Lexicographic optimal chains and manifold triangulations

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Abstract

Given a simplicial complex K, we consider the problem of finding a simplicial d-chain minimal in a given homology class. This is sometimes referred to as the *Optimal Homologous Chain Problem* (OHCP).

We consider here simplicial chains with coefficients in $\mathbb{Z}/2\mathbb{Z}$ and the particular situation where, given a total order on d-simplices $\sigma_1 < \ldots < \sigma_n$, the weight of simplex i is 2^i. In this case, the comparison of chains is a *lexicographic ordering.*

Similarly, we consider the problem of *finding a minimal chain for a prescribed boundary.* We show that, for both problems, the same matrix reduction algorithm used for the computation of homological persistence diagrams, applied to the filtration induced by the order on p-simplices, allows a $\mathcal{O}(n^3)$ worst case time complexity algorithm.

Second, we consider OHCP in the particular case where K is a $(d+1)$-pseudo-manifold, for example when it triangulates a $(d+1)$-sphere. In this case, there is a $\mathcal{O}(n \log n)$ algorithm which can be seen, by duality, as a *lexicographic minimum cut* in the dual graph of K.

Third, we introduce a total order on n-simplices for which, when the points lie in the n-Euclidean space, the support of the lexicographic-minimal chain with the convex hull boundary as boundary constraint is precisely the n-dimensional Delaunay triangulation, or in a more general setting, the regular triangulation of a set of weighted points.

Fourth, we apply the lexicographic min-cut on the dual graph of the 3-dimensional Delaunay complex of a points cloud in $\mathbb{R}^3$. This gives in practice an efficient algorithm providing minimal solutions that, while inheriting the optimality for 2-dimensional Delaunay triangulations, reveals to create pertinent and convincing meshes for surface reconstruction, in particular in the case of noisy point clouds with non uniform densities and outliers, such as those produced by physical captures or Structure From Motion algorithms.

Last, it is known that a Čech complex with the right parameter over a point cloud sampling a compact connected C^2 manifold embedded in Euclidean space, if the sampling density is high enough with respect to the manifold reach, captures the homotopy type of the manifold and in particular its fundamental class. In the particular case of 2-manifolds, we show that, under good sampling conditions, the lexicographic minimal chain representative of the image of the fundamental class in the Čech complex provides a triangulation of the manifold.

Contents

Chapter 1

Introduction

Given a point cloud sampling a subset of Euclidean space, *shape reconstruction* consists in retrieving a shape that is both geometrically and topologically close to the sampled subset. It has been shown that, when the given point cloud satisfies specific sampling conditions related to some regularity measure on the subset, simplicial complexes such as alpha shape [28], Čech or Rips complexes built on the point cloud share the homotopy type of the set [17, 6].

In this work, beyond creating a simplicial complex sharing the homotopy type, we aim to compute, when the object is a smooth manifold, a homeomorphic simplicial complex, in other words a manifold *triangulation*.

Motivation: minimal chain as manifold triangulation Indeed when the unknown shape is assumed to be a connected, compact d-manifold without boundary (respectively with a given, known boundary) it has a unique d-homology class (respectively relative d-homology class), called the *fundamental class*. In this case, the support of a minimal chain representative of this d-cycle in a simplicial complex that reproduces the homology of the shape (Čech, Rips, or alpha-shape) is a reasonable candidate for a manifold triangulation. We show indeed in Section 9 that this is the case for smooth 2-manifolds embedded in n-dimensional Euclidean space.

The L^1 norm on simplicial d-chains is defined by the weight assigned to each p-simplex and, when coefficients are taken in $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, the norm of the chain is just the sum of the weights of the simplices in the chain.

Delaunay weights and Delaunay lexicographic order For this purpose, we use a family of weights, called here *Delaunay Energy*, motivated by the lifted paraboloid construction for Delaunay triangulation. Such weights have been introduced first in [22] and then further used to optimize triangulations [1, 21]. In fact the same weights can be used to optimize directly in the affine space of homologous chains. Indeed, it has been observed ([4] and Lemma 6.13) that given a point cloud in the Euclidean space $\mathbb{R}^n$, in the *complete* n-dimensional simplicial complex, i.e. the complex containing as n-simplices all the $(n+1)$-tuples of points, the support of the n-chain minimal for the norm induced by these weights, with the cloud convex hull boundary as boundary constraint, coincides with Delaunay triangulations.

A particular choice of these weights makes the L^1 minimization of chains over coefficients in $\mathbb{Z}_2$ equivalent to a minimization along a lexicographic order on chains induced by a given order on simplices. Indeed, if there is an order on simplices such that the weight of each simplex σ is greater than the sum of weights of all simplices smaller than σ (for example when simplex i has weight 2^i), then comparing the respective L^1 norms of two chains is equivalent to comparing these two chains in lexicographic order (Section 8).

Minimal homology generators The computation of minimal simplicial homology generators has been a wide subject of interest for its numerous applications related to shape analysis, computer graphics or computer-aided design. Coined in [24], we recall the Optimal Homologous Chain Problem (OHCP):

Problem 1.1 (OHCP)**.** *Given a p-chain A in a simplicial complex K and a set of weights given by a diagonal matrix W of appropriate dimension, find the* 1*-norm minimal chain $\Gamma_{\min}$ homologous to A:*

$$\Gamma_{\min} = \min_{\Gamma, B} ||W \cdot \Gamma||_1 \text{ such that } \Gamma = A + \partial_{d+1} B \text{ and } \Gamma \in \boldsymbol{C_p}(K), B \in \boldsymbol{C_{p+1}}(K)$$

It has been shown that OHCP is NP-hard in the general case when using coefficients in $\mathbb{Z}_2$ [20, 16].

