Index-2 hybrid DAE: a case study with well-posedness and numerical analysis

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Abstract: In this work, we study differential algebraic equations with constraints defined in a piecewise manner using a conditional statement. Such models classically appear in systems where constraints can evolve in a very small time frame compared to the observed time scale. The use of conditional statements or hybrid automata are a powerful way to describe such systems and are, in general, well suited to simulation with event driven numerical schemes. However, such methods are often subject to chattering at mode switch in presence of sliding modes, or can result in Zeno behaviours.
In contrast, the representation of such systems using differential inclusions and method from non-smooth dynamics are often closer to the physical theory but may be harder to interpret. Associated time-stepping numerical methods have been extensively used in mechanical modelling with success and then extended to other fields such as electronics and system biology.
In a similar manner to the previous application of non-smooth methods to the simulation of piecewise linear ODEs, non-smooth event-capturing numerical scheme are applied to piecewise linear DAEs. In particular, the study of a 2-D dynamical system of index-2 with a switching constraint using set-valued operators, is presented.

Keywords: Hybrid Systems, ODE, DAE, Nonsmooth Dynamical Systems, Linear Complementarity Problem

1. INTRODUCTION

The aim of this work is to study hybrid differential algebraic equations (hybrid DAE), i.e., dynamical systems with some algebraic constraints switching with respect to the state variables. Such hybrid DAE systems are used in numerous field from electronics, Acary et al. (2010) to chemical process engineering, Stechlinski et al. (2018). They are especially used in model-based design through the use of language like MODELICA as in Henningsson et al. (2019).

Various work already studied hybrid DAEs. For example, DAE including complementarity constraints are a subset of differential variational inequalities (DVI). DVIs are defined and studied in Pang and Stewart (2008). In particular, they analyse the well-posedness of index one DVI and DAE of mixed index between 1 and 2. Matrosov (2006) proposes a concept of solutions, which is inspired by the one of Filippov (1960), for DAE with discontinuous constraints and differential part. He gives sufficient conditions for existence of solutions in Matrosov (2006), and sufficient condition for uniqueness in Matrosov (2007).

Hamann and Mehrmann (2008) and Mehrmann and Wunderlich (2009) provide a study of well-posedness of hybrid DAE structured as a hybrid automata. In addition, a numerical implementation of sliding modes for DAE systems is provided to avoid chattering when switching. It is interesting to note that the sliding solutions obtained in Mehrmann and Wunderlich (2009) are similar to the solutions from Matrosov (2006), assuming the solution of $\mathbf{z}(t)$ in each mode is obtained by index reduction. Furthermore, Mehrmann and Wunderlich (2009) need explicit transition functions from one mode (DAE) to another, in addition to consistent reset conditions. Trenn (2012) defines solutions of hybrid DAE with exogenous switching. In particular, he introduces the notion of distributional solutions which can also be used to efficiently solve inconsistent initial conditions of classical DAE as an exogenous switching at $t = 0$. Camlibel et al. (2016) extend results of well-posedness of differential inclusions to differential algebraic inclusions $\mathrm{P}\dot{\mathbf{x}} \in -\mathcal{F}(\mathbf{x})$ with a maximal monotone operator $\mathcal{F}(\cdot)$. Then, assuming the passivity of the Weierstrass-Kronecker form of a system (1) with $\mathcal{M}(\cdot)$ a maximal monotone operator, sufficient conditions for the well-posedness of absolutely continuous (AC) solutions of (1) are given.

$$\begin{cases} \mathrm{E}\dot{\mathbf{y}}(t) = \mathrm{A}\mathbf{y}(t) + \mathrm{B}\boldsymbol{\lambda}(t) + \mathbf{e} \\ \mathbf{w}(t) = \mathrm{C}\mathbf{y}(t) + \mathrm{D}\boldsymbol{\lambda}(t) + \mathbf{q} \\ \mathbf{w}(t) \in \mathcal{M}(-\boldsymbol{\lambda}(t)). \end{cases} \tag{1}$$

It important to note that this formalism is the closest to the example studied in this paper (see (5)), with the notable difference that in our case the operator $\mathcal{M}(\cdot)$ is not maximal monotone but hypo-monotone. Additionally, its rewriting in the form of (1) with maximal monotone operator is not a passive system.

Stechlinski et al. (2018); Barton et al. (2018), and Khan (2018) define from the Clarke jacobian a notion of generalised differential index and an associated index reduction procedure in the context of non-smooth DAE, with at least Lipschitz continuous constraints. Current implementation and theory are limited to semi-explicit index-1 non-smooth

DAE. Finally, we can cite another work for index reduction of hybrid DAE based on non standard analysis by Benveniste et al. (2017). This work uses non-standard analysis to construct well-defined transitions from one mode to another in the context of hybrid DAE even in the presence of varying index. In particular, this work pairs well with Mehrmann and Wunderlich (2009), which needs the knowledge of transition and re-initialisation maps when switching from one mode to another.

Let us now define the general framework of linear hybrid DAE that we wish to study from the point of view of non-smooth dynamics, and event-capturing numerical methods. We wish to consider hybrid linear DAE defined as:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathrm{A}\mathbf{x}(t) + \mathrm{B}\mathbf{z}(t) + \mathbf{e} \\ \quad 0 = \mathbf{g}_i(\mathbf{x}(t), \mathbf{z}(t)) = \mathrm{C}_i\mathbf{x}(t) + \mathrm{D}_i\mathbf{z}(t) + \mathbf{q}_i \\ \qquad , \forall (\mathbf{x}(t), \mathbf{z}(t)) \in \mathcal{X}_i. \end{cases} \tag{2}$$

The sets $\mathcal{X}_i = \{(\mathbf{x}, \mathbf{z}) \in \mathbb{R}^{n_1+n_2=n} \mid \mathbf{h}_i(\mathbf{x}, \mathbf{z}) = \mathrm{H}_i\mathbf{x}(t) + \mathrm{F}_i\mathbf{z}(t) + \mathbf{p}_i > 0\}$ define a partition of $\mathbb{R}^n$ such that:

- $\bigcup\limits_i \overline{\mathcal{X}}_i = \mathbb{R}^n$
- $\mathrm{int}(\mathcal{X}_i) \neq \emptyset, \quad \forall i$
- for $i \neq j$, $\partial\mathcal{X}_i \cap \partial\mathcal{X}_j = \emptyset$

where, $\mathbf{x}$, $\mathbf{z}$ are the differential and algebraic variables, respectively. We can build using step-functions [1] in a similar fashion to Acary et al. (2014) a generalised constraint.

$$\mathbf{g}(\mathbf{x}, \mathbf{z}) = \sum_i \left(\prod_{j \neq i} (1 - \mathrm{s}^+(\mathbf{h}_j(\mathbf{x}, \mathbf{z}))) \right) \mathrm{s}^+(\mathbf{h}_i(\mathbf{x}, \mathbf{z}))\mathbf{g}_i(\mathbf{x}, \mathbf{z}) \tag{3}$$
$$= 0$$

where $\mathrm{s}^+(\mathbf{y}) = 0$ if $\mathbf{y} < 0$ and $\mathrm{s}^+(\mathbf{y}) = 1$ if $\mathbf{y} > 0$, the behaviour in $\mathbf{y} = 0$ depending on later relaxations. In particular, in the context of piecewise ODE, the work of Acary et al. (2014) shows that methods for non-smooth dynamics can be efficiently applied using such reformulation. Then, depending on the concept of solutions applied on the switching surfaces (using convexification as in Filippov (1960), or using multivalued functions as in Aizerman and Pyatnitskiy (1974)), the resulting solutions may differ. Here, we study the extension of such concepts of solutions when applied to switching constraints instead of switching vector fields.

