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CONFORMALITY FOR A ROBUST CLASS OF NON-CONFORMAL ATTRACTORS

BEATRICE POZZETTI, ANDRÉS SAMBARINO, AND ANNA WIENHARD

ABSTRACT. In this paper we investigate the Hausdorff dimension of limit sets of Anosov representations. In this context we revisit and extend the framework of hyperconvex representations and establish a convergence property for them, analogue to a differentiability property. As an application of this convergence, we prove that the Hausdorff dimension of the limit set of a hyperconvex representation is equal to a suitably chosen critical exponent. In the appendix, in collaboration with M. Bridgeman, we extend a classical result on the Hessian of the Hausdorff dimension on purely imaginary directions.

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1. INTRODUCTION

In his seminal paper, Sullivan [52] describes the Hausdorff dimension of the limit set L_Γ , of a discrete group Γ acting on the real hyperbolic n -space, in terms of the Dirichlet series

$$s \mapsto \sum_{\gamma \in \Gamma} e^{-sd(o, \gamma o)}.$$

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More precisely, the *critical exponent* of such a series is

$$h_\Gamma = \inf \left\{ s : \sum_{\gamma \in \Gamma} e^{-sd(o, \gamma o)} < \infty \right\} = \sup \left\{ s : \sum_{\gamma \in \Gamma} e^{-sd(o, \gamma o)} = \infty \right\}$$

and Sullivan shows:

Theorem (Sullivan). *If Γ is a convex co-compact subgroup of $\mathrm{PSO}(1, n)$ then the Hausdorff dimension of L_Γ is h_Γ .*

This is related to understanding the Hausdorff dimension of a hyperbolic set in dynamical terms. Indeed, the non-wandering set of the geodesic flow of $\Gamma \backslash \mathbb{H}^n$ is, by definition, a maximal isolated compact hyperbolic set, h_Γ is its topological entropy and Sullivan's result can be interpreted in terms of the Ledrappier-Young formula [37].

Describing the Hausdorff dimension of a hyperbolic repeller as a dynamical quantity is today well understood in the *conformal* setting, i.e. when the derivative of the dynamics, restricted to the unstable distribution, acts as a conformal map (see Chen-Pesin's survey [14] and references therein). Analogously, Sullivan's result has been generalized to convex-cocompact groups of a $\mathrm{CAT}(-1)$ -space X (see for example Bourdon [6] and Yue [53]). The metric on the visual boundary ∂X used to compute the Hausdorff dimension is the *visual metric*, for which the action of $\mathrm{Isom} X$ is conformal (i.e. sends balls to balls).

However, other natural metrics on ∂X appear in very common situations: if X is a rank 1 symmetric space of non-compact type, then its visual boundary carries the structure of a differentiable manifold and thus one would also like to understand the Hausdorff dimension of limit sets for a (any) Riemannian metric on ∂X . Unless X is the real hyperbolic n -dimensional space, the Riemannian structure behaves differently from the visual structure: the action of $\mathrm{Isom} X$ is no longer conformal.

The dynamical characterization of Hausdorff dimension in a non-conformal setting is still not completely understood. We refer the reader again to Chen-Pesin's survey [14]. Let us also note that only very recently Bárány-Hochman-Rapaport [2] provided a complete answer for Iterated-Function-Systems on the plane. On the discrete groups side, Dufloux [20] has studied a class of Schottky subgroups of isometries of the complex hyperbolic n -space, that he calls *well positioned*, and proves the analogue of Sullivan's result for the Hausdorff dimension of the limit set with respect to any Riemannian metric.

1.1. This paper. In this paper we are interested in describing the Hausdorff dimension of the limit set of discrete subgroups of a semi-simple Lie group G , for a Riemannian structure on the flag spaces (or boundaries) of G . The groups we will consider, called *Anosov representations*, are in many ways similar to convex cocompact subgroups of $\mathrm{SO}(1, n)$, but do not act conformally on the boundaries of G .

Anosov representations were introduced by Labourie [36] for fundamental groups of negatively curved closed manifolds and the definition was extended by Guichard-W. [27] to any hyperbolic group. Such representations provide the appropriate generalization of the class of convex co-compact subgroups in the context of Lie groups of higher rank [27, 31, 32].

We will not use the original definition but follow a more recent approach, developed by Kapovich-Leeb-Porti [32], G eritaud-Guichard-Kassel-W. [24] and in particular Bochi-Potrie-S. [5], that provides a simplified definition and gives better quantitative control of Anosov representations.

Let $\mathbb{K} = \mathbb{R}$ or \mathbb{C} , consider an inner (or Hermitian if $\mathbb{K} = \mathbb{C}$) product in \mathbb{K}^d and, for $g \in \mathrm{GL}_d(\mathbb{K})$, denote by $g \mapsto g^*$ the corresponding adjoint operator. The singular values of g , i.e. the square root of the modulus of the eigenvalues of gg^* , are denoted by

$$\sigma_1(g) \geq \dots \geq \sigma_d(g).$$

Let Γ be a finitely generated discrete group, consider a finite symmetric generating set S and denote by $||$ the associated word metric on Γ . Given $p \in \llbracket 1, d-1 \rrbracket$ denote by $\mathcal{G}_p(\mathbb{K}^d)$ the Grassmannian of p -dimensional subspaces of \mathbb{K}^d . For a homomorphism $\rho : \Gamma \rightarrow \mathrm{PGL}_d(\mathbb{K})$, the following are equivalent:

- i) There exist positive constants c, μ such that for all $\gamma \in \Gamma$ one has

$$\frac{\sigma_{p+1}}{\sigma_p}(\rho(\gamma)) \leq ce^{-\mu|\gamma|},$$

- ii) The group Γ is word-hyperbolic and there exist ρ -equivariant maps $(\xi^p, \xi^{d-p}) : \partial\Gamma \rightarrow \mathcal{G}_p(\mathbb{K}^d) \times \mathcal{G}_{d-p}(\mathbb{K}^d)$ such that for every $x \neq y \in \partial\Gamma$ one has

$$\xi^p(x) \oplus \xi^{d-p}(y) = \mathbb{K}^d,$$

and a suitable associated flow is contracting.

If either condition is satisfied we will say that ρ is an $\{\mathfrak{a}_p\}$ -Anosov representation¹. For such a representation, the critical exponent $h_\rho^{\mathfrak{a}_p}$ of the Dirichlet series

$$\Phi_\rho^{\mathfrak{a}_p}(s) = \sum_{\gamma \in \Gamma} \left(\frac{\sigma_{p+1}}{\sigma_p}(\rho(\gamma)) \right)^s \quad (1)$$

is well defined. By definition, the series is convergent for every $s > h_\rho^{\mathfrak{a}_p}$ and divergent for every $0 < s < h_\rho^{\mathfrak{a}_p}$.

If ρ is furthermore $\{\mathfrak{a}_{p+1}\}$ -Anosov then $h_\rho^{\mathfrak{a}_p}$ is analytic with respect to ρ , and agrees with the entropy of a suitably defined flow (see for example Bridgeman-Canary-Labourie-S. [7] and Potrie-S. [42, Corollary 4.9]). But in general little is known about $h_\rho^{\mathfrak{a}_p}$ without this extra assumption.

We will mainly focus on $\{\mathfrak{a}_1\}$ -Anosov representations. The chosen inner product on \mathbb{K}^d induces a metric on $\mathbb{P}(\mathbb{K}^d)$, we will denote by $\mathrm{Hff}(A)$ the Hausdorff dimension of a subset $A \subset \mathbb{P}(\mathbb{K}^d)$ for this metric. As a first result we obtain the following, independently obtained by Glorieux-Monclair-Tholozan [23].

Proposition (Proposition 4.1). *Let $\rho : \Gamma \rightarrow \mathrm{PGL}_d(\mathbb{K})$ be $\{\mathfrak{a}_1\}$ -Anosov. Then*

$$\mathrm{Hff}(\xi^1(\partial\Gamma)) \leq h_\rho^{\mathfrak{a}_1}.$$

In order to discuss situations in which equality holds, we introduce the notion of *locally conformal points* of ρ (Definition 5.5), these are points of $\partial\Gamma$ designed to detect some asymptotic conformality of the non-conformal action of $\rho(\Gamma)$ when restricted to the limit set $\xi^1(\partial\Gamma)$. Using Patterson's construction we then obtain a

¹ The implication ii) \Rightarrow i) comes from Labourie [36] and Guichard-W. [27]. The implication i) \Rightarrow ii) is more recent and due to Kapovich-Leeb-Porti [32], see also Gu eritaud-Guichard-Kassel-W. [24] and Bochi-Potrie-S. [5] for different approaches. In the language of Bochi-Potrie-S. [5, Section 3.1] a representation verifying condition i) is called *p-dominated*.

(not necessarily quasi-invariant) measure $\mu_\rho^{a_1}$ on $\partial\Gamma$. Following Sullivan, we then prove the following.

Theorem (Theorem 5.14). *If the set of locally conformal points of ρ has positive $\mu_\rho^{a_1}$ -measure, then*

$$\text{Hff}(\xi^1(\partial\Gamma)) = h_\rho^{a_1}.$$

Interestingly, for a rich class of Anosov representations, a 3-point transversality condition, inspired by Labourie [36], forces asymptotic conformality:

Definition. Consider $p, q, r \in \llbracket 1, d-1 \rrbracket$ such that $p+q \leq r$. A $\{\mathbf{a}_p, \mathbf{a}_q, \mathbf{a}_r\}$ -Anosov representation $\rho : \Gamma \rightarrow \text{PGL}_d(\mathbb{K})$ is called (p, q, r) -hyperconvex if for every triple of pairwise distinct points $x, y, z \in \partial\Gamma$ one has

$$(\xi^p(x) \oplus \xi^q(y)) \cap \xi^{d-r}(z) = \{0\}.$$

(Note that p and q are not required to be distinct.)

The main result of this paper is the following.

Theorem A (Corollary 6.8 and Corollary 7.3). *Let ρ be $(1, 1, 2)$ -hyperconvex. Then*

$$h_\rho^{a_1} = \text{Hff}(\xi^1(\partial\Gamma)) \leq \text{Hff}(\mathbb{P}(\mathbb{K}^2)).$$

The aforementioned analyticity result for $h_\rho^{a_1}$, together with Theorem A, has the following consequence:

Corollary 1.1. *Let $\{\rho_u : \Gamma \rightarrow \text{PGL}_d(\mathbb{K})\}_{u \in D}$ be an analytic family of $(1, 1, 2)$ -hyperconvex representations, then $u \mapsto \text{Hff}(\xi_u^1(\partial\Gamma))$ is analytic.*

In fact Theorem A holds in greater generality. We can replace 2's by any $p \in \llbracket 2, d-1 \rrbracket$ if we additionally require that for every $\gamma \in \Gamma$ one has

$$\sigma_2(\rho(\gamma)) = \sigma_p(\rho(\gamma)),$$

see Corollary 6.9 and Corollary 7.3. This extra condition on the singular values should be interpreted as a restriction on the Zariski closure of the representation (see subsection 8.1 for situations such as $\text{PSp}(1, n)$ and $\text{PU}(1, n)$, and subsection 8.2 for the group $\text{PSO}(p, q)$).

A key ingredient for the proof of Theorem A is the following convergence property for hyperconvex representations, from the inequality readily follows.

Theorem B (Theorem 7.1). *Let ρ be (p, q, r) -hyperconvex, then for every $(w, y) \in \partial^{(2)}\Gamma$ one has*

$$\lim_{(w, y) \rightarrow (x, x)} d(\xi^p(w) \oplus \xi^q(y), \xi^r(x)) = 0.$$

We further investigate how vast the class of hyperconvex representations is. On the one hand one has the following remarks that provide many examples by the *represent and deform* method (see subsection 7.2):

- if $\rho : \Gamma \rightarrow \text{PGL}_d(\mathbb{R})$ is hyperconvex then, by complexifying, one obtains a hyperconvex representation over \mathbb{C} : this is direct from the definition;
- the space of (p, q, r) -hyperconvex representations is open in $\text{hom}(\Gamma, \text{PGL}_d(\mathbb{K}))$ (Proposition 6.2).

On the other hand there are some ‘verifiable’ restrictions imposed by the hyperconvexity condition. For example, a $(1, 1, p)$ -hyperconvex representation of Γ induces a continuous injective map

$$\partial\Gamma - \{\text{point}\} \rightarrow \mathbb{P}(\mathbb{K}^p),$$

(see Corollary 6.6), and there might be topological obstructions for the existence of such a map. More interesting restrictions arise when $\mathbb{K} = \mathbb{R}$ and $\partial\Gamma$ is a manifold:

Corollary (Proposition 7.4). *Let Γ be such that $\partial\Gamma$ is homeomorphic to a $(p - 1)$ -dimensional sphere. If $\rho : \Gamma \rightarrow \text{PGL}_d(\mathbb{R})$ is $(1, 1, p)$ -hyperconvex then $\xi^1(\partial\Gamma)$ is a \mathbb{C}^1 sphere.*

Using openness of hyperconvexity we find new explicit examples of Zariski dense groups with \mathbb{C}^1 limit set.

Corollary (Corollary 7.7). *There exist Zariski dense subgroups $\Gamma < \text{PGL}_{d(d+1)}(\mathbb{R})$ whose limit set is a \mathbb{C}^1 sphere of dimension $d - 1$.*

Sharper results of similar nature were obtained by Zhang-Zimmer [54].

We now turn to the special situation when $\partial\Gamma$ is a circle. Then Theorem A gives the following computation of $h_\rho^{\mathfrak{a}_1}$:

Corollary. *Assume $\partial\Gamma$ is homeomorphic to a circle, if $\rho : \Gamma \rightarrow \text{PGL}_d(\mathbb{R})$ is $(1, 1, 2)$ -hyperconvex then $h_\rho^{\mathfrak{a}_1} = 1$.*

This implies Potrie-S. [42, Theorem B] and further generalizes it to the Hitchin component of $\text{PSO}(p, p)$. The proof of Potrie-S. [42, Theorem A] applies then verbatim also to the Hitchin component of $\text{PSO}(p, p)$ and we thus obtain a rigid inequality for the critical exponent in the symmetric space of $\text{PSO}(p, p)$. We refer the reader to Sections 9.2 and 9.3 for more details on Hitchin representations.

While the property of having constant $h_\rho^{\mathfrak{a}_1}$ was expected to be a rare phenomenon, peculiar to Hitchin components, or possibly higher rank Teichmüller theories, we provide, in Section 6.3, many more examples of representations of fundamental groups of surfaces for which Theorem A applies. Interestingly enough, when $\partial\Gamma$ is a circle (and $\mathbb{K} = \mathbb{R}$), $(1, 1, 2)$ -hyperconvexity is not only a local condition, but it can be pushed far away. We say that an $\{\mathfrak{a}_1\}$ -Anosov representation is *weakly irreducible* if $\xi^1(\partial\Gamma)$ is not contained in a proper subspace of $\mathbb{P}(\mathbb{R}^d)$.

Proposition (Proposition 9.3). *Assume that $\partial\Gamma$ is homeomorphic to a circle. Then the space of real weakly irreducible $(1, 1, 2)$ -hyperconvex representations of Γ is closed among real weakly irreducible $\{\mathfrak{a}_1, \mathfrak{a}_2\}$ -Anosov representations.*

In Appendix B, written in collaboration with M. Bridgeman, we adopt a different point of view and consider the Hausdorff dimension $\text{Hff}(\xi_\rho^1(\partial\Gamma))$ as function on the space of $\{\mathfrak{a}_1\}$ -Anosov representations in $\mathfrak{X}(\Gamma, \text{PSL}_d(\mathbb{C}))$. We obtain a higher rank analogue of a celebrated result by Bridgeman-Taylor [10] and McMullen [40] which we now explain.

The complex structure of $\text{PSL}_d(\mathbb{C})$ provides a complex structure on the character variety $\mathfrak{X}(\Gamma, \text{PSL}_d(\mathbb{C}))$, denote by J the associated tensor squaring $- \text{id}$. If $\rho \in \mathfrak{X}(\Gamma, \text{PSL}_d(\mathbb{C}))$ is projective Anosov let us denote by

$$\text{Hff}_1(\rho) = \text{Hff}(\xi_\rho^1(\partial\Gamma)).$$

We refer the reader to subsection [B.2](#) for the definition of the Liouville pressure form $\mathbf{P}_{\mathbf{a}_1}$ from Bridgeman-Canary-Labourie-S. [\[8\]](#).

Theorem (Theorem [B.1](#)). *Assume that $\partial\Gamma$ is homeomorphic to a circle. Let $\rho : \Gamma \rightarrow \mathrm{PGL}_d(\mathbb{R})$ be a $(1, 1, 2)$ -hyperconvex representation and assume that ρ is a regular point of the character variety $\mathfrak{X}(\Gamma, \mathrm{PGL}_d(\mathbb{C}))$. Then for every differentiable curve $(\rho_t)_{t \in (-\varepsilon, \varepsilon)} \subset \mathfrak{X}(\Gamma, \mathrm{PSL}_d(\mathbb{R}))$ with $\rho_0 = \rho$ one has*

$$\mathrm{Hess} \mathrm{Hff}_1(J\dot{\rho}) = \mathbf{P}_{\mathbf{a}_1}(\dot{\rho}).$$

If S is a closed, orientable, connected genus ≥ 2 surface, [\[8, Theorem 1.6\]](#) states that $\mathbf{P}_{\mathbf{a}_1}$ is Riemannian on the Hitchin component $\mathcal{H}_d(S) = \mathcal{H}(S, \mathrm{PSL}_d(\mathbb{R})) \subset \mathfrak{X}(\pi_1 S, \mathrm{PSL}_d(\mathbb{R}))$ and thus one obtains the following.

Corollary 1.2. *The Hessian of*

$$\mathrm{Hff}_1 : \mathfrak{X}(\pi_1 S, \mathrm{PSL}_d(\mathbb{C})) \rightarrow \mathbb{R}$$

at $\rho \in \mathcal{H}_d(S)$ is strictly positive on every direction except $\mathbb{T}_\rho \mathcal{H}_d(S)$, in particular, the Hitchin locus is an isolated minimum of Hff_1 .

Throughout the paper we allow \mathbb{K} to be a local field (not necessarily Archimedean, as we required in this introduction). Originally Anosov representations were only defined over Archimedean fields as it is possible to show that if Γ admits a Anosov representation $\rho : \Gamma \rightarrow \mathrm{PSL}_d(\mathbb{K})$ for non-Archimedean \mathbb{K} , then Γ is virtually free. The main result of our paper, however, associates to such an action an interesting geometric quantity, the Hausdorff dimension of the limit set, which we are able to relate to a dynamical data, the orbit growth rate. We find this very interesting, and this justifies the extra work needed to develop the theory in this more general setting.

The main results go through in this generality, except the analyticity of Hausdorff dimension: the key step is to show that for a $\{\mathbf{a}_1\}$ -Anosov representation its *entropy*, defined by

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log \#\{\gamma \in \Gamma : \log \sigma_1(\rho(\gamma)) \leq t\}$$

is analytic with ρ . We don't know if this is true, but in [Appendix A](#) we use the thermodynamical formalism to prove that the Hausdorff dimension depends continuously on the representation (and is actually as regular as the map $\rho \mapsto \xi_\rho$ is).

Outline of the paper. The preliminaries of the paper, collected in [Section 2](#) come from three different areas: quantitative linear algebra, dynamics and geometric group theory. In [§ 2.1](#) we recall relations between the singular values of an element in $\mathrm{PGL}_d(\mathbb{K})$ and metric properties of its action on Grassmannian manifolds, in the general context of a local field \mathbb{K} . In [§ 2.2](#) we discuss the dynamical backgrounds and indicate how to extend Bochi-Gourmelon's theorem as well as the theory of dominated splittings to general local fields. [§ 2.3](#) collects the facts about hyperbolic groups and cone types that we will need in the paper.

[Section 3](#) concerns Anosov representations: we extend to the non-Archimedean setup the definition and the results we will need, particularly concerning the definition and properties of the equivariant boundary maps. Our discussion follows the lines of Bochi-Potrie-S. [\[5\]](#).

In Section 4 we prove that for any Anosov representation the Hausdorff dimension of the limit curve provides a lower bound for the critical exponent for the first root. In Section 5 we give a condition guaranteeing that such bound is optimal, namely the abundance of *locally conformal points* with respect to a suitable measure.

Section 6 concerns the notion of (p, q, r) -hyperconvexity, an open condition (Proposition 6.2) that guarantees abundance of locally conformal points: this is the content of Proposition 6.7, the main technical result of the paper. Using the theory of SL_2 representations we provide in § 6.3 many examples of hyperconvex representations of fundamental groups of surfaces and hyperbolic three manifolds.

In Section 7 we discuss another interesting consequence of hyperconvexity: such property guarantees a weak differentiability property for the limit set (Theorem 7.1) which allows us, on the one hand, to obtain good bounds on the Hausdorff dimension (Proposition 7.3), and on the other to provide examples of Zariski dense subgroups whose limit set in the projective space is a C^1 manifold: we obtain these through the *represent and deform* method explained, in a concrete example, in Proposition 7.5.

In Section 8 we discuss in detail two families of representations for which all our results apply: on the one hand we detail the geometric meaning of our notions in the case of convex cocompact subgroups of rank one groups, rediscovering and generalizing results of Dufloux (§ 8.1), on the other we give a concrete criterion that guarantees hyperconvexity for subgroups of $\mathrm{SO}(p, q)$ and provide examples of groups that satisfy it (§ 8.2).

The last section of the paper (Section 9) concerns representations of fundamental groups of hyperbolic surfaces (or more generally compact hyperbolic orbifolds). For these we show that hyperconvexity is also a closed condition (Proposition 9.3), and discuss a new proof and generalization of a result of Potrie-S. [42].

The two appendices of the paper rely on the Thermodynamical formalism: in Appendix A we extend results of S. [48] and Bridgeman-Canary-Labourie-S. [7] to show continuity of the entropy map on the character variety also in the non-Archimedean setting, this implies that the regularity of the dependence of the Hausdorff dimension on the representation is at least as good as the variation of the boundary map. In the second appendix, joint with Martin Bridgeman, we prove Theorem B.1. Our main result (Theorem A) was the last missing piece to be combined with results from Bridgeman-Canary-Labourie-S. [7, 8] and Bridgeman-Taylor [10].

2. PRELIMINARIES

In the paper we will need preliminaries from three different sources: quantitative linear algebra, dynamics and particularly the work of Bochi-Gourmelon [4] and Bochi-Potrie-S. [5] on dominated sequences, and algebraic and metric properties of hyperbolic groups. We recall the results we need here.

2.1. Quantitative linear algebra. As anticipated at the end of the introduction, in the paper we will be dealing with representations of finitely generated groups on finite dimensional vector spaces over local fields. We recall here some quantitative results we will need. More details on algebraic groups over local fields can be found in Quint [45].

2.1.1. *Angles and distances on Grassmannians.* We denote by \mathbb{K} a local field, and by $|\cdot| : \mathbb{K} \rightarrow \mathbb{R}^+$ its absolute value. Recall that if \mathbb{K} is \mathbb{R} or \mathbb{C} then $|\cdot|$ is the usual modulus, if, instead, \mathbb{K} is non-Archimedean, we require that $|\omega| = \frac{1}{q}$ where ω denotes the *uniformizing element*, namely a generator of the maximal ideal of the valuation ring \mathcal{O} , and q is the cardinality of the residue field $\mathcal{O}/\omega\mathcal{O}$ (this is finite because \mathbb{K} is, by assumption, local).

Given a finite dimensional vector space V over \mathbb{K} , we denote by $\|\cdot\| : V \rightarrow \mathbb{R}^+$ a *good norm*: for an Archimedean field \mathbb{K} this means that $\|\cdot\|$ is induced from an Hermitian product, if \mathbb{K} is non-Archimedean this means that there exists a basis $\{e_1, \dots, e_n\}$ such that $\|\sum a_i e_i\| = \max\{|a_i|\}$. In this second case we say that a decomposition $V = V_1 \oplus V_2$ is *orthogonal* if $\|v_1 + v_2\| = \max\{\|v_1\|, \|v_2\|\}$. In general, since \mathbb{K} is locally compact, any two norms on V are equivalent.

The choice of a good norm $\|\cdot\|$ on V induces a good norm on every exterior power of V (this is discussed in Quint [45]). This allows to generalize the notion of angle to the non-Archimedean setting: for $v, w \in V$, we define $\sphericalangle(v, w)$ to be the unique number in $[0, \pi]$ such that

$$\sin \sphericalangle(v, w) = \frac{\|v \wedge w\|}{\|v\| \|w\|}.$$

Observe that the angle crucially depends on the choice of the norm. Following Bochi-Potrie-S. [5] we define the angle of two subspaces $P, Q < \mathbb{K}^d$ as

$$\sphericalangle(P, Q) = \min_{v \in P^\times} \min_{w \in Q^\times} \sphericalangle(v, w),$$

where $P^\times = P \setminus \{0\}$, $Q^\times = Q \setminus \{0\}$.

The sine of the angle gives a distance, that we sometimes denote by d , on the projective space $\mathbb{P}(V)$, and more generally on every Grassmannian $\mathcal{G}_k(V)$: we set for $P, Q \in \mathcal{G}_k(V)$

$$d(P, Q) := \max_{v \in P^\times} \min_{w \in Q^\times} \sin \sphericalangle(v, w) = \min_{v \in P^\times} \max_{w \in Q^\times} \sin \sphericalangle(v, w),$$

this corresponds to the Hausdorff distance of $\mathbb{P}(P), \mathbb{P}(Q)$ regarded as subsets of $\mathbb{P}(V)$ with the aforementioned distance. Observe that

$$d(P, Q) \geq \sin \sphericalangle(P, Q)$$

and the latter inequality is, apart from very special cases, strict.

More generally we extend the distance to subspaces of possibly different dimension: for $P \in \mathcal{G}_k(V)$, $Q \in \mathcal{G}_l(V)$, $k \leq l$ we set

$$d(P, Q) := \max_{v \in P^\times} \min_{w \in Q^\times} \sin \sphericalangle(v, w) = \min_{W \in \mathcal{G}_k(Q)} d(P, W).$$

Such distance vanishes if and only if $P \subset Q$.

2.1.2. *Singular values.* Assume now that \mathbb{K} is commutative. Given a \mathbb{K} -norm on V we say that $g \in \mathrm{GL}(V, \mathbb{K})$ is a *semi-homothecy* if there exists a g -invariant \mathbb{K} -orthogonal decomposition $V = V_1 \oplus \dots \oplus V_k$ and $\sigma_1, \dots, \sigma_k \in \mathbb{R}_+$ such that for every $i \in \llbracket 1, k \rrbracket$ and every $v_i \in V_i$ one has

$$\|gv_i\| = \sigma_i \|v_i\|.$$

The numbers σ_i are called the ratios of the semi-homothecy g .

Consider a maximal abelian subgroup of diagonalizable matrices $A \subset \mathrm{GL}(V, \mathbb{K})$, let $K \subset \mathrm{GL}(V, \mathbb{K})$ be a compact subgroup such that if $N_{\mathrm{GL}}(A)$ is the normalizer of

A in $\mathbf{GL}(V, \mathbb{K})$ then $N_{\mathbf{GL}}(A) = (N_{\mathbf{GL}}(A) \cap K)A$. Following Quint [44, Théorème 6.1] there exists a \mathbb{K} -norm $\|\cdot\|$ on V such that

- $\|\cdot\|$ is preserved by K ,
- A acts on $(V, \|\cdot\|)$ by semi-homotheties.