Contributions Sections 3, 4 and 5 consider a specialization of OHCP: the Lexicographic Optimal Homologous Chain Problem (Lex-OHCP). Using coefficients in $\mathbb{Z}_2$, minimality is meant according to a lexicographic order on chains induced by a total order on simplices. Formulated in the context of OHCP, this would require ordering the simplices using a total order and taking a weight matrix W with generic term $W_{ii} = 2^i$, allowing the L^1-norm minimization to be equivalent to a minimization along the lexicographic order. We first show how an algorithm based on the matrix reduction algorithm used for the computation of persistent homology [27] allows to solve this particular instance of OHCP in $O(n^3)$ worst case complexity (Section 3). Using a very similar process, we then show that the problem of finding a minimal d-chain bounding a given $(d-1)$-cycle admits a similar algorithm with the same algorithmic complexity (Section 4). Finally, Section 5 considers Lex-OHCP in the case where the simplicial complex K is a strongly connected $(d+1)$-pseudomanifold. By formulating it as a Lexicographic Minimum Cut (LMC) dual problem, the algorithm can be improved to a quasilinear complexity.

In Section 6, we introduce a total order on triangles for which the support of the minimal chain is precisely the regular triangulation, a generalization of Delaunay triangulation, in the case of a point cloud in euclidean space. Section 7 explores the particular case of 2-chains and state some properties of lexicographic minimal 2-chains immersed in n-Euclidean space.

Equipped with this *Delaunay order*, Section 8 provides an application of the developed Lex-OHCP algorithms to point cloud triangulation. We describe an open surface algorithm given an input boundary as well as a watertight surface reconstruction algorithm given an interior and exterior information based on a lexicographic minimum cut on the dual graph of a 3-dimensional Delaunay complex.

Section 9 studies C^2 2-manifolds embedded in n-dimensional Euclidean space and proves, under good sampling condition with respect to the manifold reach, that both the lexicographic-minimal chain and a certain L^1 norm-minimal chain (Remark 9.2) realize a triangulation of the manifold (Theorem 9.1). The proof of Theorem 9.1 is given for completeness, as it supports our initial motivation of using a lexicographic order for manifold triangulation. However, the reader should considered it as work in progress, as current works are considering simpler arguments as well as an extension to higher dimensions, assuming a "protection" on the sampling set (Section 5.4 in [11]). The proof of Theorem 9.1 makes use of [12] for its final global argument, which relies heavily on the Whitney Lemma. Although we consider this argument to be the most natural, an alternative argument to the Whitney Lemma is given in Appendix F.

Related work Several authors have studied algorithm complexities for the computation of L^1-norm optimal cycles in homology classes [30, 18, 16, 19, 26, 20, 24, 25]. However, to the best of our knowledge, considering lexicographic-minimal chains in their homology classes is a new idea. When minimal cycles are of codimension 1 in a pseudo-manifold, the idea of considering the minimal cut problem on the dual graph has been previously studied. In particular, Chambers et al. [16] have considered graph duality to derive complexity results for the computation of optimal homologous cycles on 2-manifolds. Chen et al. [20] also use a reduction to a minimum cut problem on a dual graph to compute minimal non-null homologous cycles on n-complexes embedded in $\mathbb{R}^n$. Their polynomial algorithm (Theorem 5.2.3 in [20]) for computing a homology class representative of minimal radius is reminiscent of our algorithm for computing lexicographic minimal representatives (Algorithm 4). In a recent work [25], Dey et al. consider the dual graph of pseudo-manifolds in order to obtain polynomial time algorithms for computing minimal persistent cycles. Since they consider arbitrary weights, they obtain the $\mathcal{O}(n^2)$ complexity of best known minimum cut/maximum flow algorithms [39]. The lexicographic order introduced in our work can be derived from the idea of a variational formulation of the Delaunay triangulation, first introduced in [22] and further studied in [1, 21].

Many authors have designed practical algorithms for the triangulation of point set in $\mathbb{R}^3$ sampling an embedded 2-manifold. We review a few of them in section 8.1.

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Chapter 2

Definitions

Simplicial complexes Consider an independent family $A = (a_0, ..., a_d)$ of $(d+1)$-points of $\mathbb{R}^N$. We call a d-simplex σ spanned by A the set of all points $x = \sum_{i=0}^{d} t_i a_i$ where $\forall i \in [0,d], t_i \geq 0$ and $\sum_{i=0}^{d} t_i = 1$. Any simplex spanned by a subset of A is called a face of σ.
A simplicial complex K is a collection of simplices such that:

1. Every face of a simplex of K is in K.

2. The intersection of two simplices of K is either empty or a face of both simplices.

Simplicial chains. Let K be a simplicial complex of dimension at least d. The notion of chains can be defined with coefficients in any ring but we restrict here the definition to coefficients in the field $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$. A d-chain A with coefficients in $\mathbb{Z}_2$ is a formal sum of d-simplices :

$$A = \sum_i x_i \sigma_i, \text{ with } x_i \in \mathbb{Z}_2 \text{ and } \sigma_i \in K \tag{2.1}$$

Interpreting the coefficient $x_i \in \mathbb{Z}_2 = \{0,1\}$ in front of simplex σ_i as indicating the existing of σ_i in the chain A, we can view the d-chain A as a set of simplices : for a d-simplex σ and a d-chain A, we write that $\sigma \in A$ if the coefficient for σ in A is 1. With this convention, the sum of two chains corresponds, when chains are seen as set of simplices, to their symmetric difference. We denote $\boldsymbol{C_d}(K)$ the vector space over the field $\mathbb{Z}_2$ of d-chains in the complex K.

