from (2.1) that, for every α, we asociate to each $g \in \mathsf{PGL}(V_\mathbb{K})$ a basin of attraction $B_{\theta,\alpha}(g) \subset \mathcal{F}_\theta$. We will now use Lemma 2.6 to show that, for every α, there exist c_0, c_1 such that the intersection of a θ-basin of attraction $B_{\theta,\alpha}(\rho(\gamma))$ is contained in a (c_0, c_1)-coarse cone type.

Proposition 3.3. *Let $\rho : \Gamma \to \mathsf{PGL}(V_\mathbb{K})$ be projective Anosov and consider $\alpha > 0$. There exist c_0, c_1 only depending on α and ρ such that for every $\theta \subset \Pi$ containing a_1, and every $\gamma \in \Gamma$*

$$(\xi^1)^{-1}\left(\pi_{\theta,1}\left(B_{\theta,\alpha}(\rho(\gamma))\right)\right) \subset \mathcal{C}_\infty^{c_0,c_1}(\gamma).$$

Proof. It is enough to show that if $\xi^1(x) \in \pi_{\theta,1}(B_{\theta,\alpha}(\rho(\gamma)))$ and $|\gamma|$ is big enough, then there is a quasi-geodesic ray from γ^{-1} to x that passes through the identity whose constants only depend on α and ρ. Consider a quasigeodesic ray $\{\alpha_i\}$ converging to x, and fix $1 > \alpha' > \alpha$. Since, by assumption, $\xi^1(x) \in B_{\mathsf{a}_1,\alpha}(\rho(\gamma))$, we can find a constant L depending on ρ only, such that for every $i > L$ it holds $U_1(\rho(\alpha_i)) \in B_{\mathsf{a}_1,\alpha'}(\rho(\gamma))$ (the uniformity of L follows from the last statement in Proposition 2.5). By definition we have $\measuredangle(U_1(\rho(\alpha_i)), U_{d-1}(\rho(\gamma^{-1}))) > \alpha'$, and thus, in particular, $d(U_1(\rho(\alpha_i)), U_1(\rho(\gamma^{-1})) > \alpha'$. Let now $(\alpha_i)_{i=0}^{-|\gamma|_S}$ be a geodesic segment with $\alpha_0 = e$, $\alpha_{-|\gamma|_S} = \gamma$. Up to further enlarging α' and L (depending on the representation only) we have also that $d(U_1(\rho(\alpha_{-L})), U_1(\rho(\alpha_L)) > \alpha'$. Lemma 2.6 implies that the sequence $(\alpha_i)_{i=-|\gamma|_S}^{\infty}$ obtained as concatenation of the geodesic between γ^{-1} and the identity and the ray from the identity to x is a quasi geodesic ray, thus the result. □

Corollary 3.4. *Let $\rho : \Gamma \to \mathsf{PGL}(V_\mathbb{K})$ be projective Anosov and consider $\alpha > 0$. There exists C only depending on α and ρ such that for every $\theta \subset \Pi$ containing a_1, if*

$$\xi^1(\partial\Gamma) \cap \pi_{\theta,1}\Big(\rho(\gamma) \cdot B_{\theta,\alpha}(\rho(\gamma)) \cap \rho(\eta) \cdot B_{\theta,\alpha}(\rho(\eta))\Big) \neq \varnothing$$

then

$$d(\gamma,\eta) \leqslant \big||\gamma| - |\eta|\big| + C.$$

Proof. This follows immediately combining Proposition 3.3 and Proposition 3.2. □

In particular we can use basins of attraction to construct coverings of the image of the boudary map with bounded overlap:

Proposition 3.5 (Cfr. P.-S.-W. [36, Lemma 2.21])**.** *Let $\rho : \Gamma \to \mathsf{PGL}(V_{\mathbb{K}})$ be projective Anosov. There exists α small enough so that, for every $T > 0$, the family of open sets*

$$\mathcal{U}_T := \{\rho(\gamma) \cdot B_{\mathsf{a}_1,\alpha}(\rho(\gamma)) : |\gamma| = T\}$$

defines an open covering of $\xi^1(\partial\Gamma)$. Furthermore there exists a constant C depending on α (and ρ) such that for every $x \in \partial\Gamma$ and every T, $\xi(x)$ is contained in at most C elements of $\mathcal{U}_T$.

Proof. Let $x \in \partial\Gamma$, let $\{\gamma_i\}$ be a geodesic ray based at the identity representing x. Propositions 2.4 and 2.5 guarantee that there exists $\alpha = \alpha_\rho$ such that

$$\measuredangle \rho(\gamma_T^{-1})\xi^1(x), U_{d-1}(\rho(\gamma_T^{-1})) > \alpha,$$

therefore $\xi^1(x) \in \rho(\gamma_T)B_{\mathsf{a}_1,\alpha}(\rho(\gamma_T))$. The second statement is a direct consequence of Corollary 3.4. □

3.2. **Ellipses.** The purpose of this section is to prove that for a projective Anosov representation, the set $\rho(\gamma) \cdot B_{\mathsf{a}_1,\alpha}(\rho(\gamma))$ is coarsely contained in an ellipsoid with axes of size

$$\frac{\sigma_2}{\sigma_1}(\rho(\gamma)), \dots, \frac{\sigma_d}{\sigma_1}(\rho(\gamma)).$$

Definition 3.6. Let V be a d-dimensional $\mathbb{K}$-vector space with $\mathbb{K}$-norm $\|\cdot\|$. Let

$$u_1 \oplus \cdots \oplus u_d$$

be a $\mathbb{K}$-orthogonal decomposition and let $v = \sum v_i u_i$ be the associated decomposition of $v \in V$, for suitable $v_i \in \mathbb{K}$. Choose positive real numbers $a_2 \geqslant \dots a_d \geqslant 1$. If $\mathbb{K}$ is Archimedean, an *ellipsoid* about $\mathbb{K}u_1$ is the projectivisation of

$$\{v \in V : |v_1|^2 \geqslant \sum_2^d (a_i|v_i|)^2\}$$

for some $a_i > 0$. If, instead, $\mathbb{K}$ is non-Archimedean, an *ellipsoid* about $\mathbb{K}u_1$ is the projectivisation of

$$\left\{v \in V : |v_1| \geqslant \max_{2\leqslant i\leqslant d}(a_i|v_i|)\right\}$$

The vector spaces $u_1 \oplus u_i$ are *the axes* of the ellipsoid and the *size* of the axis $u_1 \oplus u_i$ is $1/a_i$. We need the following covering lemma.

Lemma 3.7. *Let E be an ellipsoid with axis of size $1 \geqslant \beta_2 \geqslant \dots \geqslant \beta_d$. For every $p \in [\![2, d]\!]$, E can be covered by*

$$2^{2p}\left(\frac{\beta_2 \cdots \beta_{p-1}}{\beta_p^{p-2}}\right)^{d_{\mathbb{K}}}$$

balls of radius $\sqrt{d}\beta_p$.

Proof. We consider the affine chart of $\mathbb{P}(V)$ corresponding to $u_1 = 1$. The ellipsoid E is contained in the product of the balls $\{|v_i| \leqslant \beta_i\} \subset \mathbb{K}$ (it agrees with such product if $\mathbb{K}$ is non-Archimedean). If $\mathbb{K}$ is Archimedean, the ball $\{|v_i| \leqslant \beta_i\}$ is contained in the union of $\left\lceil \frac{\beta_i}{\beta_p} \right\rceil^{d_{\mathbb{K}}}$ balls of radius β_p. Since the product of d balls of radius β_p is contained in a ball of radius $\sqrt{d}\beta_p$ we obtain that E can be covered by

$$\left\lceil \frac{\beta_2}{\beta_p} \right\rceil^{d_{\mathbb{K}}} \cdots \left\lceil \frac{\beta_{p-1}}{\beta_p} \right\rceil^{d_{\mathbb{K}}}$$

balls of radius $\sqrt{d}\beta_p$.

If, instead, $\mathbb{K}$ is non-Archimedean, the ball $\{|v_i| \leqslant \beta_i\}$ can be decomposed in $q^{\left\lfloor \log_q(\frac{\beta_i}{\beta_p}) \right\rfloor}$ balls of radius β_p, and hence E can be covered with

$$q^{\left\lfloor \log_q\left(\frac{\beta_2}{\beta_p}\right) \right\rfloor} \cdots q^{\left\lfloor \log_q\left(\frac{\beta_{p-1}}{\beta_p}\right) \right\rfloor}$$

balls of radius β_p.

□

Proposition 3.8. *Consider $\alpha > 0$. For every $g \in \mathsf{PGL}(V_{\mathbb{K}})$ one has that the image of the corresponding Cartan's basin of attraction $g \cdot B_{\mathsf{a}_1,\alpha}(g)$ is contained in the ellipsoid about $U_1(g)$ with axes $u_1(g) \oplus u_i(g)$ of size*

$$\frac{1}{\sin\alpha} \frac{\sigma_i}{\sigma_1}(g).$$

Proof. Assume first that $\mathbb{K}$ is Archimedean. By definition of $B_{\mathsf{a}_1,\alpha}(g)$, for every $v \in \mathbb{K}^d$ with $\mathbb{K} \cdot v \in B_{\mathsf{a}_1,\alpha}(g)$ one has

$$|v_1|^2 \geqslant (\sin\alpha)^2 \sum_1^d |v_i|^2,$$

where $(v_1, \cdots, v_d)$ are the coefficients in the decomposition of v with respect to the orthogonal splitting $V = \bigoplus g^{-1} u_i(g)$.

Since the coefficients w_i of gv in the decomposition induced by the orthogonal decomposition $V = \bigoplus u_i(g)$ satisfy $|w_i| = \sigma_i(g)|v_i|$, one has

$$\begin{aligned} |w_1|^2 = \sigma_1(g)^2 |v_1|^2 &\geqslant \sigma_1(g)^2 (\sin\alpha)^2 \sum_{i=2}^d |v_i|^2 \\ &= \sigma_1(g)^2 (\sin\alpha)^2 \sum_{i=2}^d \frac{1}{\sigma_i(g)^2} |w_i|^2. \end{aligned}$$

One concludes that gv lies on the corresponding ellipsoid. The non-Archimedean case follows analogously. □

3.3. **The lower bound on the affinity exponent.** We now have all the ingredients needed to prove Theorem 3.1:

Proof. For each $T > 0$ denote by $\mathcal{U}_T$ the covering of $\xi^1(\partial\Gamma)$ given by Proposition 3.5. By definition, $U = U_\gamma \in \mathcal{U}_T$ is of the form $\rho(\gamma) \cdot B_{\mathsf{a}_1,\alpha}(\rho(\gamma))$ for some γ satisfying

$|\gamma| = T$. Proposition 3.8 applied to $\rho(\gamma)$ implies that $\rho(\gamma) \cdot B_{\mathsf{a}_1,\alpha}(\rho(\gamma))$ is contained in an ellipsoid about $\mathbb{K}u_1(\rho(\gamma))$ of axes with sizes

$$\frac{1}{\sin\alpha}\frac{\sigma_2}{\sigma_1}(\rho(\gamma)), \dots, \frac{1}{\sin\alpha}\frac{\sigma_d}{\sigma_1}(\rho(\gamma)).$$

Furthermore, since ρ is Anosov, we deduce from Lemma 2.2 that $\sup_{U\in\mathcal{U}_T} \operatorname{diam} U$ is arbitrarily small as T goes to infinity. Recall that the s-capacity $\mathcal{H}^s$ was defined by Equation (1.1). Applying the covering Lemma 3.7 to these ellipses and any $p \in [\![2, d]\!]$, we obtain

$$\mathcal{H}^s(\xi(\partial\Gamma)) \leqslant 2^{2p}\left(\frac{\sqrt{d}}{\sin\alpha}\right)^s \inf_T \sum_{\gamma:|\gamma|\geqslant T}\left(\frac{\sigma_2}{\sigma_1}\big(\rho(\gamma)\big)\cdots\frac{\sigma_{p-1}}{\sigma_1}\big(\rho(\gamma)\big)\right)^{d_{\mathbb{K}}}\left(\frac{\sigma_p}{\sigma_1}\big(\rho(\gamma)\big)\right)^{s-d_{\mathbb{K}}(p-2)}.$$

By definition, the affinity exponent h_ρ^{Aff} is such that for all $s > h_\rho^{\mathsf{Aff}}$ the broken Dirichlet series

$$\sum_{\gamma\in\Gamma}\left(\frac{\sigma_2}{\sigma_1}\big(\rho(\gamma)\big)\cdots\frac{\sigma_{p-1}}{\sigma_1}\big(\rho(\gamma)\big)\right)^{d_{\mathbb{K}}}\left(\frac{\sigma_p}{\sigma_1}\big(\rho(\gamma)\big)\right)^{s-d_{\mathbb{K}}(p-2)} : s \in [d_{\mathbb{K}}(p-2), d_{\mathbb{K}}(p-1)]$$

is convergent and thus for all $s > h_\rho^{\mathsf{Aff}}$ we have

$$2^p\left(\frac{\sqrt{d}}{\sin\alpha}\right)^s \inf_T \sum_{\gamma:|\gamma|\geqslant T}\left(\frac{\sigma_2}{\sigma_1}\big(\rho(\gamma)\big)\cdots\frac{\sigma_{p-1}}{\sigma_1}\big(\rho(\gamma)\big)\right)^{d_{\mathbb{K}}}\left(\frac{\sigma_p}{\sigma_1}\big(\rho(\gamma)\big)\right)^{s-d_{\mathbb{K}}(p-2)} = 0.$$

As a result we conclude that for all $s > h_\rho^{\mathsf{Aff}}$ the s-capacity $\mathcal{H}^s(\xi(\partial\Gamma))$ vanishes, hence

$$h_\rho^{\mathsf{Aff}} \geqslant \dim_{\mathrm{Hff}}(\xi(\partial\Gamma)).$$

This completes the proof. □

The following generalization of Corollary 1.9 is also immediate:

Corollary 3.9. *Let $\rho : \Gamma \to \mathsf{PGL}(V_{\mathbb{K}})$ be projective Anosov. If $\dim_{\mathrm{Hff}}\big(\xi(\partial\Gamma)\big) \geqslant (p-1)d_{\mathbb{K}}$, then*

$$\dim_{\mathrm{Hff}}(\xi(\partial\Gamma)) \leqslant d_{\mathbb{K}}(p-1)h_\rho(\mathfrak{J}^u_{p-1}).$$

Proof. Observe that, for every p with $d_{\mathbb{K}}(p-1) \leqslant h_\rho^{\mathsf{Aff}}$, and for every $s \in [d_{\mathbb{K}}(p-1), d_{\mathbb{K}}p]$, the broken Dirichlet series defining the affinity exponent

$$\Phi_\rho^{\mathsf{Aff}}(s) = \sum_{\gamma\in\Gamma}\left(\frac{\sigma_2}{\sigma_1}\big(\rho(\gamma)\big)\cdots\frac{\sigma_p}{\sigma_1}\big(\rho(\gamma)\big)\right)^{d_{\mathbb{K}}}\left(\frac{\sigma_{p+1}}{\sigma_1}\big(\rho(\gamma)\big)\right)^{s-d_{\mathbb{K}}(p-1)}$$

is smaller than or equal to the series associated to the unstable Jacobian

$$\Phi_\rho^{\mathfrak{J}^u_{p-1}}(s) = \sum_{\gamma\in\Gamma}\left(\frac{\sigma_2}{\sigma_1}\big(\rho(\gamma)\big)\cdots\frac{\sigma_p}{\sigma_1}\big(\rho(\gamma)\big)\right)^{\frac{s}{d_{\mathbb{K}}(p-1)}}.$$

The result follows as $\Phi_\rho^{\mathfrak{J}^u_p}(s) \leqslant \Phi_\rho^{\mathfrak{J}^u_{p-1}}(s)$. □

4. Semi-simple algebraic groups

Let G be a connected semi-simple $\mathbb{K}$-group, $\mathsf{G}_{\mathbb{K}}$ the group of its $\mathbb{K}$-points, A a maximal $\mathbb{K}$-split torus and $\mathbf{X}(\mathsf{A})$ the group of its $\mathbb{K}^*$-characters. Consider the real vector space $\mathsf{E}^* = \mathbf{X}(\mathsf{A}) \otimes_{\mathbb{Z}} \mathbb{R}$ and E its dual. For every $\chi \in \mathbf{X}(\mathsf{A})$, we denote by χ^ω the corresponding linear form on E.

4.1. **Restricted roots and parabolic groups.** Let Σ be the set of restricted roots of A in $\mathfrak{g}$, the set Σ^ω is a root system of E^*. Let Σ^+ be a system of positive roots and Π the associated subset of simple roots. Let E^+ be the Weyl chamber determined by the positive roots $(\Sigma^\omega)^+$.