Whenever such a norm is fixed, for every $g \in \mathbf{GL}(V)$ we denote the *norm* and its *co-norm* by

$$\|g\| := \max_{v \in V^\times} \frac{\|gv\|}{\|v\|} \quad \mathbf{m}(g) = \inf_{v \in V^\times} \frac{\|gv\|}{\|v\|}.$$

Let $d = \dim V$. Keeping notation from Quint [44], we denote by $\mathbf{E} := \mathbb{R}^d$ a real vector space with a restricted root system of $\mathbf{GL}(V)$, and by

$$\mathbf{E}^+ = \{x = (x_1, \dots, x_d) \in \mathbb{R}^d \mid x_1 \geq \dots \geq x_d\}$$

a Weyl chamber of \mathbf{E} . We will denote by $\mathbf{a}_i \in \mathbf{E}^*$ the simple roots of \mathbf{E} , so that

$$\mathbf{a}_i(x) = x_i - x_{i+1} \in \mathbb{R}.$$

The choice of an ordering $e_1 \oplus \dots \oplus e_d$ of the joint eigenlines of A (the eigenlines are uniquely determined Quint [45, Lemma II.1.3]) induces a map $\nu : A \rightarrow \mathbf{E}$ given by

$$\nu(a) := (\log |\sigma_1(a)|, \dots, \log |\sigma_d(a)|),$$

where $\sigma_1(a), \dots, \sigma_d(a)$ are the semi-homothety ratios in the basis $\{e_1, \dots, e_d\}$. We set $A^+ := \nu^{-1}(\mathbf{E}^+)$, so that A^+ consists of those elements $a \in A$ whose corresponding semi-homothety ratios satisfy $\sigma_1(a) \geq \dots \geq \sigma_d(a)$.

With respect to the basis $\{e_1, \dots, e_d\}$, it holds that $K = \mathbf{GL}(d, \mathcal{O})$, and the map ν extends to the Cartan projection, still denoted ν from the whole $\mathbf{GL}(V, \mathbb{K})$: indeed $\mathbf{GL}(V, \mathbb{K}) = KA^+K$, and, given $a_1, a_2 \in A^+$, the element a_1 belongs to Ka_2K if and only if $\nu(a_1) = \nu(a_2)$. In particular we can set $\nu(g) = \nu(a_g)$ for any element $a_g \in A$ such that there exist $k_g, l_g \in K$ with $g = k_g a_g l_g$ (Bruhat-Tits [12, Section 3.3]).

For every $g \in \mathbf{GL}(V, \mathbb{K})$, we choose a Cartan decomposition $g = k_g a_g l_g$ as above and define, for $p \in \llbracket 1, d-1 \rrbracket$,

$$u_p(g) = k_g \cdot e_p \in V.$$

If \mathbb{K} is Archimedean, the set $\{u_p(g) : p \in \llbracket 1, d-1 \rrbracket\}$ is an *arbitrary* orthogonal choice of axes (ordered in decreasing length) of the ellipsoid $\{Av : \|v\| = 1\}$. Note that for every $v \in g^{-1}u_p(g)$ one has $\|gv\| = \sigma_p(g)\|v\|$.

We furthermore denote by $U_p(g)$ the *Cartan attractor* of g :

$$U_p(g) = u_1(g) \oplus \dots \oplus u_p(g) = k_g \cdot (e_1 \oplus \dots \oplus e_p).$$

Definition 2.1. An element $g \in \mathbf{GL}(V, \mathbb{K})$ is said to have a *gap of index* p if $\sigma_p(g) > \sigma_{p+1}(g)$. In that case, if \mathbb{K} is Archimedean, the p -dimensional space $U_p(g)$ is independent of the Cartan decomposition of g .

Note that if g has a gap of index p , then the decomposition

$$U_{d-p}(g^{-1}) \oplus g^{-1}(U_p(g))$$

is orthogonal: this is clear when \mathbb{K} is Archimedean (see Remark 2.4 for the general case)

Remark 2.2. If \mathbb{K} is not Archimedean, the components k_g, l_g in the Cartan decomposition are not uniquely determined even if g has gaps of every index; in particular the spaces $U_p(g)$ always depend on the choice of the Cartan decomposition. For example take $d = 2$; if $|a| > |b|$ we have

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ b/a & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

and both $\begin{pmatrix} 1 & 0 \\ b/a & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ belong to $K = \mathrm{SL}(2, \mathcal{O})$. In this example it is easy to verify that the set of possible Cartan attractors $U_1(g)$ coincides with the ball of center e_1 and radius $|b/a|$. Note that, since \mathbb{K} is non-Archimedean, any point in this ball is a center.

2.1.3. *Quantitative results.* Many of the auxiliary technical results in [5] rely on the min-max characterization of singular values of linear maps from \mathbb{R}^d to \mathbb{R}^d . This characterization in fact generalizes to any local field if one replaces the singular values with the semi-homothecy ratios:

$$\sigma_p(A) = \max_{P \in \mathcal{G}_p(V)} \mathbf{m}(A|_P) \quad \sigma_{p+1}(A) = \min_{Q \in \mathcal{G}_{d-p}(V)} \|A|_Q\|.$$

Therefore the quantitative linear algebraic facts collected in [5, Appendix 3] carry through. We now state the ones that we will use in the following.

Lemma 2.3 ([5, Lemma A.4]). *Let $g, h \in \mathrm{GL}(V, \mathbb{K})$ have a gap of index p . Then for any possible choice of Cartan attractor $U_p(g)$ (resp. $U_p(gh)$):*

$$d(U_p(gh), U_p(g)) \leq \|h\| \|h^{-1}\| \frac{\sigma_{p+1}}{\sigma_p}(g) \quad (2)$$

$$d(U_p(gh), gU_p(h)) \leq \|g\| \|g^{-1}\| \frac{\sigma_{p+1}}{\sigma_p}(h). \quad (3)$$

Remark 2.4. If \mathbb{K} is non-Archimedean, the Cartan attractors $U_p(g)$ are not uniquely defined (cfr. Remark 2.2). However it follows from Lemma 2.3 that, given two different Cartan decompositions for g , $g = k_g a_g l_g = k'_g a'_g l'_g$, and denoting $V_p = \langle e_1, \dots, e_p \rangle$, we have

$$d(k_g V_p, k'_g V_p) \leq \frac{\sigma_{p+1}}{\sigma_p}(g),$$

namely all possible different choices for $U_p(g)$ are contained in a ball of radius $\frac{\sigma_{p+1}}{\sigma_p}(g)$. As the distance d is, in this case, non-Archimedean, we deduce, also in this case, that any choice of $U_p(g)$ is orthogonal to $gU_{d-p}(g^{-1})$ for any other choice of $U_{d-p}(g^{-1})$.

Lemma 2.5 ([5, Lemma A.6]). *Let $g \in \mathrm{GL}(V, \mathbb{K})$ have a gap of index p . Then, for all $P \in \mathcal{G}_p(V)$ transverse to $U_{d-p}(g^{-1})$ we have:*

$$d(g(P), U_p(g)) \leq \frac{\sigma_{p+1}}{\sigma_p}(g) \frac{1}{\sin \sphericalangle(P, U_{d-p}(g^{-1}))}.$$

Lemma 2.6 ([5, Lemma A.7]). *Let $g, h \in \mathrm{GL}(V, \mathbb{K})$. Suppose that g and gh have gaps of index p . Let $\alpha := \sphericalangle(U_p(h), U_{d-p}(g^{-1}))$. Then:*

$$\begin{aligned} \sigma_p(gh) &\geq (\sin \alpha) \sigma_p(g) \sigma_p(h), \\ \sigma_{p+1}(gh) &\leq (\sin \alpha)^{-1} \sigma_{p+1}(g) \sigma_{p+1}(h). \end{aligned}$$

Given a subspace $P \in \mathcal{G}_p(V)$ we denote by V^\perp a chosen orthogonal complement of V ; this always exists, but is not unique if \mathbb{K} is non-Archimedean. Suppose that $P, W \in \mathcal{G}_p(V)$ satisfy $d(P, W) < 1$. Then $W \cap P^\perp = \{0\}$, and so there exists a unique linear map

$$L_{W,P}: P \rightarrow P^\perp \quad \text{such that} \quad W = \{v + L_{W,P}(v) : v \in P\}. \quad (4)$$

The association $L_{W,P} \mapsto W$ provides an affine chart for $\mathcal{G}_p(V)$. The next lemma states that this chart is 1-Lipschitz, and it is 4-biLipschitz on a sufficiently small neighbourhood of P :

Lemma 2.7 ([5, Lemma A.11]). *Let $P, P_1, P_2 \in \mathcal{G}_p(V)$, with $d(P_i, P) < 1$, then*

$$d(P_1, P_2) \leq \|L_{P_1,P} - L_{P_2,P}\|.$$

If moreover $d(P_i, P) < 1/\sqrt{2}$ then $\|L_{P_1,P} - L_{P_2,P}\| \leq 4d(P_1, P_2)$.

Proof. The proof of [5, Lemma A.11] smartly combines the triangular inequality for the distance d and the characterization

$$d(P_1, P_2) = \max_{w \in P_1^*} \min_{v \in P_2^*} \frac{\|v - w\|}{\|w\|}.$$

Since both hold when V is a vector space over a local field \mathbb{K} , the proof generalizes without modifications. In case \mathbb{K} is non-Archimedean, one could also deduce the better estimate $\|L_{P_1,P} - L_{P_2,P}\| \leq 2d(P_1, P_2)$. \square

The next lemma is a variation of [5, Lemma A.10]. In [5] there is an assumption on $d(P_i, P)$ depending on g that we replace here with the contraction assumption $d(gP_i, gP) < 1/\sqrt{2}$. Despite the proof is very similar to [5, Lemma A.10], we include it for completeness:

Lemma 2.8. *Let V be a d -dimensional \mathbb{K} -vector space, and $g \in \text{GL}(V)$. Choose $P \in \mathcal{G}_p(V)$ and $Q \in \mathcal{G}_{d-p}(V)$ such that the pairs (P, Q) and (gP, gQ) are orthogonal. Then for every $P_i \in \mathcal{G}_p(V)$, ($i = 1, 2$) with $P_i \cap Q = \{0\}$ such that $d(gP_i, gP) < 1/\sqrt{2}$ it holds*

$$d(gP_1, gP_2) \geq \frac{\mathbf{m}(g|_Q)}{4\|g|_P\|} d(P_1, P_2). \quad (5)$$

Proof of Lemma 2.8. Using the same notation as in (4), for each $i = 1, 2$, we consider the linear map $L_i = L_{P_i,P} : P \rightarrow P^\perp$ and $M_i = L_{gP_i, gP} : gP \rightarrow gP^\perp$; these are well defined since $P_i \cap Q = \{0\}$. Clearly the two maps are related by $L_i = (g^{-1}|_{gQ}) \circ M_i \circ (g|_P)$. As a consequence,

$$\|L_1 - L_2\| = \|(g^{-1}|_{gQ}) \circ (M_1 - M_2) \circ (g|_P)\| \leq \frac{\|g|_P\|}{\mathbf{m}(g|_Q)} \|M_1 - M_2\|.$$

Lemma 2.7 gives:

$$\|L_1 - L_2\| \geq d(P_1, P_2).$$

Since by assumption $d(gP_i, gP) < \delta = 1/\sqrt{2}$, Lemma 2.7 implies:

$$\|M_1 - M_2\| \leq 4d(gP_1, gP_2).$$

Putting these three estimates together, we get

$$d(gP_1, gP_2) \geq \frac{1}{4} \|M_1 - M_2\| \geq \frac{1}{4} \frac{\mathbf{m}(g|_Q)}{\|g|_P\|} \|L_1 - L_2\| \geq \frac{1}{4} \frac{\mathbf{m}(g|_Q)}{\|g|_P\|} d(P_1, P_2).$$

\square

The following corollary of Lemma 2.8 will be useful in Section 5.2:

Corollary 2.9. *Let V be a \mathbb{K} -vector space, $W < V$ a subspace of dimension 2, and $g \in \mathrm{GL}(V)$. Denote by $\sigma_i(g|_W)$ the semi-homothecy ratios of $g : W \rightarrow gW$ where the norm on W (resp. gW) is induced by the norm on V . For every $P_i \in \mathbb{P}W$ with $P_i \cap U_{d-1}(g^{-1}) = \{0\}$, and $d(gP_i, u_1(g|_W)) < 1/\sqrt{2}$ it holds*

$$d(gP_1, gP_2) \geq \frac{\sigma_2(g|_W)}{4\|g|_W\|} d(P_1, P_2).$$

Proof. This follows directly from Lemma 2.8 once we choose $P = u_2(g^{-1}|_{gL})$, $Q = u_1(g^{-1}|_{gL})$. \square

Another useful corollary of Lemma 2.8 is the following.

Corollary 2.10. *Given $\alpha > 0$, there exist positive δ and b with the following properties. Let V be a d -dimensional \mathbb{K} -vector space, and $g \in \mathrm{GL}(V)$. Suppose that $P \in \mathcal{G}_p(V)$ and $Q \in \mathcal{G}_{d-p}(V)$ satisfy*

$$\min\{\angle(P, Q), \angle(gP, gQ)\} \geq \alpha. \quad (6)$$

Then for every $P_i \in \mathcal{G}_p(V)$, ($i = 1, 2$) with $P_i \cap Q = \{0\}$ such that $d(gP_i, gP) < \delta$ one has

$$d(gP_1, gP_2) \geq b \frac{\mathbf{m}(g|_Q)}{\|g|_P\|} d(P_1, P_2). \quad (7)$$

Proof. Since all good norms are equivalent, the general case follows from Lemma 2.8 by changing the norm. \square

2.2. Dynamical background. We now turn to the dynamical preliminaries. The goal of this section is to extend the results of Bochi-Gurmelon [4] and Bochi-Potrie-S. [5] to the non-Archimedean setting.

2.2.1. Dominated splittings and Bochi-Gurmelon's Theorem. In this section we recall the definition of dominated splittings and review its connection with cone fields.

Let X be a compact metric space equipped with a continuous homeomorphism $\vartheta : X \rightarrow X$. Let V be a finite dimensional \mathbb{K} -vector space and let $\psi_0 : X \rightarrow \mathrm{GL}(V, \mathbb{K})$ be continuous. We will denote by $\psi : X \times V \rightarrow X \times V$ the induced cocycle defined by

$$\psi_x(v) = \psi(x, v) = (\vartheta(x), \psi_0(x)v).$$

Definition 2.11. Consider a good norm $\|\cdot\|$ on V . Let $\Lambda \subset X$ be a ϑ -invariant subset, then we say that $\psi|_\Lambda$ has a *dominated splitting* if the trivial bundle $\Lambda \times V$ splits as a Whitney sum of two ψ -invariant sub-bundles $V = E \oplus F$ with the following extra condition: there exist positive μ and c such that for every n positive, $x \in \Lambda$, $u \in E_x$ and $w \in F_x$ one has

$$\frac{\|\psi_x^n u\|}{\|\psi_x^n w\|} \leq ce^{-\mu n} \frac{\|u\|}{\|w\|}.$$

In this situation we say moreover that F (resp. E) is the unstable (resp. stable) bundle and that F *dominates* E .

Note that this condition is independent of the chosen norm. The dominated splitting of $\psi|_\Lambda$ is unique provided *its index*, i.e. $\dim_{\mathbb{K}} F$, is fixed and it extends to the closure $\bar{\Lambda}$ of Λ (see Crovisier-Potrie [16, Proposition 2.2 and 2.5] whose proof works verbatim in our setting). Furthermore:

Proposition 2.12 ([16]). *Suppose a linear flow ψ has dominated splittings $E^1 \oplus F^1$ and $E^2 \oplus F^2$ of index $p_1 \leq p_2$. Then $E^2 \subseteq E^1$ and $F^1 \subseteq F^2$.*

In the case when $\mathbb{K} = \mathbb{R}$ Bochi-Gourmelon [4, Theorem A] gave the following criterion for dominated splittings to exist; their the proof generalizes to every local field \mathbb{K} , as Oseledets theorem holds in this generality:

Theorem 2.13 (Bochi-Gourmelon [4]). *Let X be a compact metric space, V a \mathbb{K} -vector space and $\psi : X \times V \rightarrow X \times V$ a linear cocycle. Then the linear flow ψ has a dominated splitting $E \oplus F$ with $\dim F = p$ if and only if there exist $c > 0$, $\mu > 0$ such that for every $x \in X$ and $n \geq 0$ we have*

$$\frac{\sigma_{p+1}(\psi_x^n)}{\sigma_p(\psi_x^n)} < ce^{-\mu n}.$$

Moreover, the bundles² are given by:

$$F_x = \lim_{n \rightarrow +\infty} U_p(\psi_{\vartheta^{-n}(x)}^n) \text{ and } E_x = \lim_{n \rightarrow +\infty} U_{d-p}(\psi_{\vartheta^n(x)}^{-n}),$$

and these limits are uniform.

Proof. Bochi-Gourmelon's proof is based on the one hand on some angle estimates building upon the min-max characterization of singular values of matrices in $\mathrm{GL}_d(\mathbb{R})$, and on the other hand on the multiplicative ergodic theorem (Oseledets theorem). The former hold verbatim in the general local field setting once the singular values are replaced by the semi-homothecy ratios as defined in Section 2.1.2, the required multiplicative ergodic theorem was established (following Oseledets original proof) by Margulis [39, Theorem V.2.1], the integrability of ψ follows from its continuity and the compactness of the base X . With these ingredients at hand, the sketch of the proof explained in [5, Section A.4] applies verbatim. \square

The existence of a dominated splitting can be furthermore characterized in terms of cone fields; this will be crucial to prove openness of Anosov representations in Section 3.1 (note that the non-Archimedean case has not yet been established). Given a decomposition $V = V_1 \oplus V_2$ and a positive a , then the subset defined by

$$\{v \in V : a\|v_1\| \geq \|v_2\|\}$$

is called a a -cone (of dimension $\dim V_1$) on V .

A cone field on $\Lambda \subset X$ is a continuous choice $x \mapsto \mathcal{C}_{a(x),x}$ of a $a(x)$ -cone on V (of fixed dimension) for each $x \in \Lambda$. Cone fields can be used to characterize dominated splittings.

Proposition 2.14 (See Sambarino [50, Proposition 2.2]). *Let $\Lambda \subset X$ be ϑ -invariant. The cocycle $\psi|_\Lambda$ has a dominated splitting of index i if and only if there exists a map $a : \Lambda \rightarrow \mathbb{R}^+$ bounded away from 0 and ∞ , a cone field $\mathcal{C}_{a(x),x}$ on Λ of dimension i , a number $0 < \lambda < 1$ and a positive integer n_0 such that for every $x \in \Lambda$ the closure of $\psi_x^{n_0}(\mathcal{C}_{a(x),x})$ is contained in $\mathcal{C}_{\lambda a(\vartheta^n(x)), \vartheta^n(x)}$.*

²For completeness, let us note that the space U_p associated to an operator from a vector space equipped with a good norm to itself, can be defined for an operator between two vector spaces both equipped with good norms.

2.2.2. *Dominated sequences.* Bochi-Potrie-S. [5, Section 2] applied Bochi-Gourmelon's Theorem 2.13 to the compact space of *dominated sequences of matrices*, and got useful implications on the relative position of the axes of the ellipsoid associated to the products of such sequences: we recall now the relevant definitions and results from [5] where these were first established.

Given $C > 1$, define the following compact set:

$$\mathcal{D}(C) := \{g \in \mathrm{GL}(V, \mathbb{K}), \|g\| \leq C, \|g^{-1}\| \leq C\}.$$

If I is a (possibly infinite) interval in \mathbb{Z} , the set $\mathcal{D}(C)^I$ is endowed with the product topology, turning it into a compact metric space.

Let $p \in \llbracket 1, d-1 \rrbracket$, $\mu > 0$, $c > 0$. For each interval $I \subset \mathbb{Z}$, we denote by $\mathcal{D}(C, p, c, \mu, I)$ the set of sequences of matrices $(g_n) \in \mathcal{D}(C)^I$ such that for all $m, n \in I$ with $m \geq n$ we have

$$\frac{\sigma_{p+1}}{\sigma_p}(g_m \cdots g_{n+1} g_n) \leq ce^{-\mu(m-n+1)}.$$

Definition 2.15. An element of $\mathrm{GL}(V, \mathbb{K})^I$ is a *dominated sequence* if it belongs to $\mathcal{D}(C, p, c, \mu, I)$ for some C, p, c , and μ .

Consider the map shift : $\mathcal{D}(C)^{\mathbb{Z}} \rightarrow \mathcal{D}(C)^{\mathbb{Z}}$ defined by

$$\mathrm{shift}((g_n)_{-\infty}^{\infty}) = (g_{n+1})_{-\infty}^{\infty}$$

and let $\psi_0 : \mathcal{D}(C)^{\mathbb{Z}} \rightarrow \mathrm{GL}(V, \mathbb{K})$ be $\psi_0((g_n)) = g_0$. The subsets $\mathcal{D}(C, p, c, \mu, \mathbb{Z})$ are shift-invariant and automatically verify the hypothesis of Theorem 2.13.

Proposition 2.16 (Bochi-Potrie-S. [5, Proposition 2.4]). *For each sequence $x = (g_n) \in \mathcal{D}(C, p, \mu, c, \mathbb{Z})$, the limits:*

$$F_x := \lim_{n \rightarrow +\infty} U_p(g_{-1} g_{-2} \cdots g_{-n}),$$

$$E_x := \lim_{n \rightarrow +\infty} U_{d-p}(g_0^{-1} \cdots g_{n-2}^{-1} g_{n-1}^{-1}),$$

exist and are uniform over $\mathcal{D}(C, p, \mu, c, \mathbb{Z})$. Moreover, F dominates E and $E \oplus F$ is a dominated splitting for the linear cocycle over the shift defined above.

By a compactness argument, the theorem above ensures transversality for Cartan attractors and repellers computed in finite, but sufficiently long, sequences of matrices:

Lemma 2.17 (Bochi-Potrie-S. [5, Lemma 2.5]). *Given $C > 1$, $\mu > 0$, and $c > 0$, there exist $L \in \mathbb{N}$ and $\delta > 0$ with the following properties. Suppose that $I \subset \mathbb{Z}$ is an interval and $\{g_i\}_{i \in I}$ is an element of $\mathcal{D}(C, p, c, \mu, I)$. If $n < k < m$ all belong to I and $\min\{k-n, m-k\} > L$ then:*

$$\angle(U_p(g_{k-1} \cdots g_{n+1} g_n), U_{d-p}(g_k^{-1} g_{k+1}^{-1} \cdots g_{m-1}^{-1})) > \delta.$$

2.3. Hyperbolic groups. The last source of preliminaries comes from geometric group theory. Here we recall basic facts about hyperbolic groups and cone types.

Let Γ be a finitely generated group. We fix a finite symmetric generating set S and denote by $||$ the associated word length: for $\gamma \in \Gamma - \{e\}$ we denote by $|\gamma|$ the least number of elements of S needed to write γ as a word on S , and define the induced distance $d_\Gamma(\gamma, \eta) = |\gamma^{-1}\eta|$. A geodesic segment on Γ is a sequence $\{\alpha_i\}_0^k$ of elements in Γ such that $d_\Gamma(\alpha_i, \alpha_j) = |i-j|$.

In the paper we will be only interested in word-hyperbolic groups, namely such that the metric space $(\Gamma, ||)$ is Gromov hyperbolic. Following the footprints of [5], our analysis will be based on the study of cone types, and natural objects associated to them.

2.3.1. *Cone types.* In the paper we follow Cannon's original definition of cone types, which is more convenient for our geometric purposes, but the reader should be warned that the definition used in [5] is slightly different

Definition 2.18. The *cone type* of $\gamma \in \Gamma$ is defined by

$$\mathcal{C}(\gamma) = \{\eta \in \Gamma : |\gamma\eta| = |\eta| + |\gamma|\}.$$

Notice that if $\eta \in \mathcal{C}(\gamma)$ then

$$d_{\Gamma}(\gamma^{-1}, \eta) = |\gamma\eta| = |\eta| + |\gamma| = |\eta| + |\gamma^{-1}| = d_{\Gamma}(e, \eta) + d_{\Gamma}(e, \gamma^{-1}),$$

i.e. there exists a geodesic segment through e with endpoints γ^{-1} and η . Reciprocally, the endpoint of a geodesic segment starting at γ^{-1} and passing through e necessarily belongs to $\mathcal{C}(\gamma)$.

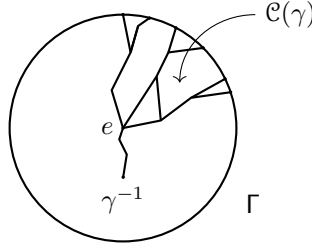


FIGURE 1. The cone type of $\gamma \in \Gamma$

A fundamental result of Cannon is that, provided Γ is hyperbolic, there are only finitely many cone types, and the cone type of a word γ only depends on a bounded suffix of γ (see for example Bridson-Haefliger [11, p. 455] or Coornaert-Delzant-Papadopoulos [15, p. 145]).

2.3.2. *The geodesic automaton.* See Bridson-Haefliger [11, p. 456].

Given a cone type \mathcal{C} and $a \in S \cap \mathcal{C}$, one easily checks that for every $\gamma \in \Gamma$ with $\mathcal{C}(\gamma) = \mathcal{C}$ one gets

$$a\mathcal{C}(\gamma a) \subset \mathcal{C}(\gamma).$$

Furthermore it is easy to verify that in such case the cone type $\mathcal{C}(\gamma a)$ doesn't depend on γ (see for example [15, Lemma 4.3]), and, with a slight abuse of notation we will denote such cone type $a \cdot \mathcal{C}$.

The *geodesic automaton* of Γ (this also depends on S) is the labelled graph \mathcal{G} defined as follows:

- the vertices are the cone types of Γ ;
- there is an edge $\mathcal{C}_1 \xrightarrow{a} \mathcal{C}_2$ from vertex \mathcal{C}_1 to vertex \mathcal{C}_2 , labelled by a generator $a \in S$, iff $a \in \mathcal{C}_1$ and $\mathcal{C}_2 = a \cdot \mathcal{C}_1$.

Since Γ is hyperbolic there are only finitely many cone types and thus the geodesic automaton has a finite number of vertices.

Let us explain the relation with geodesics. Consider a *geodesic segment* $(\gamma_0, \gamma_1, \dots, \gamma_\ell)$ that is, a sequence of elements of Γ such that $d(\gamma_n, \gamma_m) = |n - m|$, and assume that $\gamma_0 = \text{id}$. Then, there are $a_0, \dots, a_{\ell-1}$ in a generating set S such that $\gamma_n = a_0 a_1 \cdots a_{n-1}$. Note that for each n , the following is an edge of the geodesic automaton graph \mathcal{G} :

$$\mathcal{C}(\gamma_n) \xrightarrow{a_n} \mathcal{C}(\gamma_{n+1}).$$

Thus we obtain a (finite) walk on \mathcal{G} starting from the vertex $\mathcal{C}(\text{id})$. Conversely, for each such walk we may associate a geodesic segment starting at the identity.

Let us define also the *recurrent geodesic automaton* as the maximal recurrent subgraph \mathcal{G}^* of \mathcal{G} ; its vertices are called *recurrent cone types*.

Let Λ_Γ be the subset of all bi-infinite labelled sequences of \mathcal{G}^* . It is a closed shift-invariant subset of $(\mathcal{G}^*)^{\mathbb{Z}}$ and the induced dynamical system $\text{shift} : \Lambda_\Gamma \rightarrow \Lambda_\Gamma$ is a sofic shift (as in Lind-Marcus [38]).

The following concept that will be useful in Section 5.

Definition 2.19. Given an integer k we say that two cone types $\mathcal{C}_1, \mathcal{C}_2$ are *k-nested* if there is a path of length k in the geodesic automaton from \mathcal{C}_1 to \mathcal{C}_2 . In this case there an element $\beta \in \Gamma$ with $|\beta| = k$ and such that $\beta\mathcal{C}_2 \subset \mathcal{C}_1$.

Since Γ is hyperbolic, there are only finitely many cone types, therefore, for every k , there are only finitely many k -nested pairs of elements (however, as soon as Γ is non-elementary, the number of k -nested pairs grows exponentially with k).