Boundary operator. For a d-simplex $\sigma = [a_0, ...a_d]$, the boundary operator is defined as the operator:

$$\partial_d : \boldsymbol{C_d}(K) \to \boldsymbol{C_{d-1}}(K)$$
$$\partial_d \sigma \underset{\text{def.}}{=} \sum_{i=0}^{d} [a_0, ..., \widehat{a_i}, ...a_d]$$

where the symbol $\widehat{a_i}$ means the vertex is deleted from the array. The kernel of the boundary operator $Z_d = \operatorname{Ker} \partial_d$ is called the group of d-cycles and the image of the operator $B_d = \operatorname{Im} \partial_{p+1}$ is the group of d-boundaries. We say two d-chains A and A' are homologous if $A - A' = \partial_{d+1} B$, for some $(d+1)$-chain B.
In what follows, a d-simplex σ can also be interpreted as the d-chain containing only the d-simplex σ.

Lexicographic order. We assume now a total order on the d-simplices of K, $\sigma_1 < ... < \sigma_n$, where $n = \dim \boldsymbol{C_d}(K)$. From this order, we define a lexicographic total order on d-chains: Recall that wih coefficients in $\mathbb{Z}_2$ one has $C_1 + C_2 = C_1 - C_2$.

Definition 2.1 (Lexicographic total order)**.** *For $C_1, C_2 \in \boldsymbol{C_d}(K)$:*

$$C_1 \sqsubseteq_{lex} C_2 \underset{\textit{def.}}{\Longleftrightarrow} \begin{cases} C_1 + C_2 = 0 \\ \text{or} \\ \sigma_{\max} = \max\{\sigma \in C_1 + C_2\} \in C_2 \end{cases}$$

This total order naturally extends to a strict total order $\sqsubset_{lex}$ on $\boldsymbol{C_d}(K)$:

$$C_1 \sqsubset_{lex} C_2 \underset{\text{def.}}{\Longleftrightarrow} \begin{cases} C_1 + C_2 \neq 0 \\ \text{and} \\ \sigma_{\max} = \max\{\sigma \in C_1 + C_2\} \in C_2 \end{cases}$$

Chapter 3

Homologous lexicographic-minimal chains

In this section, we construct an algorithmic solution to the following problem:

Problem 3.1 (Homologous Lexicographic-Minimal Chain Problem)**.** *Given a simplicial complex K with a total order on the d-simplices and a d-chain $A \in \boldsymbol{C_d}(K)$, find the unique chain $\Gamma_{\min}$ defined by :*

$$\Gamma_{\min} = \min_{\sqsubseteq_{lex}} \{\Gamma \in \boldsymbol{C_d}(K) \mid \exists B \in \boldsymbol{C_{d+1}}(K), \Gamma - A = \partial_{d+1} B\}$$

This problem is a particular instance of OHCP (Optimal Homologous Chain Problem)[24] when the weight of i^{th} d-simplex is 2^i. We first introduce the notions of **reduction** and **total reduction** to reformulate Problem 3.1. We then recall the matrix reduction algorithm for persistence homology presented in [27] (page 153), slightly modified for the purpose of this section.

Definition 3.1. *A d-chain $A \in \boldsymbol{C_d}(K)$ is said **reducible** if there is a d-chain $\Gamma \in \boldsymbol{C_d}(K)$ (called **reduction**) and a $(d+1)$-chain $B \in \boldsymbol{C_{d+1}}(K)$ such that:*

$$\Gamma \sqsubset_{lex} A \quad \text{and} \quad \Gamma - A = \partial_{d+1} B$$

*If this property cannot be verified, the d-chain A is said **irreducible**.*
*If A is reducible, we call **total reduction** of A the unique irreducible reduction of A. If A is irreducible, A is said its own total reduction.*

Problem 3.1 can be reformulated as finding the total reduction of A.

With $m = \dim \boldsymbol{C_d}(K)$ and $n = \dim \boldsymbol{C_{d+1}}(K)$, we now consider the m-by-n boundary matrix ∂_{d+1} with entries in $\mathbb{Z}_2$. We enforce that rows of the matrix are ordered according to the strict total order on d-simplices: $\sigma_1 < \sigma_2 < \cdots < \sigma_m$, where the d-simplex σ_i is the basis element corresponding to the i^{th} row of the boundary matrix. The order of columns, corresponding to an order on $(d+1)$-simplices, is not relevant for this section and can be chosen arbitrarily.
The index of the lowest non-zero coefficient of a column R_j of matrix R is denoted by $\mathrm{low}(j)$. This index is not defined for a column whose coefficients are all zero.

Algorithm 1 is a slightly modified version of the boundary reduction algorithm presented in [27]. Indeed, for our purpose, we do not need the whole square boundary matrix $\partial : \bigoplus_j \boldsymbol{C_j}(K) \to \bigoplus_j \boldsymbol{C_j}(K)$ storing all the simplices of all dimensions and we apply the algorithm to the sub-matrix $\partial_{d+1} : \boldsymbol{C_{d+1}}(K) \to \boldsymbol{C_d}(K)$.