Let W be the Weyl group of Σ, it is isomorphic to the quotient of the normalizer $N_{\mathsf{G}_\mathbb{K}}(\mathsf{A}_\mathbb{K})$ of $\mathsf{A}_\mathbb{K}$ in $\mathsf{G}_\mathbb{K}$ by its centralizer $Z_{\mathsf{G}_\mathbb{K}}(\mathsf{A}_\mathbb{K})$. Let $\mathrm{i} : \mathsf{E} \to \mathsf{E}$ be the opposition involution: if $u : \mathsf{E} \to \mathsf{E}$ is the unique element in the Weyl group with $u(\mathsf{E}^+) = -\mathsf{E}^+$ then $\mathrm{i} = -u$.

A subset $\theta \subset \Pi$ determines a pair of opposite parabolic subgroups P_θ and $\check{\mathsf{P}}_\theta$ whose Lie algebras are defined by

$$\mathfrak{p}_\theta = \bigoplus_{\mathsf{a}\in\Sigma^+\cup\{0\}} \mathfrak{g}_\mathsf{a} \oplus \bigoplus_{\mathsf{a}\in\langle\Pi-\theta\rangle} \mathfrak{g}_{-\mathsf{a}},$$

and

$$\check{\mathfrak{p}}_\theta = \bigoplus_{\mathsf{a}\in\Sigma^+\cup\{0\}} \mathfrak{g}_{-\mathsf{a}} \oplus \bigoplus_{\mathsf{a}\in\langle\Pi-\theta\rangle} \mathfrak{g}_\mathsf{a}.$$

The group $\check{\mathsf{P}}_\theta$ is conjugated to the parabolic group $\mathsf{P}_{\mathrm{i}\,\theta}$. Let

$$\mathfrak{l}_\theta = \mathfrak{p}_\theta \cap \check{\mathfrak{p}}_\theta$$

be the Lie algebra of the associated Levi group.

The $\mathbb{K}$-*flag space* associated to θ is $\mathcal{F}_\theta(\mathsf{G}_\mathbb{K}) = \mathsf{G}_\mathbb{K}/\mathsf{P}_{\theta,\mathbb{K}}$, the $\mathsf{G}_\mathbb{K}$ orbit of the pair $([\mathsf{P}_{\theta,\mathbb{K}}], [\check{\mathsf{P}}_{\theta,\mathbb{K}}])$ is the unique open orbit for the action of $\mathsf{G}_\mathbb{K}$ in the product $\mathcal{F}_\theta(\mathsf{G}_\mathbb{K}) \times \mathcal{F}_{\mathrm{i}\,\theta}(\mathsf{G}_\mathbb{K})$. This orbit is denoted by $\mathcal{F}^{(2)}_\theta(\mathsf{G}_\mathbb{K})$.

For $y \in \mathcal{F}_{\mathrm{i}\,\theta}(\mathsf{G}_\mathbb{K})$ denote by

$$\mathrm{Ann}(y) = \{x \in \mathcal{F}_\theta(\mathsf{G}_\mathbb{K}) : (x, y) \notin \mathcal{F}_\theta(\mathsf{G}_\mathbb{K})^{(2)}\} \tag{4.1}$$

the closed submanifold of flags in $\mathcal{F}_\theta(\mathsf{G}_\mathbb{K})$ that are not transverse to y.

Denote by $(\cdot,\cdot)$ a W-invariant inner product on E, $(\cdot,\cdot)$ the induced inner product on E^* and define

$$\langle\chi, \psi\rangle = \frac{2(\chi, \psi)}{(\psi, \psi)}$$

and let $\{\omega_\mathsf{a}\}_{\mathsf{a}\in\Pi}$ be the *dual basis* of Π, i.e. $\langle\omega_\mathsf{a}, \mathsf{b}\rangle = \delta_{\mathsf{ab}}$. The linear form ω_a is *the fundamental weight* associated to a.

4.2. **Cartan decomposition.** Let $\nu : \mathsf{A}_\mathbb{K} \to \mathsf{E}$ be defined, for $z \in \mathsf{A}_\mathbb{K}$, as the unique vector in E such that for every $\chi \in \mathbf{X}(\mathsf{A})$ one has

$$\chi^\omega(\nu(z)) = \log|\chi(z)|.$$

Denote by $\mathsf{A}^+_\mathbb{K} = \nu^{-1}(\mathsf{E}^+)$.

Let $K \subset \mathsf{G}_\mathbb{K}$ be a compact group that contains a representative for every element of the Weyl group W. This is to say, such that the normaliser $N_{\mathsf{G}_\mathbb{K}}(\mathsf{A}_\mathbb{K})$ verifies $N_{\mathsf{G}_\mathbb{K}}(\mathsf{A}_\mathbb{K}) = (N_{\mathsf{G}_\mathbb{K}}(\mathsf{A}_\mathbb{K}) \cap K)\mathsf{A}_\mathbb{K}$. One has $\mathsf{G}_\mathbb{K} = K\mathsf{A}^+_\mathbb{K}K$ and if $z, w \in \mathsf{A}^+_\mathbb{K}$ are such that $z \in KwK$ then $\nu(z) = \nu(w)$. There exists thus a function

$$a : \mathsf{G}_\mathbb{K} \to \mathsf{E}^+$$

such that for every $g_1, g_2 \in \mathsf{G}_\mathbb{K}$ one has that $g_1 \in Kg_2K$ if and only if $a(g_1) = a(g_2)$. It is called the *Cartan projection* of $\mathsf{G}_\mathbb{K}$.

In the case of $G_\mathbb{K} = \mathsf{PGL}(V_\mathbb{K})$ this is nothing but the ordered list of semihomotecy ratioes defined in Section 2.1.

4.3. **Representations of $\mathsf{G}_{\mathbb{K}}$.** Let $\Lambda : \mathsf{G} \to \mathsf{PGL}(V)$ be a finite dimensional rational[3] irreducible representation and denote by $\phi_\Lambda : \mathfrak{g} \to \mathfrak{sl}(V)$ the Lie algebra homomorphism associated to Λ. Then the *weight space* associated to $\chi \in \mathbf{X}(\mathsf{A})$ is the vector space

$$V_\chi = \{v \in V : \phi_\Lambda(a)v = \chi(a)v \ \forall a \in \mathsf{A}_{\mathbb{K}}\}$$

and if $V_\chi \neq 0$ then we say that $\chi^\omega \in \mathsf{E}^*$ is a *restricted weight* of Λ. Theorem 7.2 of Tits [44] states that the set of weights has a unique maximal element with respect to the order $\chi \geqslant \psi$ if $\chi - \psi$ is positive on E^+. This is called *the highest weight* of Λ and denoted by χ_Λ.

Definition 4.1. Let θ_Λ be the set of simple roots $\mathsf{a} \in \Pi$ such that $\chi_\Lambda - \mathsf{a}$ is still a weight of Λ.

Remark 4.2. The subset θ_Λ is the subset of simple roots such that the following holds: Consider $\mathsf{a} \in \Sigma^+$, $n \in \mathfrak{g}_{-\mathsf{a}}$ and $v \in \chi_\Lambda$, then $\phi_\Lambda(n)v = 0$ if and only if $\mathsf{a} \in \langle \Pi - \theta_\Lambda \rangle$.

Definition 4.3. We denote by $\| \, \|_\Lambda$ a good norm on V invariant under ΛK and such that $\Lambda \mathsf{A}_{\mathbb{K}}$ consists on semi-homotecies, if $\mathbb{K}$ is Archimedean the existence of such a norm is classical, if $\mathbb{K}$ is non-Archimedean then this is the content of Quint [38, Théorème 6.1].

For every $g \in \mathsf{G}_{\mathbb{K}}$ one has

$$\log \|\Lambda g\|_\Lambda = \chi_\Lambda(a(g)). \tag{4.2}$$

If $g = k_g z_g l_g$ with $k, l \in K$ and $z_g \in \mathsf{A}_{\mathbb{K}}^+$ then for all $v \in \Lambda(l_g^{-1})V_{\chi_\Lambda}$ one has $\|\Lambda g(v)\|_\Lambda = \|\Lambda g\|_\Lambda \|v\|_\Lambda$.

Denote by W_{χ_Λ} the $\Lambda \mathsf{A}_{\mathbb{K}}$-invariant complement of V_{χ_Λ}. Note that the stabilizer in $\mathsf{G}_{\mathbb{K}}$ of W_{χ_Λ} is $\check{\mathsf{P}}_{\theta,\mathbb{K}}$, and thus one has a map of flag spaces

$$(\xi_\Lambda, \xi_\Lambda^*) : \mathcal{F}^{(2)}_{\theta_\Lambda}(\mathsf{G}_{\mathbb{K}}) \to \mathcal{G}^{(2)}_{\dim V_{\chi_\Lambda}}(V). \tag{4.3}$$

This is a proper embedding which is an homeomorphism onto its image. Here $\mathcal{G}^{(2)}_{\dim V_{\chi_\Lambda}}(V)$ is the open $\mathsf{PGL}(V_{\mathbb{K}})$-orbit in the product of the Grassmannian of $(\dim V_{\chi_\Lambda})$-dimensional subspaces and the Grassmannian of $(\dim V - \dim V_{\chi_\Lambda})$-dimensional subspaces.

One has the following proposition by Tits (see also Humphreys [30, Chapter XI]).

Proposition 4.4 (Tits [44])**.** *For each $\mathsf{a} \in \Pi$ there exists a finite dimensional rational irreducible representation $\Lambda_{\mathsf{a}} : \mathsf{G} \to \mathsf{PSL}(V_{\mathsf{a}})$, such that $\chi_{\Lambda_{\mathsf{a}}}$ is an integer multiple of the fundamental weight ω_{a} and $\dim V_{\chi_{\Lambda_{\mathsf{a}}}} = 1$. All other weights of Λ_{a} are of the form*

$$\chi_{\mathsf{a}} - \mathsf{a} - \sum_{\mathsf{b} \in \Pi} n_{\mathsf{b}} \mathsf{b},$$

where $n_{\mathsf{b}} \in \mathbb{N}$.

We will fix from now on such a set of representations and call them, for each $\mathsf{a} \in \Pi$, the *Tits representation associated to* a.

[3] i.e. a rational map between algebraic varieties.

4.4. **The center of the Levi group $\mathsf{P}_{\theta,\mathbb{K}} \cap \check{\mathsf{P}}_{\theta,\mathbb{K}}$.** We now consider the vector subspace

$$\mathsf{E}_\theta = \bigcap_{\mathsf{a}\in\Pi-\theta} \ker \mathsf{a}^\omega$$

together with the unique projection $\pi_\theta : \mathsf{E} \to \mathsf{E}_\theta$, that is invariant under the subgroup W_θ of the Weyl group spanned by reflections associated to roots in $\Pi - \theta$:

$$W_\theta = \{w \in W : w(v) = v \,\forall v \in \mathsf{E}_\theta\}.$$

The dual space $(\mathsf{E}_\theta)^*$ is canonically the subspace of E^* of π_θ-invariant linear forms and it is spanned by the fundamental weights of roots in θ

$$(\mathsf{E}_\theta)^* = \{\varphi \in \mathsf{E}^* : \varphi \circ \pi_\theta = \varphi\} = \langle \omega_\mathsf{a} : \mathsf{a} \in \theta\rangle.$$

Since $\pi_\theta^2 = \pi_\theta$, the pre-composition with π_θ induces a projection $\mathsf{E}^* \to (\mathsf{E}_\theta)^*$ denoted by

$$\varphi \mapsto \varphi^\theta := \varphi \circ \pi_\theta.$$

The following examples will be relevant in Section 7 and 8 respectively:

Example 4.5. Let $\mathsf{G}_\mathbb{K} = \mathsf{PGL}(V_\mathbb{K})$, consider $p \in [\![2, d-2]\!]$ and let $\theta = \{\mathsf{a}_1, \mathsf{a}_p, \mathsf{a}_{d-1}\}$, so that

$$\mathsf{E}_\theta = \{(a_1, \dots, a_d) \in \mathsf{E} : a_2 \quad = \cdots = a_p \text{ and } a_{p+1} = \cdots = a_{d-1}\}$$

is three dimensional. Using the fact that the fundamental weights ω_i (for $i = 1, p, d-1$) belong to $(\mathsf{E}_\theta)^*$ one checks that the projection is

$$\begin{aligned}
&\varepsilon_1(\pi_\theta(a)) = a_1,\\
&\varepsilon_i(\pi_\theta(a)) = \frac{a_2 + \cdots + a_p}{p-1} = \frac{\omega_p - \omega_1}{p-1}(a) \text{ for every } i \in [\![2, p]\!],\\
&\varepsilon_i(\pi_\theta(a)) = \frac{a_{p+1} + \cdots + a_{d-1}}{d-p-1} = \frac{\omega_{d-1} - \omega_p}{d-p-1}(a) \text{ for every } i \in [\![p+1, d-1]\!],\\
&\varepsilon_d(\pi_\theta(a)) = a_d.
\end{aligned}$$

One has then that

$$\mathsf{a}_p^\theta = \frac{\omega_p - \omega_1}{p-1} - \frac{\omega_{d-1} - \omega_p}{d-p-1}$$

and that $\mathsf{a}_p^\theta|_{\mathsf{E}^+ - \{0\}} \geqslant \mathsf{a}_p|_{\mathsf{E}^+ - \{0\}}$.

Example 4.6. Consider the group $\mathsf{SO}(p, q)$ of transformations in $\mathsf{PSL}_{p+q}(\mathbb{R})$ preserving a signature (p, q) bilinear form with $p < q$. One has that

$$\mathsf{E} = \{(a_1, \dots, a_p) : a_i \in \mathbb{R}\}$$

equipped with the root system

$$\Sigma^\omega = \{\varepsilon_i : i \in [\![1, p]\!]\} \cup \{a \mapsto a_i - a_j : i, j \in [\![1, p]\!]\}.$$

A Weyl chamber can be chosen as

$$\mathsf{E}^+ = \{a \in \mathsf{E} : a_i \geqslant a_{i+1} \,\forall i \in [\![1, p-1]\!] \text{ and } a_p \geqslant 0\}$$

and the associated set of simple roots

$$\Pi = \{\mathsf{a}_i : i \in [\![1, p-1]\!]\} \cup \{\varepsilon_p\}.$$

Consider then $\theta = \{\mathsf{a}_i : i \in [\![1, p-1]\!]\}$, so that $\mathsf{E}_\theta = \ker \varepsilon_p$ and thus $\mathsf{a}_i \in (\mathsf{E}_\theta)^*$ for $i \in [\![1, p-2]\!]$. Moreover,

$$\mathsf{a}^\theta_{p-1} = \varepsilon_{p-1}$$

and one has that $\mathsf{a}^\theta_{p-1}|_{\mathsf{E}^+ - \{0\}} \geqslant \mathsf{a}_{p-1}|_{\mathsf{E}^+ - \{0\}}$.

4.5. **Gromov product.** Recall from S. [42] that the *Gromov product*[4] based at K is the map

$$(\cdot|\cdot)_K : \mathcal{F}^{(2)}_\theta(\mathsf{G}_\mathbb{K}) \to \mathsf{E}_\theta$$

defined to be the unique vector $(x|y)_K \in \mathsf{E}_\theta$ such that

$$\chi_\mathsf{a}((x|y)_K) = -\log \sin \measuredangle_{\| \, \|_{\Lambda_\mathsf{a}}}(\xi_{\Lambda_\mathsf{a}} x, \xi^*_{\Lambda_\mathsf{a}} y)$$

for all $\mathsf{a} \in \theta$, where χ_a is the fundamental weight associated to the Tits representation Λ_a of a. Note that

$$\max_{\mathsf{a}\in\theta} \chi_\mathsf{a}((x|y)_K) = \max_{\mathsf{a}\in\theta} |\chi_\mathsf{a}((x|y)_K)| = -\log \min_{\mathsf{a}\in\theta} \sin \measuredangle_{\| \, \|_{\Lambda_\mathsf{a}}}(\xi_{\Lambda_\mathsf{a}} x, \xi^*_{\Lambda_\mathsf{a}} y). \tag{4.4}$$

On has the following remark from Bochi-Potrie-S. [6].

Remark 4.7 ([6, Remark 8.11]). Let $\Lambda : \mathsf{G} \to \mathsf{PGL}(V)$ be a finite dimensional rational irreducible representation, if $(x, y) \in \mathcal{F}^{(2)}_{\theta_\Lambda}(\mathsf{G}_\mathbb{K})$ then

$$(\xi_\Lambda x|\xi^*_\Lambda y)_{\| \, \|_\Lambda} = \chi_\Lambda((x|y)_K).$$

where $\| \, \|_\Lambda$ denotes the (stabilizer of the) inner product on V such that ΛK is orthogonal (see Definition 4.3).