Lemma 2.20. *If $\{\alpha_i\} \subset \Gamma$ is a geodesic, then the pair $(\mathcal{C}(\alpha_i), \mathcal{C}(\alpha_{i+k}))$ is k -nested.*

Proof. As $(\alpha_i)_i$ is a geodesic, for every i, k there exists $\beta_{i,k}$ with $\alpha_{i+k} = \alpha_i \beta_{i,k}$ and $|\alpha_{i+k}| = |\alpha_i| + |\beta_{i,k}|$. If x then belongs to $\mathcal{C}(\alpha_{i+k})$, $\beta_{i,k}x$ belongs to $\mathcal{C}(\alpha_k)$: indeed $|\alpha_i| + |\beta_{i,k}x| \geq |\alpha_i(\beta_{i,k}x)| = |\alpha_{i+k}x| = |\alpha_i| + |\beta_{i,k}| + |x| \geq |\alpha_i| + |\beta_{i,k}x|$. As the same argument shows that $|\beta_{i,k}x| = |\beta_{i,k}| + |x|$. The word $\beta_{i,k}$ labels the desired path in the geodesic automaton. \square

2.3.3. Coverings of the Gromov boundary. Recall that, as Γ is Gromov hyperbolic, its boundary $\partial\Gamma$, consisting of equivalence classes of geodesic rays, is well defined up to homeomorphism. We associate to every cone type \mathcal{C} a subset of $\partial\Gamma$, the *cone type at infinity*, by considering limit points of geodesic rays starting on e and totally contained in \mathcal{C} :

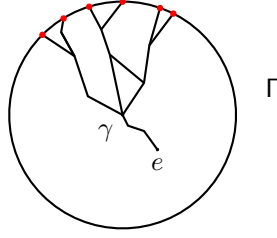
$$\mathcal{C}_\infty = \{[(\alpha_i)] | (\alpha_i) \text{ geodesic ray, } \alpha_0 = e, \alpha_i \in \mathcal{C}\}.$$

It follows from the discussion in the previous paragraph that every point in $\partial\Gamma$ is contained in at least one of the sets \mathcal{C}_∞ . As there are only finitely many cone types, we obtain a finite covering of $\partial\Gamma$ by considering $\mathcal{U} = \{\mathcal{C}_\infty(\gamma)\}$. Starting from this covering we will construct new coverings that will serve as our Sullivan shadows:

Lemma 2.21. *Given $T > 0$, the family of open sets*

$$\mathcal{U}_T := \{\gamma\mathcal{C}_\infty(\gamma) : |\gamma| \geq T\}$$

defines an open covering of $\partial\Gamma$.


 FIGURE 2. The set $\gamma \cdot \mathcal{C}_\infty(\gamma)$

Proof. We have to check that every point $x \in \partial\Gamma$ is covered, but this is evident since considering a geodesic ray $\{\alpha_i\}_0^\infty$ in Γ starting from e converging to x , one has that for all i , $x \in \alpha_i \mathcal{C}_\infty(\alpha_i)$, see Figure 2. \square

3. ANOSOV REPRESENTATIONS

Anosov representations from fundamental groups of negatively curved closed manifolds to $\mathrm{PGL}(d, \mathbb{R})$ were introduced by Labourie [36] and generalized by Guichard-W. [27] to any hyperbolic group. In this section we will generalize to non-Archimedean local fields the work of [5], which provides a simplified definition³.

3.1. Anosov representations and dominated splittings. Let Γ be a discrete group of finite type, fix a finite symmetric generating set S_Γ and denote by $||$ the associated word length.

Definition 3.1. Consider $p \in \llbracket 1, d-1 \rrbracket$. A representation $\rho : \Gamma \rightarrow \mathrm{PGL}_d(\mathbb{K})$ is $\{\mathfrak{a}_p\}$ -Anosov⁴ if there exist positive constants c, μ such that for all $\gamma \in \Gamma$ one has

$$\frac{\sigma_{p+1}}{\sigma_p}(\rho(\gamma)) \leq ce^{-\mu|\gamma|}. \quad (8)$$

A $\{\mathfrak{a}_1\}$ -Anosov representations will be called *projective Anosov*.

One has the following direct remark.

Remark 3.2 (Bochi-Potrie-S.). Let $\rho : \Gamma \rightarrow \mathrm{PGL}_d(\mathbb{K})$ be $\{\mathfrak{a}_p\}$ -Anosov. Given a geodesic $\{\alpha_i\}_{i \in \mathbb{Z}}$ let us denote by $\bar{\alpha}_i = \alpha_{i+1}^{-1} \alpha_i \in S_\Gamma$, then we have

$$(\rho(\bar{\alpha}_i))_{i \in \mathbb{Z}} \in \mathcal{D}(C, p, c, \mu, \mathbb{Z})$$

where c, μ come from equation (8) and $C = \max\{\|\rho(a)\| : a \in S_\Gamma\}$. Note also that

$$(\rho(\bar{\alpha}_i))_{i \in \mathbb{Z}} \in \mathcal{D}(C, d-p, c, \mu, \mathbb{Z}),$$

and thus Theorem 2.13 provides the following splittings of \mathbb{K}^d

$$E_{(\rho(\bar{\alpha}_i))}^p \oplus F_{(\rho(\bar{\alpha}_i))}^{d-p} \quad \text{and} \quad E_{(\rho(\bar{\alpha}_i))}^{d-p} \oplus F_{(\rho(\bar{\alpha}_i))}^p,$$

³Morse actions on Euclidean buildings (and thus in particular Anosov subgroups of $\mathrm{PGL}_d(\mathbb{K})$ when \mathbb{K} is non-Archimedean) were already defined by Kapovich-Leeb-Porti [31, Definition 5.35], the interest of such concept was also suggested in Géritaud-Guichard-Kassel-W. [24, Remark 1.6 (a)].

⁴In the language of Bochi-Potrie-S. [5, Section 3.1] a $\{\mathfrak{a}_p\}$ -Anosov representation is called *p-dominated*.

with the obvious inclusions according to dimension. By domination, these four bundles vary continuously in $\mathcal{D}(C, p, c, \mu, \mathbb{Z}) \cap \mathcal{D}(C, d - p, c, \mu, \mathbb{Z})$. Finally, Proposition 2.16 yields, for $k \in \{p, d - p\}$ and $m \geq 0$

$$U_k(\rho(\alpha_m)) = U_k(\bar{\alpha}_0^{-1} \cdots \bar{\alpha}_m^{-1}) \rightarrow E_{(\rho(\bar{\alpha}_i))}^k$$

and

$$U_k(\rho(\alpha_{-m})) = U_k(\bar{\alpha}_{-1} \cdots \bar{\alpha}_{-m}) \rightarrow F_{(\rho(\bar{\alpha}_i))}^k$$

as $m \rightarrow \infty$.

Using dominated splittings it is possible to deduce strong angle estimates between Cartan attractors along geodesic rays through the origin; for example the next result is a direct consequence of Lemma 2.17.

Proposition 3.3 (Bochi-Potrie-S.). *Let $\rho : \Gamma \rightarrow \mathrm{PGL}_d(\mathbb{K})$ be a $\{\mathfrak{a}_p\}$ -Anosov representation. Then there exists $\eta > 0$ and $L \in \mathbb{N}$ such that for every geodesic segment $(\alpha_i)_0^k$ in Γ through e with $|\alpha_0|, |\alpha_k| \geq L$ one has*

$$\angle(U_p(\rho(\alpha_k)), U_{d-p}(\rho(\alpha_0))) > \eta.$$

Bochi-Potrie-S. [5] applied the theory of dominated splittings to the sofic shift $\Lambda_\Gamma \rightarrow \Lambda_\Gamma$ induced by the recurrent geodesic automaton (see Section 2.3.2), to get an easy proof of openness of Anosov representations. Their proof easily extends to every local field:

Proposition 3.4. *The set of $\{\mathfrak{a}_p\}$ -Anosov representations is open in $\mathrm{hom}(\Gamma, \mathrm{PGL}_d(\mathbb{K}))$.*

Proof. A representation $\rho : \Gamma \rightarrow \mathrm{PGL}_d(\mathbb{K})$ induces a linear cocycle A_ρ over Λ_Γ , which admits a dominated splitting if and only if the representation ρ is $\{\mathfrak{a}_p\}$ -Anosov. Observe that the cocycle A_ρ varies continuously with the representation, since it only depends on the value of ρ on a generating set of Γ . Since, by Proposition 2.14, having a dominated splitting is an open condition on the space of cocycles, the result follows. \square

3.2. Boundary maps. From now on we will assume that Γ is a word hyperbolic group. This is not a restriction: Kapovich-Leeb-Porti proved that the only groups admitting Anosov representations are hyperbolic [31, Theorem 6.15] (cfr. also Bochi-Potrie-S. [5] for a different proof in the Archimedean case and S. [47] for the extension of this proof to the non-Archimedean case). We can thus talk freely about the Gromov boundary $\partial\Gamma$.

An important property of $\{\mathfrak{a}_p\}$ -Anosov representations is that they admit equivariant boundary maps:

Proposition 3.5 (Bochi-Potrie-S. [5, Proposition 4.9]). *If $\rho : \Gamma \rightarrow \mathrm{PGL}_d(\mathbb{K})$ is $\{\mathfrak{a}_p\}$ -Anosov, then for any geodesic ray $\{\gamma_n\}$ with endpoint x , the limits*

$$\xi_\rho^p(x) := \lim_{n \rightarrow \infty} U_p(\rho(\gamma_n)) \quad \xi_\rho^{d-p}(x) := \lim_{n \rightarrow \infty} U_{d-p}(\rho(\gamma_n))$$

exist and do not depend on the ray; they define continuous ρ -equivariant transverse maps $\xi^p : \partial\Gamma \rightarrow \mathcal{G}_p(\mathbb{K}^d)$, $\xi^{d-p} : \partial\Gamma \rightarrow \mathcal{G}_{d-p}(\mathbb{K}^d)$.

Proof. The proof in [5, Proposition 4.9] works without modification in our context: despite $U_p(\rho(\gamma_n))$ is not uniquely defined, Lemma 2.3 (2) guarantees that, for every choice of $U_p(\rho(\gamma_n))$, the sequence $\{U_p(\rho(\gamma_n))\}$ is Cauchy, and therefore has a limit; furthermore, since any pair of geodesic rays defining x is at bounded distance,

Lemma 2.3 (2) shows that the limit doesn't depend on the chosen sequence, and the maps are continuous. The equivariance follows from Lemma 2.3 (3). \square

The uniformity of the limits in Proposition 3.5 can be quantified explicitly (cfr. [5, Lemma 4.7]). This will be useful in the proof of Theorem 7.1:

Lemma 3.6. *Let $\rho : \Gamma \rightarrow \mathrm{PGL}_d(\mathbb{K})$ be $\{\mathbf{a}_p\}$ -Anosov. Then there exist constants C, μ such that for every $x \in \alpha\mathcal{C}_\infty(\alpha)$*

$$d(\xi^p(x), U_p(\alpha)) \leq C e^{-\mu|\alpha|}.$$

In particular, given $\varepsilon > 0$ there exists $L \in \mathbb{N}$ such that

$$\overline{\bigcup_{\gamma:|\gamma|\geq L} U_p(\rho(\gamma))} \subset \mathcal{N}_\varepsilon(\xi_\rho^p(\partial\Gamma)).$$

Proof. If $x \in \alpha\mathcal{C}_\infty(\alpha)$ there exists a geodesic ray $(\alpha_i)_{i \in \mathbb{N}}$ through α with endpoint x . In particular we get

$$d(U_p(\rho(\alpha)), \xi_\rho^p(x)) \leq \sum_{i \geq |\alpha|} d(U_p(\rho(\alpha_i)), U_p(\rho(\alpha_{i+1}))) \leq c \sum_{i \geq |\alpha|} e^{-\mu i}.$$

Here the first inequality is a consequence of the triangular inequality, the second follows from Lemma 2.3 (2).

The second statement follows since, as Γ is word hyperbolic, there is a constant D such that, for every $\gamma \in \Gamma$ we can choose a geodesic ray $(\alpha_i)_{i \in \mathbb{N}}$, and $k \in \mathbb{N}$ such that $d(\gamma, \alpha_k) \leq D$. This implies that $\gamma = \alpha_k h$ with $|h| \leq D$. Let x be the endpoint of the geodesic $(\alpha_i)_{i \in \mathbb{N}}$. Then

$$d(U_1(\rho(\gamma)), \xi_\rho^1(x)) \leq d(U_1(\rho(\alpha_k h)), U_1(\rho(\alpha_k))) + d(U_p(\rho(\alpha_i)), \xi^p(x)).$$

\square

Bochi-Potrie-S. [5] observed that the boundary map has an explicit characterization in term of the linear cocycle A_ρ over the sofic shift Λ_Γ (described in the proof of Proposition 3.4). Recall from Definition 2.11 that whenever a cocycle $A_\rho : \Lambda_\Gamma \times \mathbb{K}^d \rightarrow \Lambda_\Gamma \times \mathbb{K}^d$ has a dominated splitting we denote by E (resp. F) the stable (resp. unstable) bundle.

Proposition 3.7 ([5, Proposition 5.2]). *Let $\rho : \Gamma \rightarrow \mathrm{PGL}_d(\mathbb{K})$ be $\{\mathbf{a}_p\}$ -Anosov. Let $x, y \in \partial\Gamma$ and $(\bar{\alpha}_i)_{i \in \mathbb{N}}^\infty \in \Lambda_\Gamma$ a geodesic from y to x . Then one has*

$$\xi^p(x) = E_{(\rho(\bar{\alpha}_i))}^p \quad \text{and} \quad \xi^{d-p}(y) = F_{(\rho(\bar{\alpha}_i))}^{d-p}.$$

As a corollary we can follow [5] and generalize to the non-Archimedean case the following important fact originally proved by Labourie [36] and Guichard-W. [27].

Corollary 3.8. *Let $\rho : \Gamma \rightarrow \mathrm{PGL}_d(\mathbb{K})$ be $\{\mathbf{a}_p\}$ -Anosov. The boundary maps $\xi^p : \partial\Gamma \rightarrow \mathcal{G}_p(\mathbb{K}^d)$, $\xi^{d-p} : \partial\Gamma \rightarrow \mathcal{G}_{d-p}(\mathbb{K}^d)$ vary continuously with the representation.*

Proof. This follows at once from the arguments in the proof of Propositions 3.4 and 3.7, as splittings vary continuously with the cocycles by Proposition 2.14. \square

The boundary map, which is unique, gives a realization of the boundary $\partial\Gamma$ in $\mathbb{P}(\mathbb{K}^d)$, a space where the dynamics is governed by ratios of semi-homothecy ratios of elements in the projective linear group. To stress this fact and the dependence on ρ we introduce the following notation, which will be heavily used in the rest

of the paper: if $\rho : \Gamma \rightarrow \mathrm{GL}_d(\mathbb{K})$ is a $\{\mathfrak{a}_p\}$ -Anosov representation with equivariant boundary map ξ^p , and $x \in \partial\Gamma$, we will write

$$x_\rho^p := \xi^p(x).$$

We noticed that this notation improves readability of many formulas and conveniently stresses the dependence of ξ on ρ .

3.3. Geometric estimates. We conclude the section on Anosov representations collecting a number of geometric lemmas that will be useful in later on. The first result provides the quantification we will need of the following geometric principle: endpoints of a geodesics through the origin are uniformly far in the visual boundary, the same holds for their image under the boundary map associated to an Anosov representation.

Lemma 3.9. *Let $\rho : \Gamma \rightarrow \mathrm{PGL}_d(\mathbb{K})$ be $\{\mathfrak{a}_p\}$ -Anosov representation. Then there exists $\nu > 0$, depending only on ρ such that: if $\{\alpha_i\}_{i \in \mathbb{Z}} \subset \Gamma$ is a geodesic through id with endpoints $x, z \in \partial\Gamma$, then for all $i \in \mathbb{Z}$ one has*

$$\angle(\rho(\alpha_i^{-1})x_\rho^p, \rho(\alpha_i^{-1})z_\rho^{d-p}) > \nu.$$

Proof. Recall that we denote by $\mathcal{D}(C, d-p, c, \mu, \mathbb{Z})$ the compact, shift invariant space of dominated sequences (cfr. Definition 2.15). The bundle $\mathcal{D}(C, d-p, c, \mu, \mathbb{Z}) \times \mathbb{K}^d$ admits a dominated splitting $E^p \oplus F^{d-p}$ and, by compactness, we get

$$\nu = \inf_{\{g_i\} \in \mathcal{D}(C, d-p, c, \mu, \mathbb{Z})} \angle(E_{(g_i)}^p, F_{(g_i)}^{d-p}) > 0.$$

Remark 3.2 implies that, since ρ is $\{\mathfrak{a}_p\}$ -Anosov, $(\rho(\bar{\alpha}_i))_{i \in \mathbb{Z}} \in \mathcal{D}(C, d-p, c, \mu, \mathbb{Z})$, furthermore one directly computes that

$$\psi^n((\rho(\alpha_{i+1}^{-1}\alpha_i))_{i \in \mathbb{Z}}, v) = ((\rho(\alpha_{i+1}^{-1}\alpha_i))_{i-n \in \mathbb{Z}}, \rho(\alpha_n^{-1})v).$$

As we know from Proposition 2.16 that

$$x_\rho^p = \lim_{i \rightarrow \infty} U_p(\rho(\alpha_i)) = E_{(\rho(\bar{\alpha})_i)}^p$$

and

$$z_\rho^{d-p} = \lim_{i \rightarrow \infty} U_{d-p}(\rho(\alpha_{-i})) = F_{(\rho(\bar{\alpha})_i)}^{d-p},$$

we deduce that for all $i \in \mathbb{Z}$ one has

$$\angle(\rho(\alpha_i^{-1})x_\rho^p, \rho(\alpha_i^{-1})z_\rho^{d-p}) > \nu.$$

□

The next lemma will be crucial in Section 6. It quantifies how the inverse of elements in a geodesic expand the distances exponentially in neighbourhoods of their Cartan attractors; this should be compared with [5, Corollary A.14]:

Lemma 3.10. *Let $\rho : \Gamma \rightarrow \mathrm{PGL}_d(\mathbb{K})$ be $\{\mathfrak{a}_p\}$ -Anosov. There exist positive constants $\bar{c}, \bar{\mu}, \delta$ depending only on ρ , and $L \in \mathbb{N}$ such that, for every geodesic ray $\{\alpha_i\}_{i \in \mathbb{N}} \subset \Gamma$, with $\alpha_0 = \mathrm{id}$ and endpoint x , every $i \geq L$, and every $z, w \in \partial\Gamma$ satisfying $z_\rho^p, w_\rho^p \in B_\delta(x_\rho^p)$, and $\rho(\alpha_i^{-1})\{z_\rho^p, w_\rho^p\} \subset B_\delta(\rho(\alpha_i^{-1})x_\rho^p)$, we have*

$$d(\rho(\alpha_i^{-1})w_\rho^p, \rho(\alpha_i^{-1})z_\rho^p) \geq \bar{c}e^{\mu i}d(w_\rho^p, z_\rho^p).$$

Proof. We complete the ray $\{\alpha_i\}_{i \in \mathbb{N}}$ to a biinfinite geodesic $\{\alpha_i\}_{i \in \mathbb{Z}}$ with second endpoint y . The sequence $s = \{\rho(\alpha_{i+1}^{-1}\alpha_i)\}_{i \in \mathbb{Z}}$ belongs to $\mathcal{D}(C, p, c, \mu, \mathbb{Z})$. It follows from Propositions 2.16 and 3.5 that the sequence s has the dominated splitting $E \oplus F$ where $F_s = x_\rho^p$ and $E_s = y_\rho^{d-p}$. So that there exist constants $\bar{\mu}, c_1$ such that

$$\frac{\mathbf{m}(\rho(\alpha_i^{-1})|_{y_\rho^{d-p}})}{\|\rho(\alpha_i^{-1})|_{x_\rho^p}\|} \geq c_1 e^{\bar{\mu}i}.$$

Since, by Lemma 3.9, the angles $\sphericalangle(x_\rho^p, y_\rho^{d-p})$ and $\sphericalangle(\rho(\alpha_i^{-1})x_\rho^p, \rho(\alpha_i^{-1})y_\rho^{d-p})$ are bounded below by a uniform constant ν , we can apply Lemma 2.8 with $P = x_\rho^p$, $Q = y_\rho^{d-p}$ and $g = \rho(\alpha_i)^{-1}$ and get

$$d(\rho(\alpha_i^{-1})w_\rho^p, \rho(\alpha_i^{-1})z_\rho^p) \geq b \frac{\mathbf{m}(\rho(\alpha_i^{-1})|_{y_\rho^{d-p}})}{\|\rho(\alpha_i^{-1})|_{x_\rho^p}\|} d(w_\rho^p, z_\rho^p).$$

□

4. AN UPPER BOUND ON THE HAUSDORFF DIMENSION OF THE LIMIT SET

In this section we will prove the following upper bound, this result is independently obtained by Glorieux-Monclair-Tholozan [23] for Archimedean \mathbb{K} . Recall from the introduction that if $\rho : \Gamma \rightarrow \mathrm{PGL}_d(\mathbb{K})$ is projective Anosov then $h_\rho^{\mathfrak{a}_1}$ is the critical exponent of the Dirchlet series

$$s \mapsto \sum_{\gamma \in \Gamma} \left(\frac{\sigma_2}{\sigma_1}(\rho(\gamma)) \right)^s.$$

We denote by $\mathrm{Hff}(A)$ the Hausdorff dimension of a subset $A \subset \mathbb{P}(\mathbb{K}^d)$ for the metric induced by a good norm on \mathbb{K}^d .

Proposition 4.1. *Let $\rho : \Gamma \rightarrow \mathrm{PGL}_d(\mathbb{K})$ be $\{\mathfrak{a}_1\}$ -Anosov, then $\mathrm{Hff}(\xi^1(\partial\Gamma)) \leq h_\rho^{\mathfrak{a}_1}$.*

Recall that for a metric space (Λ, d) and for $s > 0$ its s -capacity is defined as

$$\mathcal{H}^s(\Lambda) = \inf_{\varepsilon} \left\{ \sum_{U \in \mathcal{U}} \mathrm{diam} U^s : \mathcal{U} \text{ is a covering of } \Lambda \text{ with } \sup_{U \in \mathcal{U}} \mathrm{diam} U < \varepsilon \right\}$$

and that

$$\mathrm{Hff}(\Lambda) = \inf\{s : \mathcal{H}^s(\Lambda) = 0\} = \sup\{s : \mathcal{H}^s(\Lambda) = \infty\}. \quad (9)$$

In order to prove Proposition 4.1 we will analyze the image, under the boundary map ξ^1 of the covering \mathcal{U}_T described in subsection 2.3.3, whose elements consist of images of cone types at infinity under sufficiently big group elements. The following crucial lemma will allow us to show that the images of the boundaries of cone types transform as expected under group elements:

Lemma 4.2. *Let $\rho : \Gamma \rightarrow \mathrm{PGL}_d(\mathbb{K})$ be a projective Anosov representation. Then there exist $\delta > 0$, $L \in \mathbb{N}$ such that for all $\gamma \in \Gamma$ with $|\gamma| > L$ and every $x \in \mathcal{C}_\infty(\gamma)$ one has*

$$\sphericalangle(x_\rho^1, U_{d-1}(\rho(\gamma^{-1}))) > \delta.$$

Proof. By definition of $\mathcal{C}(\gamma)$, for all $x \in \mathcal{C}_\infty(\gamma)$ there exists a geodesic ray $\{\alpha_i\}_0^\infty$ in Γ with $\alpha_0 = \gamma^{-1}$ and $\alpha_i \rightarrow x$ as $i \rightarrow \infty$. The lemma then follows combining Proposition 3.3 and Proposition 3.5. □

4.1. Proof of Proposition 4.1. For each $T > 0$ consider the covering \mathcal{U}_T of $\partial\Gamma$ given by Lemma 2.21. By definition, $U = U_\gamma \in \mathcal{U}_T$ is of the form $\gamma\mathcal{C}_\infty(\gamma)$ for some $\gamma \in \Gamma$ with $|\gamma| \geq T$.

Lemma 4.2 implies that there exists δ such that for every $x \in \mathcal{C}_\infty(\gamma)$ one has

$$d(x_\rho^1, U_{d-1}(\rho(\gamma^{-1}))) \geq \delta$$

and thus Lemma 2.5 applied to $\rho(\gamma)$ implies that

$$d(\rho(\gamma)x_\rho^1, U_1(\rho(\gamma))) \leq \frac{1}{\delta} \frac{\sigma_2}{\sigma_1}(\rho(\gamma))$$

which implies that

$$\text{diam } \xi(U_\gamma) \leq \frac{2}{\delta} \frac{\sigma_2}{\sigma_1}(\rho(\gamma)) \leq Ce^{-\mu T}.$$

In particular $\sup_{U \in \mathcal{U}_T} \text{diam } U$ is arbitrarily small as $T \rightarrow \infty$. Hence,

$$\mathcal{H}^s(\xi^1(\partial\Gamma)) \leq \inf_T \sum_{U \in \mathcal{U}_T} (\text{diam } \xi^1(U))^s \leq \left(\frac{2}{\delta}\right)^s \inf_T \sum_{\gamma: |\gamma| \geq T} \left(\frac{\sigma_2}{\sigma_1}(\rho(\gamma))\right)^s.$$

By definition, if $s > h_\rho^{a_1}$ then the Dirichlet series $\Phi_\rho^{a_1}(s)$ is convergent (recall eq. (1)). Hence, for every $s > h_\rho^{a_1}$ one has

$$\lim_{T \rightarrow \infty} \sum_{\gamma: |\gamma| \geq T} \left(\frac{\sigma_2}{\sigma_1}(\rho(\gamma))\right)^s = 0,$$

which implies that the s -capacity $\mathcal{H}^s(\xi^1(\partial\Gamma))$ vanishes and thus $\text{Hff}(\xi^1(\partial\Gamma)) \leq h_\rho^{a_1}$.

5. LOCAL CONFORMALITY AND HAUSDORFF DIMENSION

The goal of this section is to find a class of representations for which the equality in Proposition 4.1 holds. This happens in three steps.

- In Section 5.1 we study the *thickened cone types* $X_\infty(\alpha)$, these are a thickening, in $\mathbb{P}(\mathbb{K}^d)$, of the image by ξ^1 of $\mathcal{C}_\infty(\alpha)$ for a given α , and define the *locally conformal points*.
- In Section 5.2 we prove that if x is a locally conformal point, then there is a geodesic ray $\alpha_i \rightarrow x$ such that the sets $\rho(\alpha_i)X_\infty(\alpha_i)$ behave coarsely like balls around x ; the harder inequality is the lower containment, which is achieved in Corollary 5.10.
- In Section 5.3 we define a measure that behaves like an Ahlfors regular measure for the sets $\alpha X_\infty(\alpha)$. Putting this together with the previous section, arguments coming from Sullivan's original paper allow us to conclude the desired equality, provided we can guarantee existence of many locally conformal points, this is the purpose of Section 5.4.

5.1. Thickened cone types at infinity and locally conformal points. Let $\rho : \Gamma \rightarrow \text{PGL}_d(\mathbb{K})$ be projective Anosov, it follows from Proposition 3.3 that there is a positive lower bound on the distance of Cartan attractors and repellers of geodesic rays through the origin. Such number will play an important role in our study:

Definition 5.1. Let $\rho : \Gamma \rightarrow \text{PGL}_d(\mathbb{K})$ be projective Anosov, and let L be fixed big enough. The *least angle* δ_ρ is

$$\delta_\rho = \inf \sin \left(\sphericalangle \left(U_1(\rho(\alpha_k)), U_{d-1}(\rho(\alpha_{-m})) \right) \right)$$

where $(\alpha_i)_{i \in \mathbb{Z}}$ ranges among biinfinite geodesics through the origin, and $k, m > L$.