One checks easily that Algorithm 1 has $\mathcal{O}(mn^2)$ time complexity. We allow ourselves to consider each column R_j of matrix R, formally an element of $\mathbb{Z}_2^m$, as the corresponding d-chain in the basis $(\sigma_1, \ldots, \sigma_m)$.

Algorithm 1: Reduction algorithm for the ∂_{d+1} matrix

```
R = ∂_{d+1}
for j ← 1 to n do
    while R_j ≠ 0 and ∃j_0 < j with low(j_0) = low(j) do
        R_j ← R_j + R_{j_0}
    end
end
```

We now introduce a few lemmas useful for deriving an algorithm solving Problem 3.1.

Lemma 3.2. *After matrix reduction, a non-zero column $R_j \neq 0$ can be described as:*

$$R_j = \sigma_{\mathrm{low}(j)} + \Gamma,$$

where Γ is a reduction for $\sigma_{\mathrm{low}(j)}$.

Proof. As all matrix operations performed on R by the reduction algorithm are linear, each non-zero column R_j of R is in the image of ∂_{d+1}. Therefore, there exist a $(d+1)$-chain B such that :

$$R_j = \sigma_{\mathrm{low}(j)} + \Gamma = \partial_{d+1} B$$

which, in $\mathbb{Z}_2$, is equivalent to

$$\Gamma - \sigma_{\mathrm{low}(j)} = \partial_{d+1} B$$

By definition of the low of a column, we have immediately:

$$\Gamma \sqsubset_{lex} \sigma_{\mathrm{low}(j)}$$

We have thus shown that for each non-zero column, the largest simplex is reducible and the other d-simplices of the column are a reduction for it. $\square$

Lemma 3.3. *After the matrix reduction algorithm, there is a one-to-one correspondence between the reducible d-simplices and non-zero columns of R:*

$$\sigma_i \in \boldsymbol{C_d}(K) \text{ is reducible } \iff \exists j \in [1, n], R_j \neq 0 \text{ and } \mathrm{low}(j) = i$$

Proof. Lemma 3.2 shows immediately that the simplex corresponding to the lowest index of a non-zero column is reducible.

Suppose now that a d-simplex $\tilde{\sigma}$ is reducible and let $\tilde{\Gamma}$ be a reduction of it:

$$\tilde{\sigma} + \tilde{\Gamma} = \partial B \quad \text{and} \quad \tilde{\Gamma} \sqsubset_{lex} \tilde{\sigma}$$

Algorithm 1 realizes the matrix factorization $R = \partial_{d+1} \cdot V$, where matrix V is invertible [27]. It follows that $\operatorname{Im} R = \operatorname{Im} \partial_{d+1}$. Therefore the non-zero columns of R span $\operatorname{Im} \partial_{d+1}$ and since $\tilde{\sigma} + \tilde{\Gamma} = \partial B \in \operatorname{Im} \partial_{d+1}$, there is a family $(R_j)_{j\in J} = (\sigma_{\mathrm{low}(j)}, \Gamma_j)_{j\in J}$ of columns of R such that :

$$\tilde{\sigma} + \tilde{\Gamma} = \sum_{j\in J} \sigma_{\mathrm{low}(j)} + \Gamma_j$$

Every $\sigma_{\mathrm{low}(j)}$ represents the largest simplex of a column, and Γ_j a reduction chain for $\sigma_{\mathrm{low}(j)}$.

As observed in section VII.1 of [27], one can check that the low indexes in R are unique after the reduction algorithm. Therefore, there is a $j_{\max} \in J$ such that:

$$\forall j \in J \setminus \{j_{\max}\},\ \mathrm{low}(j) < \mathrm{low}(j_{\max})$$

$$
\begin{aligned}
\sigma_{j_{\max}} &= \max\{\sigma \in \sum_{j \in J} \sigma_{\text{low}(j)} + \Gamma_j\} \\
&= \max\left\{\sigma \in \tilde{\sigma} + \tilde{\Gamma}\right\} \\
&= \tilde{\sigma}
\end{aligned}
$$

We have shown that for the reducible simplex $\tilde{\sigma}$, there is a non-zero column $R_{j_{\max}}$ with $\tilde{\sigma} = \sigma_{\text{low}(j_{\max})}$ as its largest simplex. □

Lemma 3.4. *A d-chain A is reducible if and only if at least one of its d-simplices is reducible.*

Proof. If there is a reducible d-simplex $\sigma \in A$, A is reducible by the d-chain $A' = A - \sigma + Red(\sigma)$, where $Red(\sigma)$ is a reduction for σ.
We suppose A to be reducible. Let $\Gamma \sqsubset_{lex} A$ be a reduction for A and B the $(d+1)$-chain such that $\Gamma - A = \partial B$. We denote $\sigma_{\max} = \max\{\sigma \in A - \Gamma\}$. We show that $\sigma_{\max}$ is homologous to $\Gamma - A + \sigma_{\max}$:

$$\Gamma - A = \partial B \implies (\Gamma - A + \sigma_{\max}) - \sigma_{\max} = \partial B$$

The chain $\Gamma - A + \sigma_{\max}$ only contains simplices smaller than $\sigma_{\max}$, by definition of the lexicographic order (Definition 2.1). We have thus shown that if A is reducible, it contains at least one simplex that is reducible. □

Combining the three previous lemma give the intuition on how to construct the total reduction solving Problem 3.1: Lemma 3.4 allows to consider each simplex individually, Lemma 3.3 characterizes a reducible nature of a simplex using the reduced boundary matrix and Lemma 3.3 describes a column of the reduction boundary matrix as a simplex and its reduction. We now present Algorithm 2, referred to as the **total reduction algorithm**. For a d-chain Γ, $\Gamma[i] \in \mathbb{Z}_2$ denotes the coefficient of the i^{th} simplex in the chain Γ.