4.6. **Iwasawa cocycle and its relation to representations of G.** Another important decomposition of Lie groups that will play a role in our work is the Iwasawa decomposition:

$$\mathsf{G}_\mathbb{K} = K\mathsf{A}_\mathbb{K}\mathsf{U}_{\Pi,\mathbb{K}},$$

where $\mathsf{P}_{\Pi,\mathbb{K}}$ is the minimal parabolic subgroup, and $\mathsf{U}_{\Pi,\mathbb{K}}$ is its unipotent radical. For general local field $\mathbb{K}$ the decomposition of an element is not necessarily unique, but if $z_1, z_2 \in \mathsf{A}_\mathbb{K}$ are such that $z_1 \in Kz_2\mathsf{U}_{\Pi,\mathbb{K}}$, then $\nu(z_1) = \nu(z_2)$.

Quint used the Iwasawa decomposition to define the Iwasawa cocycle

$$\mathrm{b}_\Pi(g, x) = \nu(z)$$

where $x = k[\mathsf{P}_{\theta,\mathbb{K}}] \in \mathcal{F}_\theta(\mathsf{G}_\mathbb{K})$ with $k \in K$, $g \in \mathsf{G}_\mathbb{K}$ and gk has Iwasawa decomposition $gk = lzu$

Quint [40] proves the following lemma.

Lemma 4.8 (Quint [40, Lemmas 6.1 and 6.2])**.** *The map $p_\theta \circ \mathrm{b}_\Pi$ factors trough a map $\mathrm{b}_\theta : \mathsf{G}_\mathbb{K} \times \mathcal{F}_\theta(\mathsf{G}_\mathbb{K}) \to \mathsf{E}_\theta$. The map b_θ verifies the cocycle relation: for every $g, h \in \mathsf{G}_\mathbb{K}$ and $x \in \mathcal{F}_{\theta,\mathbb{K}}(\mathsf{G}_\mathbb{K})$ one has*

$$\mathrm{b}_\theta(gh, x) = \mathrm{b}_\theta(g, hx) + \mathrm{b}_\theta(h, x).$$

One also has the following behavior of b_θ under the representations of G.

Lemma 4.9 (Quint [40, Lemma 6.4])**.** *Let $\Lambda : \mathsf{G} \to \mathsf{PGL}(V)$ be a proximal irreducible representation, then for every $x \in \mathcal{F}_{\theta_\Lambda}(\mathsf{G}_\mathbb{K})$ and $g \in \mathsf{G}_\mathbb{K}$ one has*

$$\chi_\Lambda\big(\mathrm{b}_{\theta_\Lambda}(g, x)\big) = \log \frac{\|\Lambda(g)v\|_\Lambda}{\|v\|_\Lambda},$$

where $v \in \xi_\Lambda(x) - \{0\}$.

[4] This is the negative of the defined in S. [42].

4.7. **Cartan attractors and Cartan's attracting basins.** Consider $g \in \mathsf{G}_{\mathbb{K}}$ and let $g = k_g z_g l_g$ be a Cartan decomposition. Given $\theta \subset \Pi$, the *Cartan attractor of* g *in* $\mathcal{F}_\theta(\mathsf{G}_{\mathbb{K}})$ is defined by

$$U_\theta(g) = U_\theta^K(g) = k_g[\mathsf{P}_{\theta,\mathbb{K}}],$$

and the *Cartan basin of* g is defined, for $\alpha > 0$, by

$$B_{\theta,\alpha}(g) = \{x \in \mathcal{F}_\theta(\mathsf{G}_{\mathbb{K}}) : \big(x|U_{\mathrm{i}\,\theta}(g^{-1})\big)_K < \alpha\}.$$

Remark 4.10. If $\Lambda : \mathsf{G} \to \mathsf{PGL}(V)$ is a rational irreducible representation with $\theta_\Lambda \subset \theta$ then

$$\xi_\Lambda(U_\theta(g)) = U^{\|\,\|_\Lambda}_{\dim V_{\chi_\Lambda}}\big(\Lambda(g)\big).$$

Notice that the flag $U_\theta(g)$ is an arbitrary choice of a "most expanding" flag of type θ for g, however, it is clear from the definition that given $\alpha > 0$ there exists a constant K_α such that if $y \in \mathcal{F}_\theta(\mathsf{G}_{\mathbb{K}})$ belongs to $B_{\theta,\alpha}(g)$ then for all $\mathsf{a} \in \theta$ one has

$$|\chi_{\mathsf{a}}\big(a(g) - \mathrm{b}_\theta(g,y)\big)| \leqslant K_\alpha. \tag{4.5}$$

4.8. **The $\mathsf{PSL}_d(\mathbb{K})$ case.** Given a good norm τ on $\mathbb{K}^d$, and considering the exterior power representations of $\mathsf{PSL}_d(\mathbb{K})$, one sees that Lemma 4.9 provides the following computation for the Iwasawa cocycle $\mathrm{b} : \mathsf{PSL}_d(\mathbb{K}) \times \mathcal{F}(\mathbb{K}^d) \to \mathsf{E}$ associated to a maximal compact group stabilizing τ. For $p \in [\![1,d]\!]$ and given $g \in \mathsf{PSL}_d(\mathbb{K})$, $x \in \mathcal{F}(\mathbb{K}^d)$ one has

$$\omega_p(\mathrm{b}(g,x)) = \log \frac{\|gv_1 \wedge \cdots \wedge gv_p\|}{\|v_1 \wedge \cdots \wedge v_p\|} \tag{4.6}$$

where $\{v_1, \dots, v_p\}$ is any basis of the p-dimensional space x^p of x and $\|\,\|$ is the norm on $\wedge^p \mathbb{K}^d$ induced by τ.

Notice that, by definition, the number $\omega_p(\mathrm{b}(g,x))$ only depends on x^p, so in order to simplify notation we will also denote it by $\omega_p(\mathrm{b}(g,x^p))$.

5. Patterson-Sullivan measures in non-Anosov directions

An interesting quantity associated to a discrete subgroup $\Gamma < \mathsf{G}_{\mathbb{K}}$ is its critical exponent h_Γ^X which measures the exponential orbit growth rate of orbit points in balls (in the symmetric space of $\mathsf{G}_{\mathbb{K}}$) as the radius grows. The theory of Quint's growth indicator function, which we briefly recall in Section 5.1 allows to deduce information on h_Γ^X from information on the critical exponent of linear forms ϕ on the Weyl chamber E, that are often easier to handle with the aid of Patterson-Sullivan measures. When the discrete group $\Gamma < \mathsf{G}_{\mathbb{K}}$ is the image of an Anosov representation $\rho : \Gamma \to \mathsf{G}_{\mathbb{K}}$, and the form ϕ belongs to the dual of the Levi-Anosov subspace E_{θ_ρ}, then the thermodynamical formalism applies (see the Theorem 5.12).

In this section we will, instead, be interested in studying forms ϕ that do not belong to $(\mathsf{E}_{\theta_\rho})^*$. Our main result is Theorem 5.14 in which we show that, provided a representation ρ is Anosov with respect to some root, the existence of Patterson-Sullivan measure in any flag manifold, and thus also in non-Anosov directions ϕ, have strong implications on the critical exponent of ϕ.

5.1. **Quint's growth indicator.** We recall here some definitions from Quint [39, 40].

Let $\Gamma \subset \mathsf{G}_{\mathbb{K}}$ be a discrete subgroup, its *Quint growth indicator function* [39] $\Psi_\Gamma : \mathsf{E}^+ \to \mathbb{R}_+ \cup \{-\infty\}$ is defined as follows. Given a norm $\| \, \|$ on E and an open cone $\mathscr{C} \subset \mathsf{E}^+$ let $h_{\mathscr{C}}^{\| \, \|}$ be the critical exponent of the Dirichlet series

$$s \mapsto \sum_{g \in \Gamma : a(g) \in \mathscr{C}} e^{-s\|a(g)\|}$$

and define $\Psi_\Gamma : \mathsf{E}^+ \to \{-\infty\} \cup [0, \infty)$ by

$$\Psi_\Gamma(v) = \|v\| \inf_{v \in \mathscr{C}} h_{\mathscr{C}}^{\| \, \|},$$

where the infimum is taken over all open cones containing v. One can easily check that Ψ_Γ does not depend on the chosen norm $\| \, \|$ and is 1-positively-homogenous.

Dually one considers the growth on linear forms. The *limit (or Benoist* [4]*) cone* $\mathcal{L}_\Gamma$ of Γ is defined as the limit points of sequences $t_n a(g_n)$ where $(t_n)_{n \in \mathbb{N}} \subset \mathbb{R}_+$ converges to 0 and $(g_n)_{n \in \mathbb{N}} \subset \Gamma$. Denote its dual cone by

$$(\mathcal{L}_\Gamma)^* = \{\varphi \in \mathsf{E}^* : \varphi | \mathcal{L}_\Gamma - \{0\} \geqslant 0\},$$

and, for $\varphi \in (\mathcal{L}_\Gamma)^*$ let $h_\Gamma(\varphi)$ be the critical exponent of the Dirichlet series

$$\sum_{g \in \Gamma} e^{-s\varphi\left(a(g)\right)},$$

that is

$$h_\Gamma(\varphi) = \limsup_{t \to \infty} \frac{1}{T} \log \#\{\gamma \in \mathsf{\Gamma} | \varphi\Big(a\big(\rho(\gamma)\big)\Big) < t\}.$$

One has the following.

Lemma 5.1. *It holds*

$$h_\Gamma(\min\{\phi_1, \dots, \phi_k\}) = \max\{h_\Gamma(\phi_1), \dots, h_\Gamma(\phi_k)\}.$$

Proof. One inequality is clear. For the other one, one has

$$\begin{aligned} h_\Gamma(\min\{\phi_1, \dots, \phi_k\}) &\leqslant \lim_{t \to \infty} \tfrac{1}{t} \log \sum_{i=1}^{k} \#\{\gamma \in \mathsf{\Gamma} | \phi_i(a(\rho(\gamma))) < t\} \\ &\leqslant \lim_{t \to \infty} \tfrac{1}{t} \log k \max_i \#\{\gamma \in \mathsf{\Gamma} | \phi_i(a(\rho(\gamma))) < t\} \\ &= \max\{h_\Gamma(\phi_1), \dots, h_\Gamma(\phi_k)\} \end{aligned}$$

□

One can then define the subset

$$\mathcal{D}_\Gamma = \{\varphi \in (\mathcal{L}_\Gamma)^* : h_\Gamma(\varphi) \in (0, 1]\}.$$

The next lemma is clear from the definitions, but is very useful in applications:

Lemma 5.2. *If ϕ belongs to $\mathcal{D}_\Gamma$, then $\phi + \psi \in \mathcal{D}_\Gamma$ for every $\psi \in (\mathcal{L}_\Gamma)^*$.*

The following result from Quint [39] allows to deduce information on te critical exponent of various norms in terms of growth of linear functions, that are often easier to compute:

Proposition 5.3 (Quint [39])**.** *One has that*

$$\mathcal{D}_\Gamma = \{\varphi \in \mathsf{E}^* : \forall v \in \mathsf{E}^+ \ \varphi(v) \geqslant \Psi_\Gamma(v)\},$$

and thus it is a convex set. Moreover, for any 1*-positively-homogenous function* $\Theta : \mathsf{E}^+ \to \mathbb{R}$ *the critical exponent* $h_\Gamma(\Theta)$ *of the Dirichlet series*

$$s \mapsto \sum_{g\in\Gamma} e^{-s\Theta\left(a(g)\right)}$$

can be computed as $h_\Gamma(\Theta) = \sup\limits_{v\in\mathsf{E}^+} \dfrac{\Psi_\Gamma(v)}{\Theta(v)}$.

The importance of the set $\mathcal{D}_\Gamma$ is provided by the following theorem: it is possible to compute the orbit growth rate with respect to various norms studying properties of the set $\mathcal{D}_\Gamma$:

Theorem 5.4 (Quint [39])**.** *If the Zariski closure of* Γ *is semi-simple then* Ψ_Γ *is concave, consequently for every norm* $\| \, \|$ *on* E *one has*

$$h_\Gamma^{\| \, \|} = \inf\{\|\varphi\|^* : \varphi \in \mathcal{D}_\Gamma\}$$

where $\| \, \|^*$ *is the induced operator norm on* E^*.

Remark 5.5. Recall that, if we endow the symmmetric space (or the affine building) X associated to $\mathsf{G}_\mathbb{K}$ with a $\mathsf{G}_\mathbb{K}$-invaraint Riemannian metric, there exists an Euclidean norm $\| \, \|_X$ on E such that for every $g \in \mathsf{G}_\mathbb{K}$ one has

$$d_X([K], g[K]) = \|a(g)\|_X.$$

Theorem 5.4 provides then the following formula for the critical exponent of a discrete group with reductive Zariski-closure in the symmetric space X:

$$h_\Gamma^X = \inf\{\|\phi\|_X^* : \phi \in \mathcal{D}_\Gamma\}.$$

The topological boundary $\mathcal{Q}_\Gamma$ of $\mathcal{D}_\Gamma$ will be called *Quint's indicator set of* Γ. We will also denote by

$$\mathcal{Q}_{\Gamma,\theta} = \mathcal{Q}_\Gamma \cap (\mathsf{E}_\theta)^*.$$

Let us record here a useful direct consequence of the convexity of $\mathcal{D}_\Gamma$.

Lemma 5.6. *Let* $\Gamma \subset \mathsf{G}_\mathbb{K}$ *be a discrete subgroup and let* $\phi, \varphi \in (\mathcal{L}_\Gamma)^*$, *then*

$$h_\Gamma(\phi + \varphi) \leqslant \frac{h_\Gamma(\phi) h_\Gamma(\varphi)}{h_\Gamma(\phi) + h_\Gamma(\varphi)}$$

We end this subsection with the following definition from Quint [40].

Definition 5.7. Given $\theta \subset \Pi$ and $\varphi \in (\mathsf{E}_\theta)^*$ a (Γ, φ)*-Patterson-Sullivan measure* on $\mathcal{F}_\theta(\mathsf{G}_\mathbb{K})$ is a finite Radon measure μ such that for every $g \in \Gamma$ one has

$$\frac{\mathsf{d}g_*\mu}{\mathsf{d}\mu}(x) = e^{-\varphi\left(\mathrm{b}_\theta(g^{-1},x)\right)}.$$

5.2. **Anosov representations with values in $\mathsf{G}_{\mathbb{K}}$.** Let Γ be a discrete group and fix $\theta \subset \Pi$.

Definition 5.8. A representation $\rho : \Gamma \to \mathsf{G}_{\mathbb{K}}$ is θ-*Anosov* if there exist constants $c \geqslant 0$ and $\mu > 0$ such that for every $\gamma \in \Gamma$ and $\mathsf{a} \in \theta$ one has

$$\mathsf{a}\Big(a\big(\rho(\gamma)\big)\Big) \geqslant \mu|\gamma| - c.$$

If $\rho : \Gamma \to \mathsf{G}_{\mathbb{K}}$ is θ-Anosov and Λ_{a} is as in Proposition 4.4, then $\Lambda_{\mathsf{a}}\rho : \Gamma \to \mathsf{PGL}(V_{\mathbb{K}})$ is projective Anosov. In particular subsection 2.2 applies to arbitrary $\mathsf{G}_{\mathbb{K}}$ and one obtains the following result.

Theorem 5.9 (Kapovich-Leeb-Porti [32])**.** *If $\rho : \Gamma \to \mathsf{G}_{\mathbb{K}}$ is θ-Anosov then Γ is word-hyperbolic and there exist continuous equivariant maps $\xi^{\theta}_{\rho} : \partial\Gamma \to \mathcal{F}_{\theta}(\mathsf{G}_{\mathbb{K}})$ and $\xi^{\mathrm{i}\,\theta}_{\rho} : \partial\Gamma \to \mathcal{F}_{\mathrm{i}\,\theta}(\mathsf{G}_{\mathbb{K}})$ such that the product map $(\xi^{\theta}_{\rho}, \xi^{\mathrm{i}\,\theta}_{\rho}) : \partial^{(2)}\Gamma \to \mathcal{F}^{(2)}_{\theta}(\mathsf{G}_{\mathbb{K}})$ is transverse.*

We will sometime use the notation introduced in [36] and, if $x \in \partial\Gamma$ is a point, denote by

$$x^{\theta}_{\rho} := \xi^{\theta}_{\rho}(x) \in \mathcal{F}_{\theta}(\mathsf{G}_{\mathbb{K}})$$

the image of x via the boundary map. If $\theta = \{\mathsf{a}_k\}$ consists of a single root we will also write ξ^{k}_{ρ} and x^{k}_{ρ} instead of $\xi^{\{\mathsf{a}_k\}}_{\rho}$ and $x^{\{\mathsf{a}_k\}}_{\rho}$.