We consider coverings of $\xi(\partial\Gamma)$ obtained by translating *thickened cone types at infinity*:

$$X_\infty(\alpha) := \mathcal{N}_{\delta_\rho/2}(\xi_\rho^1(\mathcal{C}_\infty(\alpha))) \cap \xi^1(\partial\Gamma).$$

By construction the sets $X_\infty(\alpha)$ are coarsely balls of $\xi^1(\partial\Gamma)$ centered at points in $\xi^1(\mathcal{C}_\infty(\alpha))$:

Remark 5.2. For every α in Γ , and every $x \in \mathcal{C}_\infty(\alpha)$, the thickened cone type at infinity $X_\infty(\alpha)$ contains a ball centered at the point x_ρ^1 of uniform radius:

$$B(x_\rho^1, \delta_\rho/2) \cap \xi^1(\partial\Gamma) \subset X_\infty(\alpha)$$

Thanks to Proposition 3.3 we can control how thickened cone types shrink under the action of group elements:

Lemma 5.3. *Let $\rho : \Gamma \rightarrow \mathrm{PGL}_d(\mathbb{K})$ be projective Anosov. Then there exist K, L such that, for every $z_\rho^1, w_\rho^1 \in X_\infty(\alpha_i)$, and every $i > L$,*

$$d(\alpha_i z_\rho^1, \alpha_i w_\rho^1) \leq K \frac{\sigma_2}{\sigma_1}(\rho(\alpha_i)) d(z_\rho^1, w_\rho^1).$$

Proof. The proof is analogue to the one of Lemma 2.8, we borrow some notation from that section. As ρ is projective Anosov we have $d(z_\rho^1, U_{d-1}(\alpha_i^{-1})) > \delta_\rho/2$ (Lemma 4.2). In particular, if we choose $P = \alpha_i^{-1}U_1(\alpha_i)$, $Q = U_{d-1}(\alpha_i^{-1})$ and denote by $L_z := L_{z_\rho^1, P}$ and $M_z := L_{\rho(\alpha_i)z_\rho^1, \rho(\alpha_i)P}$, we have

$$d(\rho(\alpha_i)z_\rho^1, \rho(\alpha_i)w_\rho^1) \leq \|M_w - M_z\| = \|(g|_Q)\| \|L_w - L_z\| \|g|_P^{-1}\|.$$

The estimate follows since, as $d(z_\rho^1, U_{d-1}(\alpha_i^{-1})) > \delta_\rho/2$ there exists K such that $\|L_w - L_z\| \leq K d(z_\rho^1, w_\rho^1)$. \square

Corollary 5.4. *If $\rho : \Gamma \rightarrow \mathrm{PGL}_d(\mathbb{K})$ is projective Anosov, and $x \in \alpha_i \mathcal{C}_\infty(\alpha_i)$*

$$\rho(\alpha_i)X_\infty(\alpha_i) \subset B\left(x_\rho^1, K \frac{\sigma_2}{\sigma_1}(\rho(\alpha_i))\right) \cap \xi(\partial\Gamma).$$

In particular, if $\{\alpha_i\}_1^\infty$ is a geodesic ray with endpoint x , the sets $\rho(\alpha)X_\infty(\alpha)$ form a fundamental system of open neighbourhoods of x in $\xi(\partial\Gamma)$ (cfr. Figure 3).

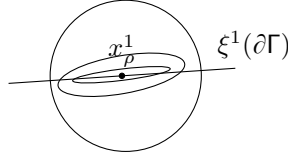


FIGURE 3. The sets of the form $\rho(\alpha_i)X_\infty(\alpha_i)$ for a geodesic ray $\{\alpha_i\}$ with endpoint x are the intersections of thinner and thinner ellipses with the limit curve.

Definition 5.5 gives conditions guaranteeing that the sets $\rho(\alpha)X_\infty(\alpha)$ are coarsely balls whose sizes we can precisely estimate. Given $g \in \mathrm{GL}_d(\mathbb{K})$ we denote by

$$1 \leq p_1(g) < \dots < p_{k(g)}(g) < d$$

the indices of the gaps of g .

Definition 5.5. Let $\rho : \Gamma \rightarrow \mathrm{PGL}_d(\mathbb{K})$ be projective Anosov. We say that $x \in \partial\Gamma$ is a (ε, L) -locally conformal point for ρ if there exists a geodesic ray $\{\alpha_i\}_0^\infty$ in Γ based at the identity and with endpoint x such that the following conditions hold:

- (i) for all big enough i one has $p_2(\alpha_i) = p_2$ does not depend on i ,
- (ii) for every $i > L$, and for every $z \in X_\infty(\alpha_i)$ one has

$$\sin \left(\sphericalangle \left(z_\rho^1 \oplus \rho(\alpha_i^{-1})x_\rho^1, U_{d-p}(\rho(\alpha_i^{-1})) \right) \right) > \varepsilon.$$

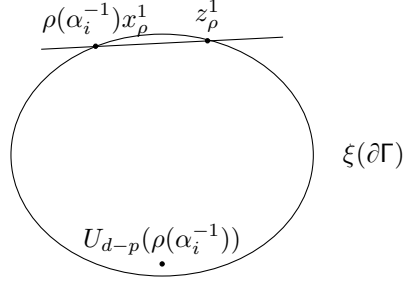


FIGURE 4. The third condition in Definition 5.5.

Note that, in general, the index p_2 might depend on the point x and we do not require that the representation ρ is $\{\mathfrak{a}_{p_2}\}$ -Anosov. In the special case when ρ is $\{\mathfrak{a}_1, \mathfrak{a}_2\}$ -Anosov, the condition (i) is automatically satisfied with $p_2 = 2$, but (ii) can only hold if the dimension of $\partial\Gamma$ is very small (cfr. Corollary 6.6).

Remark 5.6. A generic element $g \in \mathrm{PGL}(V)$ has $p_2(g) = 2$. Nevertheless there are many interesting geometric situations in which condition (i) holds for $p_2 > 2$. For example if g is a generic element in $\mathrm{SO}(m, n)$, we have that $\Lambda^m g \in \mathrm{SL}(V)$ has $p_2(\Lambda^m g) = n - m + 1$, so one can enforce $p_2 > 2$ by considering representations in smaller subgroups. In Section 8.1 we will describe another interesting class of examples.

5.2. Neighborhoods of locally conformal points that are coarsely balls.

We will now show that if x is a locally conformal point for ρ , and $\alpha_i \rightarrow x$ is a geodesic ray, then the sets $\rho(\alpha_i)X_\infty(\alpha_i)$ are coarsely balls centered at x_ρ^1 of radius $\sigma_2/\sigma_1(\rho(\alpha_i))$ for the distance on $\xi_\rho^1(\partial\Gamma)$ induced by d , this will be achieved in Corollary 5.10, and motivated the terminology *locally conformal*.

Proposition 5.7. Let $\rho : \Gamma \rightarrow \mathrm{PGL}_d(\mathbb{K})$ be projective Anosov. There exist \bar{L} such that, for every (ε, L) -locally conformal point x , there exists a geodesic ray $\{\alpha_i\}_0^\infty$ from the identity with endpoint x , such that for every $i > \bar{L}$ and every $z \in X_\infty(\alpha_i)$ it holds

$$\frac{\varepsilon}{4} \frac{\sigma_2}{\sigma_1}(\rho(\alpha_i)) \cdot d(z_\rho^1, (\alpha_i^{-1}x)_\rho^1) \leq d((\alpha_i z)_\rho^1, x_\rho^1).$$

Proof. Let $W := z_\rho^1 \oplus (\alpha_i^{-1}x)_\rho^1$. As x is (ε, L) -locally conformal, for every $i > L$, we have $d(W, U_{d-p}(\rho(\alpha_i^{-1}))) > \varepsilon$. In particular, for every $i > L$ it holds

$$\frac{\sigma_2}{\sigma_1}(\rho(\alpha_i)|_W) \geq \varepsilon \frac{\sigma_p}{\sigma_1}(\rho(\alpha_i)) = \varepsilon \frac{\sigma_2}{\sigma_1}(\rho(\alpha_i)).$$

Here the last equality is due to the fact that p is the first gap for $\rho(\alpha_i)$ and thus $\sigma_p(\rho(\alpha_i)) = \sigma_2(\rho(\alpha_i))$.

Furthermore $\rho(\alpha_i^{-1})u_2(\rho(\alpha_i)|_W) = W \cap U_{d-1}(\rho(\alpha_i^{-1}))$, and then, since ρ is projective Anosov and $z_\rho^1 \in X_\infty(\alpha)$, we have $d(z_\rho^1, \rho(\alpha_i^{-1})u_2(\rho(\alpha_i)|_W)) > \delta_\rho/2$, where δ_ρ is the constant from Definition 5.1. This implies that we can find \bar{L} depending on ρ and ε only such that for every $i > \bar{L}$

$$d(\rho(\alpha_i)z_\rho^1, u_1(\alpha_i|_W)) < d(\rho(\alpha_i)z_\rho^1, u_1(\alpha_i)) + d(u_1(\alpha_i), u_1(\alpha_i|_W)) < 1/\sqrt{2}.$$

The proposition then follows from Corollary 2.9. \square

Recall from Definition 2.19 that we say that a pair of cone types $(\mathcal{C}(\alpha_1), \mathcal{C}(\alpha_2))$ of Γ are k -nested if there exists a path in the geodesic automaton of length k between $\mathcal{C}(\alpha_1)$ and $\mathcal{C}(\alpha_2)$. In this case we say that $\beta \in \Gamma$ is a *nesting word* if β labels one such path.

Lemma 5.8. *For every L big enough (depending only on ρ) there exists a constant c (depending on ρ and L) such that for every L -nested pair $(\mathcal{C}(\alpha_1), \mathcal{C}(\alpha_2))$ and any nesting word β it holds*

- (i) $\rho(\beta)X_\infty(\alpha_2) \subset X_\infty(\alpha_1)$
- (ii) for every $z_\rho^1 \in \xi(\beta\mathcal{C}_\infty(\alpha_2))$ and every $w_\rho^1 \in X_\infty(\alpha_1) \setminus \rho(\beta)X_\infty(\alpha_2)$, it holds

$$d(z_\rho^1, w_\rho^1) > c.$$

Proof.

- (i) By definition of δ_ρ and $X_\infty(\alpha_2)$ we can find L big enough so that, if $|\beta| > L$, every point in $X_\infty(\alpha_2)$ is far from $U_{d-1}(\rho(\beta^{-1}))$. Up to possibly enlarging L we can assume, by Lemma 2.5, that $\rho(\beta)$ contracts distances on $X_\infty(\alpha_2)$ so that

$$\rho(\beta)X_\infty(\alpha_2) \subseteq \mathcal{N}_{\delta_\rho/2}(\rho(\beta)\xi_\rho^1(\mathcal{C}_\infty(\alpha_2))) \cap \xi(\partial\Gamma) \subseteq X_\infty(\alpha_1).$$

- (ii) Since, by construction, $X_\infty(\alpha_2)$ contains the intersection of $\xi_\rho^1(\partial\Gamma)$ with a ball around any point $z_\rho^1 \in \xi_\rho^1(\mathcal{C}_\infty(\alpha_2))$ of radius $\delta_\rho/2$, the set $\rho(\beta)X_\infty(\alpha_2)$ contains the intersection of $\xi_\rho^1(\partial\Gamma)$ with the ball around any point $z_\rho^1 \in \rho(\beta)\xi_\rho^1(\mathcal{C}_\infty(\alpha_2))$ of radius

$$\frac{\delta_\rho}{2} \frac{\sigma_d}{\sigma_1}(\rho(\beta)) :$$

$\frac{\sigma_d}{\sigma_1}(g)$ is the smallest contraction for the action of $g \in \mathrm{SL}(d, \mathbb{K})$ on $\mathbb{P}(\mathbb{K}^d)$. As only finitely many β can occur, the result follows. \square

Combining Proposition 5.7 and Propostion 5.8 we obtain:

Proposition 5.9. *There exists c_1 depending on ρ only such that, if L is as in Lemma 5.8 and $\{\alpha_i\} \subset \Gamma$ is a geodesic ray with endpoint x , for every y with $y_\rho^1 \in \rho(\alpha_n)X_\infty(\alpha_n) \setminus \rho(\alpha_{n+L})X_\infty(\alpha_{n+L})$, it holds*

$$d(y_\rho^1, x_\rho^1) \geq c_1 \frac{\sigma_2}{\sigma_1}(\alpha_{n+L}).$$

Proof. It follows from Lemma 2.20 that for every n, L the pair $(\mathcal{C}_\infty(\alpha_n), \mathcal{C}_\infty(\alpha_{n+L}))$ is L -nested. Furthermore, up to choosing L large enough, we can apply Lemma 5.8

to the pair $(\mathcal{C}_\infty(\alpha_n), \mathcal{C}_\infty(\alpha_{n+L}))$. If we denote by $z := \alpha_n^{-1}x$ and $w := \alpha_n^{-1}y$ we deduce that $d(z_\rho^1, w_\rho^1) > c$. Proposition 5.7 implies then that

$$d(y_\rho^1, x_\rho^1) \geq \frac{c\varepsilon}{4} \frac{\sigma_2}{\sigma_1}(\alpha_n) \geq c_1 \frac{\sigma_2}{\sigma_1}(\alpha_{n+L})$$

Where in the last inequality we used that, as L is fixed the homothety ratio gap of α_n is uniformly comparable to the one of α_{n+L} . \square

As a corollary of Proposition 5.7 we can finally get the main result of the section (cfr. Figure 3):

Corollary 5.10. *Let $\rho : \Gamma \rightarrow \mathrm{PGL}_d(\mathbb{K})$ be a projective Anosov; then for every locally conformal point $x \in \partial\Gamma$ there exists a geodesic ray $\alpha_i \rightarrow x$ with*

$$B\left(x_\rho^1, c_1 \frac{\sigma_2}{\sigma_1}(\rho(\alpha_i))\right) \cap \xi(\partial\Gamma) \subset \rho(\alpha_i)X_\infty(\alpha_i).$$

Proof. This follows from the above proposition by observing that the sets $\rho(\alpha_i)X_\infty(\alpha_i)$ form a fundamental system of neighborhoods of x_ρ^1 in $\xi(\partial\Gamma)$. \square

5.3. A regular measure for conformal points. The goal of this section is to construct, following Patterson's original idea, a measure, supported on $\xi_\rho^1(\partial\Gamma)$, for which we can get good estimates on the measure of the cone types. This will be used, in Section 5.4, to obtain the desired lower bound on the Hausdorff dimension of the limit set. ⁵

Let $\rho : \Gamma \rightarrow \mathrm{PGL}_d(\mathbb{K})$ be a projective Anosov representation. We can assume that

$$\Phi_\rho(h_\rho^{\mathfrak{a}_1}) = \sum_{\gamma \in \Gamma} \left(\frac{\sigma_2}{\sigma_1}(\rho(\gamma)) \right)^{h_\rho^{\mathfrak{a}_1}} = \infty :$$

otherwise, as it is standard in Patterson-Sullivan theory, we would carry out the same construction with the aid of the modified Poincaré series

$$\Phi_\rho(s) = \sum_{\gamma \in \Gamma} f(\mathfrak{a}_1(\rho(\gamma))) \left(\frac{\sigma_2}{\sigma_1}(\rho(\gamma)) \right)^{h_\rho^{\mathfrak{a}_1}},$$

where $f(s)$ is the function constructed (for example) in Quint [46, Lemma 8.5].

We will therefore assume from now on that the Poincaré series diverges at its critical exponent; for every $s > h_\rho^{\mathfrak{a}_1}$, we define

$$\mu_\rho^s = \frac{1}{\Phi_\rho(s)} \sum_{\gamma \in \Gamma} \left(\frac{\sigma_2}{\sigma_1}(\rho(\gamma)) \right)^s \delta_{U_1(\rho(\gamma))}.$$

Recall from Section 2.1 that, for every element $\gamma \in \Gamma$ we chose a Cartan decomposition of $\rho(\gamma)$ and therefore a 1-dimensional subspace $U_1(\rho(\gamma))$.

One easily checks that for every $s > h_\rho^{\mathfrak{a}_1}$ the functional $f \mapsto \int f d\mu_\rho^s$ is continuous on $C(\mathbb{P}(\mathbb{K}^d), \mathbb{R})$ with the uniform topology and hence one can take a weak* accumulation point of μ_ρ^s , as $s \rightarrow h_\rho^{\mathfrak{a}_1}$, in the space of Radon probability measures on $\mathbb{P}(\mathbb{K}^d)$. We will denote such Radon measure by $\mu_\rho^{\mathfrak{a}_1}$, (note that we do not show, nor require, that $\mu_\rho^{\mathfrak{a}_1}$ is the only accumulation point of μ_ρ^s).

⁵See Remark 5.15 for a comparison with the work of Quint [46]

Lemma 5.11. *For any $\eta \in \Gamma$ the (signed) measure*

$$\varepsilon(\eta, s) := \eta_* \mu_\rho^s - \frac{1}{\Phi_\rho(s)} \sum_{\gamma \in \Gamma} \left(\frac{\sigma_2}{\sigma_1}(\rho(\gamma)) \right)^s \delta_{U_1(\rho(\eta\gamma))}$$

weakly converges to zero as $s \rightarrow h_\rho^{\mathfrak{a}_1}$.*

Proof. Indeed by definition

$$\eta_* \mu_\rho^s = \frac{1}{\Phi_\rho(s)} \sum_{\gamma \in \Gamma} \left(\frac{\sigma_2}{\sigma_1}(\rho(\gamma)) \right)^s \delta_{\rho(\eta)U_1(\rho(\gamma))}.$$

Furthermore, Lemma 2.3 (3) implies that

$$d(\rho(\eta)U_1(\rho(\gamma)), U_1(\rho(\eta\gamma))) \leq \|\eta\| \|\eta^{-1}\| \frac{\sigma_2}{\sigma_1} \rho(\gamma).$$

In order to show that $\varepsilon(\eta, s)$ converges to zero, it is enough to show that for every continuous function $f : \mathbb{P}(V) \rightarrow \mathbb{R}$ the integral of f on $\varepsilon(\eta, s)$ tends to zero as s converges to $h_\rho^{\mathfrak{a}_1}$. However every such function f is uniformly continuous, and therefore for every ε we can find δ such that $|f(x) - f(y)| < \varepsilon/2$ if $d(x, y) < \delta$. It is then enough to choose s close enough to $h_\rho^{\mathfrak{a}_1}$ so that the mass of μ_ρ^s of the elements γ such that $\sigma_1/\sigma_2(\gamma) < \|\eta\| \|\eta^{-1}\|/\delta$ is smaller than $\frac{\varepsilon}{2\|f\|}$. \square

One has the following proposition (compare with Sullivan's shadow Lemma [52]).

Proposition 5.12. *Let $\rho : \Gamma \rightarrow \mathrm{PGL}_d(\mathbb{K})$ be a projective Anosov representation, then for all $\eta \in \Gamma$ one has*

$$\left(\frac{\sigma_d}{\sigma_1}(\rho(\eta)) \right)^{h_\rho^{\mathfrak{a}_1}} \leq \frac{\mu_\rho^{\mathfrak{a}_1}(\rho(\eta)X_\infty(\eta))}{\mu_\rho^{\mathfrak{a}_1}(X_\infty(\eta))} \leq \frac{4}{\delta_\rho^2} \left(\frac{\sigma_2}{\sigma_1}(\rho(\eta)) \right)^{h_\rho^{\mathfrak{a}_1}}.$$

Recall that there are finitely many cone types, so the number $\mu_\rho^{\mathfrak{a}_1}(X_\infty(\eta))$ is an irrelevant constant.

Proof. Consider $s > h_\rho^\sigma$, $\eta \in \Gamma$ and a continuous function $f : \mathbb{P}(\mathbb{K}^d) \rightarrow \mathbb{R}$. One has

$$\begin{aligned} \mu_\rho^s(f \circ \rho(\eta)^{-1}) &= \frac{1}{\Phi_\rho(s)} \sum_{\gamma \in \Gamma} \left(\frac{\sigma_2}{\sigma_1}(\rho(\gamma)) \right)^s f(\rho(\eta^{-1})U_1(\rho(\gamma))) \\ &= \varepsilon(\eta^{-1}, s)(f) + \frac{1}{\Phi_\rho(s)} \sum_{\gamma \in \Gamma} \left(\frac{\sigma_2}{\sigma_1}(\rho(\eta\gamma)) \right)^s f(U_1(\rho(\gamma))) \quad (10) \\ &= \varepsilon(\eta^{-1}, s)(f) + \frac{1}{\Phi_\rho(s)} \sum_{\gamma \in \Gamma} \left(\frac{\sigma_2}{\sigma_1}(\rho(\eta\gamma)) \frac{\sigma_1}{\sigma_2}(\rho(\gamma)) \right)^s \left(\frac{\sigma_2}{\sigma_1}(\rho(\gamma)) \right)^s f(U_1(\rho(\gamma))), \end{aligned}$$

where $\varepsilon(\eta^{-1}, s)$ is the term estimated in Lemma 5.11, so that $\varepsilon(\eta^{-1}, s)(f)$ converges to zero when $s \rightarrow h_\rho^{\mathfrak{a}_1}$.

Assume that the support of f contains $X_\infty(\eta)$ in its interior and s is close enough to $h_\rho^{\mathfrak{a}_1}$. Then only big enough γ 's are relevant, and together with Proposition 3.3 this implies that if $f(U_1(\rho(\gamma))) > 0$ then $\sin(\angle(U_1(\rho(\gamma)), U_{d-1}(\rho(\eta)))) > \varepsilon_f$, for some ε_f depending on the support of f . Note that ε_f approaches $\delta_\rho/2$ as $\mathrm{supp} f \rightarrow X_\infty(\eta)$.

Choosing a sequence $s_k \rightarrow h_\rho^{a_1}$ such that $\mu_\rho^{s_k} \rightarrow \mu_\rho^{a_1}$ one has, using Lemma 2.6 and equation (10), that, for any such f

$$\mu_\rho^{a_1}(f \circ \rho(\eta)^{-1}) = \lim_{s_k \rightarrow h_\rho^{a_1}} \mu_\rho^{s_k}(f \circ \rho(\eta)^{-1}) \leq \frac{4}{\delta_\rho^2} \left(\frac{\sigma_2}{\sigma_1}(\rho(\eta)) \right)^{h_\rho^{a_1}} \mu_\rho^{a_1}(f).$$

By continuity of $f \mapsto \mu_\rho^{a_1}(f)$, one concludes the desired upper bound. The lower bound follows similarly. \square

Since cone types shrink to any given point of $\partial\Gamma$ one has the following consequences of Proposition 5.12.

Corollary 5.13. *The measure $\mu_\rho^{a_1}$ has total support and no atoms.*

Proof. If α_i is a geodesic ray converging to x then $\rho(\alpha_i)X_\infty(\alpha_i)$ is a family of open neighborhoods decreasing to x , and since ρ is projective Anosov one has $(\sigma_2/\sigma_1)(\rho(\alpha_i)) \rightarrow 0$ as $i \rightarrow \infty$. As $\mu_\rho^{a_1}$ is a Radon measure we have on the one hand

$$\mu_\rho^{a_1}(\{x\}) = \inf\{\mu_\rho^{a_1}(\rho(\alpha_i)X_\infty(\alpha_i))\} \leq \frac{4}{\delta_\rho^2} \left(\frac{\sigma_2}{\sigma_1}(\rho(\eta)) \right)^{h_\rho^{a_1}},$$

on the other hand for every open set A intersecting $\xi(\partial\Gamma)$ we can find α such that $\rho(\alpha_i)X_\infty(\alpha_i)$ is contained in A , and thus

$$\mu_\rho^{a_1}(A) \geq \mu_\rho^{a_1}(\rho(\alpha_i)X_\infty(\alpha_i)) \geq \left(\frac{\sigma_d}{\sigma_1}(\rho(\eta)) \right)^{h_\rho^{a_1}}.$$

\square

5.4. When conformal points are abundant. Denote by

$$\text{LC}(\rho) = \{x \in \partial\Gamma : x \text{ is locally conformal for } \rho\}.$$

We can now prove the following.

Theorem 5.14. *Let $\rho : \Gamma \rightarrow \text{PGL}_d(\mathbb{K})$ be a projective Anosov representation. If $\mu_\rho^{a_1}(\text{LC}(\rho)) > 0$, then*

$$\text{Hff}(\xi_\rho^1(\partial\Gamma)) = h_\rho^{a_1}.$$

Proof. As we already established in Proposition 4.1, $\text{Hff}(\xi_\rho^1(\partial\Gamma)) \leq h_\rho^{a_1}$, so we only need to show the reverse inequality. The proof will follow the main ideas in Sullivan's original work [52], using Corollary 5.10 and Proposition 5.12 as key replacement for the conformality of a Kleinian group action on its boundary, and Sullivan's shadow lemma.

Given $x \in \text{LC}(\rho)$ and a geodesic ray $\{\alpha_i\}$ on Γ converging to x , Corollary 5.10 implies that for all $i \geq N_0(x)$ the set $\rho(\alpha_i)X_\infty(\alpha_i)$ is coarsely (with constants independent of x) a ball of radius

$$r_i(x) = \frac{\sigma_2}{\sigma_1}(\rho(\alpha_i))$$

about x_ρ^1 (for the induced metric on $\xi_\rho^1(\partial\Gamma)$). For convenience, let us denote

$$r_0(x) = \frac{\sigma_2}{\sigma_1}(\rho(\alpha_{N_0}))$$

the *first time* where the sets $\rho(\alpha_i)X_\infty(\alpha_i)$ become balls compatible with our chosen thresholds.

Proposition 5.12 then states that

$$\mu_\rho^{\mathfrak{a}_1}(B(x, r_i(x))) \leq cr_i(x)^{h_\rho^{\mathfrak{a}_1}}. \quad (11)$$

Observe that we can extend equation (11) for any $0 < r \leq r_0(x)$, up to possibly worsening the constant c : Since ρ is projective Anosov, the word length of $\gamma \in \Gamma$ is coarsely $\log \sigma_2/\sigma_1(\rho(\gamma))$, thus

$$r_i(x)/r_{i+1}(x) \leq K_\rho$$

for some constant K_ρ only depending on ρ . Given r it suffices to consider $r_{i+1}(x) \leq r \leq r_i(x)$ and thus

$$\mu_\rho^{\mathfrak{a}_1}(B(x, r)) \leq c \left(\frac{r_i(x)}{r_{i+1}(x)} \right)^{h_\rho^{\mathfrak{a}_1}} r_{i+1}(x)^{h_\rho^{\mathfrak{a}_1}} \leq L_\rho r^{h_\rho^{\mathfrak{a}_1}}.$$

Furthermore, there exists ε such that the set $X_\varepsilon = \{x \in \text{LC}(\rho) : r_0(x) \geq \varepsilon\}$ has positive $\mu_\rho^{\mathfrak{a}_1}$ -mass: follows from the general fact that countable union of sets with measure 0 has measure 0, since we assumed $\mu_\rho^{\mathfrak{a}_1}(\text{LC}(\rho)) > 0$,

The remainder arguments are verbatim as in Haïssinsky [28, Théorème F.4]. We include them for completeness: as X_ε is a subset of $\xi(\partial\Gamma)$, it is enough to verify that $\text{Hff}(\xi(X_\varepsilon)) \geq h_\rho^{\mathfrak{a}_1}$; we will show that, denoting by $\sigma := h_\rho^{\mathfrak{a}_1}$, we have $\mathcal{H}^\sigma(X_\varepsilon) > 0$. Indeed let us denote by $d := \frac{\mu_\rho^{\mathfrak{a}_1}(X_\varepsilon)}{2L_\rho}$. By definition of σ -capacity we can find an open covering $\mathcal{B} = \{B(x_i, r_i)\}$ of X_ε consisting of balls of radius $r_i < \varepsilon$ and such that

$$\sum r_i^\sigma \leq \mathcal{H}^\sigma(X_\varepsilon) + d.$$

On the other hand we have

$$\mu_\rho^{\mathfrak{a}_1}(X_\varepsilon) \leq \mu_\rho^{\mathfrak{a}_1} \left(\bigcup_i B(x_i, r_i) \right) \leq \sum_i \mu_\rho^{\mathfrak{a}_1}(B(x_i, r_i)) \leq \sum_i L_\rho r_i^\sigma.$$

This shows that $\mathcal{H}^\sigma(X_\varepsilon)$ is positive, and concludes the proof. \square

Remark 5.15. Patterson-Sullivan measures in a setup close to ours were extensively studied by Quint [46]. For our geometric applications it is crucial to have an Ahlfors regular measure of exponent $h_\rho^{\mathfrak{a}_1}$. Let us denote by G the Zariski closure of $\rho(\Gamma)$, assume that G is reductive (despite this is not always the case in the examples we have in mind), and let \mathcal{F}_G denote the full flag space associated to G . Quint [46, Theorem 8.4] provides a quasi-invariant measure μ on \mathcal{F}_G called a $(\rho(\Gamma), h_\rho^{\mathfrak{a}_1} \mathfrak{a}_1)$ -Patterson-Sullivan, with the desired transformation rule, as long as a technical condition is satisfied, namely that the form $h_\rho^{\mathfrak{a}_1} \mathfrak{a}_1$ is tangent to the growth indicator function $\psi_{\rho(\Gamma)}$. In order to guarantee that this is the case we would have to further assume that the representation ρ is $\{\mathfrak{a}_p\}$ -Anosov, and that $p_2(\rho(\gamma)) = p$ for every $\gamma \in \Gamma$. The measure μ could then be pushed forward via the projection $\mathcal{F}_G \rightarrow \mathbb{P}(\mathbb{K}^d)$ and the fact that ρ is $\{\mathfrak{a}_1, \mathfrak{a}_p\}$ -Anosov would imply that the new measure on $\mathbb{P}(\mathbb{K}^d)$ would still be quasi-invariant. However deducing the analogue of Proposition 5.12 in that setting would require some work as our representations are, in most interesting cases, not Zariski dense.