In this work, and its associated working example, we restrain ourselves to the simple case with only two algebraic constraints $\mathrm{C}_1\mathbf{x} + \mathrm{D}_1\mathbf{z} + \mathbf{q}_1 = 0$ and $\mathrm{C}_2\mathbf{x} + \mathrm{D}_2\mathbf{z} + \mathbf{q}_1 = 0$ and one switching condition depending on the sign of $\mathbf{h}(\mathbf{x}, \mathbf{z}) = \mathrm{H}\mathbf{x}(t) + \mathrm{F}\mathbf{z}(t)$. Then, we construct a relaxation of these two constraints along the switching surface $\mathbf{h}(\mathbf{x}, \mathbf{z}) = 0$ by "filling the graph". Such relaxed constraint can be designed in $\mathbf{h}(\mathbf{x}, \mathbf{z}) = 0$ by considering the convex hull of the left and right limit of $\mathbf{g}(\mathbf{x}, \mathbf{z})$ when $\mathbf{h}(\cdot) < 0$ and $\mathbf{h}(\cdot) > 0$, respectively. We could also consider multi-valued step functions in (3) in a similar fashion to Aizerman and Pyatnitskiy (1974) approach for discontinuous ODEs. For this working example, we consider the convexification of the constraints along the switching surface.

As we have seen, most results consider either a high index hybrid DAE framework with event-driven numerical methods and explicit transition functions, or mainly index-1 DAE with non-smooth constraints aiming at rewriting the system as a differential inclusion [2] into a Lipschitz function, or a maximal monotone operator. In this paper, a bridge between the hybrid DAE formalism and the non-smooth DAE formalism is established. We show on a simple working example the difficulties arising with such relaxations, as well as how classical non-smooth numerical methods perform in this context. We also propose some modification to the numerical scheme to overcome the associated troubles. The article is organized as follows: in Section 2.1 we study the necessary and sufficient conditions for existence of absolutely continuous solutions to our working example. In Section 2.3.1, we study the well-posedness of the jumps dynamics associated with this example, and in particular the resulting generalised equation. In Section 2.4.1, we analyse the well-posedness of the implicit Euler numerical scheme when applied on our case study and we propose variation of the implicit Euler scheme to solve problems arising in this example. Finally, in Section 2.5 we give some numerical simulations using our proposed numerical scheme, and we show experimentally on some cases a convergence in $O(h)$. Conclusions, and final thoughts are given in Section 3.

2. ANALYSIS OF A HYBRID DAE EXAMPLE

Let us first introduce some notations. Vectors of real variables $\mathbf{x} = (x_1, \ldots, x_i, \ldots, x_n)$ in $\mathbb{R}^n$ are denoted in **bold**. In the context of algebraic differential systems, the variables $\mathbf{x}$ will denote differential variables, while $\mathbf{z}$ will denote algebraic variables. In the context of non-smooth expression, we will in general note $\boldsymbol{\lambda}$ the Lagrange multipliers. We now consider the working example of this paper:

$$\begin{cases} \dot{x}_1(t) = 1 + \mathrm{B}_1 z(t) \\ \dot{x}_2(t) = \mathrm{B}_2 z(t) \\ \texttt{if } x_1 > 0 \texttt{ then } : \\ \qquad 0 = 1 + x_1(t) - x_2(t) \\ \texttt{if } x_1 < 0 \texttt{ then } : \\ \qquad 0 = -1 - x_1(t) - x_2(t) \end{cases} \tag{4}$$

which is a particular case of (2) with $\mathrm{A} = 0$, $\mathrm{B} = (\mathrm{B}_1, \mathrm{B}_2)^{\mathrm{T}}$, $C_1 = (1, -1)$, $C_2 = (-1, -1)$, $H_1 = (-1, 0)$, $H_2 = (1, 0)$, $\mathrm{D}_1 = 0$, and $\mathrm{D}_2 = 0$. In $x_1 = 0$, the system does not have continuous solutions whatever the active constraint, so we keep strict inequalities in (4).

As exposed in the introduction, the hybrid constraints can be embedded into a set-valued constraint obtained by convexification. We construct the hybrid DAE system (5) where the constraint is $0 = -x_2 + \lambda(x_1 + 1)$ with $\lambda \in \mathrm{sign}(x_1)$. This set-valued algebraic constraint equals the ones of (4) when $x_1 < 0$ (respectively $x_1 > 0$), and is a convex relaxation of both in $x_1 = 0$: that is $x_2 \in \mathrm{convexHull}(\{x_2 = -1\}, \{x_2 = 1\}) = [-1, 1]$ (see Fig. 1). This yields the non-smooth DAE system (5):

$$\begin{cases} \dot{x}_1(t) = 1 + \mathrm{B}_1 z(t) \\ \dot{x}_2(t) = \mathrm{B}_2 z(t) \end{cases} \tag{5a}$$
$$\begin{cases} \quad 0 = \lambda(t)(1 + x_1(t)) - x_2(t) \\ \lambda(t) \in \mathrm{sign}(x_1(t)), \end{cases} \tag{5b}$$

[1] or using sign functions

[2] differential algebraic inclusion in the case of Camlibel et al. (2016)

where sign($\cdot$) is a set-valued operator, sign : $\mathbb{R} \rightrightarrows \mathbb{R}$, such that:

$$\operatorname{sign}(x) = \begin{cases} \{-1\}, & \text{if } x < 0 \\ [-1,1], & \text{if } x = 0 \\ \{1\}, & \text{if } x > 0. \end{cases}$$