If $\theta \subset \Pi$ contains the root a, we denote by $\pi_{\mathsf{a}} : \mathcal{F}_{\theta}(\mathsf{G}_{\mathbb{K}}) \to \mathcal{F}_{\mathsf{a}}(\mathsf{G}_{\mathbb{K}})$ the natural projection. It is easy to deduce from Corollary 3.4 the following more general statement:

Corollary 5.10. *Let $\rho : \Gamma \to \mathsf{G}_{\mathbb{K}}$ be a-Anosov and consider $\alpha > 0$. There exists C only depending on α and ρ such that for every $\theta \subset \Pi$ containing a, if*

$$\xi^{\mathsf{a}}_{\rho}(\partial\Gamma) \cap \pi_{\mathsf{a}}\Big(\rho(\gamma) \cdot B_{\theta,\alpha}(\rho(\gamma)) \cap \rho(\eta) \cdot B_{\theta,\alpha}(\rho(\eta))\Big) \neq \varnothing$$

then

$$d(\gamma, \eta) \leqslant \big||\gamma| - |\eta|\big| + C.$$

Definition 5.11. Given a representation $\rho : \Gamma \to \mathsf{G}_{\mathbb{K}}$ we define its *Anosov-Levi space* as $(\mathsf{E}_{\theta_\rho})^*$ where

$$\theta_{\rho} = \{\mathsf{a} \in \Pi : \rho \text{ is } \mathsf{a}\text{-Anosov}\}.$$

It is spanned by the fundamental weights $\{\omega_{\mathsf{a}} : \mathsf{a} \in \theta_{\rho}\}$.

A more precise description of the indicator set of ρ can be given on its Anosov-Levi space. The following is a combination of Bridgeman-Canary-Labourie-S. [8, Theorem 1.3], Potrie-S. [35, Proposition 4.11] and S. [41].

Theorem 5.12. *Let $\rho : \Gamma \to \mathsf{G}_{\mathbb{K}}$ be a representation, then $\mathcal{Q}_{\rho(\Gamma),\theta_\rho}$ is an analytic co-dimension 1 embedded sub-manifold of $(\mathsf{E}_{\theta_\rho})^*$ that varies analytically with ρ; moreover its restriction to the dual of the vector space spanned by the periods is strictly convex.*

5.3. **When some wall is not attained.** The purpose of this subsection is to explore $\mathcal{Q}_{\rho(\Gamma)}$ in directions that are not controlled by the roots with respect to which ρ is Anosov.

Definition 5.13. Let $\rho : \Gamma \to \mathsf{G}_{\mathbb{K}}$ be an a-Anosov representation. Consider $\theta \subset \Pi$ with $\mathsf{a} \in \theta$ and let μ^{φ} be a $\big(\rho(\Gamma), \varphi\big)$-Patterson-Sullivan measure on $\mathcal{F}_{\theta}(\mathsf{G}_{\mathbb{K}})$ for some $\varphi \in (\mathsf{E}_{\theta})^*$. We say that ρ is μ^{φ}-*irreducible* if for every $y \in \mathcal{F}_{\mathrm{i}\,\theta}(\mathsf{G}_{\mathbb{K}})$ one has

$$\mu^{\varphi}\big(\operatorname{Ann}(y)\big) < \mu^{\varphi}\big(\mathcal{F}_{\theta}(\mathsf{G}_{\mathbb{K}})\big).$$

It is clear that if $\rho(\Gamma)$ is Zariski dense in $\mathsf{G}_{\mathbb{K}}$ then it is μ^{φ}-irreducible for any Patterson-Sullivan measure. Even assuming Zariski-density, the following result is a refinement of Quint [40, Théorème 8.1] when θ contains a root with respect to which ρ is Anosov. Indeed, in the general case treated by Quint, one needs to control the mass of shadows on the flag space associated to $\Pi - \theta$, and, as a result, the existence of a $(\rho(\Gamma), \varphi)$-Patterson Sullivan measure only ensures that $\varphi + \rho_{\theta^c}$ is in $\mathcal{D}_{\rho(\Gamma)}$, where ρ_{θ^c} is a suitable form that is non-negative on the Weyl chamber. In our case, the Anosov condition with respect to one root in θ permits to control φ directly.

Theorem 5.14. *Let $\rho : \Gamma \to \mathsf{G}_{\mathbb{K}}$ be an a-Anosov representation. Consider $\theta \subset \Pi$ with $\mathsf{a} \in \theta$ and let μ^{φ} be a $\big(\rho(\Gamma), \varphi\big)$-Patterson-Sullivan measure on $\mathcal{F}_{\theta}(\mathsf{G}_{\mathbb{K}})$ for some $\varphi \in (\mathsf{E}_{\theta})^*$. Assume ρ is μ^{φ}-irreducible, then*

$$\varphi \in \mathcal{D}_{\rho(\Gamma)}.$$

The rest of the section is devoted to the proof of this result. We begin with the following lemma from Quint [40]. Quint assumes that the representation is Zariski dense, an hypothesis that is too strong for the appliations we have in mind. We observe however that for the proof to work only μ^{φ}-irreducibility is needed. We sketch the proof for completeness.

Lemma 5.15 ([40, Lemme 8.2])**.** *Let $\rho : \Gamma \to \mathsf{G}_{\mathbb{K}}$ be a representation, μ^{φ} be a $\big(\rho(\Gamma), \varphi\big)$-Patterson-Sullivan measure on $\mathcal{F}_{\theta}(\mathsf{G}_{\mathbb{K}})$. Assume ρ is μ^{φ}-irreducible, then there exists $\alpha_0 > 0$ such that for every given $0 < \alpha < \alpha_0$ there exist $k > 0$ only depending on α, such that for every $\gamma \in \Gamma$ one has*

$$k^{-1} e^{-\varphi\left(a(\rho(\gamma))\right)} \leqslant \mu^{\varphi}\Big(\rho(\gamma) B_{\theta,\alpha}(\rho(\gamma))\Big) \leqslant k e^{-\varphi\left(a(\rho(\gamma))\right)}.$$

Proof. Observe that μ^{φ}-irreducibility guarantees that there exist $\alpha, k > 0$ such that for every $\gamma \in \Gamma$, $\mu^{\phi}(B_{\theta,\alpha}(\rho(\gamma))) \geqslant k$: indeed otherwise there would be a sequence of reals $\alpha_n \to 0$ and elements $\gamma_n \in \Gamma$ with $\mu^{\phi}(B_{\theta,\alpha_n}(\rho(\gamma_n))) \leqslant 1/n$. We can assume, up to extracting a subsequence, that the complement of $B_{\theta,\alpha_n}(\rho(\gamma))$ converges to $\operatorname{Ann}(y)$ for some $y \in \mathcal{F}_{\mathrm{i}\,\theta}$, and this contraddicts μ^{φ}-irreducibility.

The result then follows from the definition of $(\rho(\Gamma), \phi)$-Patterson-Sullivan measure using Equation (4.5). $\square$

The rest of the proof of Theorem 5.14 is similar to the argument showing that if there exists a Patterson-Sullivan density of a given exponent, then this exponent must be greater than the critical exponent (see for example Sullivan [43] and Quint's notes [37, Theorem 4.11]):

Proof of Theorem 5.14. We have to show that for every $s > 0$ one has

$$\sum_{\gamma\in\Gamma} e^{-(1+s)\varphi\left(a(\rho(\gamma))\right)} < \infty.$$

Corollary 5.10 implies that given $\alpha > 0$ there exists $N \in \mathbb{N}$, such that if $t > 0$ and

$$\Gamma_t = \{\gamma \in \Gamma : t \leqslant |\gamma| \leqslant t+1\},$$

then for every $x \in \partial\Gamma$ one has

$$\#\Big\{\gamma \in \Gamma_t : \pi_{\mathsf{a}}^{-1}(\xi_\rho^{\mathsf{a}}(x)) \cap \rho(\gamma)B_{\theta,\alpha}(\rho(\gamma)) \neq \varnothing\Big\} \leqslant N,$$

Lemma 5.15 now yields for every $t \geqslant 0$

$$\infty > \mu^\varphi\Big(\pi_{\mathsf{a}}^{-1}\big(\xi_\rho^{\mathsf{a}}(\partial\Gamma)\big)\Big) \geqslant C \sum_{\gamma\in\Gamma_t} e^{-\varphi\left(a(\rho(\gamma))\right)}, \tag{5.1}$$

where C is independent of t. This is to say, there exists $K > 0$ independent of $t \in \mathbb{R}_+$ such that

$$\sum_{\gamma\in\Gamma_t} e^{-\varphi\left(a(\rho(\gamma))\right)} < K.$$

Since $\varphi \in (\mathcal{L}_{\rho(\Gamma)})^*$ and ρ is a-Anosov there exist positive δ, δ' and C such that

$$\varphi\big(a(\rho(\gamma))\big) \geqslant \delta'\mathsf{a}\big(\rho(\gamma)\big) \geqslant \delta|\gamma| - C.$$

One concludes that for every $s > 0$ one has

$$\begin{aligned}\sum_{\gamma\in\Gamma} e^{-(1+s)\varphi\left(a(\rho(\gamma))\right)} &\leqslant \sum_{n=0}^{\infty}\sum_{\gamma\in\Gamma_n} e^{-\varphi\left(a(\rho(\gamma))\right)} e^{-s\mathsf{a}(\rho(\gamma))} \\ &\leqslant Ke^C \sum_{n=0}^{\infty} e^{-\delta sn} < \infty,\end{aligned}$$

as desired.

□

6. Anosov representations with Lipschitz limit set

In this section we will prove Theorem A. We will hence fix some notation throughout this section.

Assumption 6.1. The group Γ will be a word-hyperbolic group whouse boundary $\partial\Gamma$ is homeomorphic to a sphere of dimension d_Γ. We will also fix a projective Anosov representation $\rho : \Gamma \to \mathsf{PSL}_d(\mathbb{R})$ such that the sphere $\xi_\rho^1(\partial\Gamma)$ is a Lipschitz submanifold of $\mathbb{P}(\mathbb{R}^d)$, i.e. it is locally the graph of a Lipschitz map. Note that we have restricted ourselves to $\mathbb{K} = \mathbb{R}$.

6.1. **The p-th Jacobian.** Given a line ℓ contained in a $p+1$-dimensional subspace V of $\mathbb{R}^d$, the space of infinitesimal deformations of ℓ inside V

$$\mathsf{T}_\ell\mathbb{P}(V) \subset \mathsf{T}_\ell\mathbb{P}(\mathbb{R}^d)$$

carries a natural volume form induced by the choice of a scalar product τ on $\mathbb{R}^d$. Namely, if one considers the τ-orthogonal decomposition $V = \ell \oplus \ell_V^\perp$, then one canonically identifies $\mathsf{T}_\ell\mathbb{P}(V) = \hom(\ell, \ell_V^\perp)$ and thus one can define $\Omega_{\ell,V} \in \wedge^p(\mathsf{T}_l\mathbb{P}(V))$ by

$$\Omega_{\ell,V}(\varphi_1, \dots, \varphi_p) = \frac{v \wedge \varphi_1(v) \wedge \cdots \wedge \varphi_p(v)}{\|v\|^{p+1}}$$

for any $v \in \ell - \{0\}$.

Definition 6.2. The linear form $\mathcal{J}_p^u \in (\mathsf{E}_{\{\mathsf{a}_1,\mathsf{a}_{p+1}\}})^*$ defined by

$$\mathcal{J}_p^u = (p+1)\omega_1 - \omega_{p+1}$$

is called *the p-th unstable Jacobian.*

Lemma 6.3. *Given $g \in \mathsf{PSL}_d(\mathbb{R})$ and a partial flag $(\ell, V) \in \mathcal{F}_{\{\mathsf{a}_1,\mathsf{a}_{p+1}\}}(\mathbb{R}^d)$, one has*

$$g^*\Omega_{g\ell,gV} = \exp\Big(- \mathcal{J}_p^u\big(\mathrm{b}_{\{\mathsf{a}_1,\mathsf{a}_{p+1}\}}(g,(\ell,V))\big)\Big)\Omega_{\ell,V}.$$

Proof. This is an explicit computation using equation (4.6) and the definition of $\Omega_{\ell,V}$.

Indeed, whenever $\varphi_1, \dots, \varphi_p \in \hom(\ell, \ell_V^\perp)$ are linearly independent, the vectors $\{v, \varphi_1(v), \cdots, \varphi_p(v)\}$ form a basis of V and thus:

$$\begin{aligned}
g^*\Omega_{g\ell,gV}(\varphi_1, \dots, \varphi_p) &= \Omega_{g\ell,gV}(g\varphi_1, \dots, g\varphi_p)\\
&= \frac{gv \wedge (g\varphi_1)(gv) \wedge \cdots \wedge (g\varphi_p)(gv)}{\|gv\|^{p+1}}\\
&= \frac{gv \wedge g(\varphi_1(v)) \wedge \cdots \wedge g(\varphi_p(v))}{\|gv\|^{p+1}}\\
&= \frac{gv \wedge g(\varphi_1(v)) \wedge \cdots \wedge g(\varphi_p(v))}{v \wedge \varphi_1(v) \wedge \cdots \wedge \varphi_p(v)} \frac{v \wedge \varphi_1(v) \wedge \cdots \wedge \varphi_p(v)}{\|v\|^{p+1}} \frac{\|v\|^{p+1}}{\|gv\|^{p+1}}\\
&= \exp\big(\omega_{p+1}(\mathrm{b}_{\{\mathsf{a}_1,\mathsf{a}_{p+1}\}}(g,V)) - (p+1)\omega_1(\mathrm{b}_{\{\mathsf{a}_1,\mathsf{a}_{p+1}\}}(g,\ell))\big)\Omega_{\ell,V}.
\end{aligned}$$

□

6.2. **Existence of a $\mathcal{J}_{d_\Gamma}^u$-Patterson-Sullivan measure.** Let us prove the following proposition.

Proposition 6.4. *Under assumption 6.1, there exists a $(\rho(\Gamma), \mathcal{J}_{d_\Gamma}^u)$-Patterson-Sullivan measure on $\mathcal{F}_{\{\mathsf{a}_1,\mathsf{a}_{d_\Gamma}\}}$.*

Proof. It follows from Rademacher's theorem [17, Theorem 3.2] that $\xi_\rho^1(\partial\Gamma)$ has a well defined Lebesgue measure class (cfr. [19, Section 3.2]), and that Lebesgue almost every point $\xi_\rho^1(x) \in \xi_\rho^1(\partial\Gamma)$ has a well defined tangent space, this defines a $d_\Gamma + 1$ dimensional vector subspace $x_\rho^{d_\Gamma+1} \in \mathcal{F}_{\{\mathsf{a}_{d_\Gamma+1}\}}(\mathbb{R}^d)$ such that

$$\mathsf{T}_{\xi_\rho^1(x)}\big(\xi_\rho^1(\partial\Gamma)\big) = \hom(\xi_\rho^1(x), x_\rho^{d_\Gamma+1}/\xi_\rho^1(x)). \tag{6.1}$$

Consider the ρ-equivariant measurable map $\zeta_\rho : \xi^1_\rho(\partial\Gamma) \to \mathcal{F}_{\{\mathsf{a}_1,\mathsf{a}_{d_\Gamma+1}\}}(\mathbb{R}^d)$ defined by

$$\zeta_\rho(\xi^1_\rho(x)) = (\xi^1_\rho(x), x^{d_\Gamma+1}_\rho). \tag{6.2}$$

We can then define a volume form on $\xi^1_\rho(\partial\Gamma)$ via

$$\xi^1_\rho(x) \mapsto \Omega_{\zeta_\rho(\xi^1_\rho(x))}.$$

This form is defined Lebesgue almost everywhere and thus defines a Lebesgue measure on $\xi^1_\rho(\partial\Gamma)$, which we will denote by ν_ρ. Lemma 6.3 implies directly that the push-forward $(\zeta_\rho)_*\nu_\rho$ is the desired measure. □

6.3. **When $\partial\Gamma$ is a circle.** Recall from the introduction that we say that ρ is *weakly irreducible* if the vector space span $\big(\xi^1_\rho(\partial\Gamma)\big)$ is the whole space.

Lemma 6.5. *Under assumption 6.1 together with weakly irreducibility of ρ and $d_\Gamma = 1$, one has that ρ is μ^φ-irreducible for any $(\rho(\Gamma), \varphi)$-Patterson-Sullivan measure on $\mathcal{F}_{\{\mathsf{a}_1,\mathsf{a}_2\}}(\mathbb{R}^d)$ whose projection is absolutely continuous with ν_ρ.*

Proof. If this were not the case, there would exist $(W_0, P_0) \in \mathcal{F}_{\{\mathsf{a}_{d-2},\mathsf{a}_{d-1}\}}(\mathbb{R}^d)$ such that $\mathrm{Ann}(W_0, P_0)$ would have full μ^φ-mass; as ρ is projective Anosov we can furthermore assume that $P_0 = \xi^{d-1}_\rho(x)$ for some $x \in \partial\Gamma$ and thus the condition $\xi^1_\rho(y) \subset P_0$ only occurs for $y = x$.