6. (p, q, r) -HYPERCONVEXITY

In this section we introduce (p, q, r) -hyperconvex representations, establish geometric properties and provide the link with local conformality.

6.1. Hyperconvex representations. The following definition is inspired from Labourie [36] for surface groups. Let Γ be a word-hyperbolic group and denote by

$$\partial^{(3)}\Gamma = \{(x, y, z) \in (\partial\Gamma)^3 : \text{pairwise distinct}\}.$$

Definition 6.1. Consider $p, q, r \in \llbracket 1, d-1 \rrbracket$ such that $p+q \leq d$. We say that a representation $\rho : \Gamma \rightarrow \mathrm{PGL}_d(\mathbb{K})$ is (p, q, r) -hyperconvex if it is $\{\mathbf{a}_p, \mathbf{a}_q, \mathbf{a}_r\}$ -Anosov and for every triple $(x, y, z) \in \partial^{(3)}\Gamma$ one has

$$(x_\rho^p \oplus y_\rho^q) \cap z_\rho^{d-r} = \{0\}.$$

Note that, since $p+q < d$ and the representation is $\{\mathbf{a}_p, \mathbf{a}_q\}$ -Anosov, the sum $x_\rho^p + y_\rho^q$ is necessarily direct. Hence, hyperconvexity implies that $p+q \leq r$. We will observe in Corollary 6.6 that ρ can only be (p, q, r) -hyperconvex if $r-p-q \geq \dim(\partial\Gamma) - 1$. Note that we do not require p and q to be different.

Proposition 6.2. *The space of (p, q, r) -hyperconvex representations is open in*

$$\mathrm{hom}(\Gamma, \mathrm{PGL}_d(\mathbb{K})).$$

Proof. The proof follows the same lines as Labourie [36, Proposition 8.2]. Since the action of Γ on $\partial^{(3)}\Gamma$ is properly discontinuous and co-compact, given a triple $(x, y, z) \in \partial^{(3)}\Gamma$ there exists $\gamma \in \Gamma$ such that the the points $\gamma x, \gamma y$ and γz are pairwise far apart. Considering a (p, q, r) -hyperconvex representation ρ , one concludes that the angles between any pair of the spaces $(\gamma x)_\rho^p, (\gamma y)_\rho^q$ and $(\gamma z)_\rho^{d-r}$ are bounded away from zero. Corollary 3.8 states that the Anosov condition is open and that equivariant maps vary continuously with the representation, hence, since the map $\partial^{(2)}\Gamma \rightarrow \mathcal{G}_{p+q}(\mathbb{K}^d)$

$$(a, b) \mapsto a_\rho^p \oplus b_\rho^q$$

is continuous away from the diagonal the result follows. \square

Since hyperconvexity is an open property, one can provide interesting examples of hyperconvex representations by looking at representations of the form $\Gamma \rightarrow G \rightarrow \mathrm{GL}_d(\mathbb{K})$, where the first arrow is convex co-compact (see Section 7.2) furthermore hyperconvexity behaves well with field extensions:

Lemma 6.3. *Let $\mathbb{K} \subset \mathbb{F}$ be a field extension, if $\rho : \Gamma \rightarrow \mathrm{PGL}_d(\mathbb{K})$ is (p, q, r) -hyperconvex then so is $\rho : \Gamma \rightarrow \mathrm{PGL}_d(\mathbb{F})$.*

We conclude the subsection providing obstructions to the existence of $(1, 1, r)$ -hyperconvex representations. A useful tool for this is the stereographic projection:

Definition 6.4. Let $\rho : \Gamma \rightarrow \mathrm{PGL}_d(\mathbb{K})$ be $\{\mathbf{a}_1, \mathbf{a}_r\}$ -Anosov. Given $z \in \partial\Gamma$, the *stereographic projection* defined by z (and ρ) is the continuous map

$$\pi_{z, \rho} : \partial\Gamma - \{z\} \rightarrow \mathbb{P}(\mathbb{K}^d / z_\rho^{d-r})$$

defined as follows: since ρ is $\{\mathbf{a}_1\}$ -Anosov, for every point $x \in \partial\Gamma$ different from z , the vector space $x_\rho^1 \oplus z_\rho^{d-r}$ has dimension $d-r+1$ and projects to a line in the quotient space $\mathbb{K}^d / z_\rho^{d-r}$; we define $\pi_{z, \rho}(x) \in \mathbb{P}(\mathbb{K}^d / z_\rho^{d-r})$ to be the projectivisation of this line.

The following is immediate from the definitions:

Lemma 6.5. *If the representation ρ is $(1, 1, r)$ -hyperconvex then for every $z \in \partial\Gamma$ the map $\pi_{z, \rho}$ is continuous and injective.*

Proof. The stereographic projection $\pi_{z, \rho}$ is the composition of the boundary map $\xi^1 : \partial\Gamma \setminus \{z\} \rightarrow \mathbb{P}(\mathbb{K}^d)$ with the projection $\mathbb{P}(\mathbb{K}^d \setminus z_\rho^{d-r}) \rightarrow \mathbb{P}(\mathbb{K}^d / z_\rho^{d-r})$, which is algebraic outside the $(d-r)$ -dimensional subspace z_ρ^{d-r} ; it is well defined as ρ is \mathfrak{a}_1 -Anosov, and is therefore continuous. Injectivity follows directly from the definition of hyperconvexity. \square

Corollary 6.6. *If there is no continuous injective map $\partial\Gamma - \{z\} \rightarrow \mathbb{P}(\mathbb{K}^r)$, then there is no $(1, 1, r)$ -hyperconvex representation $\rho : \Gamma \rightarrow \text{PGL}_d(\mathbb{K})$.*

6.2. From hyperconvexity to local conformality. We now find a link between hyperconvexity and local conformality. The following statement is the main technical result of Section 6, and will be crucial in the proof of Theorem 7.1.

Recall from Section 5.1 that we defined, for every projective Anosov representation $\rho : \Gamma \rightarrow \text{PGL}_d(\mathbb{K})$, the thickened cone type at infinity $X_\infty(\alpha)$ as the intersection of the $\delta_\rho/2$ -neighbourhood of $\xi_\rho^1(\mathcal{C}_\infty(\alpha))$ with the image of the boundary map. In a similar way, if ρ is $\{\mathfrak{a}_\rho\}$ -Anosov, we set

$$X_\infty^p(\alpha) := \mathcal{N}_{\delta_\rho/2} \xi_\rho^p(\mathcal{C}_\infty(\alpha)) \cap \xi_\rho^p(\partial\Gamma),$$

where δ_ρ^p is as estimated in Proposition 3.3.

Proposition 6.7. *Let $\rho : \Gamma \rightarrow \text{PGL}_d(\mathbb{K})$ be (p, q, r) -hyperconvex. Then there exist constants L, ε such that for every $\alpha \in \Gamma$ with $|\alpha| > L$, for every $x \in \mathcal{C}_\infty(\alpha)$ and every $y \in (\xi^q)^{-1}X_\infty^q(\alpha)$, it holds*

$$\sin \sphericalangle(x^p \oplus y^q, U_{d-r}(\rho(\alpha^{-1}))) > \varepsilon. \quad (12)$$

Proof. Choose an auxiliary metric d on $\partial\Gamma$ inducing the correct topology, and observe that there exists ν_0 such that, if x, z are endpoints of a geodesic $\{\alpha_i\}_{i \in \mathbb{Z}}$ through the origin, for every $y \in \partial\Gamma \setminus \{x, z\}$ there exists $n \in \mathbb{Z}$ such that

$$\begin{aligned} d(\alpha_n^{-1}x, \alpha_n^{-1}z) &> \nu_0, \\ d(\alpha_n^{-1}z, \alpha_n^{-1}y) &> \nu_0, \\ d(\alpha_n^{-1}x, \alpha_n^{-1}y) &> \nu_0; \end{aligned}$$

in this case we say that the triple $\alpha_n^{-1}(x, y, z)$ is ν_0 -separated. Indeed we can for example choose the distance d induced by our chosen distance on $\mathcal{G}_q(\mathbb{K}^d)$ through the boundary map ξ_ρ^q , the fact that there is a lower bound on the values $d(\alpha_n^{-1}x, \alpha_n^{-1}z)$ follows from Lemma 3.9, and the fact that we can find a suitable n , such that both $d(\alpha_n^{-1}z, \alpha_n^{-1}y) > \nu_0$ and $d(\alpha_n^{-1}x, \alpha_n^{-1}y) > \nu_0$ is a consequence of Lemma 3.10 combined with the fact that the action of the images of the generators on $\mathcal{G}_q(\mathbb{K}^d)$ is uniformly Lipschitz. Observe that it follows from the north-south dynamics of the action of Γ on $\partial\Gamma$ that the set of n for which the triple is ν_0 -separated is coarsely an interval of uniformly bounded length.

As the representation $\rho : \Gamma \rightarrow \text{PGL}_d(\mathbb{K})$ is (p, q, r) -hyperconvex we can find ε_0 such that if the triple (s, w, t) is ν_0 -separated one has

$$\sin \sphericalangle(s_\rho^p \oplus t_\rho^q, w_\rho^{d-r}) > \varepsilon_0 :$$

this is guaranteed since the set of ν_0 -separated triples is compact as the group is hyperbolic.

Let us first show that if y is close enough to x (depending on ε_0 , as well as the representation ρ), we can find ε_1, L_1 for which Equation (12) holds.

In order to do so, observe that there exists K depending on Γ only such that, if $x \in \mathcal{C}_\infty(\alpha)$, there exists a bi-infinite geodesic $\{\alpha_i\}_{i \in \mathbb{Z}}$ and an integer M such that $d(\alpha_{-M}, \alpha^{-1}) < K$; of course in this case $||\alpha| - M| < K$. We denote by z be the second endpoint of such geodesic.

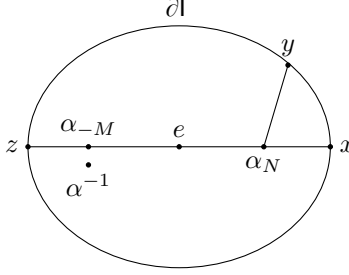


FIGURE 5. The first step in the proof of Proposition 6.7

We furthermore choose $N \in \mathbb{N}$ such that $\alpha_N^{-1}(x, y, z)$ is ν_0 -separated. The size of N measures how close y is to x . Using the triangular inequality we get

$$\sin \angle(x^p \oplus y^q, U_{d-r}(\rho(\alpha^{-1}))) \geq \sin \angle(U_r(\rho(\alpha_N)), U_{d-r}(\rho(\alpha_{-M})))$$

$$-d(U_{d-r}(\rho(\alpha_{-M})), U_{d-r}(\rho(\alpha))) - d((\rho(\alpha_N)((\alpha_N^{-1}x)^p \oplus (\alpha_N^{-1}y)^q), U_r(\rho(\alpha_N)))$$

the first term of the expression is bigger than δ_ρ^r provided $|\alpha|$ is big enough by Lemma 3.3, the second term is smaller than $\delta_\rho^r/3$ if $|\alpha|$ is big enough by Lemma 2.3 (2): indeed $\alpha^{-1} = \alpha_{-M}a$ for some $a \in \Gamma$ with $|a| < K$. We chose L_1 so that these two conditions are satisfied. In order to prove our claim it is enough to verify that we can find N_0 big enough, depending on the representation only, such that for every $N \geq N_0$, it holds

$$d(\rho(\alpha_N)((\alpha_N^{-1}x)^p \oplus (\alpha_N^{-1}y)^q), U_r(\rho(\alpha_N))) < \delta_\rho/3.$$

Since z_ρ^{d-r} and x_ρ^r are far apart and $U_r(\rho(\alpha_N)) \rightarrow x_\rho^r$ as $N \rightarrow \infty$ uniformly in N , the angle between z_ρ^{d-r} and $U_r(\rho(\alpha_N))$ is bounded below for all positive big enough N depending only on the representation ρ ; using Lemma 2.5 we deduce that

$$d(U_{d-r}(\rho(\alpha_N^{-1})), \rho(\alpha_N^{-1})z_\rho^{d-r}) \leq \frac{\sigma_{d-r+1}(\rho(\alpha_N^{-1}))}{\sigma_{d-r}} \frac{1}{\sin \angle(z_\rho^{d-r}, U_r(\rho(\alpha_N)))}$$

and, since the representation is $\{\mathbf{a}_{d-r}\}$ -Anosov, the latter quantity is smaller than $\varepsilon_0/2$ for big enough positive N . Thus

$$d(\rho(\alpha_N)((\alpha_N^{-1}x)^p \oplus (\alpha_N^{-1}y)^q), U_r(\rho(\alpha_N))) < \frac{\sigma_{r+1}(\rho(\alpha_N))}{\sigma_r} \frac{2}{\varepsilon_0},$$

by Lemma 2.5. This concludes the first step, we can chose $\varepsilon_1 = \delta_\rho^r/3$, but the required N might be big.

We are thus left to verify that, up to possibly shrinking ε_1 and enlarging L_1 , Equation (12) is also verified in case the n for which $\alpha_n^{-1}(x, y, z)$ is ν_0 -far is smaller than N . Observe that, since the group Γ is finitely generated, and the contraction ratio of every generator is bounded, we can find C , depending on ρ , such that $d(\alpha_n^{-1}x, \alpha_n^{-1}y) \leq C^n d(x, y)$, and therefore we can find ν_1 depending on N only

such that $d(y, x) > \nu_1$. Since furthermore $y_\rho^q \in X_\rho^q(\alpha)$, and thus we have a lower bound on $d(y, z)$, we deduce, up to further shrinking ν_1 , that the triple (x, y, z) is ν_1 -far. The same argument as above let us deduce that there exists ε_2 such that

$$\sin \sphericalangle(x_\rho^p \oplus y_\rho^q, z_\rho^{d-r}) > \varepsilon_2.$$

It is then enough to chose L_2 big enough so that $d(z_\rho^{d-r}, U_{d-r}(\alpha^{-1})) < \varepsilon_2/2$. The lemma holds with $L = \max\{L_1, L_2\}$ and $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. \square

Proposition 6.7 combined with Theorem 5.14 yields the following Hausdorff dimension computations.

Corollary 6.8. *Let $\rho : \Gamma \rightarrow \mathrm{PGL}_d(\mathbb{K})$ be $(1, 1, 2)$ -hyperconvex, then*

$$\mathrm{Hff}(\xi^1(\partial\Gamma)) = h_\rho^{a_1}.$$

Corollary 6.9. *Let $\rho : \Gamma \rightarrow \mathrm{PGL}_d(\mathbb{K})$ be $(1, 1, r)$ -hyperconvex. Assume moreover that for every $\gamma \in \Gamma$ one has $\sigma_2(\rho(\gamma)) = \sigma_r(\rho(\gamma))$, then every point of $\partial\Gamma$ is locally conformal for ρ and thus*

$$h_\rho^{a_1} = \mathrm{Hff}(\xi^1(\partial\Gamma)).$$

6.3. Examples: (ir)reducible SL_2 . The easiest examples of hyperconvex representations are induced from representations of $\mathrm{SL}_2(\mathbb{K})$ (see for example Humphreys's book [29] for standard basic facts on the representation theory of SL_2).

Recall that for every $d \in \mathbb{N} - \{0, 1\}$ there is a (unique up to conjugation) irreducible representation $\iota_d : \mathrm{SL}_2(\mathbb{K}) \rightarrow \mathrm{SL}_d(\mathbb{K})$. This representation is given by the action of $\mathrm{SL}_2(\mathbb{K})$ on the symmetric powers $\mathcal{S}^{d-1}(\mathbb{K}^2)$, which can be identified with the space of homogenous polynomials on two variables of degree $d - 1$ with coefficients in \mathbb{K} . If we denote by $\mathbf{E}_{\mathrm{SL}_2(\mathbb{K})}^*$ the weight space, the representation ι_d has highest weight $\chi_{\iota_d} \in \mathbf{E}_{\mathrm{SL}_2(\mathbb{K})}^*$ given by $\chi_{\iota_d}(x) = (d - 1)x$.

Let $\mathcal{F}(\mathcal{S}^{d-1}(\mathbb{K}^2))$ denote the full flag space associated to $\mathrm{SL}(\mathcal{S}^{d-1}(\mathbb{K}^2))$. The Veronese map $\zeta : \mathbb{P}(\mathbb{K}^2) \rightarrow \mathcal{F}(\mathcal{S}^{d-1}(\mathbb{K}^2))$ is defined by

$$\zeta(x) = \{\zeta^k(x)\}_{k=1}^{d-1}$$

where $\zeta^k(\ell)$ is the k -dimensional vector subspace of $\mathcal{S}^{d-1}(\mathbb{K}^2)$ consisting of polynomials that have x^{d-k} as a factor. It is easy to check that ζ is ι_d -equivariant and the image of an attractor in $\mathbb{P}(\mathbb{K}^2)$ is an attractor in $\mathcal{F}(\mathcal{S}^{d-1}(\mathbb{K}^2))$.

Remark 6.10. Note that for every pair of distinct points $x \neq y$ in $\mathbb{P}(\mathbb{K}^2)$ the flags $\zeta(x)$ and $\zeta(y)$ are in general position, i.e. for every $k \in \llbracket 1, d - 1 \rrbracket$, it holds $\zeta^k(x) \cap \zeta^{d-k}(y) = \{0\}$.

Moreover, using the transitivity of the $\mathrm{SL}_2(\mathbb{K})$ -action on transverse pairs, it is easy to check the following:

Proposition 6.11. *Let $\zeta = \{\zeta^i\}_{i=1}^{d-1}$ be the Veronese embedding of $\mathbb{P}(\mathbb{K}^2)$ into $\mathcal{F}(\mathcal{S}_{d-1}(\mathbb{K}^2))$, then for every triple $p + q + r = d$ and pairwise distinct $x, y, z \in \mathbb{P}(\mathbb{K}^2)$ one has*

$$\zeta^p(x) \oplus \zeta^q(y) \oplus \zeta^r(z) = \mathbb{K}^d.$$

Corollary 6.12. *For every convex cocompact⁶ subgroup $\Gamma < \mathrm{SL}_2(\mathbb{K})$, the representation $\iota_d|_\Gamma : \Gamma \rightarrow \mathrm{SL}_d(\mathbb{K})$ is (p, q, r) -hyperconvex for every (p, q, r) such that $r \geq p + q$. The same holds for small deformations.*

⁶For non-Archimedean fields \mathbb{K} , in analogy with the Archimedean case, we say that a representation is *convex cocompact* if it is Anosov, as in Definition 3.1.

We can obtain many more examples of hyperconvex representations by considering direct sums of irreducible representations. A representation $\pi : \mathrm{SL}_2(\mathbb{K}) \rightarrow \mathrm{SL}(V, \mathbb{K})$ decomposes in irreducible modules

$$\pi = \bigoplus_1^k \iota_{d_i},$$

where we have ordered $d_1 \geq \dots \geq d_k$. The highest weight $\chi_\pi \in \mathbf{E}_{\mathrm{SL}_2(\mathbb{K})}^*$ is $\chi_\pi(x) = (d_1 - 1)x$. Let us denote by

$$\chi_\pi^{(2)} \geq \dots \geq \chi_\pi^{(\dim V)}$$

the remaining weights in decreasing order.

Definition 6.13. Given $k \in \llbracket 2, \dim V \rrbracket$, we say that π is *k-coherent* if $\chi_\pi^{(k)} > d_2 - 1$, equivalently if $d_1 > d_2 + 2(k - 1)$.

An important example of 2-coherent representations are exterior powers:

Example 6.14. For every $p \in \llbracket 1, d - 1 \rrbracket$ the representation

$$\wedge^p \iota_d : \mathrm{SL}_2(\mathbb{K}) \rightarrow \mathrm{SL}(\wedge^p \mathbb{K}^d)$$

is 2-coherent.

Proof. Considering a diagonalizable element in $\mathrm{SL}_2(\mathbb{K})$ one explicitly checks that the top 3 weights of $\wedge^p \iota_d$ are

$$\begin{aligned} - \chi_{\wedge^p \iota_d} &= d - 1 + \dots + d - 1 - 2p = p(d - p), \\ - \chi_{\wedge^p \iota_d}^{(2)} &= \chi_{\wedge^p \iota_d} - 2, \\ - \chi_{\wedge^p \iota_d}^{(3)} &= \chi_{\wedge^p \iota_d}^{(4)} = \chi_{\wedge^p \iota_d} - 4. \end{aligned}$$

□

Observe that a representation π is *k-coherent* if and only if the top k eigenlines of a diagonalizable element in $\pi(\mathrm{SL}_2(\mathbb{K}))$ belong to the top irreducible factor. Therefore Definition 6.13 guarantees some hyperconvexity:

Proposition 6.15. *Let $\rho : \Gamma \rightarrow \mathrm{SL}_2(\mathbb{K})$ be convex co-compact. If $\pi : \mathrm{SL}_2(\mathbb{K}) \rightarrow \mathrm{SL}(V, \mathbb{K})$ is k-coherent, then $\pi \circ \rho$ is (p, q, k) -hyperconvex for every p, q with $p + q \leq k$.*

Proof. Since $\chi^{(k)} > d_2 - 1$ one has that $\chi^{(k)} > \chi^{(k+1)}$ and thus $\pi \circ \rho$ is $\{\mathbf{a}_k\}$ -Anosov. Coherence implies thus that $\chi^{(l)} > \chi^{(l+1)}$ for every $l \in \llbracket 1, k \rrbracket$ and thus $\pi \circ \rho$ is also $\{\mathbf{a}_p, \mathbf{a}_q\}$ -Anosov since both p and q are smaller than k . The remainder of the statement follows from Lemma 6.11: if N denotes the dimension of V , $\zeta_{d_1}^p : \partial\Gamma \rightarrow \mathcal{G}_p(\mathcal{S}^{d-1}(\mathbb{K}^2)) \subset \mathcal{G}_p(V)$ is the ι_d -equivariant map induced by ζ , and $\xi_\pi^l : \partial\Gamma \rightarrow \mathcal{G}_l(V)$ denotes the boundary map associated to π , we have, for every $l \leq k$, that $\xi_\pi^l = \zeta_{d_1}^l$ and

$$\xi_\pi^{N-l} = \zeta_{d_1}^{d_1-l} \oplus \bigoplus_{i=2}^k \mathcal{S}^{d_i-1}(\mathbb{K}^2).$$

□

In particular Proposition 6.15 can be used to construct example of representations of Kleinian groups satisfying the assumptions of Theorem 5.14.

7. DIFFERENTIABILITY PROPERTIES

7.1. Convergence on pairs and bounds on the Hausdorff dimension. The following result, which follows from Proposition 6.7 is inspired by Guichard [26, Proposition 21], however, Guichard's proof relies heavily on the fact that $\partial\Gamma$ is a circle, and that the representation is (p, q, r) -hyperconvex for *every* triple p, q, r with $p + q = r$.

Theorem 7.1. *Let $\rho : \Gamma \rightarrow \mathrm{PGL}_d(\mathbb{K})$ be (p, q, r) -hyperconvex then for every $(w, y) \in \partial^{(2)}\Gamma$ one has*

$$\lim_{(w,y) \rightarrow (x,x)} d(w_\rho^p \oplus y_\rho^q, x_\rho^r) = 0.$$

More precisely there exist constants C, μ such that, if $\{\alpha_i\}$ is a geodesic ray with endpoint x , for every $w, y \in \alpha_i \mathcal{C}_\infty(\alpha_i)$ it holds

$$d(w_\rho^p \oplus y_\rho^q, x_\rho^r) \leq C e^{-\mu i}.$$

Proof. The first claim is a direct consequence of the second, as the sets $\alpha_i \mathcal{C}_\infty(\alpha_i)$ form a fundamental system of neighbourhoods of the point x .

As the representation ρ is (p, q, r) -hyperconvex, and $w, y \in \alpha_i \mathcal{C}_\infty(\alpha_i)$, we deduce from Proposition 6.7 that

$$\sin \angle((\alpha_i^{-1} w)_\rho^p \oplus (\alpha_i^{-1} y)_\rho^q, U_{d-r}(\rho(\alpha_i^{-1}))) > \varepsilon.$$

In particular Lemma 2.5 implies

$$d(w_\rho^p \oplus y_\rho^q, U_r(\alpha_i)) \leq \frac{\sigma_{r+1}(\rho(\alpha_i))}{\sigma_r} \frac{1}{\varepsilon} \leq \frac{C_1}{\varepsilon} e^{-\mu_1 i}.$$

Where C_1, μ_1 are the constants provided by the fact that ρ is $\{\mathbf{a}_r\}$ -Anosov. The result now follows, via triangular inequality, from Lemma 3.6, which guarantees that

$$d(x_\rho^r, U_r(\alpha_i)) \leq C_2 e^{-\mu_2 i}.$$

□

The following easy converse is useful for applications:

Proposition 7.2. *Consider $p, q, r \in \llbracket 1, d-1 \rrbracket$ with $p + q \leq r$. If $\rho : \Gamma \rightarrow \mathrm{PGL}_d(\mathbb{K})$ is $\{\mathbf{a}_p, \mathbf{a}_q, \mathbf{a}_r\}$ -Anosov and for every $x \in \partial\Gamma$ one has*

$$\lim_{(w,y) \rightarrow (x,x)} d(w_\rho^p \oplus y_\rho^q, x_\rho^r) = 0, \quad (13)$$

then ρ is (p, q, r) -hyperconvex.

Proof. Since ρ is $\{\mathbf{a}_p, \mathbf{a}_q\}$ -Anosov and $p + q \leq r \leq d-1$, for every pair of distinct points w, y the sum $w_\rho^p + y_\rho^q$ is direct. Since ρ is $\{\mathbf{a}_r\}$ -Anosov there is a lower bound on $\sin \angle(x_\rho^r, z_\rho^{d-r})$ if x, z are the endpoints of a geodesic through the origin. Combining this fact with (13) we can find ε, δ such that

$$(x_\rho^p \oplus y_\rho^q) \cap z_\rho^{d-r} = \{0\}$$

for every triple with $d(x, y) < \varepsilon$ and $d(x, z) > \delta > \varepsilon$. Any triple in $\partial^{(3)}\Gamma$ can be transformed in such a triple by an element of Γ and thus the claim follows. □

Using the stereographic projection (see Definition 6.4) combined with Theorem 7.1 it is possible to deduce the following estimate on Hausdorff dimension:

Proposition 7.3. *Let $\rho : \Gamma \rightarrow \mathrm{PGL}_d(\mathbb{K})$ be $(1, 1, r)$ -hyperconvex, then*

$$\mathrm{Hff}(\xi^1(\partial\Gamma)) \leq \mathrm{Hff}(\mathbb{P}(\mathbb{K}^r)).$$

Proof. The Hausdorff dimension of a compact set is the maximum of the Hausdorff dimension of the sets in a finite open cover. It is thus enough to verify that if $\rho : \Gamma \rightarrow \mathrm{PGL}_d(\mathbb{K})$ is $(1, 1, r)$ -hyperconvex, then the stereographic projection $\pi_{z, \rho}$ is Lipschitz on a sufficiently small open neighbourhood of a point x . But we know that as x and z are far and ρ is $\{\mathbf{a}_r\}$ -Anosov, the subspaces x_ρ^r and z_ρ^{d-r} make a definite angle. The result is then a consequence of Theorem 7.1: Indeed, it implies we can find an open neighbourhood \mathcal{U} of x such that for every pair $w, y \in \mathcal{U}$ the angle that $w_\rho^1 \oplus y_\rho^1$ makes with z_ρ^{d-r} is bigger than a fixed constant. This is enough to guarantee that the stereographic projection doesn't distort distances too much. \square

7.2. When $\partial\Gamma$ is a manifold and $\mathbb{K} = \mathbb{R}$. A classical result of Benoist [3] states that if a word hyperbolic group of projective transformations divides a convex set, then the boundary of this set has to be C^1 . These, together with Hitchin representations, have become the paradigm of Zariski-dense projective Anosov representations whose limit set is a regular manifold. The purpose of this section is to provide new examples of such phenomena. Sharper results of similar nature have recently been obtained independently by Zhang-Zimmer [54].