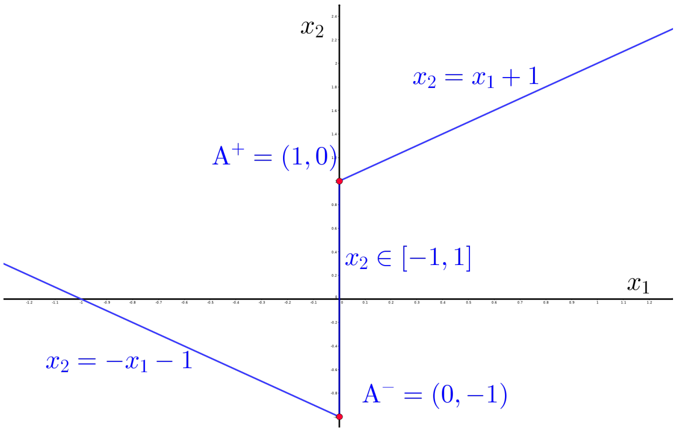

Fig. 1. Phase-space representation of the constraint of (4) and (5b)

2.1 Analysis of absolutely continuous solutions

Let us study the conditions on the differential part of the DAE (5), and in particular on the parameters B_1, B_2, for the existence existence of absolutely continuous solution $x_1(t)$, $x_2(t)$ to (5), and in particular a sliding mode along the switching surface $x_1 = 0$, for arbitrary time interval and initial conditions. Let us also observe that solutions $(x_1(t), x_2(t))$ of the non-smooth constraint of system (5) are such that:

$$0 \in -x_2(t) + |x_1(t)| + \operatorname{sign}(x_1(t)). \qquad (6)$$

Definition 1. (Global AC solutions). We say there exists global AC solutions if for some initial conditions $\mathbf{x}(t_0)$ satisfying (6), there exists a solution to (5a) almost everywhere and to (5b) for all $t \in [t_0, T[$, and any $T > t_0$.

Let us state the main result of this section.

Proposition 2. (AC solutions). The system defined in (5) has global AC solutions on an arbitrary interval $[t_0, T[$, $T > t_0$ for any consistent initial condition $\mathbf{x}(0)$, if and only if B_1, B_2 are chosen such that:

$$\mathrm{B}_1 \neq 0, \ \frac{\mathrm{B}_2}{\mathrm{B}_1} \leq 0, \ (\mathrm{B}_1 + \mathrm{B}_2) \neq 0. \qquad (7)$$

For the sake of conciseness, we will give in this paper the sketch, and the main ideas of the proof. For the complete proof we refer to Rocca et al. (2019).

Proof. Let us consider first the existence of solutions to the piecewise DAE system (5), we first consider the conditions of existence of solutions in each mode: mode 1 ($x_1 < 0$), mode 2 ($x_1 > 0$), mode 3 ($x_1 = 0$). Then, we consider the conditions where solutions can switch from one mode to another. For example, let us study the solution which are AC in the mode 1. Solutions in mode 1 are such that $x_1(t) < 0$, $\lambda(t) = -1$ for all $t \in [t', t'+\varepsilon] \triangleq I_\varepsilon$, with $\varepsilon > 0$. Therefore, for all $t \in I_\varepsilon$ such that $x_1(t) < 0$, solutions of (5) in mode 1 must also satisfy:

$$-\dot{x}_2(t) - \dot{x}_1(t) = 0 \Leftrightarrow (\mathrm{B}_2 + \mathrm{B}_1) z(t) + 1 = 0$$
$$\Leftrightarrow z(t) = \frac{-1}{\mathrm{B}_2 + \mathrm{B}_1},$$

which has a solution if and only if $(\mathrm{B}_2 + \mathrm{B}_1) \neq 0$. We deduce that there exist AC solutions in mode 1 if and only if $(\mathrm{B}_2 + \mathrm{B}_1) \neq 0$, and we say that mode 1 is not feasible if $(\mathrm{B}_2 + \mathrm{B}_1) = 0$. We note (see (4) and (5)) that this corresponds to $\mathrm{C}_1\mathrm{B}$ non-singular with $\mathrm{C}_1 = (-1, -1)$. This condition on B is the same as: the system must be of differentiation index 2 in mode 1 (similarly for mode 2 and 3). Then, the following result holds:

$$\dot{x}_1(t) = \frac{\mathrm{B}_2}{\mathrm{B}_2 + \mathrm{B}_1}.$$

We deduce that either $\dot{x}_1(t) \leq 0$ and $x_1(t) < 0$ for all $t > t'$, or $\dot{x}_1(t) > 0$ and there exists $t_1 > t'$ such that $x_1(t_1) = 0$:

- If $(\mathrm{B}_1 + \mathrm{B}_2)\mathrm{B}_2 \leq 0$ then $\dot{x}_1(t) \leq 0$, for all $t \in [t', +\infty[\triangleq I_\infty$: the system stays in mode 1 forever.
- If $(\mathrm{B}_1 + \mathrm{B}_2)\mathrm{B}_2 > 0$ then $\dot{x}_1(t) > 0$, for $t > t'$. Moreover, for $t_1 = -x_1(t')\frac{\mathrm{B}_1+\mathrm{B}_2}{\mathrm{B}_2} + t'$ we have $\mathbf{x}(t_1) = (0, -1)$ and the trajectory reaches mode 3.

We can already provide a necessary [3] condition on $\mathrm{B} = (\mathrm{B}_1, \mathrm{B}_2)^{\mathrm{T}}$ for the existence of global AC solutions in mode 1:

$$\mathrm{B}_1 + \mathrm{B}_2 \neq 0. \qquad \text{(Cond A}_1\text{)}$$

We can proceed in a similar fashion for mode 2, and 3. Then, conditions on B_1, B_2 for existence of global AC solution for solutions staying in any of this three modes are given by:

$$\mathrm{B}_1 + \mathrm{B}_2 \neq 0, \ \mathrm{B}_1 - \mathrm{B}_2 \neq 0, \ \mathrm{B}_1 \neq 0. \qquad \text{(Cond A)}$$

Let us consider now the conditions for existence of solutions such that there exists a continuous transition from one mode to another. For example, let us study the condition for a transition from mode 3 to mode 1 (case 4.3 in Rocca et al. (2019)). Let us assume $x_1(t') = 0$ and $\frac{\mathrm{B}_2}{\mathrm{B}_1} > 0$: then there exists some time $t_1 > t'$ such that $x_2(t_1) = -1$. Furthermore, we need $\dot{x}_1(t_1^+) < 0$ to switch to mode 1: this is equivalent to the condition $(\mathrm{B}_1 + \mathrm{B}_2)\mathrm{B}_2 < 0$. However, there is no $\mathrm{B} = (\mathrm{B}_1, \mathrm{B}_2)^{\mathrm{T}}$ such that $(\mathrm{B}_1 + \mathrm{B}_2)\mathrm{B}_2 < 0$ and $\frac{B_2}{B_1} > 0$ and solutions cannot continue further than point A^-. Consequently, the condition (Cond A) must be further restricted to:

$$\mathrm{B}_1 + \mathrm{B}_2 \neq 0, \ \frac{\mathrm{B}_2}{\mathrm{B}_1} \leq 0, \ \mathrm{B}_1 \neq 0. \qquad \text{(Cond B)}$$

We note that $\mathrm{B}_1 - \mathrm{B}_2 \neq 0$ always holds for $\frac{\mathrm{B}_2}{\mathrm{B}_1} \leq 0$. In a similar fashion, considering the conditions for existence of a continuous transition taking the modes two by two, we can conclude that (Cond B) is a necessary and sufficient condition for the existence [4] of global AC solutions assuming an arbitrary consistent initial condition. □

In Fig. 2, we summarise the conditions of Proposition 2 for existence of global AC solutions for any initial

[3] Note this is a sufficient and necessary condition for existence of local solution in mode 1, however this is only a necessary condition for global solutions of the hybrid DAE (5).