Hence, since the projection of μ^φ is absolutely continuous w.r.t. to ν_ρ one has that for μ^φ-almost every $\xi^1_\rho(x) \in \xi^1_\rho(\partial\Gamma)$ the vector space x^2_ρ from subsection 6.2 intersects W_0.

Let us choose a scalar product τ on $\mathbb{R}^d$, and the induced distance function of $\mathbb{P}(\mathbb{R}^d)$. Let us denote by $[W_0]$ the quotient vector space $\mathbb{R}^d/W_0$, it is a 2-dimensional vector space and every line $\ell \notin W_0$ defines a line $[\ell \oplus W_0]$ in $[W_0]$. Moreover, for every $\delta > 0$ the double quotient projection

$$\pi : \big\{\ell \in \mathbb{P}(\mathbb{R}^d) : \measuredangle_\tau(\ell, W_0) > \delta\big\} \to \mathbb{P}\big([W_0]\big),$$

defined by $\pi(\ell) = \big[[\ell \oplus W_0]\big]$, is Lipschitz.

We denote by $U_\delta \subset \xi^1_\rho(\partial\Gamma)$ the relative open subset defined by

$$U_\delta = \{\ell \in \xi^1_\rho(\partial\Gamma) : \measuredangle_\tau(\ell, W_0) > \delta\}$$

and consider the Lipschitz map $\pi|U_\delta : U_\delta \to \mathbb{P}([W_0])$. Since, by assumption, for μ^φ-almost every $\xi^1_\rho(x) \in \xi^1_\rho(\partial\Gamma)$ the plane x^2_ρ intersects W_0, one concludes from equation (6.1) that $\pi|U_\delta$ has zero derivative ν_ρ-almost everywhere.

Since Lipschitz maps are absolutely continuous, and in particular satisfy the fundamental theorem of calculus, we deduce that $\pi|\xi^1_\rho(\partial\Gamma)$ is constant. This implies that

$$\xi^1_\rho(\partial\Gamma) \subset W_0 \oplus \xi^1_\rho(x),$$

for any $x \in \partial\Gamma$, which contradicts the weak irreducibility assumption. □

We can now prove Theorem A when $d_\Gamma = 1$:

Corollary 6.6. *Let Γ be a word-hyperbolic group such that $\partial\Gamma$ is homeomorphic to a circle. Let $\rho : \Gamma \to \mathsf{PGL}_d(\mathbb{R})$ be a weakly irreducible a_1-Anosov representation such that $\xi^1_\rho(\partial\Gamma)$ is a Lipschitz curve. Then*

$$\mathsf{a}_1 \in \mathcal{Q}_{\rho(\Gamma)}.$$

Proof. Note that $\mathsf{a}_1 = \mathcal{J}_1^u$ is the first unstable Jacobian. Since $\xi_\rho^1(\partial\Gamma)$ is a Lipschitz circle, it has Hausdorff dimension 1 and thus Corollary 1.9 implies that $h_\rho^{\mathsf{a}_1} \geqslant 1$.

On the other hand, Lemma 6.4 provides a $(\rho(\Gamma), \mathcal{J}_1^u)$-Patterson-Sullivan measure $\mu^{\mathcal{J}_1^u}$ on $\mathcal{F}_{\{\mathsf{a}_1,\mathsf{a}_2\}}(V_\mathbb{R})$ that projects to the Lebesgue measure on $\xi_\rho^1(\partial\Gamma)$. Since ρ is weakly irreducible, Lemma 6.5 implies that it is $\mu^{\mathcal{J}_1^u}$-irreducible, thus Theorem 5.14 applies to give

$$\mathsf{a}_1 = \mathcal{J}_1^u \in \mathcal{D}_{\rho(\Gamma)},$$

this is to say $h_\rho(\mathsf{a}_1) \leqslant 1$ which concludes the proof.

□

Before proceeding to arbitrary d_Γ let us record a direct consequence of Corollary 6.6. Let us say that ρ is *coherent* if the first root arising in span $\left(\xi_\rho^1(\partial\Gamma)\right)$ is a_1.

Corollary 6.7. *Let Γ be a word-hyperbolic group such that $\partial\Gamma$ is homeomorphic to a circle. Let $\rho : \Gamma \to \mathsf{G}_\mathbb{K}$ be an a-Anosov representation and assume there exists proximal, real representation $\Lambda : \mathsf{G}_\mathbb{K} \to \mathsf{PGL}(V_\mathbb{R})$ with first root a, such that $\Lambda \circ \rho$ is coherent, then*

$$\mathsf{a} \in \mathcal{Q}_{\rho(\Gamma)}.$$

6.4. **When $\partial\Gamma$ has arbitrary dimension.** Recall that a subgroup $\Gamma \subset \mathsf{PGL}(V_\mathbb{K})$ is *strongly irreducible* if any finite index subgroup acts irreducibly. It is well known that this is equivalent to the fact that the connected component of the identity of the Zariski closure of Γ acts irreducibly on $\mathbb{K}^d$.

We will need the following lemma (that does not require assumption 6.1).

Lemma 6.8. *Let $\eta : \Gamma \to \mathsf{PGL}_d(\mathbb{R})$ be a strongly irreducible a_1-Anosov representation. Assume that there exists $p \in [\![1, d-1]\!]$ and a measurable η-equivariant section $\zeta : \partial\Gamma \to \mathcal{F}_{\{\mathsf{a}_1,\mathsf{a}_p\}}(\mathbb{R}^d)$. Then η is μ^φ-irreducible for any $(\rho(\Gamma), \varphi)$-Patterson-Sullivan measure on $\mathcal{F}_\theta(\mathbb{K}^d)$.*

Proof. Otherwise we would be able to find a subspace $W_0 \in \mathcal{F}_{\{\mathsf{a}_{d-p}\}}(\mathbb{R}^d)$ such that for almost every[5] $\xi_\rho^1(x) \in \xi_\rho^1(\partial\Gamma)$ one has $\zeta(x)^p \cap W_0 \neq \{0\}$. Since ζ is η-equivariant, we would find a p-dimensional subspace V such that for every $\gamma \in \Gamma$,

$$\eta(\gamma)V \cap W_0 \neq \{0\}.$$

This implies that for every g in the Zariski closure of $\eta(\Gamma)$ it holds that $\dim gV \cap W_0 \geqslant 1$. The contradiction comes from Labourie [33, Proposition 10.3] stating that the identity component of such a Lie group cannot act irreducibly. □

We can now prove Theorem A for arbitrary d_Γ.

Corollary 6.9. *Under assumption 6.1 together with strong irreducibility of ρ one has*

$$\mathcal{J}_{d_\Gamma}^u \in \mathcal{Q}_\rho(\Gamma).$$

Proof. Since $\xi_\rho^1(\partial\Gamma)$ is a Lipschitz sphere, it has Hausdorff dimension d_Γ and thus Corollary 1.9 implies that $h_\rho(\mathcal{J}_{d_\Gamma}^u) \geqslant 1$. Lemma 6.4 guarantees the existence of a $(\rho(\Gamma), \mathcal{J}_{d_\Gamma}^u)$-Patterson-Sullivan measure. Moreover, the equivariant map from equation (6.2) allows us to apply Lemma 6.8 and thus we are in the hypothesis of Theorem 5.14, consequently $h_\rho(\mathcal{J}_{d_\Gamma}^u) \leqslant 1$, which concludes the proof. □

[5] with respect to the pushed forward measure $\pi_*\mu^\varphi$, where $\pi : \mathcal{F}_{\{\mathsf{a}_1,\mathsf{a}_p\}}(\mathbb{R}^d) \to \mathbb{P}(\mathbb{R}^d)$ consist con forgetting the p-th coordinate,

7. $(1,1,p)$-hyperconvex representations and a C^1-dichotomy for surface groups

In this section we will consider projective Anosov representations whose image of the boundary map is a C^1 submanifold. In the second part of the section we will prove Corollary 1.8 providing a a C^1-dichotomy for surface groups.

7.1. (1,1,p)-hyperconvex representations.

Definition 7.1. A $\{\mathsf{a}_1, \mathsf{a}_p\}$-Anosov representation $\rho : \Gamma \to \mathsf{PGL}_d(\mathbb{R})$ is $(1,1,p)$-*hyperconvex* if, for every pairwise distinct $x, y, z \in \partial\Gamma$, the sum

$$\xi^1(x) + \xi^1(y) + \xi^{d-p}(z)$$

is direct.

Example 7.2. Examples of Zariski dense hyperconvex representations can be obtained by deforming $\mathsf{S}^k \circ \iota$, where S^k denotes the k-th symmetric power and $\iota : \Gamma \to \mathsf{PO}(1,p)$ is the inclusion of a co-compact lattice, see P.-S.-W. [36, Corollary 7.6].

Hyperconvex representations were introduced by Labourie [33] for surface groups and further studied by Zhang-Zimmer [45] when the boundary of Γ is a topologically a sphere and by P.-S.-W. [36] for arbitrary hyperbolic groups. In both [36, Proposition 7.4] and [45, Theorem 1.1] one finds the following result.

Theorem 7.3 (P.-S.-W. and Zhang-Zimmer)**.** *Assume that $\partial\Gamma$ is topologically a sphere of dimension $p-1$ and let $\rho : \Gamma \to \mathsf{PGL}_d(\mathbb{R})$ be a $(1,1,p)$-hyperconvex representation. Then $\xi^1_\rho(\partial\Gamma)$ is a C^1-sphere.*

Theorem A then gives:

Corollary 7.4. *Assume that $\partial\Gamma$ is topologically a sphere of dimension $p-1$ and let $\rho : \Gamma \to \mathsf{PSL}_d(\mathbb{R})$ be strongly irreducible and $(1,1,p)$-hyperconvex. Then $h_\rho(\mathcal{J}^u_p) = 1$.*

Remark 7.5. This generalizes Potrie-S. [35, Corollary 7.1]. Observe however that, since the limit set $\xi^1(\partial\Gamma)$ is a C^1-submanifold of $\mathbb{P}(\mathbb{R}^d)$, the arguments of [35] adapt directly to give a version of Corollary 7.4 without requiring strong irreducibility.

Glorieux-Monclair-Tholozan [22] recently showed the following.

Theorem 7.6 (Glorieux-Monclair-Tholozan [22])**.** *Let $\rho : \Gamma \to \mathsf{PGL}_d(\mathbb{R})$ be an a_1-Anosov representation that preserves a propery convex domain, then*

$$2h_\rho(\omega_1 + \omega_{d-1}) \leqslant \dim_{\mathrm{Hff}}\big((\xi^1, \xi^{d-1})(\partial\Gamma)\big),$$

where $(\xi^1, \xi^{d-1}) : \partial\Gamma \to \mathbb{P}(\mathbb{R}^d) \times \mathbb{P}\big((\mathbb{R}^d)^*\big)$.

As an application of Corollary 7.4 we show that, for $(1,1,p)$-hyperconvex representations with $p < d-1$ such bound can never be acheived (note that we do <u>not</u> require the representation to preserve a convex set):

Proposition 7.7. *Assume that $\partial\Gamma$ is topologically a sphere of dimension $p-1$ and let $\rho : \Gamma \to \mathsf{PGL}_d(\mathbb{R})$ be strongly irreducible and $(1,1,p)$-hyperconvex. If $p < d-1$, then*

$$2h_\rho(\omega_1 + \omega_{d-1}) < (1-\varepsilon)(p-1),$$

where $\varepsilon > 0$ only depends on the $\{\mathsf{a}_1, \mathsf{a}_p\}$-Anosov constants of ρ.

Proof. Since $p < d-1$ the functional $\phi \in \mathsf{E}^*$

$$\phi = \frac{\omega_p - \omega_1}{p-1} - \frac{\omega_{d-1} - \omega_1}{d-2}$$

is non-zero, moreover observe that, for every $v \in \mathsf{E}^+$ one has

$$\phi(v) \geqslant \left(\frac{d-p-1}{d-2}\right) \mathsf{a}_p(v).$$

Since ρ is a_p-Anosov, the last computation implies that $\ker \phi \cap \mathcal{L}_{\rho(\Gamma)} = \{0\}$ this is to say that $\phi \in (\mathcal{L}_{\rho(\Gamma)})^*$, in particular ϕ has a well defined entropy $h_\rho(\phi) \in (0, \infty)$. Moreover,

$$\begin{aligned} h_\rho\left(\frac{p-1}{d-2}((d-1)\omega_1 - \omega_{d-1})\right) &= h_\rho\big(\mathcal{J}^u_{p-1} + (p-1)\phi\big) && (7.1)\\ &\leqslant \frac{h_\rho(\phi)}{h_\rho(\phi) + p - 1}, && (7.2) \end{aligned}$$

where the equality comes from the equality between the corresponding linear forms and the inequality follows from Lemma 5.6 together with Corollary 7.4 stating that $h_\rho(\mathcal{J}^u_{p-1}) = 1$.

Finally, observe that

$$\begin{aligned} \frac{(p-1)}{2}(\omega_1 - \omega_{d-1}) &= \frac{1}{2}\Big(\frac{p-1}{d-2}((d-1)\omega_1 - \omega_{d-1}) + \frac{p-1}{d-2}((d-1)\omega_{d-1} - \omega_1)\Big)\\ &= \frac{1}{2}\big(\mathcal{J}^u_{p-1} + (p-1)\phi + \big(\mathcal{J}^u_{p-1} + (p-1)\phi\big) \circ \mathrm{i}\,\big), \end{aligned}$$

where $\mathrm{i} : \mathsf{E} \to \mathsf{E}$ is the opposition involution. Together with equation 7.1 and Lemma 5.6, this yields

$$\begin{aligned} \frac{2}{(p-1)} h_\rho(\omega_1 - \omega_{d-1}) &\leqslant 2\frac{h_\rho\big(\mathcal{J}^u_{p-1} + (p-1)\phi\big) h_\rho\big((\mathcal{J}^u_{p-1} + (p-1)\phi) \circ \mathrm{i}\,\big)}{h_\rho\big(\mathcal{J}^u_{p-1} + (p-1)\phi\big) + h_\rho\big((\mathcal{J}^u_{p-1} + (p-1)\phi) \circ \mathrm{i}\,\big)}\\ &\leqslant h_\rho\big(\mathcal{J}^u_{p-1} + (p-1)\phi\big)\\ &\leqslant \frac{h_\rho(\phi)}{h_\rho(\phi) + p - 1} < 1, \end{aligned}$$

since entropy is i-invariant.

To conclude the proof we observe that the functional ϕ belongs to the Anosov-Levi space of every $\{\mathsf{a}_1, \mathsf{a}_p\}$-Anosov representation, its entropy thus varies continuously (Theorem 5.12) and hence

$$\eta \mapsto \frac{h_\eta(\phi)}{h_\eta(\phi) + p - 1}$$

is bounded away from 1 on compact subsets of $\mathfrak{X}_{\{\mathsf{a}_1, \mathsf{a}_p\}}\big(\Gamma, \mathsf{PGL}_d(\mathbb{R})\big)$.

□

C^1-dychotomy. Now we prove Corollary 1.8. As we will later see (Section 9 and Section 10) there are many projective Anosov representations of surface groups where the image of the boundary map is Lipschitz. However, when we embed the surface group into $\mathsf{PSL}_2(\mathbb{R})$ and look small deformations of representations

$$\Gamma \to \mathsf{PSL}_2(\mathbb{R}) \to \mathsf{PSL}_d(\mathbb{R}),$$

where the $\mathsf{PSL}_2(\mathbb{R})$ representation satisfies additional proximality assumptions, then the image of the boundary map is never Lipschitz. We restate the dichotomy for the reader's convenience.

Corollary 7.8. *Let $\Lambda : \mathsf{PSL}_2(\mathbb{R}) \to \mathsf{PSL}_d(\mathbb{R})$ be a (possibly reducible) proximal representation such that $\wedge^2\Lambda$ is also proximal. Let S be a closed connected surface of genus $\geqslant 2$ and let $\rho_0 : \pi_1 S \to \mathsf{PSL}_2(\mathbb{R})$ be discrete and faithful. Then we have the following dichotomy:*

i) If the top two weights spaces of Λ belong to the same irreducible factor, then for every small deformation $\rho : \pi_1 S \to \mathsf{PSL}_d(\mathbb{R})$ of $\Lambda\rho_0$ the curve $\xi^1_\rho(\partial\pi_1 S)$ is C^1.

ii) Otherwise, for every weakly irreducible small deformation $\rho : \pi_1 S \to \mathsf{PSL}_d(\mathbb{R})$ of $\Lambda\rho_0$ the curve $\xi^1_\rho(\partial\pi_1 S)$ is not Lipschitz.