We begin by observing that Theorem 7.1 has the following interesting consequence.

Proposition 7.4. *Let $\rho : \Gamma \rightarrow \mathrm{PGL}_d(\mathbb{R})$ be a $(1, 1, r)$ -hyperconvex representation and assume that $\partial\Gamma$ is topologically a sphere of dimension $r - 1$, then $\xi_\rho^1(\partial\Gamma)$ is a C^1 manifold with $\mathbb{T}_{x_\rho^1} \xi_\rho^1(\partial\Gamma) = \mathbb{T}_{x_\rho^1} \mathbb{P}(x_\rho^r)$.*

Proof. Theorem 7.1 implies that the set $\xi_\rho^1(\partial\Gamma)$ is differentiable at x_ρ^1 with tangent space $\mathbb{T}_{x_\rho^1} \mathbb{P}(x_\rho^r)$. The continuity of $x \mapsto x_\rho^r$ completes the proof. \square

Proposition 7.4 can be applied to many different situations to produce interesting examples through the *represent and deform* method, we now explain how this works in a specific situation. Denote by

$$S^k : \mathrm{PGL}_{d+1}(\mathbb{R}) \rightarrow \mathrm{PGL}(S^k(\mathbb{R}^{d+1}))$$

the k -symmetric power.

Note that in $\mathrm{PGL}_{d+1}(\mathbb{K})$ a $(1, 1, d)$ -hyperconvex representation is a projective Anosov representation ρ such that for each triple $(x, y, z) \in \partial^{(2)}\Gamma$ the sum $x_\rho^1 + y_\rho^1 + z_\rho^1$ is direct.

Proposition 7.5. *Let $\rho : \Gamma \rightarrow \mathrm{PGL}_{d+1}(\mathbb{R})$ be a $(1, 1, d)$ -hyperconvex representation and assume that there exist $c > 0, \mu > 1$ such that, for every $\gamma \in \Gamma$,*

$$\frac{\sigma_1(\rho(\gamma))\sigma_d(\rho(\gamma))}{\sigma_2(\rho(\gamma))^2} > ce^{\mu|\gamma|}. \quad (14)$$

Then the composition

$$S^k \circ \rho : \Gamma \rightarrow \mathrm{PGL}(S^k(\mathbb{R}^{d+1}))$$

is $(1, 1, d)$ -hyperconvex.

Proof. We endow $S^k(\mathbb{R}^{d+1})$ with the norm induced by our choice of norm on \mathbb{R}^{d+1} . For this choice, and for every $g \in \mathrm{PGL}_{d+1}(\mathbb{R})$, the semi-homotecy ratios of $S^k g$ are just the products of k -tuples of semihomotecy ratios of g . Assumption (14) then gives

$$\begin{aligned} - \mathbf{a}_1(\nu(S^k g)) &= \mathbf{a}_1(\nu(g)), \\ - \mathbf{a}_d(\nu(S^k g)) &= \min\{\mathbf{a}_d(\nu(g)), \log \frac{\sigma_1(\rho(g))\sigma_d(\rho(g))}{\sigma_2(\rho(g))^2}\} \end{aligned}$$

Since ρ is $\{\mathbf{a}_1, \mathbf{a}_d\}$ -Anosov, we deduce from Definition 3.1 that $S^k \rho$ is also $\{\mathbf{a}_1, \mathbf{a}_d\}$ -Anosov.

Observe that the map S^k is equivariant with respect to the map between the partial flags

$$S^k : \{\text{line} \subset \text{hyperplane}\} \rightarrow \{\text{line} \subset d\text{-dimensional subspace}\}$$

defined by

$$S^k(l, H) = (l^{\odot k}, l^{\odot k-1} \odot H).$$

Here we denote by \odot the symmetric tensors.

It is immediate to verify that Assumption (14) also implies that $S^k \circ \xi$ sends attractors to attractors, therefore, by continuity of $S^k \circ \xi$, we have, for every $x \in \partial\Gamma$,

$$S^k(x_\rho^1, x_\rho^d) = (x_{S^k \rho}^1, x_{S^k \rho}^d).$$

Finally, the convergence property (Theorem 7.1) for ρ , together with the differentiability of $S^2 : \mathbb{P}(\mathbb{R}^{d+1}) \rightarrow \mathbb{P}(S^2(\mathbb{R}^{d+1}))$ implies that

$$\lim_{(w,y) \rightarrow (x,x)} \not\prec (w_{S^2 \rho}^1 \oplus y_{S^2 \rho}^1, x_{S^2 \rho}^d) = 0.$$

Proposition 7.2 yields the result. \square

As a direct corollary we get:

Corollary 7.6. *If $\rho : \Gamma \rightarrow \mathrm{PSO}(d, 1)$ is cocompact, every small deformation*

$$\eta : \Gamma \rightarrow \mathrm{PGL}(S^k(\mathbb{R}^{d+1}))$$

of $S^k \rho$ is $(1, 1, d)$ -hyperconvex. Any such η will have a C^1 -sphere as limit set in $\mathbb{P}(S^k(\mathbb{R}^{d+1}))$.

Applying Johnson-Millson's [30] bending technique we obtain the announced Zariski dense subgroups whose limit set is a C^1 sphere:

Corollary 7.7. *There exists a Zariski dense subgroup $\Gamma < \mathrm{PGL}(S^2(\mathbb{R}^{d+1}))$ whose limit set is a C^1 sphere.*

Proof. Let M be a d -dimensional closed hyperbolic manifold that has a totally geodesic, co-dimension one, closed submanifold N . The inclusion $\Gamma = \pi_1 M \subset \mathrm{SO}(d, 1) \rightarrow \mathrm{SL}(S^2(\mathbb{R}^{d+1}))$ satisfies the hypothesis of Proposition 7.5. Without loss of generality we can assume that $\pi_1 N \subset \mathrm{SO}(d-1, 1)$. Observe that the centralizer of

$$S^2(\mathrm{SO}(d-1, 1)) \subset \mathrm{SL}(S^2(\mathbb{R}^{d+1}))$$

is non-trivial and strictly contains that of $S^2(\mathrm{SO}(d, 1))$: as an $S^2(\mathrm{SO}(d, 1))$ -module, $S^2(\mathbb{R}^{d+1})$ splits as a direct sum of an irreducible representation (usually denoted $\mathbb{S}_{[2]}(\mathbb{R}^{d,1})$) and a trivial representation, its centralizer is thus reduced to \mathbb{R}^* . The decomposition as a $S^2(\mathrm{SO}(d-1, 1))$ -module splits as the sum $\mathbb{S}_{[2]}(\mathbb{R}^{d-1,1}) \oplus \mathbb{R}^{d-1,1} \oplus \mathbb{R}^2$ where the action on the second factor is the standard action, while the action on

\mathbb{R}^2 is trivial. In particular the centralizer of $S^2(\mathrm{SO}(n-1, 1))$ is $\mathrm{GL}(2, \mathbb{R}) \times \mathbb{R}^*$. By bending the representation along N with a nontrivial element in $\mathrm{GL}(2, \mathbb{R})$ which doesn't leave invariant the factor \mathbb{R} , we obtain the desired representation. \square

8. EXAMPLES OF LOCALLY CONFORMAL REPRESENTATIONS

The purpose of this section is to discuss some of the many examples in which restricting the Zariski closure of a representation to a non-split real form of $\mathrm{SL}_d(\mathbb{K})$ gives room for $(1, 1, p)$ hyperconvex representations for which we can also guarantee that the second gap p_2 is strictly bigger than 2.

8.1. Hyperconvex representations in $\mathrm{PU}(1, d)$ and $\mathrm{PSp}(1, d)$. The first interesting setting in which Theorem 5.14 applies for large classes of representations is given by considering representations in the rank one groups $\mathrm{PU}(1, d)$ or $\mathrm{PSp}(1, d)$. To unify the treatment we will write $\mathrm{PO}_{\mathbb{K}}(1, d)$ for either $\mathrm{PU}(1, d)$ if $\mathbb{K} = \mathbb{C}$ or $\mathrm{PSp}(1, d)$ if $\mathbb{K} = \mathbb{H}$ and regard $\mathrm{PO}_{\mathbb{K}}(1, d)$ as a subgroup of $\mathrm{PGL}(d+1, \mathbb{K})$.

Remark 8.1. Unfortunately, as \mathbb{H} is non-commutative, we don't have the setup of Section 2.2.1 at our disposal (as the exterior algebra over a non-commutative field is not well defined), however the issue can be easily solved by considering $\mathrm{SL}(d+1, \mathbb{H})$ as a subgroup of $\mathrm{SL}(2d+2, \mathbb{C})$. Given an element $g \in \mathrm{SL}(d+1, \mathbb{H})$ we denote by $g^{\mathbb{C}}$ the corresponding element in $\mathrm{SL}(2d+2, \mathbb{C})$; it is then immediate to verify that we can choose a Cartan decomposition of $g^{\mathbb{C}}$ so that, for every p , the subspace $U_{2p}(g^{\mathbb{C}})$ is a quaternionic vector space, and we thus set $U_p(g) := U_{2p}(g^{\mathbb{C}})$. Similarly we say that a sequence $(\alpha_i)_{i \in \mathbb{Z}}$ in $\mathrm{SL}(d+1, \mathbb{H})$ is p -dominated if $(\alpha_i^{\mathbb{C}})_{i \in \mathbb{Z}}$ is $2p$ -dominated in $\mathrm{SL}(2d+2, \mathbb{C})$, and that a representation $\rho : \Gamma \rightarrow \mathrm{SL}(d+1, \mathbb{H})$ is (p, q, r) -hyperconvex if the induced representation $\rho : \Gamma \rightarrow \mathrm{SL}(2d+2, \mathbb{C})$ is $(2p, 2q, 2r)$ -hyperconvex. With this at hand it is easy to verify that Theorem 5.14 holds for representations with values in $\mathrm{SL}(d+1, \mathbb{H})$.

Recall that $\mathrm{PO}_{\mathbb{K}}(1, d)$ has rank one, therefore we have at our disposal a good notion of convex co-compactness: a representation $\rho : \Gamma \rightarrow \mathrm{PO}_{\mathbb{K}}(1, d)$ is convex co-compact if and only if there is a convex $\rho(\Gamma)$ -invariant subspace of $\mathbb{H}_{\mathbb{K}}^d$ whose quotient is compact. The induced representation $\rho : \Gamma \rightarrow \mathrm{PGL}(d+1, \mathbb{K})$ is $\{\mathbf{a}_1\}$ -Anosov if and only if ρ is convex co-compact.

Observe that $\mathrm{PO}_{\mathbb{K}}(1, d)$ preserves the closed codimension 1 submanifold $\partial\mathbb{H}_{\mathbb{K}}^d \subset \mathbb{P}(\mathbb{K}^{d+1})$, furthermore one has the following.

Lemma 8.2. *For every $g \in \mathrm{PO}_{\mathbb{K}}(1, d)$, we have $U_1(g) \in \partial\mathbb{H}_{\mathbb{K}}^d \subset \mathbb{P}(\mathbb{K}^{d+1})$ and $U_d(g) = U_1(g)^{\perp}$, where the orthogonal is defined with respect to the Hermitian form defining the group $\mathrm{PO}_{\mathbb{K}}(1, d)$.*

In particular, considering for every point $x \in \partial\mathbb{H}_{\mathbb{K}}^d$ the subspace $x^{\perp} \subset \mathbb{T}_x\partial\mathbb{H}_{\mathbb{K}}^d$, one obtains a non-integrable distribution that has (real) codimension 1 if $\mathbb{K} = \mathbb{C}$ and 3 if $\mathbb{K} = \mathbb{H}$. In the complex case this is the standard contact structure on the sphere. We will refer to this distribution also in the quaternionic case as the *generalized contact distribution*. Given a distinct pair $x, y \in \partial\Gamma$ we will denote by $\mathcal{C}_{x,y}$ the intersection $\mathbb{P}(\langle x, y \rangle) \cap \partial\mathbb{H}_{\mathbb{K}}^d$. Of course if \mathbb{K} is \mathbb{C} then $\mathcal{C}_{x,y}$ is a circle, while if $\mathbb{K} = \mathbb{H}$ it is a 3-sphere. In the complex case the sets $\mathcal{C}_{x,y}$ are often referred to as *chains*, and their geometry was extensively studied by Cartan. The incidence geometry of chains (and of suitable generalizations) played an important role in Burger-Iozzi [13] and P. [43].

With these definition at hand we can rephrase our main results in the rank 1 setting:

Proposition 8.3. *A convex cocompact action $\rho : \Gamma \rightarrow \mathrm{PO}_{\mathbb{K}}(1, d)$ is $(1, 1, d)$ -hyperconvex if and only if for every distinct pair $x, y \in \partial\Gamma$, the chain $\mathcal{C}_{x_\rho^1, y_\rho^1}$ intersects $\xi(\partial\Gamma)$ only in x_ρ^1, y_ρ^1 . In this case $\mathrm{LC}(\rho) = \xi(\partial\Gamma)$, and $\xi(\partial\Gamma)$ is tangent to the generalized contact distribution.*

Proof. The first statement follows directly from the definitions: for every triple x, y, z the sum $x_\rho^1 + y_\rho^1 + z_\rho^1$ is direct if and only if z_ρ^1 doesn't belong to $\mathcal{C}_{x_\rho^1, y_\rho^1}$. The second statement follows then from Proposition 6.7, and the last is a consequence of Theorem 7.1. \square

There are many interesting examples of representations satisfying the assumption of Proposition 8.3, a natural class of examples can be obtained deforming totally real embeddings.

Lemma 8.4. *Let $\Gamma < \mathrm{PO}_{\mathbb{R}}(1, d)$ be a convex cocompact subgroup and let $\rho : \Gamma \rightarrow \mathrm{PO}_{\mathbb{K}}(1, d)$ be obtained extending the coefficients. Then ρ is $(1, 1, d)$ -hyperconvex.*

Corollary 8.5. *Every $\{\mathbf{a}_1\}$ -Anosov representation $\beta : \Gamma \rightarrow \mathrm{PO}_{\mathbb{K}}(1, d)$ sufficiently close to a totally real representation ρ is $(1, 1, d)$ -hyperconvex. In particular for each such representation*

$$\dim_{\mathrm{Hff}}(\xi(\partial\Gamma)) = h_\beta^{a_1} \leq (d-1) \dim \mathbb{K}.$$

Proof. The first statement is a direct consequence of Propositions 6.2 and 8.3. Furthermore we know that for every element $g \in \mathrm{PO}_{\mathbb{K}}(1, d)$, we have $p_2(g) = d$, and hence every point in $\partial\Gamma$ is locally conformal for β . Theorem 5.14 then applies and gives the second statement. \square

Another class of examples was studied by Dufloux in his thesis [19, 20]. He says that a Schottky subgroup $\Gamma < \mathrm{PU}(1, d)$ generated by a symmetric set W is *well positioned* if, for every $w \in W$ there is an open subsets $B(w) \subset \partial\mathbb{H}_{\mathbb{C}}^d$ such that

- the closures $\overline{B(w)}$ are pairwise disjoint;
- $w(\partial\mathbb{H}_{\mathbb{C}}^d \setminus B(w^{-1})) \subset B(w)$;
- no chain passes through three of these open subsets $B(w)$.

Similarly one can define well positioned Schottky subgroups of $\mathrm{PSp}(1, d)$ replacing chains with quaternionic three spheres (recall that in $\partial\mathbb{H}_{\mathbb{H}}^d$ any pair of points uniquely determines a 3 sphere, the boundary of a totally geodesic copy of $\mathbb{H}_{\mathbb{H}}^1$). We will denote also these subspaces of $\partial\mathbb{H}_{\mathbb{H}}^d$ *chains* for notational ease.

Arguments analogue to the ones presented in [20, Section 7.2] imply that well positioned Schottky groups are hyperconvex representations:

Proposition 8.6. *Let $\rho : \Gamma \rightarrow \mathrm{PO}_{\mathbb{K}}(1, d)$ be a well positioned Schottky subgroup. Then $\rho : \Gamma \rightarrow \mathrm{SL}_{d+1}(\mathbb{K})$ is $(1, 1, d)$ -hyperconvex. Furthermore $\mathrm{LC}(\rho) = \xi(\partial\Gamma)$.*

Proof. Observe that since $\mathrm{PO}_{\mathbb{K}}(1, d)$ is a rank one group, $p_2(\alpha)$ doesn't depend on i . Furthermore, as soon as the sequence $\{\alpha_i\}_{i=1}^\infty$ forms a geodesic ray, the sequence is d dominated by a classical ping pong argument, and it follows from Lemma 8.2 that $E_{p_2}^\rho(x) = x^\perp \subset \mathbb{P}(\mathbb{K}^{d+1})$.

In order to verify that every point $x \in \partial\Gamma$ is locally conformal, we need to check that there exists a constant c such that $\angle(\xi(y) \oplus \xi(z), U_1(\rho(\alpha^{-1}))) > c$ for

all $y, z \in X_\infty(\alpha)$. Since Γ is a well positioned Schottky group, we can choose δ_ρ as the smallest distance between two sets $B(w)$. Let w_α be the first letter of α . It follows from Lemma 3.6 that if $|\alpha|$ is big enough $U_1(\rho(\alpha^{-1})) \in B(w_\alpha^{-1})$ and $X_\infty(\alpha) \subset \bigcup_{s \neq w_\alpha} B(s)$: by construction $\mathcal{C}_\infty(\alpha) \subset \bigcup_{s \neq w_\alpha} B(s)$ and the intersection of the $\delta_\rho/2$ neighbourhood of $\bigcup_{s \neq w_\alpha} B(s)$ with the image of the boundary map is already contained in $\bigcup_{s \neq w_\alpha} B(s)$.

Since the chain $\mathcal{C}_{y,z}$ through y_ρ^1 and z_ρ^1 is the intersection of $\partial\mathbb{H}_\mathbb{K}^d$ with $\mathbb{P}(y_\rho^1 \oplus z_\rho^1)$, and, by assumption, $\mathcal{C}_{y,z}$ doesn't intersect the open subset $B(w_\alpha) \subset \partial\mathbb{H}_\mathbb{K}^d$, the result follows. The fact that the representation $\rho : \Gamma \rightarrow \mathrm{SL}_{d+1}(\mathbb{K})$ is $(1, 1, d)$ -hyperconvex is a consequence of Theorem 7.1. \square

Corollary 8.7 (cfr. [19, Corollary 43]). *Let $\rho : \Gamma \rightarrow \mathrm{PO}_\mathbb{K}(1, d)$ be a well positioned Schottky subgroup. Then*

$$\mathrm{Hff}(\xi_\rho^1(\partial\Gamma)) = h_\rho^{a_1}.$$

Proof. If $\mathbb{K} = \mathbb{C}$ this follows directly from Theorem 5.14. For $\mathbb{K} = \mathbb{H}$ it is enough to observe that in the construction of the measure μ^{a_1} performed in Section 5.3 we never used the commutativity of the field \mathbb{K} . \square

We conclude the discussion on convex cocompact subgroups of $\mathrm{PO}_\mathbb{K}(1, d)$ by showing that the set of $(1, 1, d)$ -hyperconvex representations is, in general, not closed within the space of projective Anosov representations. We will prove in Proposition 9.3, that, instead, $(1, 1, 2)$ -hyperconvex representations of fundamental groups of surfaces are closed in the space of Anosov representations. Denote by \mathbb{F}_2 the free group on two generators.

Proposition 8.8. *There exists a continuous path of $\{\mathbf{a}_1\}$ -Anosov representations $\rho_t : \mathbb{F}_2 \rightarrow \mathrm{PU}(1, d)$ such that ρ_0 is $(1, 1, d)$ -hyperconvex and ρ_1 is not $(1, 1, d)$ -hyperconvex.*

Proof. As $\mathrm{PU}(1, d)$ has rank 1, for every 4-tuple (a^+, a^-, b^+, b^-) of pairwise distinct points in $\partial\mathbb{H}_\mathbb{C}^d$ we can find elements $a, b \in \mathrm{PU}(1, d)$ with prescribed attractive and repulsive fixed points and with translation length big enough so that the group generated by a, b is free and convex cocompact on $\mathbb{H}_\mathbb{C}^d$: this follows from a classical ping pong argument. Furthermore, if $(a_t^+, a_t^-, b_t^+, b_t^-)$ vary continuously in t we can also arrange for the elements a_t, b_t to vary continuously in t ; in this way we can define a continuous path $\rho_t : \mathbb{F}_2 \rightarrow \mathrm{PU}(1, d)$ of $\{\mathbf{a}_1\}$ -Anosov representation.

Our claim follows if we choose a_0, b_0 contained in $\mathrm{PO}(1, d)$ (so that the representation is $(1, 1, d)$ -hyperconvex by Lemma 8.4), and $(a_1^+, a_1^-, b_1^+, b_1^-)$ so that (a_1^+, a_1^-, b_1^+) belong to a single chain, but b_1^- doesn't. In this case the representation ρ_1 is clearly not $(1, 1, d)$ -hyperconvex as the sum $\xi(a^+) + \xi(a^-) + \xi(b^+)$ is not direct. \square

8.2. Locally conformal representations in $\mathrm{SO}(p, q)$. We now turn our attention to the group $\mathrm{SO}(p, q)$. Every semi-simple element $g \in \mathrm{SO}(p, q)$ has $|p - q|$ eigenvalues equal to 1. In this subsection, considering suitable exterior representations of $\mathrm{SO}(p, q)$ we will produce examples of hyperconvex representations for which every point is locally conformal, and thus Corollary 6.9 applies. For these representations, the Hausdorff dimension of the limit set computes the critical exponent for the first simple root.

The following generalization of Labourie's property (H) [36, Section 7.1.4] guarantees that a suitable exterior power is hyperconvex:

Proposition 8.9. *Let $\rho : \Gamma \rightarrow \mathrm{SO}(p, q)$ be $\{\mathbf{a}_{p-1}, \mathbf{a}_p\}$ -Anosov (here $p \leq q$). Then $\wedge^p \rho : \Gamma \rightarrow \mathrm{PGL}(\wedge^p(\mathbb{R}^{p,q}))$ is $\{\mathbf{a}_1, \mathbf{a}_{q-p+1}\}$ -Anosov. It is $(1, 1, q-p+1)$ -hyperconvex if and only if for every $x, y, z \in \partial\Gamma$ pairwise distinct, the sum*

$$x_\rho^p + (z_\rho^p \cap y_\rho^{q+1}) + y_\rho^p$$

is direct. In this case every point in $\partial\Gamma$ is locally conformal.

Proof. Observe that the singular values of an element $g \in \mathrm{SO}(p, q) \subset \mathrm{SL}_{p+q}(\mathbb{R})$ have the form $\sigma_1(g) \geq \dots \geq \sigma_p(g) \geq 1 = \dots = 1 \geq \sigma_p(g)^{-1} \geq \dots \geq \sigma_1(g)^{-1}$, where 1 has multiplicity at least $q-p$ (higher if $\sigma_p(g) = 1$). If $\rho : \Gamma \rightarrow \mathrm{SO}(p, q)$ is $\{\mathbf{a}_{p-1}, \mathbf{a}_p\}$ -Anosov, then, for every γ big enough it holds $\sigma_{p-1}(\rho(\gamma)) > \sigma_p(\rho(\gamma)) > 1$, hence in particular

$$\begin{aligned} \sigma_1(\wedge^p \rho(\gamma)) &= \sigma_1(\rho(\gamma)) \dots \sigma_p(\rho(\gamma)), \\ \sigma_2(\wedge^p \rho(\gamma)) &= \sigma_{q-p+1}(\wedge^p \rho(\gamma)) = \frac{\sigma_1(\wedge^p \rho(\gamma))}{\sigma_p(\rho(\gamma))} \text{ and} \\ \sigma_{q-p+2}(\wedge^p \rho(\gamma)) &= \max \left\{ \frac{\sigma_1(\wedge^p \rho(\gamma))}{\sigma_{p-1}(\rho(\gamma))}, \frac{\sigma_1(\wedge^p \rho(\gamma))}{\sigma_p(\rho(\gamma))^2} \right\}, \end{aligned}$$

which implies that $\wedge^p \rho$ is $\{\mathbf{a}_1, \mathbf{a}_{q-p+1}\}$ -Anosov.

Denote by $\mathcal{F}_{p-1,q}(\mathbb{R}^{p,q})$ the partial flag manifold consisting of pairs of $(p-1, q)$ -dimensional isotropic subspaces and consider the map

$$\begin{aligned} L : \mathcal{F}_{p-1,p}(\mathbb{R}^{p,q}) &\rightarrow \mathcal{F}_{1,q-p+1}(\wedge^p \mathbb{R}^{p,q}) \\ (P, Q) &\mapsto (\wedge^p(Q), \wedge^{p-1}(P) \wedge Q^\perp) \end{aligned}$$

where the orthogonal is considered with respect to the bilinear form defining the group $\mathrm{SO}(p, q)$. The map L is clearly equivariant with the homomorphism $\wedge^p : \mathrm{SO}(p, q) \rightarrow \mathrm{SL}(\wedge^p(\mathbb{R}^{p,q}))$; furthermore, if $g \in \mathrm{SO}(p, q)$ is $\mathcal{F}_{p-1,q}(\mathbb{R}^{p,q})$ -proximal, namely g has an attractive fixedpoint g^+ in $\mathcal{F}_{p-1,q}(\mathbb{R}^{p,q})$, then $L(g^+) = (\wedge^p g^+)^+$. Thus if $(\xi^{p-1}, \xi^p) : \partial\Gamma \rightarrow \mathcal{F}_{p-1,q}(\mathbb{R}^{p,q})$ denote the boundary maps associated to $\rho : \Gamma \rightarrow \mathrm{SO}(p, q)$, the boundary maps associated to $\wedge^p \rho$ have the form $L \circ (\xi^{p-1}, \xi^p)$.

Let N denote the dimension of $\wedge^p(\mathbb{R}^{p,q})$. In order to check if the representation $\wedge^p \rho$ is $(1, 1, q-p+1)$ -hyperconvex, it is enough to verify that for every distinct triple $x, y, z \in \partial\Gamma$, the subspace $x_{\wedge^p \rho}^1 + z_{\wedge^p \rho}^1$ intersects transversely $y_{\wedge^p \rho}^{N-q+p-1}$, or, equivalently, the image of $x_{\wedge^p \rho}^1 + z_{\wedge^p \rho}^1$ in $\wedge^p \mathbb{R}^{p,q} / y_{\wedge^p \rho}^{N-q+p-1}$ is two dimensional.