[4] However, this does not prove the uniqueness of solutions.

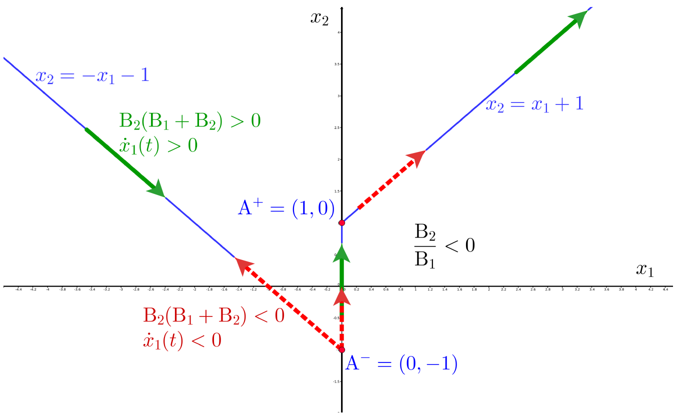

Fig. 2. In red (dashed vectors) the "sliding-repulsive" solutions, and in green (full vectors) the "sliding-crossing" solutions.

conditions satisfying (6). The set of parameters B_1, B_2 that satisfy these conditions can be separated in two subsets: the subset yielding "sliding-crossing" solutions which are unique with respect to the initial conditions, and the subset leading to "sliding-repulsive" solutions, which are not unique in $\mathbf{x}(t_0) = (0, -1)^{\mathrm{T}}$.

2.2 Analysis of the index reduced system Filippov solutions.

We can compare the sliding mode obtained in the proof of Proposition 2 (motion in the segment $[A^-, A^+]$), with the study of the solutions from Filippov (1960) in $x_1 = 0$ associated with the switching ODE obtained by index reduction of the DAE in $x_1 < 0$ and $x_1 > 0$. Such solutions are defined, for example in Mehrmann and Wunderlich (2009). The left vector field $\mathbf{f}^-(\cdot)$ (for $x_1 < 0$) and right vector field $\mathbf{f}^+(\cdot)$ (for $x_1 > 0$) are given by:

$$\mathbf{f}^-(t, \mathbf{x}(t)) = \begin{pmatrix} \dfrac{B_2}{B_1 + B_2} \\ \dfrac{-B_2}{B_1 + B_2} \end{pmatrix} \quad \mathbf{f}^+(t, \mathbf{x}(t)) = \begin{pmatrix} \dfrac{-B_2}{B_1 - B_2} \\ \dfrac{-B_2}{B_1 - B_2} \end{pmatrix}.$$

In the case of "sliding-repulsive" solutions, the switching ODE resulting from the index reduction yields the sliding motion $f_0(\mathbf{x}) = \text{convexHull}(f^-(\mathbf{x}), f^+(\mathbf{x})) \cap \{\mathbf{x} \in \mathbb{R}^2 | x_1 = 0\}$. The associated solutions correspond to the same solutions we obtain by our relaxation of the switching constraint by a generalised equation.

In the particular case of "sliding-crossing" solutions, the index reduced system does not lead to any sliding motions as convexHull$(f^-(\mathbf{x}), f^+(\mathbf{x}))$ does not intersect the switching surface. The solutions do not stay on the surface $x_1 = 0$, and due to the index reduction the constraint in $x_1 > 0$ is not satisfied anymore if $x_1(t_0) < 0$. Following the guidelines for sliding mode detection given by Mehrmann and Wunderlich (2009), an explicit transition function is necessary for continuation. The sliding solutions we obtain using our relaxation is not retrieved by Mehrmann and Wunderlich (2009) approach.

We note that the approach of convexifying the left and right reduced DAEs has already show to be wrong for some cases in Matrosov (2007) with an index-1 non-smooth example.

2.3 Analysis of solutions with bounded discontinuities.

Let us study the existence of discontinuous solutions when the trajectory cannot continue anymore with an absolutely continuous solution after some time t_j. For example, this is the case if the trajectory reaches the point A^- in Fig. 1 with $\frac{B_2}{B_1} > 0$. Solutions with discontinuities make sense in a context where the real system evolves in a time scale much smaller than the one considered in the model.

Analysis of jump dynamics Assume there is a jump at some time t_j. We first introduce the measure differential inclusions (MDI) (8) associated with (5). Let us note $\mathrm{d}\mathbf{x}$ the differential measure of $\mathbf{x}(t)$. Let us notice that both sides of the dynamics are considered as Schwartz distributions. Hence z is to be considered as a measure. In term of equality of measures we obtain:

$$\begin{cases} \mathrm{d}x_1 = \mathrm{d}t + B_1 \mathrm{d}\Lambda_z \\ \mathrm{d}x_2 = B_2 \mathrm{d}\Lambda_z \\ \quad 0 \in \; -x_2(t) + |x_1(t)| + \text{sign}(x_1(t)), \end{cases} \tag{8}$$

with $\mathrm{d}\Lambda_z(t) = z(t)\mathrm{d}t + \sigma_z \delta_{t_j}\ t_j$ with $\mathrm{d}t$ the Lebesgue measure associated with time, δ_{t_j} the Dirac distribution at $t = t_j$, and σ_z the amplitude of the jump. Then, at t_j the system (8) becomes the algebraic system:

$$\begin{cases} x_1(t_j^+) - x_1(t_j^-) = B_1 \sigma_z \\ x_2(t_j^+) - x_2(t_j^-) = B_2 \sigma_z \\ \qquad 0 \in \lambda(t_j^+) + |x_1(t_j^+)| - x_2(t_j^+) \\ \qquad \lambda(t_j^+) \in \text{sign}(x_1(t_j^+)). \end{cases} \tag{9}$$

We assume $\mathbf{x}(t)$ is an AC solution for all $t < t_j$, that is $\mathbf{x}(t_j^-)$ is satisfying the constraint (6) and there exists $\lambda(t_j^-) \in \text{sign}(x_1(t_j^-))$ such that $0 = \lambda(t_j^-) + |x_1(t_j^-)| - x_2(t_j^-)$. Multiplying the first and second lines of (9) by B_2 and B_1, respectively, one can eliminate σ_z in (9) which can be rewritten as:

$$\begin{cases} B_2 \left(x_1(t_j^+) - x_1(t_j^-)\right) = B_1 \left(x_2(t_j^+) - x_2(t_j^-)\right) \\ \qquad 0 \in \lambda(t_j^+) + |x_1(t_j^+)| - x_2(t_j^+), \\ \qquad \lambda(t_j^+) \in \text{sign}(x_1(t_j^+)) \end{cases}$$

Note that for now we do not enforce conditions for existence of a continuous solutions at t_j^+ immediately after the jump: we only define jump solutions respecting both formulations with the measures, and the constraint at t_j^+. Let us analyse the jump dynamics in (9). In a similar fashion to Proposition 2, we give the solutions of (9) depending on the parameters B_1, B_2.