Proof. By the proximality assumptions on Λ, the representation

$$\rho := \Lambda\rho_0 : \pi_1 S \to \mathsf{PSL}_d(\mathbb{R})$$

is $\{\mathsf{a}_1, \mathsf{a}_2\}$-Anosov.

Furthermore, if the first two weights of Λ belong to the same irreducible factor, the representation ρ is also $(1,1,2)$-hyperconvex, this is an open property in $\mathfrak{X}\big(\pi_1 S, \mathsf{PSL}_d(\mathbb{R})\big)$ (P.-S.-W [36]) and thus Theorem 7.3 implies that every small deformation of ρ has C^1 limit set.

If, instead, the two top weights of Λ were belonging to different irreducible factors, then it follows from the representation theory of $\mathsf{SL}(2,\mathbb{R})$ that

$$h_\rho(\mathsf{a}_1) = h_\rho(\mathcal{J}^u_1) = 2.$$

Note that the entropy if $\mathcal{J}^u_1$ is continuous on $\mathfrak{X}_{\{\mathsf{a}_1,\mathsf{a}_2\}}\big(\pi_1 S, \mathsf{PSL}_d(\mathbb{R})\big)$ (Theorem 5.12), in particular there exists a neighborhood $\mathcal{U}$ of ρ such that $h_\eta(\mathcal{J}^u_1) > 1$ for every $\eta \in \mathcal{U}$. Theorem A implies that no weakly irreducible representation in $\mathcal{U}$ can have Lipschitz limit set. □

Along the same lines we can deduce that some natural Anosov representations of hyperbolic lattices do not have Lipschitz boundary maps:

Corollary 7.9. *Let $\Gamma < \mathsf{PO}(1,n)$ be a lattice, $n \geqslant 3$ and $\rho_1 : \Gamma \to \mathsf{PO}(1,m)$ strictly dominated by the lattice embedding ρ_0. Then for any Zariski dense small deformation of $\rho_0 \oplus \rho_1^{n-1}$, the limit set $\xi^1_\rho(\partial\Gamma)$ is not Lipschitz.*

Examples of lattices Γ admitting such representations were constructed by Danciger-Gueritaud-Kassel [15, Proposition 1.8].

8. $\mathbb{H}^{p,q}$ convex-cocompact representations

Generalizing work of Mess [34] and Barbot-Mérigot [2], Danciger–Guéritaud–Kassel [16] introduced a class of representations called *$\mathbb{H}^{p,q}$-convex cocompact*. These form another interesting class of representations with Lipschitz boundary map where Theorem A apply.

Let $d = p + q$ with $p, q \geqslant 1$ and let Q be a symmetric bilinear form on $\mathbb{R}^d$ of signature (p,q). The subspace of $\mathbb{P}(\mathbb{R}^d)$ consisting on negative definite lines is called the *pseudo-Riemannian hyperbolic space* and denoted by

$$\mathbb{H}^{p,q-1} = \{\ell \in \mathbb{P}(\mathbb{R}^d) : Q|_{\ell-\{0\}} < 0\}.$$

The cone of isotropic lines is usually denoted by $\partial\mathbb{H}^{p,q-1}$.

Instead of the original definition of convex-cocompactness, we recall the characterization given by [16, Theorem 1.11].

Definition 8.1. An a_1-Anosov representation $\rho : \Gamma \to \mathsf{PO}(p,q)$ is $\mathbb{H}^{p,q-1}$-*convex cocompact* if for every pairwise distinct triple of points $x, y, z \in \partial\Gamma$, the restriction $Q|_{\xi^1_\rho(x)\oplus\xi^1_\rho(y)\oplus\xi^1_\rho(z)}$ has signature $(2,1)$.

When Γ_0 is a cocompact lattice in $\mathsf{SO}(p,1)$, $\mathbb{H}^{p,1}$-convex cocompact representations of Γ_0 are usually referred to as AdS-*quasi-Fuchsian groups*. Barbot [3] proved that these groups form connected components of the character variety $\mathfrak{X}\big(\Gamma_0, \mathsf{SO}(p,2)\big)$ only consisting of Anosov representations. In [21] Glorieux-Monclair prove that the limit set of an AdS-quasi-Fuchsian group is never a C^1-submanifold, except for Fuchsian groups.

The following is well known and easy to verify, see for example Glorieux-Monclair [20, Proposition 5.2].

Proposition 8.2. *Assume that $\partial\Gamma$ is homeomorphic to a $p-1$-dimensional sphere. If $\rho : \Gamma \to \mathsf{PO}(p,q)$ is $\mathbb{H}^{p,q}$-convex cocompact, then $\xi^1_\rho(\partial\Gamma)$ is a Lipschitz submanifold of $\partial\mathbb{H}^{p,q-1}$.*

Proof. The space $\partial\mathbb{H}^{p,q-1}$ admits a twofold cover that splits as the product $\mathbb{S}^{p-1} \times \mathbb{S}^{q-1}$. It is furthermore immediate to verify that, since for every pairwise distinct triple $(x,y,z) \in \partial\Gamma$, $Q|_{\xi^1_\rho(x)\oplus\xi^1_\rho(y)\oplus\xi^1_\rho(z)}$ has signature $(2,1)$, each one of the two lifts of $\xi^1_\rho(\partial\Gamma)$ to $\mathbb{S}^{p-1} \times \mathbb{S}^{q-1}$ is the graph of a 1-Lipschitz function $f : \mathbb{S}^{p-1} \to \mathbb{S}^{q-1}$, and, as such, is a Lipschitz submanifold of $\partial\mathbb{H}^{p,q-1}$. □

Theorem A then yields:

Corollary 8.3. *Assume that $\partial\Gamma$ is homeomorphic to a $p-1$-dimensional sphere and let $\rho : \Gamma \to \mathsf{PO}(p,q)$ be $\mathbb{H}^{p,q-1}$-convex cocompact, then*

- *if $p = 2$ and ρ is weakly irreducible then $h_\rho(\mathcal{J}^u_1) = 1$,*
- *if $p \geqslant 3$ and ρ is strongly irreducible then $h_\rho(\mathcal{J}^u_{p-1}) = 1$.*

One concludes the following upper bound for the entropy of the spectral radius inspired by Glorieux-Monclair [20].

Corollary 8.4. *Assume that $\partial\Gamma$ is homeomorphic to a $p-1$-dimensional sphere and let $\rho : \Gamma \to \mathsf{PO}(p,q)$ be $\mathbb{H}^{p,q-1}$-convex cocompact. Then*

- *if $p = 2$ and ρ is weakly irreducible then $h_\rho(\omega_1) \leqslant 1$,*
- *for $p \geqslant 3$ and ρ strongly irreducible, $h_\rho(\omega_1) \leqslant p-1$.*

Proof. Assume first $p \leqslant q$ and note that for every $g \in \mathsf{PO}(p,q)$ one has

$$\omega_p - \omega_1(\lambda(g)) = \lambda_2(g) + \cdots + \lambda_p(g) \geqslant 0.$$

By definition, $\mathcal{J}^u_{p-1} = p\omega_1 - \omega_p$ and thus

$$\frac{h_\rho(\omega_1)}{p-1} = h_\rho\big((p-1)\omega_1\big) \leqslant h_\rho(\mathcal{J}^u_{p-1}) = 1,$$

by Corollary 8.3. The only difference in the case $q < p$ is that $\mathcal{J}^u_{p-1} = p\omega_1 - \omega_q$, but the same argument applies verbatim. □

The entropy for the first fundamental weight has a particular meaning for projective Anosov representations into $\mathsf{PO}(p,q)$, notably for $q \geqslant 2$. Fix $o \in \mathbb{H}^{p,q-1}$ and consider

$$S^o = \{W < \mathbb{R}^d : o \subset W, \dim W = q \text{ and } Q|W \text{ is negative definite}\}.$$

This is a totally geodesic embedding of the symmetric space $X_{p,q-1}$ of $\mathsf{PO}(p,q-1)$ in the symmetric space $X_{p,q}$.

Given a projective Anosov representation $\rho : \Gamma \to \mathsf{PO}(p,q)$ one defines the open subset of $\mathbb{H}^{p,q-1}$

$$\Omega_\rho = \{o \in \mathbb{H}^{p,q-1} : Q(o, \xi^1_\rho(x)) \neq 0 \, \forall x \in \partial\Gamma\}.$$

Carvajales [12] shows that, assuming $\Omega_\rho \neq \varnothing$, for every $o \in \Omega_\rho$ one has

$$\lim_{t\to\infty} \frac{\log \#\{\gamma \in \Gamma : d_{X_{p,q}}(S^o, \rho(\gamma)S^o)\}}{t} = h_\rho(\omega_1)$$

and provides an asymptotic for this counting function ([12, Theorem A]).

When ρ is moreover $\mathbb{H}^{p,q-1}$-convex-cocompact, Glorieux-Monclair [20, Section 1.2] introduce a *pseudo-Riemannian critical exponent* δ_ρ, and show, in particular, that

$$\delta_\rho \leqslant p-1$$

([20, Theorem 1.2]). Carvajales proves [12, Remarks 6.9 and 7.15] that $\delta_\rho = h_\rho(\omega_1)$ so Corollary 8.4 provides a different proof of [20, Theorem 1.2] when Γ is assumed to have boundary homeomorphic to a $p-1$-dimensional sphere.

We finish the section with a direct application of Theorem 5.4 and Corollary 8.3 allowing us to get a bound for the Riemannian critical exponent. We use freely the notation from Remark 5.5.

Consider a representation $\Lambda : \mathsf{PO}(p,1) \to \mathsf{PO}(p,q)$ such that its imagie stabilizes a $p+1$-dimensional subspace V of $\mathbb{R}^d$ where $Q|V$ has signature $(p,1)$. Endow the symmetric space $X_{p,q}$ with a $\mathsf{PO}(p,q)$-invariant Riemannian metric such that the totally geodesic copy of $\mathbb{H}^p$ in $X_{p,q}$ induced be Λ has constant curvature -1. In particular, if $\iota : \Gamma \to \mathsf{PO}(p,1)$ is the lattice embedding, $h^X_{\Lambda\circ\iota} = p-1$. We show that this is an upper bound for any strongly irreducible, $\mathbb{H}^{p,q-1}$-convex-cocompact representation:

Proposition 8.5. *Assume that $\partial\Gamma$ is homeomorphic to a $p-1$-dimensional sphere and let $\rho : \Gamma \to \mathsf{PO}(p,q)$ be strongly irreducible and $\mathbb{H}^{p,q-1}$-convex-cocompact and, then*

$$h^X_\rho \leqslant p-1.$$

Proof. In view of Theorem 5.4 (or more precisely Remark 5.5), it suffices to recall that $\mathcal{D}_{\rho(\Gamma)}$ is convex (Lemma 5.3) and that, by Corollary 8.3,

$$\mathcal{J}^u_{p-1} \in \mathcal{Q}_{\rho(\Gamma)}.$$

See Potrie-S. [35, Section 1.1] for more details. □

9. Maximal Representations

An important class of representations that are in general only Anosov with respect to one maximal parabolic subgroup, but admit boundary maps with Lipschitz

image are maximal representations into Hermitian Lie groups. In this case the Lipschitz property for the image of the boundary map is a consequence of a positivity/causality property of the boundary map. We first describe the causal structure on the Shilov boundary of a Hermitian symmetric space of tube type, introduce the notion of a positive curve and show that the image of any positive curve (that is not necessarily equivariant with respect to a representation) is a Lipschitz submanifold. We then show how this applies to maximal representations and allows us to prove Theorem 9.8, the main result of this section. We also deduce consequences for the orbit growth rate on the symmetric space.

9.1. **Causal structure and positive curves.** Let $\mathsf{G}_{\mathbb{R}}$ be a simple Hermitian Lie group of tube type. Examples to keep in mind are the symplectic group $\mathsf{G}_{\mathbb{R}} = \mathsf{Sp}(2n, \mathbb{R})$ or the orthogonal group $\mathsf{G}_{\mathbb{R}} = \mathsf{SO}_0(2, n)$. The Shilov boundary $\check{S}$ of the bounded domain realization of the symmetric space associated to $\mathsf{G}_{\mathbb{R}}$ is a flag variety $\mathsf{G}_{\mathbb{R}}/\check{P}$, where $\check{P}$ is a maximal parabolic subgroup determined by a specific simple root $\{\check{\mathsf{a}}\}$. In the two cases that serve as our main examples, $\mathsf{G}_{\mathbb{R}} = \mathsf{Sp}(2n, \mathbb{R})$ and $\mathsf{G}_{\mathbb{R}} = \mathsf{SO}_0(2, n)$, the parabolic subgroup $\check{P}$ in question is, respectively, the stabilizer of a Lagrangian subspace $L \in \mathscr{L}(\mathbb{R}^{2n})$ and the stabilizer of an isotropic line $l \in \mathsf{Is}_1(\mathbb{R}^{2,n})$, so that $\check{\mathsf{a}} = \mathsf{a}_n$, resp. $\check{\mathsf{a}} = \mathsf{a}_1$.

In general, for a simple Hermitian Lie group of rank n, there is a special set of n *strongly orthogonal roots* $\mathsf{b}_1, \cdots \mathsf{b}_n$ of the complexification $\mathfrak{g}_{\mathbb{C}}$, see [29, p.582-583]. The set of strongly orthogonal roots give rise to a (holomorphic) embedding of a maximal polydisk. If the symmetric space is of tube type, the simple root $\check{\mathsf{a}}$ is the smallest strongly orthogonal root $\check{\mathsf{a}} = \mathsf{b}_n$. All the other strongly orthogonal roots are of the form $\mathsf{b}_i = \mathsf{b}_n + \varphi$, where $\varphi \in \mathsf{E}^*$ is non-negative on the Weyl-chamber. We record the following for later use:

Lemma 9.1. *Let $a \in \mathsf{E}^+$ then $\check{\mathsf{a}}(a) = \min_{i=1,\dots,n} \mathsf{b}_i(a)$.*

For Hermitian groups of tube type, the Shilov boundary carries a natural causal structure: for every $p \in \check{S}$ there is an open convex acute cone $C_p \subset \mathsf{T}_p\check{S}$ which we now define.

Recall that $\mathsf{G}_{\mathbb{R}}/\check{P}$ can be identified as the space of parabolic subgroups of $\mathsf{G}_{\mathbb{R}}$ that are conjugate to $\check{P}$. Let us fix a point $\check{p} = \check{P} \in \check{S}$, which one should think of as a point at infinity. Then at any point $p = P \in \check{S}$ that is transverse to $\check{p}$, i.e. such that the parabolic groups P and $\check{P}$ are opposite, the tangent space $\mathsf{T}_p\check{S}$ is identified with the Lie algebra $\check{\mathfrak{n}}$ of the unipotent radical of $\check{P}$, and the cone C_p is an open convex acute cone $\check{C} \subset \check{\mathfrak{n}}$ invariant under the action of the connected component of $P \cap \check{P}$.

In the case of $\mathsf{Sp}(2n, \mathbb{R})$ this is the cone of positive definite symmetric matrices, and in the case of $\mathsf{SO}_0(2, n)$ it is the cone of vectors with positive first entry, that are positive for the induced conformal class of Lorentzian inner products on $\mathsf{T}_P\,\mathsf{Is}_1(\mathbb{R}^{2,n})$.

This invariant cone $\check{C} \subset \check{\mathfrak{n}}$ in fact also gives rise to the notion of maximal triples in $\check{S}$ via the exponential map. A triple $(P, Q, \check{P})$ is said to be maximal if there exists an $s \in \check{C}$ such that $Q = \exp s \cdot P$. Extending this by the action of G leads to a notion of maximal triples in $\check{S}$, which actually coincides exactly with those triples which have maximal (generalized) Maslov index as introduced by Clerc-Ørsted [13].

Definition 9.2. Let $\check{S}$ be the Shilov boundary of a Hermitian symmetric space of tube type. A curve $\xi : \mathbb{S}^1 \to \check{S}$ is *positive* if the image of any positively oriented triple is a maximal triple.

Proposition 9.3. *Let $\xi : \mathbb{S}^1 \to \check{S}$ be a positive curve. Then $\xi(\mathbb{S}^1)$ is a Lipschitz submanifold of $\check{S}$.*

Proof. Note that whenever we pick two points $p_1 = P_1, p_2 = P_2$ on the image of ξ, the image $\xi(\mathbb{S}^1)$ can be covered by the two charts consisting of parabolic subgroups that are transverse to p_1 respectively p_2.