Recall that if $\rho : \Gamma \rightarrow \mathrm{SO}(p, q)$ is \mathbf{a}_p -Anosov, then for every distinct pair $(x, y) \in \partial\Gamma^2$ it holds $x_\rho^p \oplus y_\rho^q = \mathbb{R}^d$, furthermore we can interpret any other point z_ρ^p as a linear map $z_\rho^p : x_\rho^p \rightarrow y_\rho^q$. With this notation the condition that the sum $x_\rho^p + (z_\rho^p \cap y_\rho^{q+1}) + y_\rho^p$ is direct is equivalent to requiring that

$$z_\rho^p(x_\rho^p \cap y_\rho^{q+1}) \cap y_\rho^p = \{0\}.$$

Let us then choose a basis $\{b_1 \dots b_p\}$ of x_ρ^p such that $\{b_1, \dots, b_{p-1}\}$ forms a basis of x_ρ^{p-1} and $b_p = x_\rho^p \cap y_\rho^{q+1}$, then we have that a basis of z_ρ^p is given by $c_i = b_i + z_\rho^p(b_i)$. Furthermore the only term of the explicit expression of $c_1 \wedge \dots \wedge c_p$ that might not belong to $y_{\wedge^p \rho}^{N-q+p-1}$ is $b_1 \wedge \dots \wedge b_{p-1} \wedge c_p$. This last vector doesn't belong to $y_{\wedge^p \rho}^{N-q+p-1} + x_{\wedge^p \rho}^1$ if and only if $z_\rho^p(x_\rho^p \cap y_\rho^{q+1}) \cap y_\rho^p = \{0\}$. \square

Proposition 8.10. *Assume that there are convex cocompact representations $\rho_1 : \Gamma \rightarrow \mathrm{SO}(1, k)$, $\rho_2 : \Gamma \rightarrow \mathrm{SO}(1, l)$ such that ρ_1 strictly dominates ρ_2 , namely there*

exists constants c, μ such that $\sigma_1(\rho_1(\gamma)) > c\sigma_1(\rho_2(\gamma))^\mu$. Then the representation $\rho := \rho_1 \oplus \dots \oplus \rho_1 \oplus \rho_2 : \Gamma \rightarrow \mathrm{SO}(p, (p-1)k + l + s)$ satisfies the hypothesis of Proposition 8.9.

Proof. The representation ρ is \mathfrak{a}_p -Anosov as ρ_2 is convex cocompact, and is \mathfrak{a}_{p-1} -Anosov as ρ_1 strictly dominates ρ_2 . Explicitly writing down the boundary map ξ^p associated to ρ in term of the boundary maps $\xi_1^1 : \Gamma \rightarrow \partial\mathbb{H}_{\mathbb{C}}^k, \xi_2^1 : \Gamma \rightarrow \partial\mathbb{H}_{\mathbb{C}}^l$ associated to ρ_1, ρ_2 one verifies that $z_\rho^p(x_\rho^p \cap y_\rho^{q+1}) \cap y_\rho^p \cong z_{\rho_2}^1 \cap y_{\rho_2}^1$ and the latter intersection is empty as the representation is Anosov. \square

Danciger-Gueritaud-Kassel [17, Proposition 1.8] gave an explicit construction of convex cocompact actions ρ_1, ρ_2 on $\mathbb{H}_{\mathbb{R}}^8$ of the the group Γ generated by reflections in the faces of a 4-dimensional regular right-angled 120-cell, such that ρ_1 strictly dominates ρ_2 and therefore Proposition 8.10 applies. In this case the boundary $\partial\Gamma$ is a 3 sphere. It is also easy to construct representations satisfying the assumption of Proposition 8.10 when the group Γ is free, and in this case it one can deform the representation $\rho : \mathbb{F}_n \rightarrow \mathrm{SO}(p, q)$ to obtain a Zariski dense representation whose image under \wedge^p is locally conformal. We also expect that many more convex cocompact subgroups in rank one have the same property, and it is probably possible to give further examples of situations in which Proposition 8.9 applies for more complicated groups, as, for example, hyperbolic Coxeter groups.

The same argument as in the proof of Proposition 8.9 gives the following

Proposition 8.11. *Let $\rho : \Gamma \rightarrow \mathrm{SL}_d(\mathbb{K})$ be $\{\mathfrak{a}_{p-1}, \mathfrak{a}_p, \mathfrak{a}_s\}$ -Anosov. Assume that*

(i) *there exist constants c, μ such that*

$$\frac{\sigma_{p-1}(\rho(\gamma))\sigma_s(\rho(\gamma))}{\sigma_p(\rho(\gamma))\sigma_{p+1}(\rho(\gamma))} > ce^{\mu|\gamma|},$$

(ii) *for every $x, y, z \in \partial\Gamma$ pairwise distinct, the sum*

$$x_\rho^p + (z_\rho^p \cap y_\rho^{d-p-1}) + y_\rho^{d-s}$$

is direct,

then $\wedge^p \rho$ is $(1, 1, s - p + 1)$ -hyperconvex.

Observe that the second condition, which guarantees that the map

$$\begin{aligned} L : \mathcal{F}_{p-1, p, s}(\mathbb{K}^d) &\rightarrow \mathcal{F}_{1, s-p+1}(\wedge^p \mathbb{K}^d) \\ (P, Q, R) &\mapsto (\wedge^p(Q), \wedge^{p-1}(P) \wedge R) \end{aligned}$$

is proximal, is automatic if $s = p + 1$.

9. FUNDAMENTAL GROUPS OF SURFACES

Let us denote by Γ_S a word-hyperbolic group such that⁷ $\partial\Gamma_S$ is homeomorphic to \mathbb{S}^1 . One has the following direct consequence of Proposition 7.4 and Corollary 6.8.

Corollary 9.1. *Let $\rho : \Gamma_S \rightarrow \mathrm{PGL}_d(\mathbb{R})$ be $(1, 1, 2)$ -hyperconvex, then $h_\rho^{a_1} = 1$.*

⁷A celebrated Theorem of Gabai [22] states that a hyperbolic group Γ_S such that $\partial\Gamma_S$ is a circle is virtually the fundamental group of a connected, closed genus ≥ 2 surface. We will not use this fact.

9.1. Weak irreducibility and closedness. A projective Anosov representation $\rho : \Gamma \rightarrow \mathrm{PGL}_d(\mathbb{K})$ is *weakly irreducible* if the image of its boundary map is not contained in a proper subspace of $\mathbb{P}(\mathbb{K}^d)$. Clearly if ρ is irreducible, then ρ is weakly irreducible, but it is possible to construct examples of weakly irreducible Anosov representations with non reductive image.

The assumption of weak irreducibility can be used to study properties of the stereographic projection $\pi_{z,\rho}$ defined in Definition 6.4.

Lemma 9.2. *Let $\rho : \Gamma \rightarrow \mathrm{PGL}_d(\mathbb{K})$ be $\{\mathbf{a}_1, \mathbf{a}_p\}$ -Anosov. If the map $\pi_{z,\rho} : \partial\Gamma \rightarrow \mathbb{P}()$ collapses an open set $U \subset \partial\Gamma$, then $\pi_{z,\rho}$ is constant. In particular the representation ρ is not weakly irreducible.*

Proof. Indeed, as fixed points of attractive elements are dense in $\partial\Gamma$ we can find $\gamma \in \Gamma$ with $\gamma^+ \in U$. Up to shrinking U we can assume that $\gamma \cdot U \subset U$. Let $V \subset \mathbb{K}^d$ be the smallest subspace containing $\xi(t)$ for every t in U . As $\pi_{z,\rho}|_U$ is constant, the subspace V is proper, furthermore $\rho(\gamma)V = V$, since if $\xi(x_1), \dots, \xi(x_k)$ is a basis of V then $\xi(\gamma x_1), \dots, \xi(\gamma x_k)$ are also linearly independent vectors contained in V . In particular, for every n , $\pi_{z,\rho}(\gamma^{-n}U)$ is constant. As the union of the sets of the form $\gamma^{-n}U$ is the complement of a point in $\partial\Gamma$, the first result follows by continuity of $\pi_{z,\rho}$.

If the map $\pi_{z,\rho}$ is constant then, for every $x \in \partial\Gamma - \{z\}$, the image of the boundary map is contained in the proper subspace $x_\rho^1 + z_\rho^{d-r}$, hence the representation is not weakly irreducible. \square

Lemma 9.2 is particularly useful to analyze properties of $(1, 1, 2)$ -representations of groups Γ_S . The following argument is very similar to Labourie [36, Proposition 8.3].

Proposition 9.3. *The space of real weakly irreducible $\{\mathbf{a}_1, \mathbf{a}_2\}$ -Anosov representations of Γ_S that are not $(1, 1, 2)$ -hyperconvex is open.*

Proof. Let $\rho : \Gamma_S \rightarrow \mathrm{PGL}_d(\mathbb{R})$ be $\{\mathbf{a}_1, \mathbf{a}_2\}$ -Anosov and not $(1, 1, 2)$ -hyperconvex. By definition, there exists a triple of pairwise distinct points $x, y, z \in \partial\Gamma_S$ such that

$$(x_\rho^1 \oplus y_\rho^1) \cap z_\rho^{d-2} \neq 0, \quad (15)$$

and thus the stereographic projection $\pi_{z,\rho}$ is not injective.

Note that $\mathbb{P}(\mathbb{R}^d/z_\rho^{d-2})$ is topologically a circle. Therefore the stereographic projection $\pi_{z,\rho}$ is a map from an interval with a point removed to a circle that:

- does not collapse intervals,
- is not injective.

One can therefore, using the intermediate value theorem, find an interval $I \subset \partial\Gamma_S \setminus \{z\}$ and a point $w \in \partial\Gamma_S \setminus (\{z\} \cup I)$ such that $\pi_{z,\rho}(w)$ belongs to the interior of $\pi_{z,\rho}(I)$.

This last property will hold for any map close enough to $\pi_{z,\rho}$, in particular for the stereographic projection $\pi_{z,\eta}$ for some η close to ρ . Thus, $\pi_{z,\eta}$ is not injective and hence η is not $(1, 1, 2)$ -hyperconvex, as desired. \square

Recall from Definition 6.13 that a reducible representation $\pi : \mathrm{SL}_2(\mathbb{K}) \rightarrow \mathrm{SL}_d(\mathbb{K})$ is k -coherent if its highest k weights belong to the same irreducible factor. Combining results from previous sections one has the following.

Corollary 9.4. *Let $\pi : \mathrm{SL}_2(\mathbb{R}) \rightarrow \mathrm{SL}_d(\mathbb{R})$ be a 2-coherent representation and $\rho : \Gamma_S \rightarrow \mathrm{PSL}_2(\mathbb{R})$ be co-compact, then any deformation η of $\pi\rho$ among weakly irreducible $\{\mathbf{a}_1, \mathbf{a}_2\}$ -Anosov representations into $\mathrm{PSL}_d(\mathbb{R})$ is $(1, 1, 2)$ -hyperconvex. In particular, verifies:*

- has C^1 -limit set in $\mathbb{P}(\mathbb{R}^d)$,
- the exponential growth rate $h_\eta^{\mathbf{a}_1} = 1$.

Proof. Proposition 6.15 states that $\pi\rho$ is $(1, 1, 2)$ -hyperconvex. Proposition 6.2 states hyperconvexity is an open property and, since $\partial\Gamma_S$ is topologically a circle, Proposition 9.3 implies that $(1, 1, 2)$ -hyperconvex is closed among weakly irreducible $\{\mathbf{a}_1, \mathbf{a}_2\}$ -Anosov representations. The remaining statements follow from Proposition 7.4 and Corollary 6.8 for $\mathbb{K} = \mathbb{R}$. \square

This result can be useful to distinguish some components of weakly irreducible Anosov representations (similar bounds on the number of connected components of Anosov representations were obtained with different techniques by Stecker-Treib [51, Corollary 8.2]).

9.2. The Hitchin component of $\mathrm{PSL}_d(\mathbb{R})$. Let S be a closed connected oriented surface of genus ≥ 2 . The *Hitchin component* of $\mathrm{PSL}_d(\mathbb{R})$ is a connected component of the character variety $\mathfrak{X}(\pi_1 S, \mathrm{PSL}_d(\mathbb{R}))$ that contains a *Fuchsian representation*, i.e. a representation that factors as

$$\pi_1 S \rightarrow \mathrm{PSL}_2(\mathbb{R}) \xrightarrow{\iota_d} \mathrm{PSL}_d(\mathbb{R}),$$

where the first arrow is a the choice of a hyperbolic metric on S . Such a connected component is usually denoted by $\mathcal{H}_d(S)$ and an element $\rho \in \mathcal{H}_d(S)$ is called a *Hitchin representation*.

Recall from Labourie [36] that a map $\xi : \partial\Gamma \rightarrow \mathcal{F}(\mathbb{R}^d)$ satisfies Property (H) if for every triple of distinct points x, y, z and every integer k one has

$$\xi^{k+1}(y) + (\xi^{k+1}(z) \cap \xi^{n-k}(x)) + \xi^{n-k-2}(x) = \mathbb{R}^d.$$

One has the following central result by Labourie [36].

Theorem 9.5 (Labourie [36]). *Every Hitchin representation $\rho : \Gamma \rightarrow \mathrm{PSL}_d(\mathbb{R})$ is (p, q, r) -hyperconvex, for every triple with $p + q = r$. The equivariant boundary map $\xi : \partial\Gamma \rightarrow \mathcal{F}(\mathbb{R}^d)$ has property (H).*

Thus, one concludes the following for deformations of the exterior powers.

Proposition 9.6. *Let $\rho \in \mathcal{H}_d(S)$ and consider any $k \in \llbracket 1, d-1 \rrbracket$. Then any weakly irreducible $\{\mathbf{a}_1, \mathbf{a}_2\}$ -Anosov representation $\eta : \pi_1 S \rightarrow \mathrm{PSL}(\wedge^k \mathbb{R}^d)$ connected by weakly irreducible $\{\mathbf{a}_1, \mathbf{a}_2\}$ -Anosov representations to $\wedge^k \rho$ is $(1, 1, 2)$ -hyperconvex and consequently verifies:*

- has C^1 -limit set in $\mathbb{P}(\wedge^k \mathbb{R}^d)$,
- the exponential growth rate $h_\eta^{\mathbf{a}_1} = 1$.

Proof. Observe that for every s , the representation $\wedge^s \rho$ is $\{\mathbf{a}_1, \mathbf{a}_2\}$ -Anosov, furthermore Proposition 8.11 ensures that $\wedge^s \rho$ is $(1, 1, 2)$ -hyperconvex as the boundary curve satisfies Property (H) (for $k = s - 1$). The result then follows from Proposition 7.4, Corollary 6.8 and Corollary 9.4. \square

When no deformation is applied one recovers the following result from Potrie-S. [42, Theorem B].

Theorem 9.7 (Potrie-S. [42]). *For every $\rho \in \mathcal{H}_d(S)$ and every $k \in \llbracket 1, d-1 \rrbracket$ one has $h_\rho^{a_k} = 1$.*

9.3. Hitchin representations in other groups. More generally, let $G_{\mathbb{R}}$ be a simple real-split Lie group. These have been classified, i.e. up to finite coverings $G_{\mathbb{R}}$ is a group in the following list: $\mathrm{PSL}_d(\mathbb{R})$, $\mathrm{PSp}(2n, \mathbb{R})$, $\mathrm{SO}(n, n+1)$ $\mathrm{SO}(n, n)$, or it is the split real forms of the exceptional groups $F_{4, \mathrm{split}}$, $G_{2, \mathrm{split}}$, $E_{6, \mathrm{split}}$, $E_{7, \mathrm{split}}$ and $E_{8, \mathrm{split}}$.

The work of Kostant [34] provides a subalgebra $\iota_{\mathfrak{g}_{\mathbb{R}}} : \mathfrak{sl}_2(\mathbb{R}) \rightarrow \mathfrak{g}_{\mathbb{R}}$, unique up to conjugation and called *the principal \mathfrak{sl}_2* , such that the centralizer of $\iota_{\mathfrak{g}_{\mathbb{R}}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ has minimal dimension. Denote by $\iota_{G_{\mathbb{R}}} : \mathrm{PSL}_2(\mathbb{R}) \rightarrow G_{\mathbb{R}}$ the induced morphism. For example, $\iota_{\mathrm{PSL}_d(\mathbb{R})} = \iota_d$ is the (unique up to conjugation) irreducible representation of $\mathrm{SL}_2(\mathbb{R})$ in \mathbb{R}^d defined in Subsection 6.3.

Let S be a closed connected genus ≥ 2 surface. The *Hitchin component* of $G_{\mathbb{R}}$ is the connected component of the character variety $\mathfrak{X}(\pi_1 S, G_{\mathbb{R}})$ that contains a *Fuchsian representation*, i.e. a representation that factors as

$$\pi_1 S \rightarrow \mathrm{PSL}_2(\mathbb{R}) \xrightarrow{\iota_{G_{\mathbb{R}}}} G_{\mathbb{R}},$$

where the first arrow is a the choice of a hyperbolic metric on S . We will denote this connected component by $\mathcal{H}(S, G_{\mathbb{R}})$ and an element $\rho \in \mathcal{H}(S, G_{\mathbb{R}})$ is called a *Hitchin representation*.

Remark 9.8 (Canonical inclusions). By construction, one sees that the irreducible representation $\iota_d : \mathrm{SL}_2(\mathbb{R}) \rightarrow \mathrm{SL}_d(\mathbb{R})$ factors, depending on the parity of d , as

$$\begin{aligned} \mathrm{SL}_2(\mathbb{R}) &\xrightarrow{\iota_{\mathrm{Sp}(2n, \mathbb{R})}} \mathrm{Sp}(2n, \mathbb{R}) \rightarrow \mathrm{SL}_{2n}(\mathbb{R}), \\ \mathrm{SL}_2(\mathbb{R}) &\xrightarrow{\iota_{\mathrm{SO}(n, n+1)}} \mathrm{SO}(n, n+1) \rightarrow \mathrm{SL}_{2n+1}(\mathbb{R}), \\ \mathrm{SL}_2(\mathbb{R}) &\xrightarrow{G_{2, \mathbb{R}}} G_{2, \mathbb{R}} \rightarrow \mathrm{SO}(3, 4) \rightarrow \mathrm{SL}_7(\mathbb{R}), \end{aligned}$$

where, in each case, the first arrows is the principal inclusion $\iota_{G_{\mathbb{R}}}$. Thus

$$\begin{aligned} \mathcal{H}(S, \mathrm{PSp}(2n, \mathbb{R})) &\subset \mathcal{H}_{2n}(S), \\ \mathcal{H}(S, \mathrm{PSO}(n, n+1)) &\subset \mathcal{H}_{2n+1}(S), \\ \mathcal{H}(S, \mathbb{P}G_{2, \mathbb{R}}) &\subset \mathcal{H}(S, \mathrm{PSO}(3, 4)) \subset \mathcal{H}_7(S). \end{aligned}$$

On the other hand if we consider the embedding $\mathrm{SO}(n-1, n) \subset \mathrm{SO}(n, n)$ as the stabilizer of a positive definite line. The morphism $\iota_{\mathrm{SO}(n, n)}$ is the composition of $\iota_{\mathrm{SO}(n-1, n)}$ with such inclusion. Hence the induced action of $\iota_{\mathrm{SO}(n, n)}$ on $\mathbb{R}^{n, n}$ decomposes in $\mathrm{SL}_2(\mathbb{R})$ -irreducible modules as

$$\iota_{2n-1} \oplus \iota_1,$$

and in particular $\mathcal{H}(S, \mathrm{PSO}(n, n))$ is not a subset of a $\mathrm{PSL}_{2n}(\mathbb{R})$ -Hitchin component.

It is known to experts that every Hitchin representation is Anosov with respect to the minimal parabolic of G , see for example Fock-Goncharov [21].

Recall that the simple roots of the group $\mathrm{PSO}(n, n)$ are given by $\{\mathbf{a}_1, \dots, \mathbf{a}_{n-1}, \mathbf{b}_n\}$ where, as above, $\mathbf{a}_i(x) = x_i - x_{i+1}$ and \mathbf{b}_n is defined by

$$\mathbf{b}_n(x) = x_{n-1} + x_n.$$

Thus every representation $\rho \in \mathcal{H}(S, \mathrm{PSO}(n, n))$, when considered as a representation in $\mathrm{SL}_{2n}(\mathbb{R})$ under the canonical inclusion, is $\{\mathbf{a}_p\}$ -Anosov for every $p \leq n-1$.

Furthermore it is easy to check that the n -th exterior power

$$\wedge^n : \mathrm{PSO}(n, n) \rightarrow \mathrm{PSL}(\wedge^n \mathbb{R}^{2n})$$

splits as the direct sum of two irreducible $\mathrm{PSO}(n, n)$ -modules, which have respectively \mathfrak{a}_{n-1} and \mathfrak{b}_n as first root (see for example Danciger-Zhang [18]). In particular we obtain the following result, independently announced by Labourie [35].

Theorem 9.9. *For every $\rho \in \mathcal{H}(S, \mathrm{PSO}(n, n))$ and every $p \leq n - 2$ the exterior power $\wedge^p \rho$ is $(1, 1, 2)$ -hyperconvex, and the same holds for each one of the two irreducible submodules of $\wedge^n \rho$. Thus the associated limit curve of ρ on the p -Grassmannian for $p \leq n - 2$, as well as each one of the two limit curves in the n -Grassmannian, is C^1 and one has $h_p^{\mathfrak{a}} = 1$ for every simple root \mathfrak{a} .*

Proof. Considering a diagonalizable element in $\mathrm{SL}_2(\mathbb{R})$ as in the proof of Lemma 6.14 we obtain that $\wedge^k(\iota_{2n-1} \oplus \iota_1)$ is 2-coherent for every $k \in \llbracket 1, n - 2 \rrbracket$. Similarly a direct computation shows that the 5 highest weights of $\wedge^n(\iota_{2n-1} \oplus \iota_1)$ are

$$\begin{aligned} - \chi_{\wedge^n(\iota_{2n-1} \oplus \iota_1)}^{(2)} &= \chi_{\wedge^n(\iota_{2n-1} \oplus \iota_1)}^{(2)} = 2n + \dots + 2 = n(n + 1), \\ - \chi_{\wedge^n(\iota_{2n-1} \oplus \iota_1)}^{(3)} &= \chi_{\wedge^n(\iota_{2n-1} \oplus \iota_1)}^{(4)} = 2n + \dots + 4 = n(n + 1) - 2 \\ - \chi_{\wedge^n(\iota_{2n-1} \oplus \iota_1)}^{(5)} &= 2n + \dots + 6 = n(n + 1) - 4 \end{aligned}$$

and each of the first four weights appears with multiplicity one in each irreducible $\mathrm{SO}(n, n)$ -submodules of $\wedge^n \mathbb{R}^{2n}$. We deduce that the restriction of the representation $\wedge^n(\iota_{2n-1} \oplus \iota_1)$ to each of the two submodules is also $(1, 1, 2)$ -hyperconvex. The result is then a consequence of Corollary 9.4 together with the classification of Zariski closures due to Guichard [25]. \square

Remark 9.10. Danciger-Zhang [18] recently proved that when a representation $\rho \in \mathcal{H}(S, \mathrm{PSO}(n, n))$ is regarded as a representation in $\mathrm{PSL}_{2n}(\mathbb{R})$, it is, instead, never $\{\mathfrak{a}_n\}$ -Anosov and the limit curve in the $n - 1$ -Grassmannian is never C^1 .

APPENDIX

Throughout both appendices, the *Jordan projection* of $\mathrm{PGL}_d(\mathbb{K})$ will be needed. Recall that the Jordan decomposition states that every $g \in \mathrm{PGL}_d(\mathbb{K})$ has a power⁸ g^k that can be written as a commuting product $g = g_e g_h g_n$, where g_e is elliptic, g_h is diagonalizable over \mathbb{K} and g_n is unipotent. The Jordan projection $\lambda : \mathrm{PGL}_d(\mathbb{K}) \rightarrow \mathbb{E}^+$

$$\lambda(g) = (\lambda_1(g), \dots, \lambda_d(g))$$

is defined by the ratio of the logarithm of the modulus of the eigenvalues of g_h by k , in decreasing order and (possibly) with repetition.

APPENDIX A. CONTINUITY OF ENTROPY FOR NON-ARCHIMEDEAN ANOSOV REPRESENTATIONS

The purpose of this section is to prove that the *entropy map* $\mathfrak{X}_{\{\mathfrak{a}_1\}}(\Gamma, \mathrm{PGL}_d(\mathbb{K})) \rightarrow \mathbb{R}_+$, defined by

$$\rho \mapsto h_\rho^{\lambda_1} = \limsup_{t \rightarrow \infty} \frac{1}{t} \log \#\{[\gamma] \in [\Gamma]^t : \lambda_1(\rho(\gamma)) \leq t\},$$

⁸($k = 1$ if \mathbb{K} is Archimedean)

is continuous also when \mathbb{K} is non-Archimedean. The argument follows the same outline of S. [48] and Bridgeman-Canary-Labourie-S. [7] for $\mathbb{K} = \mathbb{R}$ and will actually show that this map is as regular as the association $\rho \mapsto \xi_\rho^1$ is. As it is usually the case, the non-Archimedean situation is easier than the Archimedean one.

We need this continuity result to show that Corollary 6.8 will then imply a regularity statement for Hausdorff dimension of $\xi^1(\partial\Gamma)$ when ρ is $(1, 1, 2)$ -hyperconvex, see Corollary A.9.

A.1. Pressure. Let us recall some needed standard facts on thermodynamical formalism. The main reference is Parry-Pollicott [41].

Let Σ be a two-sided irreducible subshift of finite type and let $\text{shift} : \Sigma \rightarrow \Sigma$ the shift map (see [41, Chapter 1]). If $f : \Sigma \rightarrow \mathbb{R}$ is a measurable function then the *pressure* of f is defined by

$$\mathbf{P}(f) = \sup_{\mu \in \mathcal{M}} \left(h_\mu + \int f d\mu \right)$$

where \mathcal{M} is the set of shift-invariant probability measures on Σ and h_μ is the metric entropy of $\mu \in \mathcal{M}$, (see [41, Chapter 3]).

Let d be the metric on Σ defined by

$$d(\{x_i\}, \{y_i\}) = \sum_{i \in \mathbb{Z}} \frac{|x_i - y_i|}{2^i}$$

and let $\text{Hol}_\alpha(\Sigma)$ be the space of Hölder-continuous functions from Σ to \mathbb{R} with exponent $\alpha < 1$. This is naturally a Banach space. One has the following proposition, see [41, Proposition 4.7].

Proposition A.1. *The function $\mathbf{P} : \text{Hol}_\alpha(\Sigma) \rightarrow \mathbb{R}$ is analytic.*

If $X \subset \Sigma$ is a periodic orbit of shift with period $p(X)$ and $f : \Sigma \rightarrow \mathbb{R}$, is continuous, we define

$$S(f, X) = \sum_{i=0}^{p(X)-1} f(\text{shift}^i(x_0))$$

for any $x_0 \in X$. If moreover $f : \Sigma \rightarrow \mathbb{R}_{>0}$ is strictly positive, then the *entropy* of f is defined by

$$h_f = \limsup_{t \rightarrow \infty} \frac{\log \#\{X \subset \Sigma \text{ periodic orbit} : S(f, X) \leq t\}}{t}$$

Two functions $f, g : \Sigma \rightarrow \mathbb{R}$ are *Livšic-cohomologous* if there exists $u : \Sigma \rightarrow \mathbb{R}$ such that for every $x \in \Sigma$ one has

$$f(x) - g(x) = u(\text{shift}(x)) - u(x).$$

Observe that if this is the case and $\mu \in \mathcal{M}$ then $\int g d\mu = \int f d\mu$. In particular, two such functions have the same pressure, same function $S(f, \cdot)$, and if either one is positive, then $h_g = h_f$.

Finally, the entropy verifies *Bowen's equation*:

Lemma A.2. *Assume $f \in \text{Hol}_\alpha(\Sigma)$ is strictly positive, then*

$$s \mapsto \mathbf{P}(-sf)$$

has a unique zero and it is h_f . Moreover, if $(f_u)_{u \in (-\varepsilon, \varepsilon)} \subset \text{Hol}_\alpha(\Sigma)$ is an analytic family with f_0 strictly positive, then

$$\frac{\partial}{\partial u} \Big|_{u=0} \mathbf{P}(-h_{f_u} f_u) \neq 0.$$

Proof. See for example S. [49, Lemma 2.4] and B.-C.-L.-S. [7, subsection 3.3] for the first statement, the derivative computation follows directly from [41, Proposition 4.10], see also [7, Proposition 3.12]. \square

Remark A.3. The Implicit Function Theorem combined with Lemma A.2 and Proposition A.1 gives analyticity of $f \mapsto h_f$ as a function from strictly positive Holder-continuous functions with a given exponent to \mathbb{R} .