Proposition 3. (Jump Dynamics analysis). Let consider the case $B_1 \neq 0$: if $B_2/B_1 < -1$ there is a unique solution to (9), otherwise if $B_2/B_1 \geq -1$ there are either one or several solutions depending on $\mathbf{x}(t_j^-)$. Let us now consider $B_1 \neq 0$: if $x_2(t_j^-) \in [-1, 1]$ there are infinitely many solutions, otherwise there is only one solution.

Proof.

- Assume $\underline{B_1 \neq 0}$, then (9) can be further reduced into:

$$0 \in \; \text{sign}\left(x_1(t_j^+)\right) + |x_1(t_j^+)| - \frac{B_2}{B_1}\left(x_1(t_j^+) - x_1(t_j^-)\right) - x_2(t_j^-), \tag{10}$$

which is an equation of the form:

$$0 \in f(x) + \mathcal{F}(x)\,, \tag{11}$$

where $\mathcal{F} : \mathbb{R} \rightrightarrows \mathbb{R}$ is the maximal monotone operator $\mathrm{sign}(x)$, and $f : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function. In particular, here we have:

$$f(x) = \mathrm{a}x + \mathrm{b}|x| + \mathrm{c}\,, \tag{12}$$

with $a = \frac{-\mathrm{B}_2}{\mathrm{B}_1}$, $b = 1$, and $c = \frac{\mathrm{B}_2}{\mathrm{B}_1} x_1(t_j^-) - x_2(t_j^-)$. We can first notice that by assumption $\mathbf{x}(t_j^-)$ is solution of this generalised equation and for jump dynamics there is always existence of solutions. We now give a more in-depth study of the solutions of (11) in a general context, as this will prove to be useful for the study of numerical solutions in Section 2.4. Let now analyse the conditions for existence and/or uniqueness of solutions to such generalised equation. Proofs will be given in a succinct manner using Fig. 3 as a support. Again a more detailed study can be found in the associated report Rocca et al. (2019).

First, let us assume $a - b \neq 0$ and $a + b \neq 0$ [5]. Then, depending on the sign of $a - b$ and $a + b$ we define an associated continuous piecewise linear function $h(y)$.

(a) Let first consider $a - b > 0$ and $a + b > 0$ which is equivalent to $\mathrm{B}_2/\mathrm{B}_1 < -1$. We define $h(y) = f^{-1}(\mathbf{x})$ with $h(y) = \frac{y-c}{a-b}$ if $y \leq c$ and $h(y) = \frac{y-c}{a+b}$ if $y \geq c$. We note that $\mathrm{sign}(x) = \partial|x|$ and the conjugate of $g(x) = |x|$ is $g^*(x) = \psi_{[-1,1]}(y)$ with $\psi_{\mathrm{K}}(\cdot)$ the indicator function of the set K. It follows from Rockafellar and Wets (2009) that the inverse of $\mathrm{sign}(x)$ is $\mathcal{N}_{[-1,1]}(y) = \partial\psi_{[-1,1]}(y)$. Consequently, the generalised equation (11) is equivalent to:

$$0 \in h(y) + \mathcal{N}_{[-1,1]}(y)\,, \tag{13}$$

with $\mathcal{N}_{[-1,1]}(y)$ the normal cone to $[-1, 1]$ in y. Finally, $\mathrm{h}(\cdot)$ is continuous on $\mathbb{R}$ and as $[-1, 1]$ is a compact set in $\mathbb{R}$, it follows from (Facchinei and Pang, 2007, Corollary 2.2.5) that there is always existence of solutions to the general equation (13) (and equivalently (10)). Additionally, (Facchinei and Pang, 2007, Theorem 2.3.3) provides sufficient conditions for uniqueness: if $h(\cdot)$ is strictly monotone. Strict monotonicity of $h(\cdot)$ holds if and only if $a + b > 0$ and $a - b > 0$, as it can be seen on Fig. 3. We recall that if there is a unique solution then this solution is $\mathbf{x}(t_j^+) = \mathbf{x}(t_j^-)$, and $\sigma_z = 0$. We have proved that $a - b > 0$ and $a + b > 0$ are sufficient conditions for existence and uniqueness of solutions to (12). In addition, these sufficient conditions do not depend of c. However, they are not necessary conditions.

(b) Let us study the case where $a - b < 0$ and $a + b > 0$, which is equivalent to $\mathrm{B}_2/\mathrm{B}_1 \in]-1, 1[$. Then, we can try to build another "piecewise function" $h(y)$ by inverting the equation $y = ax + b|x| + c$ for all $x \in \mathbb{R}$. However, it follows that the resulting $h(\cdot)$ is a multi-valued operator defined on $[c, +\infty[$. Consequently, if $c > 1$ there are no solutions. If $c \leq 1$, there is either a unique solution for $c = 1$, or multiple solutions if $c < 1$. A similar reasoning can be done for the case with $a - b > 0$ and $a + b < 0$. However, the latter cannot occur if $b = 1$ as in (10).

(c) Let now consider the case where $a-b < 0$ and $a+b < 0$, which is equivalent to $\mathrm{B}_2/\mathrm{B}_1 > 1$. Then, the associated

[5] We remark the case $a - b = 0$ or $a + b = 0$ corresponds to B parallel to the right or left constraint. This means that DAEs in modes 1 and/or 2 are non-regular with $\mathrm{C}_1\mathrm{B}$ and $\mathrm{C}_2\mathrm{B}$ singular.