In any of these charts the inverse image of ξ, under the exponential map

$$\begin{array}{ccc} \mathfrak{n}_i & \to & \mathsf{G}_{\mathbb{R}}/P_i \\ s & \mapsto & \exp(s)\check{P}_j \end{array}$$

gives a map $\overline{\xi} : \mathbb{R} \to \mathfrak{n}_i$ such that for every $t_1 < t_2$ we have $\overline{\xi}(t_2) - \overline{\xi}(t_1)$ is contained in the open convex acute cone $\check{C}$, it then follows (see for example Burger-Iozzi-Labourie-W. [9, Lemma 8.10]) that the restriction of $\overline{\xi}$ to any bounded interval has finite length. As a result $\xi(\mathbb{S}^1) \subset \check{S}$ is rectifiable. It is thus possible to reparametrize $\mathbb{S}^1$ so that ξ is a Lipschitz map.

□

Remark 9.4. Note that we did not assume that the positive map is equivariant with respect to a representation. This will be important in Section 10, where we will apply Proposition 9.3 in this generality.

9.2. **Maximal representations.** Let now G denote an Hermitian semisimple Lie group and let Γ denote the fundamental group of a closed hyperbolic surface S. We consider representations $\rho : \Gamma \to \mathsf{G}$ that are maximal, i.e. they maximize the Toledo invariant, whose definition was recalled in the introduction. Important for us is that they can be characterized in terms of boundary maps by the following theorem.

Theorem 9.5 (Burger-Iozzi-W. [10, Theorem 8])**.** *A representation $\rho : \Gamma \to \mathsf{G}$ is maximal if and only if there exists a continuous, ρ-equivariant, positive map $\phi : \partial\Gamma \to \check{S}$.*

In order to apply Corollary 6.7 we need to verify some weak irreducibility assumption. Let us first treat the case when the Zariski closure of $\rho(\Gamma)$ is simple.

Corollary 9.6. *Let G be a simple Hermitian Lie group of tube type and let $\check{\mathsf{a}}$ be the root associated the Shilov boundary of G. If $\rho : \Gamma \to \mathsf{G}$ is a Zariski-dense maximal representation then*

$$\check{\mathsf{a}} \in \mathcal{Q}_{\rho(\Gamma)},$$

this is to say, $h_\rho(\check{\mathsf{a}}) = 1$.

Proof. Follows from Corollary 6.7 and Proposition 9.3 by considering the representation $\Lambda_{\check{a}}$ from Proposition 4.4. □

In the remainder of this section we show how the case of maximal representations with semi-simple target group that are not necessarily Zariski-dense, can be reduced to Corollary 9.6. To this aim, we will use a result from Burger-Iozzi-W. [11] describing the Zariski closure H of a maximal representation: H splits as $\mathsf{H}_1 \times \cdots \times \mathsf{H}_n$, each factor is Hermitian, and the inclusion in $\mathsf{H} \to \mathsf{G}$ is *tight*. In the

following we will not need the definition of a tight homomorphism, and therefore refer the interested reader to [11, Definition 1] for it.

The following lemma will then be useful:

Lemma 9.7. *Let* G *be a classical simple Hermitian Lie group of tube type and consider a tight embedding* $\iota : \mathsf{H} = \mathsf{H}_1 \times \cdots \times \mathsf{H}_n \to \mathsf{G}$. *If we denote by* $\iota_* : \mathsf{E}_{\mathsf{H}}^+ \to \mathsf{E}_{\mathsf{G}}^+$ *the induced map, then*

$$\check{\mathsf{a}}_{\mathsf{G}} \circ \iota_* = \min_i \check{\mathsf{a}}_{\mathsf{H}_i}.$$

Proof. Denote by $\pi : \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_n \to \mathfrak{g}$ the associated Lie algebra homomorphism. Let E_i be a Cartan subspace of H_i and E_{G} a Cartan subspace of G such that $\pi(\mathsf{E}_i) \subset \mathsf{E}_{\mathsf{G}}$.

As ι is tight, and G is classical, the classification of Hamlet-Pozzetti [28] applies and gives that we have an orthogonal decomposition $\mathsf{E}_{\mathsf{G}} = \mathsf{B}_1 \oplus \cdots \oplus \mathsf{B}_k$ so that $\pi | \bigoplus \mathsf{E}_i$ is a direct sum of maps $\pi_i : \mathsf{E}_i \to \mathsf{B}_i$; furthermore, there are only few possibilities for the linear map π_i: if H_i has rank greater than one, then $\mathsf{B}_i = \mathsf{E}_i^{m_i}$ for some m_i and π_i is a diagonal inclusion; instead, if E_i is one dimensional, or equivalently $\mathsf{H}_i \cong \mathsf{PSL}_2(\mathbb{R})$, then π_i is induced from a direct sum of non-trivial irreducible representations (of varying degrees). It is easy to check that the subspace B_i is then the span of the real vectors in $\mathfrak{p}$ associated to the strongly orthogonal roots that do not vanish on $\pi(\mathsf{E}_i)$. Setting $\mathsf{b}_i = \min_{j, \mathsf{b}_j|_{\mathsf{E}_i} \neq 0} \mathsf{b}_j$, we have $\mathsf{b}_i|_{\pi(\mathsf{E}_i)} = \check{\mathsf{a}}_{\mathsf{H}_i}$. And hence, with Lemma 9.1, we have $\check{\mathsf{a}}_{\mathsf{G}} = \min_i(\check{\mathsf{a}}_{\mathsf{H}_i})$.

□

We can now prove the following:

Theorem 9.8. *Let* G *be a Hermitian semi-simple Lie group such that all factors of* G *that are of tube type are classical. Let* $\theta \subset \Delta$ *be the subset of simple roots associated to the Shilov boundary of* G. *Then for every maximal representation* $\rho : \Gamma \to \mathsf{G}$ *one has*

$$\theta \subset \mathcal{Q}_{\rho(\Gamma)}.$$

Proof. If $\mathsf{G} = \mathsf{G}_1 \times \cdots \times \mathsf{G}_n$ then $\check{S} = \check{S}_1 \times \cdots \times \check{S}_n$, and therefore $\theta = \{\check{\mathsf{a}}_{\mathsf{G}_1}, \cdots \check{\mathsf{a}}_{\mathsf{G}_n}\}$ (see Burger-Iozzi-W. [11, Lemma 3.2 (1)]). Furthermore $\rho : \Gamma \to \mathsf{G}$ is maximal if and only if all $\rho_i : \Gamma \to \mathsf{G}_i$ are maximal (Burger-Iozzi-W. [10, Lemma 6.1 (3)]). Therefore we can restrict to the case that G is simple.

Since every maximal representation factors through a representation into the normalizer of a maximal tube type subgroup $\mathsf{H} < \mathsf{G}$ (Burger-Iozzi-W. [10, Theorem 5 (3)]), which is simple, has the same rank as G, and is such that $\check{\mathsf{a}}_{\mathsf{G}} = \check{\mathsf{a}}_{\mathsf{H}}$, we can restrict to the tube type case as the limit set in $\check{S}_{\mathsf{G}}$ is contained in $\check{S}_{\mathsf{H}}$ and coincides with the limit set in $\check{S}_{\mathsf{H}}$. The maximal tube type domains are always classical Hermitian symmetric spaces, except for the one exceptional Hermitian symmetric space of tube type.

If now ρ is not Zariski dense, then the Zariski closure is reductive and of tube type, so it is of the form $\mathsf{H}_1 \times \cdots \times \mathsf{H}_n$ and the representations into H_i are Zariski dense and maximal. Therefore we have $h_\rho(\check{\mathsf{a}}_{\mathsf{H}_i}) = 1$ for all i. As the inclusion $\mathsf{H}_1 \times \cdots \times \mathsf{H}_n \to \mathsf{G}$ is tight, the result follows from Lemma 9.7 and Lemma 5.1. □

9.3. **Application to the Riemannian critical exponent.** Any simple Hermitian Lie group G admits a diagonal embedding $\iota^\Delta : \mathsf{SL}(2, \mathbb{R}) \to \mathsf{G}$, which is equivariant with the inclusion of a diagonal disk in a maximal polydisk. We say that a

representation $\rho : \Gamma \to \mathsf{G}$ is *diagonal-Fuchsian* if it has the form $\rho = \iota^{\Delta} \circ \rho_0$ where $\rho_0 : \Gamma \to \mathsf{SL}(2, \mathbb{R})$ is the lift of the holonomy of a hyperbolization.

Let $K_\Delta < \mathsf{G}$ be the centralizer of the image of ι^{Δ}, which is compact. Then a diagonal Fuchsian representation ρ can be twisted by a representation $\chi : \Gamma \to K_\Delta$. We call the corresponding representation $\rho_\chi : \Gamma \to \mathsf{G}$ a *twisted diagonal* representation. Observe that the Riemannian critical exponent h^X is constant on twisted diagonal representations (the exact value h^X_{diag} depends on the choice of the normalization of the Riemannian metric)

Proposition 9.9. *Let Γ be the fundamental group of a closed surface and let $\rho : \Gamma \to \mathsf{G}$ be a maximal representation, then $h^X_\rho \leqslant h^X_{\mathrm{diag}}$.*

Proof. Let $\mathsf{b}_1, \cdots \mathsf{b}_n$ be the set of strongly orthogonal roots for $\mathsf{G}_{\mathbb{C}}$. It is immediate to verify that the limit cone $\mathcal{L}_{\rho_0(\Gamma)}$ of a representation ρ_0 in the Fuchsian locus is concentrated in the span of the vertex of the Weyl chamber is $\sum_{i=1}^n \mathsf{b}_i^*$, where b^* is the basis of E dual to $\{\mathsf{b}_1 \cdots \mathsf{b}_n\}$. We know from Corollary 9.6 that, for every ρ, the growth rate $h_\rho(\check{\mathsf{a}}) = 1$. Thus, if we denote by $(\mathsf{E}^+)^*$ the cone of functionals that are non-negative on the Weyl chamber, we get that $\check{\mathsf{a}} + (\mathsf{E}^+)^* \subset \mathcal{D}_{\rho(\Gamma)}$, and in particular all the strongly orthogonal roots are in $\mathcal{D}_{\rho(\Gamma)}$. A simple computation shows that the affine simplex determined by the strongly orthogonal roots meets the ray $\mathbb{R}\sum_{i=1}^n \mathsf{b}_i$ orthogonally in a point (it is just the diagonal in a positive quadrant meeting the span of the basis vectors), whose norm has to compute the Riemannian orbit growth rate of any representation ρ_0 in the Fuchsian locus: $\mathcal{Q}_{\rho_0(\Gamma)}$ is the affine hyperplane orthogonal to $\mathbb{R}\sum_{i=1}^n \mathsf{b}_i$ that contains $\check{\mathsf{a}}$. Remark 5.5 concludes the proof. □

Remark 9.10. Note that when G is $\mathsf{Sp}(4, \mathbb{R})$, or more generally $\mathsf{SO}_\circ(2, n)$, it follows from Collier-Tholozan-Toulisse [14] that the bound is furthermore rigid: the equality is strict unless ρ is equal to ρ_0 up to a character in the compact centralizer of its image.

Note that for maximal representations into $\mathsf{Sp}(2n, \mathbb{R})$ $n \geqslant 3$, every connected component of the space of maximal representations contains a twisted diagonal representation. However for $\mathsf{Sp}(4, \mathbb{R})$ there are exceptional components, discovered by Gothen, where every representation is Zariski dense (see Bradlow-Garcia-Prada-Gothen [7] and Guichard-W. [25]). In these components it is easy to verify that the bound we provide is sharp, despite not being acheived.

In the special case of the Hitchin component of $\mathsf{Sp}(2n, \mathbb{R})$, the bound of Proposition 9.9 is never attained, as the irreducible representations provide a better bound that is furthermore rigid (Potrie-S. [35]).

10. θ-positive representations

Throughout this section we will write

$$G = \mathsf{SO}(p, q)$$

with $p < q$. We consider the subset $\theta = \{\mathsf{a}_1, \ldots, \mathsf{a}_{p-1}\}$ of the simple roots discussed in Example 4.6 and denote by P_θ the corresponding parabolic group, by L_θ its Levi factor and by U_θ its unipotent radical.

The group G admits a *θ-positive structure* as defined by Guichard-W. [27]. This means that for every $\mathsf{b} \in \theta$ there exists an L^0_θ-invariant sharp convex cone c_b in

$$\mathfrak{u}_\mathsf{b} = \sum_{\mathsf{a} \in \Sigma^+_\theta,\, \mathsf{a} = \mathsf{b} \bmod \mathrm{Span}(\Pi - \theta)} \mathfrak{g}_\mathsf{a}.$$

Here $\Sigma_\theta^+ = \Sigma^+ \setminus \mathrm{Span}(\Pi - \theta)$. For $\mathsf{b} \in \{\mathsf{a}_1, \dots, \mathsf{a}_{p-2}\}$, the space $\mathfrak{u}_\mathsf{b}$ is one dimensional and the sharp convex cone $c_\mathsf{b} = \mathbb{R}^+ \subset \mathbb{R}$ consists of the positive elements, while $\mathfrak{u}_{\mathsf{a}_{p-1}} = \mathbb{R}^{q-p+2}$ endowed with a form q_J of signature $(1, q-p+1)$ preserved by the action of $L_\theta^0 = \mathbb{R}^{p-2} \times \mathsf{SO}^0(1, q-p+1)$. The cone $c_{\mathsf{a}_{p-1}}$ consists precisely of the positive vectors for q_J whose first entry is positive.

Following [27, §4.3] we denote by $W(\theta)$ the subgroup of the Weyl group W generated by the reflections $\{\sigma_i\}_{i=1}^{p-2}$ together with the longest element σ_{p-1} of the Weyl group $W_{\mathsf{a}_{p-1},\mathsf{a}_p}$ of the subroot system generated by the last two simple roots. $W(\theta)$ is, in our case, a Weyl group of type B_{p-1}. We denote by w_θ^0 the longest element of $W(\theta)$, and choose a reduced expression $w_\theta^0 = \sigma_{i_1} \dots \sigma_{i_l}$. Of course every reflection σ_i appears at least once among the σ_{i_k}. We consider the map

$$\begin{array}{rccc} F_{\sigma_{i_1}\dots\sigma_{i_l}} : & c^0_{\mathsf{a}_{i_1}} \times \dots \times c^0_{\mathsf{a}_{i_l}} & \to & U_\theta \\ & (v_1, \dots, v_l) & \mapsto & \exp(v_1)\dots\exp(v_l) \end{array}$$

The θ-positive semigroup U_θ^+ is defined as the image of the map $F_{\sigma_{i_1}\dots\sigma_{i_l}}$, and doesn't depend on the choice of the reduced expression [27, Theorem 4.5].

A θ-positive structure on G gives rise to the notion of a positive triple in G/P_θ. A pairwise transverse triple $(F_1, F_2, F_3) \in (G/P_\theta)^3$ such that $\mathrm{Stab}(F_3) = P_\theta$ is *θ-positive* if $F_2 = u \cdot F_1$ for some $u \in U_\theta^+$ [27, Definition 4.6], and more generally a triple is θ-positive if it lies in the G-orbit of a θ-positive triple. Let now Γ_g be the fundamental group of a hyperbolic surface. A representation $\rho : \Gamma_g \to G$ is *θ-positive* if there exists a ρ-equivariant map $\partial\Gamma_g \to G/P_\theta$ sending positive triples to θ-positive triples [27, Definition 5.3]. Guichard-Labourie-W. show that every θ-positive representation is necessarily θ-Anosov [27, Conjecture 5.4], but since the proof did not yet appear in print, in this section we will freely add this last assumption, and only discuss θ-positive Anosov representations.

Theorem 10.1. *Let $\rho : \Gamma \to \mathsf{SO}(p,q)$ be θ-positive and θ-Anosov. For every $1 \leqslant k \leqslant p-2$ the representation $\wedge^k\rho$ is $(1,1,2)$-hyperconvex.*

Proof. We denote by $\xi : \partial\Gamma_g \to G/P_\theta$ the θ-positive continuous equivariant boundary map, and by $\xi^i : \partial\Gamma_g \to \mathsf{Is}_i(\mathbb{R}^{p,q})$ the induced maps. By assumption, $\xi(y) = s \cdot \xi(x)$ for some element s in the positive semigroup of the unipotent radical of the stabilizer of $\xi(z)$. In turn $s = \exp(v_1)\dots\exp(v_l)$ with $v_t \in c^0_{\mathsf{a}_{i_t}}$ (recall that $i_t \in \{1, \dots, p-1\}$.