Algorithm 2: Total reduction algorithm

```
Inputs: A d-chain Γ, the reduced boundary matrix R
for i ← m to 1 do
    if Γ[i] ≠ 0 and ∃j ∈ [1, n] with low(j) = i in R then
        Γ ← Γ + R_j
    end
end
```

Proposition 3.5. *After the reduction algorithm, Algorithm 2 finds the total reduction of a given d-chain in $\mathcal{O}(m^2)$ time complexity.*

Proof. In Algorithm 2, we denote by Γ_{i-1} the value of variable Γ after iteration i. Since iteration counter i is decreasing from m to 1, the input and output of the algorithm are respectively Γ_m and Γ_0. At each iteration, Γ_{i-1} are either equal to Γ_i or $\Gamma_i + R_j$. Since $R_j \in \operatorname{Im} \partial_{d+1}$, Γ_{i-1} is in both cases homologous to Γ_i. Therefore, Γ_0 is homologous to Γ_m. We are left to show that Γ_0 is irreducible. From Lemma 3.4, it is enough to check that it does not contain any reducible simplex.
Let σ_i be a reducible simplex and let us show that $\sigma_i \notin \Gamma_0$. Two possibilities may occur:

- if $\sigma_i \in \Gamma_i$, then $\Gamma_{i-1} = \Gamma_i + R_j$. Since $\text{low}(j) = i$, we have $\sigma_i \in R_j$ and therefore $\sigma_i \notin \Gamma_{i-1}$.
- if $\sigma_i \notin \Gamma_i$, then $\Gamma_{i-1} = \Gamma_i$ and again $\sigma_i \notin \Gamma_{i-1}$.

Furthermore, from iterations $i-1$ to 1, the added columns R_j contain only simplices smaller than σ_i and therefore $\sigma_i \notin \Gamma_{i-1} \Rightarrow \sigma_i \notin \Gamma_0$. Observe that using an auxiliary array allows to compute the correspondence $low(j) \rightarrow i$ in time $\mathcal{O}(1)$. The column addition nested inside the loop lead to a $\mathcal{O}(m^2)$ time complexity for Algorithm 2. □

It follows that Problem 3.1 can be solved in $\mathcal{O}(mn^2)$ time complexity, by applying successively algorithms 1 and 2, i.e. $\mathcal{O}(N^3)$ if N is the size of the simplicial complex.

The theoretical complexity required to solve Problem 3.1 might seem prohibitive. Recall, however, that we are dealing with a specialization of the OHCP problem that has been shown to be NP-Hard in general [24].

Chapter 4

Lexicographic-minimal chains under imposed boundary

This section will construct the algorithmic solution to the two following problems:

Problem 4.1. *Given a simplicial complex K with a total order on the d-simplices and a d-chain $\Gamma_0 \in \boldsymbol{C_d}(K)$, find :*

$$\Gamma_{\min} = \min_{\sqsubseteq_{lex}} \{\Gamma \in \boldsymbol{C_d}(K) \mid \partial_d \Gamma = \partial_d \Gamma_0\}$$

Problem 4.2. *Given a simplicial complex K with a total order on the d-simplices and a $(d-1)$-cycle A, find out if A is a boundary, i.e. if:*

$$B_A \underset{def.}{=} \{\Gamma \in \boldsymbol{C_d}(K) \mid \partial_d \Gamma = A\} \neq \varnothing$$

If it is the case, find the minimal d-chain Γ bounded by A:

$$\Gamma_{min} = \min_{\sqsubseteq_{lex}} B_A$$

Problem 4.2 a slight variant of the Problem 4.1: in case we can find a representative Γ_0 in the set $B_A \neq \varnothing$, such that $\partial_d \Gamma_0 = A$, we are then taken back to Problem 4.1 to find the minimal d-chain $\Gamma_{\min}$ such that $\partial_d \Gamma_{\min} = \partial_d \Gamma_0 = A$.

This section is still strongly linked to the matrix reduction algorithm introduced in Section 3. Note that, unlike previous section that used the ∂_{d+1} boundary operator, we are now considering ∂_d, meaning the given total order on d-simplices applies to the greater dimension of the matrix. An arbitrary order can be taken for the $(d-1)$-simplices to build the matrix reduction of ∂_d. Indeed, as indicated in [27], the performed reduction can be written in matrix notation as $R = \partial_d \cdot V$ and the minimization steps in this section will be performed on the matrix V, whose rows do follow the given simplicial ordering.

The number of zero columns of R is the dimension of $Z_d = \operatorname{Ker} \partial_d$ [27]. Let's denote $n^Z = \dim(Z_d)$. By selecting all columns in V corresponding to zero columns in R, we obtain the matrix Z, whose columns $Z_1, \ldots, Z_{n^Z}$ form a basis of Z_d. We first show a useful property on the matrix Z.