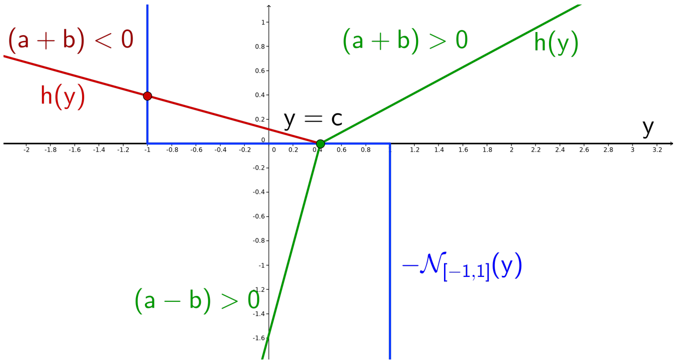

Fig. 3. solutions of the the generalised equation (13) for various sign of $(a + b)$ and $(a - b)$.

piecewise function $h(y)$ is continuous over $\mathbb{R}$ and again using (Facchinei and Pang, 2007, Corollary 2.2.5) we can prove there is always existence of solutions, independently of c. Additionally, there is uniqueness if and only if $c > 1$ or $c < 1$.

(d) Finally, if $a - b = 0$ or $a + b = 0$, which correspond to $\mathrm{B}_2/\mathrm{B}_1 = 1$ or -1, then one part of $h(y)$ is multi-valued. For example if $(a - b) = 0$ and $(a + b) > 0$, $h(\cdot)$ is defined on $[c, +\infty[$ with $h(y) \in \mathbb{R}^-$ if $y = c$, and $h(y) = \frac{y-c}{a+b}$ if $y \geq c$: if $c > 1$ there is no solutions. In a similar way, if $(a + b) < 0$ then $h(\cdot)$ is defined on $]-\infty, c]$ and there is no solution if $c < -1$. Similar reasoning can be done for the cases with $a + b = 0$.

- Assume $\underline{\mathrm{B}_1 = 0}$, then from (9) it follows:

$$\begin{cases} x_1(t_j^+) = x_1(t_j^-) \\ x_2(t_j^+) = x_2(t_j^-) + \mathrm{B}_2\sigma_z \\ x_2(t_j^+) \in |x_1(t_j^-)| + \mathrm{sign}(x_1(t_j^-)) \end{cases} \tag{14}$$

We note that if $x_1(t_j^-) \neq 0$, there is a unique solution since $|x_1(t_j^-)| + \mathrm{sign}(x_1(t_j^-))$ is uniquely defined on $\mathbb{R}\backslash\{0\}$. If $x_1(t_j^-) = 0$, then there are multiple solutions (infinitely many) with $x_2(t_j^+) \in [-1, 1]$. □

For the sake conciseness we will not enter into the detailed analysis of all the consistent jumps in this paper. In Rocca et al. (2019), we provide a detailed study of the AC consistent initialisation associated with this example by crossing the information from the complete proof of Proposition 2, and from the analysis of the jump dynamics.

2.4 Analysis of a time-stepping Backward Euler scheme

Let now consider the backward Euler discretization of system (5):

$$\begin{cases} x_{1,k+1} - x_{1,k} = h(1 + \mathrm{B}_1 z_{k+1}) \\ x_{2,k+1} - x_{2,k} = h\mathrm{B}_2 z_{k+1} \\ \qquad 0 \in \mathrm{sign}(x_{1,k+1}) + |x_{1,k+1}| - x_{2,k+1} \end{cases}, \tag{15}$$

with h a fixed time step.

Well-posedness of backward Euler discretization One sees that (15) has a structure quite close to (8), which puts the backward scheme as favourable perspective for the computations of solutions with jumps. This is the object

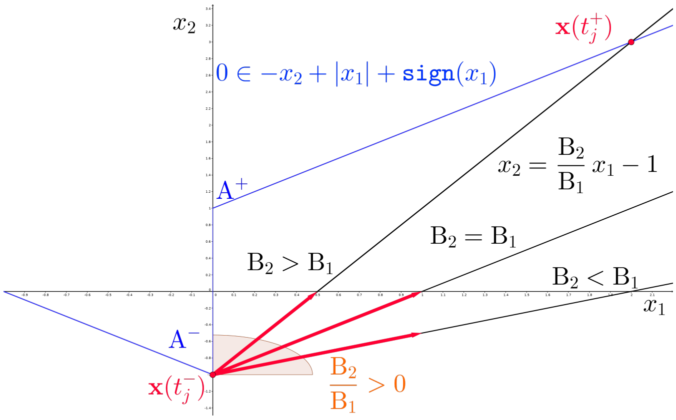

Fig. 4. Solutions of the generalised Equation (10) associated with jumps from $\mathbf{x}(t_j^-) = (0, -1)$ for various choices of $B = (B_1, B_2)^T$.

of the next analysis. Using the same method as in the previous section, we can eliminate z_{k+1} and write:

$$\begin{cases} B_2(x_{1,k+1} - x_{1,k}) = hB_2 + B_1(x_{2,k+1} - x_{2,k}) \\ \qquad\qquad 0 \in \operatorname{sign}(x_{1,k+1}) + |x_{1,k+1}| - x_{2,k+1} \end{cases}$$

- If $\underline{B_1 \neq 0}$, then we obtain the generalised equation:

$$0 \in -\frac{B_2}{B_1} x_{1,k+1} + |x_{1,k+1}| + \left(-x_{2,k} + \frac{B_2}{B_1}(x_{1,k} + h)\right) + \operatorname{sign}(x_{1,k+1}) \tag{16}$$

Sufficient conditions for uniqueness are the same as in case **(a)** in Section 2.3.1, identifying:

$$f(x_{1,k+1}) = a x_{1,k+1} + b|x_{1,k+1}| + c\,,$$

with $a = -\frac{B_2}{B_1}$, $b = 1$, and $c = \left(-x_{2,k} + \frac{B_2}{B_1}(x_{1,k} + h)\right)$, we need $h(y) = f^{-1}(y)$ to be strictly monotone.
Consequently, *sufficient conditions* for uniqueness are: $a - b = -(1 + \frac{B_2}{B_1}) > 0$, and $a + b = 1 - \frac{B_2}{B_1} > 0$, which is the previously studied case (a). We note that this corresponds to a subset of the valid $B = (B_1, B_2)^T$ for Proposition 2: $B_1 \neq 0$, $\frac{B_2}{B_1} < 0$ and $(B_1 + B_2)B_2 > 0$, the subset of "sliding-crossing" solutions. In this subset of B there is also uniqueness of the continuous solution. Furthermore, the condition $a - b > 0$ excludes the constant solutions with $B_2 = 0$ where there is not necessarily uniqueness of the solution. An important remark when $(a - b) > 0$ and $(a + b) > 0$ is that well-posedness of the discrete solution is independent of the time step h as there is no condition on c.

However, using similar arguments we can show that when $(a - b) < 0$, $(a + b) > 0$ (case **(c)** in Section 2.3.1) and $\frac{B_2}{B_1} \leq 0$ (corresponding to the previously described sliding-repulsive and the constant solutions), the implicit Euler scheme can output multiple solutions for any $h \geq 0$. For example, this is the case if $\mathbf{x}_k = (-2, 1)$ and $B_2/B_1 < 0$. Then, for any $h \leq 2$ there are 2 solutions one with $x_{1,k+1} < 0$ and one with $x_{1,k+1} = 0$ (if $h \geq 2$ the second solution is in $x_{1,k+1} > 0$). Consequently, we need a method to separate the solution approximating the continuous time solution from the others.