We set $d = p+q$. It follows from [36, Proposition 8.11] that, in order to check that $\wedge^k\rho$ is $(1,1,2)$-hyperconvex, it is enough to verify that the sum

$$\xi_\rho^k(x) + \left(\xi_\rho^k(y) \cap \xi_\rho^{d-k+1}(z)\right) + \xi_\rho^{d-k-1}(z)$$

is direct, or, equivalently that the sum

$$\xi_\rho^k(x) + s \cdot \left(\xi_\rho^k(x) \cap \xi_\rho^{d-k+1}(z)\right) + \xi_\rho^{d-k-1}(z)$$

is direct (recall that s belongs to the stabilizer of $\xi_\rho(z)$). Without loss of generality we can assume that the form Q defining the group $\mathsf{SO}(p,q)$ is represented by

$$Q = \begin{pmatrix} 0 & 0 & K \\ 0 & J & 0 \\ (-1)^p K & 0 & 0 \end{pmatrix}$$

with

$$K = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \iddots & 0 \\ (-1)^p & 0 & 0 \end{pmatrix} \text{ and } J = \begin{pmatrix} 0 & 0 & (-1)^{p-1} \\ 0 & -\mathrm{Id}_{q-p} & 0 \\ (-1)^{p-1} & 0 & 0 \end{pmatrix}$$

We can furthermore assume that $\xi^l(z) = \langle e_1, \dots, e_l \rangle$ and $\xi^l(x) = \langle e_d, \dots, e_{d-l+1} \rangle$, so that $\xi^k(x) \cap \xi^{d-k+1}(z) = e_{d-k+1}$. In order to check that the representation is $(1,1,2)$-hyperconvex, we only have to verify that, given s as above, writing $s \cdot e_{d-k+1} = \sum \alpha_i e_i$, the coefficient α_{d-k} never vanishes. But we claim that such coefficient is just $\sum_{i_t = k} v_t > 0$. Indeed, by construction, if $v_t \in c^0_{\mathsf{a}_m}$ with $m \in \{1, \dots, p-2\}$, then $\exp(v_t) \in \mathsf{SO}(p,q)$ differs from the identity only in the positions $(t, t+1)$ and $(d-t, d-t+1)$ where it is equal to v_t (cfr. [27, §4.5]), while if $v_t \in c^0_{\mathsf{a}_{p-1}}$ we have

$$\exp(v_t) = \begin{pmatrix} \mathrm{Id}_{p-2} & 0 & 0 & 0 & 0 \\ 0 & 1 & v^t & q_J(v) & 0 \\ 0 & 0 & \mathrm{Id}_{q-p+2} & Jv & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{Id}_{p-2} \end{pmatrix}.$$

The result is then immediate. □

In particular we deduce from [36, Proposition 7.4] the following

Corollary 10.2. *Let $\rho : \Gamma \to \mathsf{SO}(p,q)$ be θ-positive Anosov. For every $1 \leqslant k \leqslant p-2$ the image of $\xi^k_\rho(\partial\Gamma)$ is a C^1 submanifold of $\mathsf{Is}_k(\mathbb{R}^{p,q})$.*

We now turn to the proof of the last statement in Theorem 1.3. Instead of directly verifying that the map ξ^{p-1}_ρ has Lipschitz image, we will study properties of the map $\xi^{\theta_0}_\rho : \partial\Gamma_g \to G/P_{\theta_0}$ where

$$\theta_0 = \{\mathsf{a}_{p-2}, \mathsf{a}_{p-1}\}.$$

The flag manifold G/P_{θ_0} consists of nested pairs of isotropic subspaces of dimension $p-2$ and $p-1$.

Proposition 10.3. *Let $\rho : \Gamma \to \mathsf{SO}(p,q)$ be θ-positive Anosov. The image of the map $\xi^{\theta_0}_\rho : \partial\Gamma_g \to G/P_{\theta_0}$ is a Lipschitz submanifold of G/P_{θ_0}.*

Proof. We fix a point $z \in \partial\Gamma$ and we assume without loss of generality that $\xi^k_\rho(z) = \langle e_1, \dots, e_k \rangle$. We denote by $\mathcal{A} \subset G/P_{\theta_0}$ the set of points transverse to $\xi^{p-2,p-1}_\rho(z)$. We will show that the image of $\xi^{\theta_0}_\rho|_{\partial\Gamma\setminus\{z\}}$ is a Lipschitz submanifold of $\mathcal{A}$. Denote by $\mathcal{A}_{p-2} \subset G/P_{\mathsf{a}_{p-2}}$ the set of isotropic subspaces of dimension $p-2$ transverse to $\xi^{p-2}_\rho(z) = \langle e_1, \dots, e_{p-2} \rangle$, by Z_{p-1} the $(p-1)$-isotropic subspace $Z_{p-1} := \xi^{p-1}_\rho(z) = \langle e_1, \dots, e_{p-1} \rangle$, and by $Z^\perp_{p-1}$ its orthogonal with respect to the form Q defining $\mathsf{SO}(p,q)$. Observe that we have a smooth map

$$\begin{array}{ccc} \mathcal{A} & \mapsto & \mathcal{A}_{p-2} \times \mathsf{Is}_1(Z^\perp_{p-2}/Z_{p-2}) \\ (Y_{p-2}, Y_{p-1}) & \mapsto & (Y_{p-2}, [Y_{p-1} \cap Z^\perp_{p-2}]) \end{array}$$

whose image is the product of $\mathcal{A}_{p-2}$ with the set $\mathcal{I}_Z$ of isotropic lines transverse to the image of $Z^\perp_{p-1}$. Indeed for every pair $(Y_{p-2}, v) \in \mathcal{A}_{p-2} \times \mathcal{I}_Z$, the subspace $v + Z_{p-2}$ has dimension $p-1$ and, $\dim\big((v + Z_{p-2}) \cap Y^\perp_{p-2}\big) = 1$ as $Y^\perp_{p-2}$ and Z_{p-2} are transverse. We then have $Y_{p-1} = Y_{p-2} + ((v + Z_{p-2}) \cap Y_{p-2})$.

We denote by $\xi_Z : \partial\Gamma\backslash\{z\} \to \mathcal{I}_Z$ the composition of the map $\xi^{p-2,p-1}$ and the projection to the second factor in the product decomposition. The form Q induces a form of signature $(2, q-p+2)$ on $Z_{p-2}^{\perp}/Z_{p-2}$, which gives rise to the notion of positive curves (as introduced in Section 9). We claim that ξ_Z is a positive curve. This amounts to showing that, if $(x, y, z) \in \partial\Gamma$ is positively oriented, then $\xi_Z(y) = s^Z \xi_Z(x)$ for some positive element s^Z in the unipotent radical of the stabilizer of $[Z_{p-1}] \in \mathsf{Is}_1(Z_{p-2}^{\perp}/Z_{p-2})$. Since the representation ρ is θ-positive, we know that $\xi(y) = s \cdot \xi(x)$ for some element in the positive semigroup U_θ^+, and, as in the proof of Proposition 10.1 we can write $s = \exp(v_1)\dots\exp(v_l)$ with $v_t \in c^0_{\mathsf{a}_{i_t}}$. Observe that, for every $v_t \in c^0_{\beta_{i_t}}$, $\exp(v_t)$ induces an element $\exp(v_t)^Z$ in the unipotent radical of the stabilizer of $[Z_{p-1}] \in \mathsf{Is}_1(Z_{p-2}^{\perp}/Z_{p-2})$, and the element $\exp(v_t)^Z$ is trivial unless $\beta_{i_t} = \mathsf{a}_{p-1}$, in which case $\exp(v_t)^Z$ belongs to the positive semigroup of the unipotent radical of the stabilizer of $[Z_{p-1}]$. As at least one of the v_t in the decomposition of s belongs to such subgroup, we deduce that ξ_Z is positive, as we claimed. It follows from Proposition 9.3 that $\xi_Z(\partial\Gamma\backslash\{z\})$ is a Lipschitz submanifold of $\mathsf{Is}_1(Z_{p-2}^{\perp}/Z_{p-2})$.

As we know from Proposition 10.1 that ξ^{p-2} is a C^1-curve, we deduce that the curve $\xi^{p-2,p-1}$ is Lipschitz, being the image of a monotone map between a C^1-submanifold and a Lipschitz submanifold. This concludes the proof. □

10.1. **The critical exponent on the symmetric space is rigid.** Let $\iota_{2p-1} : \mathsf{PO}(1,2) \to \mathsf{PO}(p,p-1) \to \mathsf{PO}(p,q)$ be the composition of the the irreducible representation of dimension $2p-1$ with the standard embedding of $\mathsf{PO}(p,p-1) \to \mathsf{PO}(p,q)$. We call any representation $\rho : \Gamma \to \mathsf{PO}(p,q)$, which is the composition of a Fuchsion representation with ι_{2p-1}, a $(p,p-1)$-*Fuchsian* representation.

Lemma 10.4. *Let $\rho : \Gamma \to \mathsf{PO}(p,q)$ be θ-positive Anosov. The barycenter of the affine simplex in E_θ^* determined by $\{\mathsf{a}_1, \dots, \mathsf{a}_{p-2}, \varepsilon_{p-1}\}$ belongs to $\mathcal{D}_{\rho(\Gamma),\theta}$.*

Proof. Recall that, in the case of θ-positive representations in $\mathsf{PO}(p,q)$, the Levi-Anosov subspace is $\mathsf{E}_\theta := \ker(\mathsf{a}_p)$. In particular, for every $k \leqslant p-2$ we have that a_k belongs to the dual of E_θ, and belongs to the boundary of $\mathcal{D}_{\rho(\Gamma),\theta}$ by Corollary 10.2. Furthermore $\varepsilon_{p-1} = \mathsf{a}_{p-1} + \mathsf{a}_p$ belongs to $\mathcal{D}_{\rho(\Gamma),\theta}$ being the sum of a linear form with entropy one (the form a_{p-1} has entropy one by Proposition 10.3) and a linear form positive on the Weyl chamber (the root a_p). In particular the form corresponding to the barycenter of the affine simplex they determine in E_θ^* belongs to $\mathcal{D}_{\rho(\Gamma),\theta}$. □

Theorem 10.5. *Let Γ be the fundamental group of a surface and let $\rho : \Gamma \to \mathsf{PO}(p,q)$ be θ-positive Anosov. Then $h_\rho^{\mathfrak{X}} \leqslant h_{\rho_0}^{\mathfrak{X}}$ for any $(p,p-1)$-Fuchsian representation ρ_0.*

If equality is achieved at a totally reducible representation η then η splits as $W \oplus V$ where

(i) *W has signature $(p,p-1)$ and $\eta|W$ has Zariski closure the irreducible $\mathsf{PO}(2,1)$ in $\mathsf{PO}(p,p-1)$*
(ii) *$\eta|V$ lies in a compact group.*

Proof. The inequality follows from Lemma 10.4, together with convexity of $\mathcal{D}_{\rho(\Gamma),\theta}$ established by Theorem 5.12.

Assume now that η is a totally reducible representation such that equality holds. We can assume that $p \geqslant 3$, as the result for $p = 2$ was proven by Collier-Tholozan-Toulisse [14, Theorem 4].

Let $\mathsf{G} = \overline{\eta(\Gamma)}^Z$ be its Zariski closure. By definition, G is a real reductive group. We consider G as an abstract group, denote by $\Lambda : \mathsf{G} \to \mathsf{SO}(p,q)$ the inclusion representation, and by

$$\phi : \mathfrak{g} \to \mathfrak{so}(p,q)$$

the associated Lie algebra morphism. Denote by $\mathfrak{a}_{\mathsf{G}}$ a Cartan subspace of $\mathfrak{g}$.

Since $h_\eta^{\mathcal{X}}$ attains it maximal value, Theorem 5.12 forces the Quint indicator set $\mathcal{Q}_{\eta(\Gamma),\theta}$ to be the affine hyperplane of $(\mathsf{E}_\theta)^*$ spanned by Δ. The strict convexity guaranteed by Theorem 5.12 implies that G has real rank at most 2. Moreover we have that $\phi(\mathfrak{a}_{\mathsf{G}}) = \langle (2(p-1), 2(p-2), \dots, 2, 0), (0, \dots, 0, 1) \rangle$.

Denote by $T = \langle \xi_\eta^1(\partial\Gamma) \rangle$ the vector space spanned by the projective limit curve of η. Since η is totally reducible, the action of $\eta(\Gamma)$, and hence that of G, on T is irreducible.

Fix then a Weyl chamber $\mathfrak{a}_{\mathsf{G}}^+$ and let $\chi \in \mathfrak{a}_{\mathsf{G}}^*$ be the highest weight of $\phi(\mathfrak{g})|T$. Since η is a_1-Anosov, the attracting eigenvector of every element in $\eta(\Gamma)$, and hence of every purely loxodromic element of G, belongs to V. We therefore conclude that for every $a \in \mathfrak{a}_{\mathsf{G}}^+$

$$\chi(a) = \lambda_1\big(\phi(a)\big).$$

We denote by $\mathcal{L}_\eta^{\mathsf{G}} \subset \mathfrak{a}_{\mathsf{G}}^+$ Benoist's limit cone of $\eta(\Gamma)$ in G. As the representation η is a_2-Anosov, and thus $\mathcal{L}_\eta^{\mathsf{G}}$ avoids the only wall not ortogonal to the kernel of a_1, there exists a linear form $\mu \in \mathfrak{a}_{\mathsf{G}}^*$ such that for every $a \in \mathcal{L}_\eta^{\mathsf{G}}$ one has

$$\mu(a) = \mathsf{a}_1\big(\lambda\big(\phi(a)\big)\big).$$

Furthermore, as η is $(1,1,2)$-hyperconvex, for every $x \in \partial\Gamma$ the 2-dimensional space $\xi^{\mathsf{a}_2}(x)$ lies in T, and therefore $(\chi - \mu)(a) = \lambda_2(\phi(a))$, which implies that μ is a simple root, and $\chi = (p-1)\mu$.

For a weight ψ of the representation $\phi(\mathfrak{g})|T$ or of an irreducible factor of $\phi(\mathfrak{g})|T^\perp$, denote by V^ψ the associated weight space. We obtain from the description of $\phi(\mathfrak{a}_{\mathsf{G}})$ that the weight spaces $V^{\chi - i\mu}$ for $i \in [\![0, 2p-2]\!]$ are also 1-dimensional and contained in T. The weight space decomposition of T has thus the form

$$T = \bigoplus_{i=0}^{2p-2} V^{\chi - i\mu} \oplus V^0 \oplus V^q \oplus V^{-q},$$

where V^0 consists on vectors in the kernel of $\phi(\mathfrak{a}_{\mathsf{G}}^+)$ (except $V^{\chi-(p-1)\mu}$) and V^q corresponds to the eigenvalue $\varepsilon_p\big(\lambda\big(\phi(a)\big)\big)$. Here, V^0 as well as V^q and V^{-q} could be instead contained in $T^\perp$, and therefore not appear in the decomposition.

Let now W denote the Weyl group of $\mathfrak{g}$. As the weight lattice of $\eta|T$ is W-invariant, and there is no other weight of $\eta|T$ at distance $p-1$ from the origin, we deduce that W is reducible, and $\mathfrak{g}$ splits as $\mathfrak{g}_1 + \mathfrak{g}_2$. If μ is the root associated to $\mathfrak{g}_1$ we deduce from the fact that $V^{\chi-\mu}$ and thus $\mathfrak{g}_\mu$ is one dimensional that $\mathfrak{g}_1 = \mathfrak{sl}(2,\mathbb{R})$. As the action of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ commute, and the highest weight space for the restricted action of $\mathfrak{g}_1$ is one dimensional, we furthermore deduce that $\mathfrak{g}_2$ acts trivially on T. In particular T is an irreducible $\mathfrak{sl}(2,\mathbb{R})$ module of dimension $2p-1$ and the signature of $T^\perp$ of the (p,q)-quadratic form preserved by $\mathfrak{so}(p,q)$

is thus either negative or $(1, q-p)$. In the first case we conclude that $\phi(\mathfrak{g})|T^{\perp}$ is compact. Which is the desired result.

In order to conclude the proof we need to exclude the second case. We know from Theorem 10.1 that for every $1 \leqslant k \leqslant p-2$ and for every distinct $x, y, z \in \partial\Gamma$ the sum

$$\xi^k(x) + \left(\xi^k(y) \cap \xi^{d-k+1}(z)\right) + \xi^{d-k-1}(z)$$

is direct. With an inductive argument we deduce that for every $1 \leqslant k \leqslant p-2$, and for every $\gamma \in \Gamma$ the k-th eigenline belongs to T, and therefore the Anosov map ξ would be the boundary of a Fuchsian representation composed with an embedding of $\mathsf{PO}(1,2) \to \mathsf{PO}(p-1,p) \to \mathsf{PO}(p,q)$. However, such an embedding can never be positive because it has non-compact centralizer. □

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