Lemma 4.1. *Indexes $\{\operatorname{low}(Z_i)\}_{i \in [1, n^Z]}$ are unique:*

$$i \neq j \Rightarrow \operatorname{low}(Z_i) \neq \operatorname{low}(Z_j)$$

and if $A \in \operatorname{Ker} \partial_d \setminus \{0\}$, there exists a unique column $Z_{\max}$ of Z with:

$$\operatorname{low}(Z_{\max}) = \operatorname{low}(A)$$

Proof. In the reduction algorithm, the initial matrix V is the identity: the low indexes are therefore unique. During iterations of the algorithm, the matrix V is right-multiplied by an column-adding elementary matrix $L_{j_0,j}$, adding column j_0 to j with $j_0 < j$.

$$L_{j_0,j} = \begin{array}{cc} \begin{array}{cccccc} & & & j & & \end{array} & \\ \begin{bmatrix} 1 & & & & & \\ & 1 & & 1 & & \\ & & \ddots & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{bmatrix} & \begin{array}{c} \\ j_0 \\ \\ \\ \\ \\ \end{array} \end{array}$$

Therefore, the indexes $\{\text{low}(V_i), V_i \in V\}$ stay on the diagonal during the reduction algorithm: $\text{low}(V_i) = i$. They are therefore unique. The restriction of V to Z does not change this property.

If $A \in \operatorname{Ker} \partial_d \setminus \{0\}$, A can be written as a non-zero linear combination of non-zero columns $(Z_i)_{i\in I}$ of Z. As the lows of $(Z_i)_{i\in I}$ are unique, there is a unique column $Z_{i_{\max}}$ such that:

$$\text{low}(A) = \text{low}(Z_{i_{\max}}) = \max_{i\in I} \text{low}(Z_i)$$

□

We apply a similar total reduction algorithm as previously introduced in Section 3 but using the matrix Z. In the following algorithm, $m = \dim \boldsymbol{C_d}(K)$.

Algorithm 3: Total reduction variant

Inputs : A d-chain Γ and Z
for $i \leftarrow m$ **to** 1 **do**
  if $\Gamma[i] \neq 0$ **and** $\exists j \in [1, n^Z]$ *with* $\text{low}(j) = i$ *in* Z **then**
    $\Gamma \leftarrow \Gamma + Z_j$
  end
end

Proposition 4.2. *After matrix reduction, Algorithm 3 computes the solution for Problem 4.1 in $\mathcal{O}(m^2)$ time complexity.*

Proof. The proof is similar to the one of Lemma **??**.

In Algorithm 3, we denote by Γ_{i-1} the value of variable Γ after iteration i. Since iteration counter i is decreasing from m to 1 the input and output of the algorithm are respectively is Γ_m and Γ_0. Since $V_j^{\text{Ker}} \in \operatorname{Ker}\partial_d$, at each iteration $\partial\Gamma_{i-1} = \partial\Gamma_i$ therefore $\partial\Gamma_0 = \partial\Gamma_m$. We are left to show the algorithm's result is the minimal solution.

Suppose there is $\Gamma^\star$ such that $\partial_d\Gamma^\star = \partial\Gamma$ and $\Gamma^\star \sqsubset_{lex} \Gamma_0$. Let's consider the difference $\Gamma_0 - \Gamma^\star$, and its largest element index $\text{low}(\Gamma_0 - \Gamma^\star) = i$, with $\sigma_i \in \Gamma_0$ and $\sigma_i \notin \Gamma^\star$ by definition 2.1 of the lexicographic order. As $\Gamma_0 - \Gamma^\star \in \operatorname{Ker}\partial_d$, there has to be a column V_j^{Ker} in V^{Ker} where $\text{low}\left(V_j^{\text{Ker}}\right) = i$, from Lemma 4.1. Two possibilities may occur at iteration i:

- if $\sigma_i \in \Gamma_i$, then $\Gamma_{i-1} = \Gamma_i + V_j^{\text{Ker}}$. Since $i = \text{low}(j)$, we have $\sigma_i \in V_j^{\text{Ker}}$ and therefore $\sigma_i \notin \Gamma_{i-1}$.

- if $\sigma_i \notin \Gamma_i$, then $\Gamma_{i-1} = \Gamma_i$ and again $\sigma_i \notin \Gamma_{i-1}$.

However, from iterations $i-1$ to 1, the added columns V_j^{Ker} contains only simplices with indices smaller than i and therefore we obtain $\sigma_i \notin \Gamma_{i-1} \Rightarrow \sigma_i \notin \Gamma_0$, a contradiction to the definition of σ_i as largest element of $\Gamma^\star - \Gamma_0$. □

As previously mentioned, solving Problem 4.2 requires deciding if the set B_A is empty and, in case it is not empty, finding an element of the set B_A. Algorithm 3 can then be used to minimize this element under imposed boundary.

In the following algorithm, $m = \dim \boldsymbol{C_{d-1}}(K)$ and $n = \dim \boldsymbol{C_d}(K)$.

Algorithm 4: Finding a representative

```
Inputs : A (d−1)-chain A, a boundary matrix R reduced by V
Γ0 ← ∅
A0 ← A
for i ← m to 1 do
    if A0[i] ≠ 0 then
        if ∃j ∈ [1, n] with low(j) = i in R then
            A0 ← A0 + Rj
            Γ0 ← Γ0 + Vj
        else
            The set B_A is empty.
        end
    end
end
```

Proposition 4.3. *Algorithm 4 decides if the set B_A is non-empty, and in that case, finds a representative Γ_0 such that $\partial\Gamma_0 = A$ in $\mathcal{O}(m^2)$ time complexity.*

Proof. We start by two trivial observations following from the definition of a reduction. First, A is a boundary if and only if its total reduction is the null chain. Second, if a non-null chain is a boundary, then its greatest simplex is reducible.