- If $\underline{B_1 = 0}$ for all $x_{1,k} \neq -h$ there is a unique solution corresponding to either the continuous solution or the jump. If $x_{1,k} = -h$, there is a non unique solution in $[-1, 1]$.

We conclude that in this particular example the set of vectors B for which there is always uniqueness of solutions for the discrete scheme, is a subset of the set of B where there are globally continuous solutions. Additionally, (with the exception of $B_2 = 0$) this corresponds to the subset where there is uniqueness of the continuous solution (w.r.t. the initial condition). Apart from this sufficient conditions well-posedness, the Euler scheme outputs either no solutions, one solution, or several solutions [6], depending on the initial condition $\mathbf{x}_k$ and the time step h. In particular, it can be shown on this example that the lack of solution to the discretization always corresponds to the lack of solutions to the MDI (8) on an interval of size h.

Even though the Euler discretization outputs spurious results that do not correspond to the correct behaviour of the continuous time system, it seems to always contain, for h sufficiently small, the one approximating at O(h) the continuous solution. In particular, this implicit Euler scheme provides a consistent initialisation on an index-2 DAE with a generalised equation corresponding to a finite union of strongly connected hyper-planes of the form $C_i\mathbf{x} + e_i = 0$ (see Proposition 3 in Rocca et al. (2019)). Such generalised equation corresponds to our current example.

Minimal implicit Euler discretization As we have seen in the previous Section, the classical implicit Euler discretization may output multiple solutions and this for $h > 0$ as small as wanted (for example in some cases of $(a-b) < 0$ and $(a+b) > 0$). One needs to refine the results of the implicit Euler discretization to select the discrete solution close the continuous time solution. To this aim, we propose a minimisation over the results of the backward Euler scheme in order to keep the solutions minimal in the Euclidean norm.

Proposition 4. (Minimal Backward Euler). Consider a nonsmooth DAE system:

$$\begin{cases} \dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{z} + \mathbf{b} \\ 0 \in \mathcal{F}(\mathbf{x}, \mathbf{z})\,, \end{cases} \tag{17}$$

with $\mathbf{x}$ the differential variables, $\mathbf{z}$ the algebraic variables, and where the solutions of the generalized equation [7] $0 \in \mathcal{F}(\cdot)$ can be represented as a finite union of strongly connected hyper-planes of the form $C_i\mathbf{x} + D_i\mathbf{z} + e_i = 0$ such that (17) is of differentiation index less or equal to two. If there exists a unique solution $Y(t) = (\mathbf{x}(t), \mathbf{z}(t))$ such that $\mathbf{x}(t)$ is AC on an interval $[t_0, t_0 + \varepsilon]$ then there exists a time step $h > 0$ such that the minimal backward Euler scheme:

$$\begin{aligned} p^*_{k+1} :=& \min_{\mathbf{x}_{k+1}, \mathbf{z}_{k+1}, \boldsymbol{\lambda}_{k+1}} \frac{1}{2}\left\|\mathbf{x}_{k+1} - \mathbf{x}_k\right\|^2, \\ &\text{s.t} \quad \mathbf{x}_{k+1} - \mathbf{x}_k = hA\mathbf{x}_{k+1} + hB\mathbf{z}_{k+1} + h\mathbf{b} \\ &\qquad\quad 0 \in \mathcal{F}(\mathbf{x}_{k+1}, \mathbf{z}_{k+1})\,, \end{aligned} \tag{18}$$

provides a consistent discrete solution to (17). This means that given $Y_k = Y(t_k)$ and $Y^*_{k+1} = \operatorname{argmin}(p^*_{k+1})$ then $\left\|Y^*_{k+1} - Y(t_k + h)\right\| \to 0$ when $h \to 0$, which can be simplified to $\left\|Y^*_{k+1} - Y_k\right\| = O(h)$.

[6] possibly infinitely many if $B_1 = 0$ and $x_{1,k} = -h$.
[7] In our working example, the generalized equation can be expressed as a Mixed Linear Complementarity Problem.

The idea is to notice that the Euler method gives an $O(h)$ approximation of the solution of an index-2 linear DAE with constant coefficients (Brenan et al., 1996, Theorem 3.1.1). Consequently, if there exists a local continuous solutions for $t \in [t_0, t_0 + \varepsilon]$ on a hypersurface $\mathrm{C}_i\mathbf{x} + \mathrm{D}_i\mathbf{z} + e_i = 0$ and if the implicit Euler scheme, for $h < \varepsilon$, is such that $\mathrm{C}_i\mathbf{x}_{k+1} + \mathrm{D}_i\mathbf{z}_{k+1} + e_i = 0$, then this solution is a O(h) approximation of $\mathbf{x}(t_0 + \varepsilon)$. Although the implicit Euler scheme outputs multiple solutions, if one of the solutions is still on the same constraint as $\mathbf{x}_k$, we know this solution is an $O(h)$ approximation. A proof on the above particular example is given in the associated report Rocca et al. (2019).

2.5 Implementation and numerical results

In this section we expose some simulation results of the implicit Euler scheme (18) on the example studied in the previous sections. In particular, numerical experiments demonstrate that for example (5), if there exists at least one continuous solution, then (18) converges in $O(h)$ to one of these solutions. Furthermore, if the discretization (15) yields a unique solution for any step size, then it converges in $O(h)$ to this unique solution of (5). Implementation has been performed using the software SICONOS 4.2.0 (see Acary and Pérignon (2007)), a platform for numerical simulation of non-smooth dynamical systems. The code of these simulations can be found in the github repository associated with the SICONOS examples [8]. In this section, performance results are not discussed as the optimisation problem in (18) is currently solved by enumeration of all [9] the solutions of the generalised equation associated with the classical implicit Euler scheme (15).