A.2. Application to non-Archimedean Anosov representations. The key ingredient for continuity of entropy is Lemma A.6.

Remark A.4. If \mathbb{K} is non-Archimedean and $\rho : \Gamma \rightarrow \text{PGL}_d(\mathbb{K})$ is $\{\mathbf{a}_1\}$ -Anosov then Γ is virtually free.

Proof. Indeed, if \mathbb{K} is non-Archimedean then $\mathbb{P}(\mathbb{K}^d)$ is perfect and totally disconnected. The boundary map $\xi^1 : \partial\Gamma \rightarrow \mathbb{P}(\mathbb{K}^d)$ is a homeomorphism onto its image and thus $\partial\Gamma$ is homeomorphic to a Cantor set. One concludes from \square that such Γ is necessarily virtually free. \square

Lemma A.5. *If Γ is virtually free then its geodesic automaton Λ_Γ is a subshift of finite type.*

Given $\{g_i\}_{i \in \mathbb{Z}} \in \Lambda_\Gamma$ consider the point $x_{\{g_i\}} \in \partial\Gamma$ defined by

$$x_{\{g_i\}} = \lim_{i \rightarrow +\infty} g_0 \cdots g_i$$

and define $f_\rho^{\lambda_1} : \Lambda_\Gamma \rightarrow \mathbb{R}$ by

$$f_\rho^{\lambda_1}(\{g_i\}_{i \in \mathbb{Z}}) = \log \frac{\|\rho(g_0)v\|}{\|v\|}$$

for any $v \in (x_{\{g_i\}})_\rho^1$. Since ξ^1 is Hölder continuous, the same holds for $f_\rho^{\lambda_1}$.

Lemma A.6. *If $X = \{\text{shift}^k \{g_i\}_{i \in \mathbb{Z}}\} \subset \Lambda_\Gamma$ is a periodic orbit with period p , then letting $\gamma \in \Gamma$ be $\gamma = g_0 \cdots g_p$ one has*

$$S(f_\rho^{\lambda_1}, X) = \lambda_1(\rho(\gamma)). \quad (16)$$

Moreover, $f_\rho^{\lambda_1}$ is Livšic-cohomologous to a strictly positive function.

Proof. The first statement is a direct computation. Let us prove the second statement. Observe that for every $\gamma \in \Gamma$ one has

$$\lambda_1(\rho(\gamma)) \geq (1/2)(\lambda_1 - \lambda_2)(\rho(\gamma))$$

and moreover, since ρ is $\{\mathbf{a}_1\}$ -Anosov, equation (8) yields

$$(\lambda_1 - \lambda_2)(\rho(\gamma)) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \frac{\sigma_1}{\sigma_2}(\rho(\gamma^n)) \geq \mu \ell(\gamma), \quad (17)$$

where $\mu > 0$ is given and independent of γ and $\ell(\gamma)$ is the translation length of γ on the Cayley graph of Γ .

If $X \subset \Lambda_\Gamma$ is a periodic orbit and $\gamma_X \in \Gamma$ is its associated element one observes that its period $p(X)$ coincides with $\ell(\gamma_X)$. Equation 16 and the last paragraph yield

$$\inf_{X \subset \Lambda_\Gamma \text{ periodic}} \frac{S(f_\rho^{\lambda_1}, X)}{p(X)} > 0. \quad (18)$$

For each n , Livšic's Theorem [33, Theorem 19.2.1] implies that

$$S_n(f_\rho^{\lambda_1}, x) = \frac{1}{n} \sum_{i=0}^{n-1} f(\text{shift}^i(x))$$

is Livšic-cohomologous to $f_\rho^{\lambda_1}$. Moreover, Anosov's closing Lemma together with equation (18) imply that there exists N such that $S_N(f_\rho^{\lambda_1}, \cdot)$ is strictly positive (see for example S. [49, Lemma 3.8]). This completes the proof. \square

Remark A.7. Observe that, up to conjugation, every primitive element $\gamma \in \Gamma$ is of the form γ_X for some periodic orbit $X \subset \Lambda_\Gamma$. This implies thus that the entropy of $f_\rho^{\lambda_1}$ coincides with the entropy of ρ , i.e.

$$h_{f_\rho^{\lambda_1}} = h_\rho^{\lambda_1}.$$

Thus, Lemma A.6 and Remark A.3 yield:

Corollary A.8. *The function on $\mathfrak{X}_{\{\mathfrak{a}_1\}}(\Gamma, \text{PGL}_d(\mathbb{K}))$ defined by $\rho \mapsto h_\rho^{\lambda_1}$ is continuous.*

Proof. The only missing argument is continuity of $\rho \mapsto f_\rho^{\lambda_1}$, which follows at once from continuity of $\rho \mapsto \xi_\rho^1$ (Corollary 3.8). \square

A.3. Continuity of Hausdorff dimension. We will now prove the following.

Corollary A.9. *The Hausdorff dimension of $\xi^1(\partial\Gamma)$ is continuous on the space of $(1, 1, 2)$ -hyperconvex representations of Γ .*

Proof. Given a $(1, 1, 2)$ -hyperconvex representation consider the Hölder continuous function $f_\rho^{\mathfrak{a}_1} : \Lambda_\Gamma \rightarrow \mathbb{R}$ defined by

$$f_\rho^{\mathfrak{a}_1}(x) = 2f_\rho^{\lambda_1} - f_{\wedge^2 \rho}^{\lambda_1}.$$

This is well defined since ρ is $\{\mathfrak{a}_1, \mathfrak{a}_2\}$ -Anosov.

Note that, by definition, if $X \subset \Lambda_\Gamma$ is a periodic orbit, then

$$S(f_\rho^{\mathfrak{a}_1}, X) = (\lambda_1 - \lambda_2)(\rho(\gamma)),$$

and thus, equation (17) together with the same argument in Lemma A.6 give that $f_\rho^{\mathfrak{a}_1}$ is Livšic-cohomologous to a strictly positive function.

One concludes, as in the proof of Corollary A.8, that $f_\rho^{\mathfrak{a}_1}$ is Livšic-cohomologous to a strictly positive function and thus, the same argument as Corollary A.8 yields that $\rho \mapsto h_{f_\rho^{\mathfrak{a}_1}}$ is continuous on $\mathfrak{X}_{\{\mathfrak{a}_1, \mathfrak{a}_2\}}(\Gamma, \text{PGL}_d(\mathbb{K}))$.

The proof is concluded with the following two facts:

- $h_{f_\rho^{\mathfrak{a}_1}} = h_\rho^{\mathfrak{a}_1}$: this fact holds for any $\rho \in \text{hom}_{\{\mathfrak{a}_1, \mathfrak{a}_2\}}(\Gamma, \text{PGL}_d(\mathbb{K}))$ and the proof follows the exact same argument as S. [49, Theorem A].
- $h_\rho^{\mathfrak{a}_1} = \text{Hff}(\xi_\rho^1(\partial\Gamma))$: since ρ is $(1, 1, 2)$ -hyperconvex, this is the content of Corollary 6.8.

\square

APPENDIX B. HESSIAN OF HAUSDORFF DIMENSION ON PURE IMAGINARY DIRECTIONS

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Corollary 6.8 applied to small deformations of Hitchin representations inside $\text{hom}(\pi_1 S, \text{PSL}_d(\mathbb{C}))$ is the missing piece that, together with results from B.-Canary-Labourie-S. [7, 8] and B.-Taylor [10], generalizes a beautiful result from B.-Taylor [10] and McMullen [40]. The purpose of this section is to explain how one should glue the pieces.

Let us begin by stating the result. Let S be a closed connected oriented surface of genus ≥ 2 . The construction from B.-Canary-Labourie-S. [7] provides a $\text{Out}(\pi_1 S)$ -invariant, positive semi-definite symmetric bilinear form $\mathbf{P}_{\mathfrak{a}_1}$ on the space of regular points of $\{\mathfrak{a}_1, \mathfrak{a}_2\}$ -Anosov representation of $\pi_1 S$. This is usually called a *pressure form* associated to \mathfrak{a}_1 and its construction will be explained in subsection B.1.

The complex structure of $\text{PSL}_d(\mathbb{C})$ provides a complex structure on the character variety $\mathfrak{X}(\pi_1 S, \text{PSL}_d(\mathbb{C}))$, denote by J the associated tensor squaring $-\text{id}$. If $\rho \in \mathfrak{X}(\pi_1 S, \text{PSL}_d(\mathbb{C}))$ is projective Anosov let us denote by

$$\text{Hff}_1(\rho) = \text{Hff}(\xi_\rho^1(\partial\pi_1 S)).$$

Theorem B.1. *Let $\rho : \pi_1 S \rightarrow \text{PGL}_d(\mathbb{R})$ be a $(1, 1, 2)$ -hyperconvex representation and assume that ρ is a regular point of $\mathfrak{X}(\pi_1 S, \text{PGL}_d(\mathbb{C}))$. Then for every differentiable curve $(\rho_t)_{t \in (-\varepsilon, \varepsilon)} \subset \mathfrak{X}(\Gamma, \text{PSL}_d(\mathbb{R}))$ with $\rho_0 = \rho$ one has*

$$\text{Hess Hff}_1(J\dot{\rho}) = \mathbf{P}_{\mathfrak{a}_1}(\dot{\rho}).$$

Let us explain some situations where the theorem applies. Let $G_{\mathbb{R}}$ be a simple real-split Lie group (recall subsection 9.3) and denote by $\mathcal{H}(S, G_{\mathbb{R}})$ the Hitchin component of $G_{\mathbb{R}}$. As in subsection 9.3 we will assume that every element of $\mathcal{H}(S, G_{\mathbb{R}})$ is Π -Anosov.

The construction from B.-Canary-Labourie-S. [7] (see again subsection B.1) provides thus, for each simple root $\mathfrak{a} \in \Pi$ a $\text{Out}(\pi_1 S)$ -invariant, positive semi-definite symmetric bilinear form $\mathbf{P}_{\mathfrak{a}}$ on $\mathcal{H}(S, G_{\mathbb{R}})$, this is usually called *the pressure form of \mathfrak{a}* .

Denote by $G_{\mathbb{C}}$ the complexification of $G_{\mathbb{R}}$. Since Anosov is an open condition and every Hitchin representation is Π -Anosov, there is a neighbourhood \mathcal{U} of $\mathcal{H}(S, G_{\mathbb{R}}) \subset \mathfrak{X}(\pi_1 S, G_{\mathbb{C}})$ consisting of $\Pi_{G_{\mathbb{C}}}$ -Anosov representations. Given $\mathfrak{a} \in \Pi_{G_{\mathbb{C}}}$ let us denote by $\text{Hff}_{\mathfrak{a}} : \mathcal{U} \rightarrow \mathbb{R}$ the Hausdorff dimension of the limit curve on the flag space $\mathcal{F}_{\mathfrak{a}}(G_{\mathbb{C}})$ associated to the maximal parabolic $\mathbf{P}_{\{\mathfrak{a}\}}$,

$$\text{Hff}_{\mathfrak{a}}(\rho) = \text{Hff}(\xi_\rho^{\mathfrak{a}}(\partial\pi_1 S)).$$

The complex structure of $G_{\mathbb{C}}$ induces a complex structure on the character variety $\mathfrak{X}(\pi_1 S, G_{\mathbb{C}})$, let us denote by J the corresponding tensor which squares to $-\text{id}$.

Corollary B.2. *Let $G_{\mathbb{R}}$ be either $\text{PSL}_d(\mathbb{R})$, $\text{PSO}(n, n-1)$, $\text{PSO}(n, n)$, $\text{PSp}(2n, \mathbb{R})$ or $\mathbf{G}_{2, \text{split}}$. Then for every $v \in \mathcal{T} \mathcal{H}(S, G_{\mathbb{R}})$ and every $\mathfrak{a} \in \Pi_{G_{\mathbb{R}}}$ one has*

$$\text{Hess Hff}_{\mathfrak{a}}(Jv) = \mathbf{P}_{\mathfrak{a}}(v). \quad (19)$$

Proof. The corollary follows from Theorem B.1 applied to the exterior power representation \wedge^k , via Proposition 9.6 for the groups $\text{PSL}_d(\mathbb{R})$, $\text{PSO}(n, n-1)$, $\text{PSp}(2n, \mathbb{R})$ or $\mathbf{G}_{2, \text{split}}$, and via Theorem 9.9 for $\text{PSO}(n, n)$. \square

In [8, Theorem 1.6] B.-Canary-Labourie-S. prove that the pressure form \mathbf{P}_{a_1} , called *the Liouville pressure metric*, is Riemannian on $\mathcal{H}_d(S) = \mathcal{H}(S, \mathrm{PSL}_d(\mathbb{R}))$ (i.e. $\mathbf{P}_{a_1}(v) > 0$ whenever $v \neq 0$), thus the theorem implies the following corollary.

Corollary B.3. *The Hessian of*

$$\mathrm{Hff}_1 : \mathfrak{X}(\pi_1 S, \mathrm{PSL}_d(\mathbb{C})) \rightarrow \mathbb{R}$$

at $\rho \in \mathcal{H}_d(S)$ is strictly positive on every direction except $\mathbb{T}_\rho \mathcal{H}_d(S)$, in particular, the Hitchin locus is an isolated minimum of Hff_1 .

The remainder of the section consists on the proof of Theorem B.1.

B.1. Dynamical intersection. Let us give a (very short) account on some definitions and main results needed throughout this section. We refer the reader to B.-C.-L.-S. [7] or to the survey B.-C.-S. [9] and references therein for a more detailed exposition. These facts are consequence of deep results on the thermodynamical formalism for hyperbolic systems developed by Bowen, Ruelle, Parry-Pollicott and others.

Let X be a closed manifold equipped with a transitive Anosov flow $\phi = (\phi_t : X \rightarrow X)_{t \in \mathbb{R}}$ and consider a positive Hölder-continuous function $f : X \rightarrow \mathbb{R}_{>0}$. Given a periodic orbit τ of ϕ with period $p(\tau)$ define

$$\ell_\tau(f) = \int_0^{p(\tau)} f(\phi_s(x)) ds,$$

for any $x \in \tau$, and for $T > 0$ let $R_T(f) = \{\tau \text{ periodic} : \ell_\tau(f) \leq T\}$. The *entropy* of f is defined by

$$h(f) = \lim_{T \rightarrow \infty} \frac{\log \#R_T(f)}{T} \in (0, \infty).$$

Given another positive Hölder-continuous function $g : X \rightarrow \mathbb{R}_{>0}$ define its *dynamical intersection* (or simply *intersection*) with f by

$$\mathbf{I}(f, g) = \lim_{T \rightarrow \infty} \frac{1}{\#R_T(f)} \sum_{\tau \in R_T(f)} \frac{\ell_\tau(g)}{\ell_\tau(f)}. \quad (20)$$

These functions are well defined and vary analytically [7, Section 3.6]. Hence, so does the *renormalized intersection* of g with f defined by

$$\mathbf{J}(f, g) = \frac{h(g)}{h(f)} \mathbf{I}(f, g).$$

Proposition B.4 ([7, Prop. 3.8]). *For every pair of positive Hölder-continuous functions f and g one has $\mathbf{J}(f, g) \geq 1$.*

In particular $\mathbf{J}(f, \cdot)$ is critical at f and its Hessian is positive semidefinite, i.e. if $(f_t)_{t \in (-\varepsilon, \varepsilon)}$ is a C^1 curve of positive Hölder-continuous functions, then

$$\left. \frac{\partial^2}{\partial t^2} \right|_{t=0} \mathbf{J}(f_0, f_t) \geq 0.$$

Proposition B.5 ([7, Lemma 9.3]). *Let $(f_t)_{t \in (-\varepsilon, \varepsilon)}$ be a C^1 curve of positive Hölder-continuous functions. Then $(\partial^2 / \partial t^2)|_{t=0} \mathbf{J}(f_0, f_t) = 0$ if and only if for every periodic orbit τ one has*

$$\left. \frac{\partial}{\partial t} \right|_{t=0} h(f_t) \ell_\tau(f_t) = 0.$$

B.2. The Liouville pressure form. Recall that fixing a hyperbolic metric on S endows the unitary tangent space \mathbf{US} with an Anosov flow whose periodic orbits correspond to conjugacy classes of $\pi_1 S$. We will fix such a metric from now on, this is an (irrelevant) auxiliary choice.

One has the following result from S. [49], see also B.-Canary-Labourie-S. [7, Section 4] for an explicit construction that works for any hyperbolic group. The second part of the statement is proved in [7, Section 6].

Proposition B.6. *Let $\rho : \pi_1 S \rightarrow \mathrm{PGL}_d(\mathbb{R})$ be a projective Anosov representation. Then there exists a positive Hölder-continuous function $f_\rho^{\lambda_1} : \mathbf{US} \rightarrow \mathbb{R}_{>0}$ such that for every conjugacy class $[\gamma] \in [\pi_1 S]$ one has*

$$\ell_\gamma(f_\rho^{\lambda_1}) = \lambda_1(\rho(\gamma)).$$

Moreover, if $\{\rho_u\}_{u \in D}$ is an analytic family of such representations, then one can chose $f_{\rho_u}^{\lambda_1}$ so that the function $u \mapsto f_{\rho_u}^{\lambda_1}$ is analytic.

Proposition B.6 has the following well known consequence.

Proposition B.7. *Let $\rho : \pi_1 S \rightarrow \mathrm{PGL}_d(\mathbb{R})$ be a $\{\mathbf{a}_1, \mathbf{a}_2\}$ -Anosov representation. Then there exists a positive Hölder-continuous function $f_\rho^{\mathbf{a}_1} : \mathbf{US} \rightarrow \mathbb{R}_{>0}$ such that for every conjugacy class $[\gamma] \in [\pi_1 S]$ one has*

$$\ell_\gamma(f_\rho^{\mathbf{a}_1}) = \mathbf{a}_1(\lambda(\rho(\gamma))) = (\lambda_1 - \lambda_2)(\rho(\gamma)).$$

Moreover, if $\{\rho_u\}_{u \in D}$ is an analytic family of such representations, then one can chose $f_{\rho_u}^{\mathbf{a}_1}$ so that the function $u \mapsto f_{\rho_u}^{\mathbf{a}_1}$ is analytic.

Thus, considering two $\{\mathbf{a}_1, \mathbf{a}_2\}$ -Anosov representations ρ, η of $\pi_1 S$ (with possibly different target groups) one can define their *Liouville intersection* as

$$\mathbf{I}_{\mathbf{a}_1}(\rho, \eta) = \mathbf{I}(f_\rho^{\mathbf{a}_1}, f_\eta^{\mathbf{a}_1})$$

and analogously one defines $\mathbf{J}_{\mathbf{a}_1}(\rho, \eta)$. Moreover, given a \mathbf{C}^1 curve $(\rho_t)_{t \in (-\varepsilon, \varepsilon)}$ of such representations one defines the *Liouville pressure norm* of $\dot{\rho}_0$ by

$$\mathbf{P}_{\mathbf{a}_1}(\dot{\rho}_0) = \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} \mathbf{J}_{\mathbf{a}_1}(\rho_0, \rho_t) \geq 0.$$

This induces a symmetric bilinear 2-form on smooth points of $\{\mathbf{a}_1, \mathbf{a}_2\}$ -Anosov representations. However positive semi-definiteness is as far as thermodynamics goes, and one needs geometric arguments to prove the following.

Theorem B.8 (B.-C.-L.-S. [8, Thm. 1.6]). *The Liouville pressure form is positive definite when defined in the Hitchin component $\mathcal{H}_d(S)$. It provides thus an $\mathrm{Out}(\pi_1 S)$ -invariant Riemannian metric on $\mathcal{H}_d(S)$.*

B.3. Pluriharmonic length functions. Let us begin by stating an analogue of B.7 over \mathbf{C} .

Proposition B.9. *Let $\rho : \pi_1 S \rightarrow \mathrm{PGL}_d(\mathbf{C})$ be a $\{\mathbf{a}_1, \mathbf{a}_2\}$ -Anosov representation. Then there exists a positive Hölder-continuous function $f_\rho^{\mathbf{a}_1} : \mathbf{US} \rightarrow \mathbb{R}_{>0}$ such that for every conjugacy class $[\gamma] \in [\pi_1 S]$ one has*

$$\ell_\gamma(f_\rho^{\mathbf{a}_1}) = \mathbf{a}_1(\lambda(\rho(\gamma))) = (\lambda_1 - \lambda_2)(\rho(\gamma)).$$

Moreover, if $\{\rho_u\}_{u \in D}$ is an analytic family of such representations, then one can chose $f_{\rho_u}^{\mathbf{a}_1}$ so that the function $u \mapsto f_{\rho_u}^{\mathbf{a}_1}$ is analytic.

Proof. To find such $f_\rho^{a_1}$ it suffices to apply Proposition B.6 to the real representations $\wedge_{\mathbb{R}}^2 \rho$ and $\wedge_{\mathbb{R}}^2 \wedge_{\mathbb{C}}^2 \rho$. Positivity follows from equation (8) in the definition of Anosov. \square

The Liouville intersection $\mathbf{I}_{a_1}(\rho, \eta)$ between two such representations is hence well defined. The argument from Bridgeman-Taylor [10, Section 5] applies directly and one has the following result. Denote by

$$\text{hom}_{\{a_1, a_2\}}(\pi_1 S, \text{PSL}_d(\mathbb{C})) = \{\rho : \pi_1 S \rightarrow \text{PSL}_d(\mathbb{C}) \text{ } \{a_1, a_2\}\text{-Anosov}\}.$$

Proposition B.10. *Let $\rho \in \text{hom}_{\{a_1, a_2\}}(\pi_1 S, \text{PSL}_d(\mathbb{C}))$, then the function*

$$\mathbf{I}_{a_1}(\rho, \cdot) : \text{hom}_{\{a_1, a_2\}}(\pi_1 S, \text{PSL}_d(\mathbb{C})) \rightarrow \mathbb{R}$$

is locally the real part of a holomorphic function.

Let us make some small comments on the proof.

Proof. Let ρ be an $\{a_1, a_2\}$ -Anosov representation. By openness of dominated splittings, there exists a neighbourhood \mathcal{U} of ρ such that the constants in Definition 3.1 hold for every $\eta \in \mathcal{U}$. This implies that the convergence in the definition of $\mathbf{I}_{a_1}(\rho, \cdot)$ is uniform on compact sets of $\text{hom}_{\{a_1, a_2\}}(\pi_1 S, \text{PSL}_d(\mathbb{C}))$. For each $T > 0$, the truncated sum in equation (20) is the real part of a holomorphic function and thus Theorem 1.23 from Axler-Bourdon-Ramey [1] yields the result. \square

B.4. Vanishing directions. Complex conjugation of matrices is an external automorphism of $\text{PSL}_d(\mathbb{C})$ and thus induces an involution $\tau : \mathfrak{X}(\Gamma, \text{PSL}_d(\mathbb{C})) \rightarrow \mathfrak{X}(\Gamma, \text{PSL}_d(\mathbb{C}))$ whose fixed point set contains $\mathfrak{X}(\Gamma, \text{PSL}_d(\mathbb{R}))$. If $\rho \in \mathfrak{X}(\Gamma, \text{PSL}_d(\mathbb{R}))$ is a regular point, then the differential $d_\rho \tau$ splits the tangent space as a sum of *pure imaginary vectors* and the tangent space to the real characters:

$$\text{T}_\rho \mathfrak{X}(\Gamma, \text{PSL}_d(\mathbb{C})) = \ker(d_\rho \tau + \text{id}) \oplus \text{T}_\rho \mathfrak{X}(\Gamma, \text{PSL}_d(\mathbb{R}));$$

the almost complex structure J of $\mathfrak{X}(\pi_1 S, \text{PSL}_d(\mathbb{C}))$ interchanges this splitting.

Lemma B.11. *Let $\rho : \pi_1 S \rightarrow \text{PSL}_d(\mathbb{R})$ be $\{a_1, a_2\}$ -Anosov and let v be a pure imaginary direction at ρ . Then $\mathbf{P}_{a_1}(v) = 0$.*

This is a standard symmetry argument, see for example B.-C.-S. [9, Section 5.8].

Proof. Consider a differentiable curve $(\rho_t)_{t \in (-\varepsilon, \varepsilon)} \subset \text{hom}_{\{a_1, a_2\}}(\pi_1 S, \text{PSL}_d(\mathbb{C}))$ such that $\rho_0 = \rho$, $\dot{\rho}_0 = v$ and $\tau \rho_t = \rho_{-t}$. The functions

$$t \mapsto \ell_\gamma(f_{\rho_t}^{a_1}) = (\lambda_1 - \lambda_2)(\rho_t(\gamma)),$$

for every conjugacy class $[\gamma] \in [\pi_1 S]$, and $t \mapsto h(f_{\rho_t}^{a_1})$ are invariant under $t \mapsto -t$ and are thus critical at 0. Consequently, for every conjugacy class, the function $t \mapsto h(f_{\rho_t}^{a_1}) \ell_\gamma(f_{\rho_t}^{a_1})$ is critical at 0 and hence Proposition B.5 implies that

$$\mathbf{P}_{a_1}(v) = \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} \mathbf{J}_{a_1}(\rho_0, \rho_t) = 0.$$

\square

B.5. Proof of Theorem B.1. Let $\rho \in \mathfrak{X}(\pi_1 S, \mathrm{PSL}_d(\mathbb{R}))$ be $(1, 1, 2)$ -hyperconvex and regular. Consider $v \in \mathbb{T}_\rho \mathfrak{X}(\pi_1 S, \mathrm{PSL}_d(\mathbb{R}))$ and note that Jv is pure imaginary. Thus, Lemma B.11 implies that

$$0 = \mathbf{P}_{\mathbf{a}_1}(Jv) = \frac{\partial^2}{\partial t^2} \Big|_{t=0} \mathbf{J}_{\mathbf{a}_1}(\rho_0, \rho_t), \quad (21)$$

where $(\rho_t)_{t \in (-\varepsilon, \varepsilon)}$ is a curve such that $\rho_0 = \rho$, $\dot{\rho}_0 = v$ and $\tau\rho_t = \rho_{-t}$. Recall that $\mathbf{I}(f, f) = 1$ and that if ρ is $(1, 1, 2)$ -hyperconvex then Corollary 9.1 states that $h_\rho^{\mathbf{a}_1} = 1$, moreover, as observed in the proof of Lemma B.11, $\dot{h}^{\mathbf{a}_1}(\dot{\rho}_0) = 0$, so developing the last term of equation (21) one obtains

$$0 = \mathrm{Hess}_\rho(h^{\mathbf{a}_1})(Jv) + \mathrm{Hess}_\rho(\mathbf{I}_{\mathbf{a}_1}(\rho, \cdot))(Jv).$$

Proposition B.10 states that $\mathbf{I}_{\mathbf{a}_1}(\rho, \cdot)$ is locally the real part of a holomorphic function, so

$$\mathrm{Hess}_\rho(\mathbf{I}_{\mathbf{a}_1}(\rho, \cdot))(Jv) = -\mathrm{Hess}_\rho(\mathbf{I}_{\mathbf{a}_1}(\rho, \cdot))(v)$$

and thus

$$\mathrm{Hess}_\rho(h^{\mathbf{a}_1})(Jv) = \mathrm{Hess}_\rho(\mathbf{I}_{\mathbf{a}_1}(\rho, \cdot))(v).$$

Lemma 6.3 implies that, at least for small t , ρ_t is $(1, 1, 2)$ -hyperconvex (over \mathbb{C}) and thus Corollary 6.8 yields $h^{\mathbf{a}_1}(\rho_t) = \mathrm{Hff}(\rho_t)$. Finally, since $h^{\mathbf{a}_1} \equiv 1$ in a neighbourhood of ρ in $\mathfrak{X}(\pi_1 S, \mathrm{PSL}_d(\mathbb{R}))$ one has

$$\mathrm{Hess}_\rho(\mathbf{I}_{\mathbf{a}_1}(\rho, \cdot))(v) = \mathbf{P}_{\mathbf{a}_1}(v).$$

The result then follows.

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