If, at iteration i, $A_0[i] \neq 0$, then σ_i is the greatest simplex in A_0. In the case R has no column R_j such that $low(j) = i$, σ_i is not reducible by Lemma 3.3 and therefore A_0 is not a boundary. Since A and A_0 differ by a boundary (added columns of R), A is not a boundary either. This means the set B_A is empty.

The main difference with the previous chain reduction is we keep track of the column operations in Γ_0. If the total reduction of A is null, we have found a linear combination $(R_j)_{j\in J}$ such that $A = \sum_{j\in J} R_j$. By computing Γ_0 as the sum of the corresponding columns in V:

$$\Gamma_0 = \sum_{j\in J} V_j$$

We can verify, as $R = \partial_d \cdot V$:

$$\partial_d \Gamma_0 = \partial_d \left(\sum_{j\in J} V_j \right) = \sum_{j\in J} R_j = A$$

□

Chapter 5

Efficient algorithm in codimension 1

In this section we focus on Problem 5.1, a specialization of Problem 3.1, namely when K is a subcomplex of a $(d+1)$-pseudomanifold. For simplicity we remain in the context of simplicial complexes but this section could be adapted to the more general context of CW complexes: roughly speaking, we could replace d-simplices by d-dimensional topological disks.

5.1 Dual graph of a pseudomanifold

Recall that a d-dimensional simplicial complex is said *pure* if it is of dimension d and any simplex has at least one coface of dimension d.

Definition 5.1 (Pseudomanifold)**.** *A* d-pseudomanifold *is a pure d-dimensional simplicial complex for which each $(d-1)$-face has exactly two d-dimensional cofaces.*

In particular any combinatorial k-manifold is a k-pseudomanifold.

Definition 5.2 (Dual Graph of a pseudomanifold)**.** *The dual graph of a d-pseudomanifold $\mathcal{M}$ is the graph whose vertices are in one-to-one correspondence with the d-simplices of $\mathcal{M}$ and whose edges are in one-to-one correspondence with $(d-1)$-simplices of $\mathcal{M}$: an edge e connects two vertices v_1 and v_2 of the graph if and only if e corresponds to the $(d-1)$-face with cofaces corresponding to v_1 and v_2.*

Definition 5.3 (Strongly connected pseudomanifold)**.** *A* strongly connected d-pseudomanifold *is a d-pseudomanifold whose dual graph is connected.*

Given a strongly connected $(d+1)$-pseudomanifold $\mathcal{M}$ and $\tau_1 \neq \tau_2$ two $(d+1)$-simplices of $\mathcal{M}$, we consider a special case of problem 3.1 where $K = \mathcal{M} \setminus \{\tau_1, \tau_2\}$ and $A = \partial\tau_1$:

Problem 5.1. *Given a strongly connected $(d+1)$-pseudomanifold $\mathcal{M}$ with a total order on the d-simplicies and two distinct $(d+1)$-simplices (τ_1, τ_2) of $\mathcal{M}$, find:*

$$\Gamma_{\min} = \min_{\sqsubseteq_{lex}} \left\{ \Gamma \in \boldsymbol{C_p}(\mathcal{M}) \mid \exists B \in \boldsymbol{C_{d+1}}(\mathcal{M} \setminus \{\tau_1, \tau_2\}), \Gamma - \partial\tau_1 = \partial B \right\}$$

Definition 5.4 (Coboundary[1] operator ∂^0)**.** *Seeing a graph G as a 1-dimensional simplicial complex, we define the coboundary operator $\partial^0 : \boldsymbol{C_0}(G) \to \boldsymbol{C_1}(G)$ as the linear operator defined by the transpose of the matrix of the boundary operator $\partial_1 : \boldsymbol{C_1}(G) \to \boldsymbol{C_0}(G)$ in the canonical basis of simplices.*

[1] In order to avoid to introduce non essential formal definitions, the coboundary operator is defined over chains since, for finite simplicial complexes, the canonical inner product defines a natural bijection between chains and cochains.

For a given graph $G = (\mathcal{V}, \mathcal{E})$, $\mathcal{V}$ and $\mathcal{E}$ respectively denote its vertex and edge sets. We easily see that:

Observation 5.5. *For a set of vertices $V_0 \subset \mathcal{V}$, $\partial^0 V_0$ is exactly the set of edges in G that connect vertices in V_0 with vertices in $\mathcal{V} \setminus V_0$.*

For a d-chain $\alpha \in \boldsymbol{C_d}(\mathcal{M})$ and a $(d+1)$-chain $\beta \in \boldsymbol{C_{d+1}}(\mathcal{M})$, $\tilde{\alpha}$ and $\tilde{\beta}$ denote their respective dual 1-chain and dual 0-chain in $G(\mathcal{M})$. With this convention, one has also:

Observation 5.6. *Let $\mathcal{M}$ be a $(d+1)$-pseudomanifold. If $\alpha \in \boldsymbol{C_d}(\mathcal{M})$ and $\beta \in \boldsymbol{C_{d+1}}(\mathcal{M})$, then:*

$$\tilde{\alpha} = \partial^0 \tilde{\beta} \iff \alpha = \partial_{d+1} \beta$$

5.2 Codimension 1 and Lexicographic Min Cut (LMC)

The order on d-simplices of a $(d+1)$-pseudomanifold $\mathcal{M}$ naturally defines a corresponding order on the edges of the dual graph: $\tau_1 < \tau_2 \iff \tilde{\tau}_1 < \tilde{\tau}_2$. This order extends similarly to a lexicographic order $\sqsubseteq_{lex}$ on sets of edges (or, equivalently, 1-chains) in the graph.