We implement this problem by formulating (6) as a Linear Complementarity Problem (LCP) with an equality constraint, also called Mixed Linear Complementarity Problem (MLCP).

$$\begin{cases} 0 = -x_2 + |x_1| + \alpha \\ 0 \le |x_1| + x_1 \perp |x_1| - x_1 \ge 0 \\ 0 \le x_1^- \perp \alpha \ge -1 \\ \alpha \le 1 \perp x_1^+ \ge 0 \end{cases} \tag{19}$$

with $\alpha \in \operatorname{sign}(x_1)$. This yields a linear complementarity system (LCS) [10]:

$$\begin{cases} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{z}(t) \end{pmatrix} = \begin{pmatrix} 1 + \mathrm{B}_1 z(t) \\ \mathrm{B}_2 z(t) \\ x_1 - x_2 + 1 + \mathbf{r}(\boldsymbol{\lambda}(t)) \end{pmatrix} \\ \mathbf{r}(\boldsymbol{\lambda}(t)) = \lambda_1(t) - \lambda_2(t) \\ 0 \le 2x_1(t) + \lambda_1(t) \perp \lambda_1(t) \ge 0 \\ 0 \le \lambda_3(t) + x_1(t) \perp \lambda_2(t) \ge 0 \\ 0 \le 2 - \lambda_2(t) \perp \lambda_3(t) \ge 0 \end{cases} \tag{20}$$

with $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3) = \left(|x_1| - x_1, 1 - \alpha, x_1^-\right)$. Some numerical results can be found in Fig. 5 and Fig. 6. In these figures, we consider the particular case of sliding-crossing solutions (here $\mathrm{B} = (-0.5, 1)^{\mathrm{T}}$) where uniqueness of AC solutions and discrete solutions is guaranteed. We

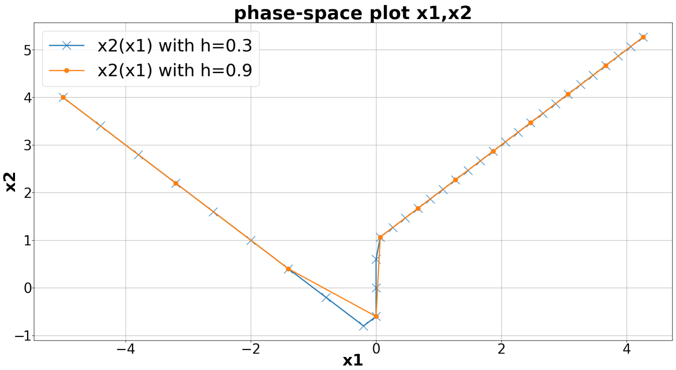

Fig. 5. Phase space plot in (x_1, x_2) of the numerical solutions for $\mathrm{B} = (-0.5, 1)$ and $h = 0.9$ or $h = 0.3$. Initial condition is $\mathbf{x}_0 = (-5, 4)$

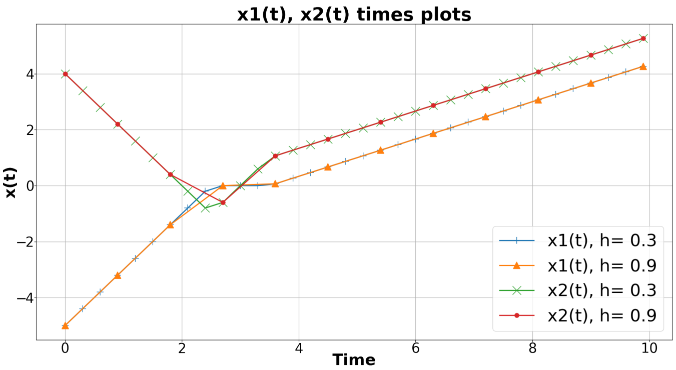

Fig. 6. Time plots of the solutions $x_1(t)$ and $x_2(t)$ for $\mathrm{B} = (-0.5, 1)$ and $h = 0.9$ or $h = 0.3$. Initial condition is $\mathbf{x}_0 = (-5, 4)$

notice that the resulting solutions in $\mathbf{x}(t)$ are Lipschitz continuous, and run through all the modes (the initial condition is taken with $x_1(t_0) < 0$). In Fig. 7, the error term $\|Y(T) - Y_k\|$ in function of the step size h is depicted. The term Y_k is the numerical approximation of $Y(T)$ by the minimal implicit Euler numerical scheme when the interval $[t_0, T]$ is subdivided in k steps of size h. In the case of sliding-crossing solutions, we choose $\mathrm{B} = (-0.5, 1)^{\mathrm{T}}$, $\mathbf{x}_0 = \mathbf{x}(t_0) = (-5, 4)$, and $T = 10$. In the case of sliding-repulsive solutions, we choose $\mathrm{B} = (-1, 0.5)^{\mathrm{T}}$, $\mathbf{x}_0 = \mathbf{x}(t_0) = (0, 0)$, and $T = 10$. Time steps are taken linearly spaced in log scale. We observe the linear convergence rate of the implicit Euler scheme when there is uniqueness of the numerical solutions. In addition, we also observe a linear convergence rate of the minimal implicit Euler scheme when there is non-uniqueness of the discrete solution (for any time step) as it is shown in Fig 7 on the curve associated with the sliding-repulsive case.

3. CONCLUSION AND FUTURE WORK

In a first time, we analysed the AC solution of a 2D example of hybrid DAE. We show that in the context of piecewise linear constraints, we can observe multiplicity of AC solutions. Furthermore, it is not enough to study each mode independently to conclude on the well-posedness of AC solutions. In a second time, we studied the generalised equation resulting from the example jump dynamic. We conclude on the conditions for well-posedness of such equation, and build a framework for the study

[8] `https://github.com/siconos/siconos-tutorials/tree/master/.sandbox/code_IFAC`
[9] In most cases limited to maximum 3 solutions.
[10] Please note that this LCS formulation is not unique as it depends of the naming convention for the λ_i variables.

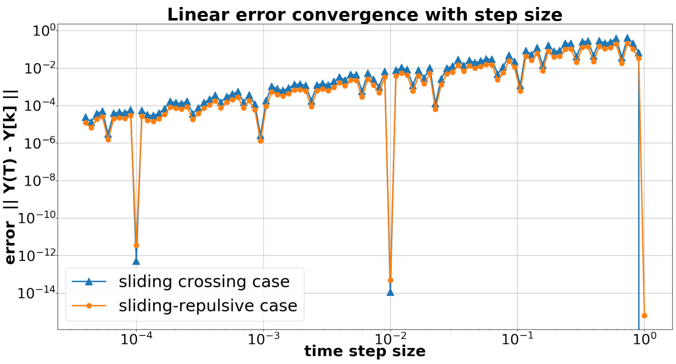

Fig. 7. Error $\|Y(t) - Y_k\|$ with respect to time step h. We consider the two kind of AC solutions: the sliding-crossing solutions and the sliding-repulsive solutions.

of numerical solutions. Indeed, in the two last sections, we show that solutions of implicit Euler scheme, which is classically used as an event-capturing scheme for non-smooth dynamical systems, are solutions of an equation with the same structure as the jump dynamics. It follows that such numerical scheme can have either none, a unique, several, or infinitely many solutions depending of small variations on the considered problem. In particular, consistence of the numerical scheme is not preserved in some cases. However, on this example it can be proven that a "correct" discrete solution can always be retrieved from the numerous solutions of the implicit Euler scheme. Consequently, we propose a minimal implicit Euler numerical scheme to select the correct solutions assuming a time step sufficiently small. In future work, we will extend these results and observations to more general dynamics and switching constraint. Another, interesting research direction would be to make the link with Camlibel et al. (2016).

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