In the following, we say a set of edges $\tilde{\Gamma}$ **disconnects** $\tilde{\tau}_1$ and $\tilde{\tau}_2$ in the graph $(\mathcal{V}, \mathcal{E})$ if $\tilde{\tau}_1$ and $\tilde{\tau}_2$ are not in the same connected component of the graph $(\mathcal{V}, \mathcal{E} \setminus \tilde{\Gamma})$.

Given a graph with weighted edges and two vertices, the min-cut/max-flow problem [29, 39] typically consists in finding the minimum cut (i.e. set of edges) disconnecting the two vertices, where minimum is meant as minimal sum of weights of cut edges. We consider a similar problem where the minimum is meant in term of lexicographic order, or equivalently when weight of edge i is 2^i.

Problem 5.2 (Lexicographic Min Cut (LMC))**.** *Given a connected graph $G = (\mathcal{V}, \mathcal{E})$ with a total order on $\mathcal{E}$ and two vertices $\tilde{\tau}_1, \tilde{\tau}_2 \in \mathcal{V}$, find the set $\tilde{\Gamma}_{\mathrm{LMC}} \subset \mathcal{E}$ minimal for the lexicographic order $\sqsubseteq_{lex}$, that disconnects $\tilde{\tau}_1$ and $\tilde{\tau}_2$ in G.*

Proposition 5.7. *$\Gamma_{\min}$ is solution of Problem 5.1 if and only its dual 1-chain $\tilde{\Gamma}_{\min}$ is solution of Problem 5.2 on the dual graph $G(\mathcal{M})$ of $\mathcal{M}$ where $\tilde{\tau}_1$ and $\tilde{\tau}_2$ are respective dual vertices of τ_1 and τ_2.*

Proof. Problem 5.1 can be equivalently formulated as:

$$\Gamma_{\min} = \min_{\sqsubseteq_{lex}} \{\partial_{d+1}(\tau_1 + B) \mid B \in \boldsymbol{C_{d+1}}(\mathcal{M} \setminus \{\tau_1, \tau_2\})\} \tag{5.1}$$

Using Observation 5.6 we see that $\Gamma_{\min}$ is the minimum in Equation (5.1) if and only if its dual 1-chain $\tilde{\Gamma}_{\min}$ satisfies:

$$\tilde{\Gamma}_{\min} = \min_{\sqsubseteq_{lex}} \left\{\partial^0(\tilde{\tau}_1 + \tilde{B}) \mid \tilde{B} \subset \mathcal{V} \setminus \{\tilde{\tau}_1, \tilde{\tau}_2\}\right\} \tag{5.2}$$

Denoting $\tilde{\Gamma}_{\mathrm{LMC}}$ the minimum of Problem 5.2, we need to show that $\tilde{\Gamma}_{\mathrm{LMC}} = \tilde{\Gamma}_{\min}$.
As $\tilde{\Gamma}_{\mathrm{LMC}}$ disconnects $\tilde{\tau}_1$ and $\tilde{\tau}_2$ in $(\mathcal{V}, \mathcal{E})$, $\tilde{\tau}_2$ is not in the connected component of $\tilde{\tau}_1$ in $(\mathcal{V}, \mathcal{E} \setminus \tilde{\Gamma}_{\mathrm{LMC}})$. We define $\tilde{B}$ such that $\{\tilde{\tau}_1\} \notin \tilde{B}$ and $\{\tilde{\tau}_1\} \cup \tilde{B}$ is the connected component of $\tilde{\tau}_1$ in $(\mathcal{V}, \mathcal{E} \setminus \tilde{\Gamma}_{\mathrm{LMC}})$. We have that $\tilde{B} \subset \mathcal{V} \setminus \{\tilde{\tau}_1, \tilde{\tau}_2\}$. Consider an edge $e \in \partial^0(\tilde{\tau}_1 + \tilde{B})$. From Observation 5.5, e connects a vertex $v_a \in \{\tilde{\tau}_1\} \cup \tilde{B}$ with a vertex $v_b \notin \{\tilde{\tau}_1\} \cup \tilde{B}$. From the definition of $\tilde{B}$, $\tilde{\Gamma}_{\mathrm{LMC}}$ disconnects v_a and v_b in G, which in turn implies $e \in \tilde{\Gamma}_{\mathrm{LMC}}$. We have therefore shown that $\partial^0(\tilde{\tau}_1 + \tilde{B}) \subset \tilde{\Gamma}_{\mathrm{LMC}}$. Using Equation (5.2) we get:

$$\tilde{\Gamma}_{\min} \sqsubseteq_{lex} \partial^0(\tilde{\tau}_1 + \tilde{B}) \sqsubseteq_{lex} \tilde{\Gamma}_{\mathrm{LMC}} \tag{5.3}$$

Now we claim that if there is a $\tilde{B} \subset \mathcal{V} \setminus \{\tilde{\tau}_1, \tilde{\tau}_2\}$ with $\tilde{\Gamma} = \partial^0(\tilde{\tau}_1 + \tilde{B})$, then $\tilde{\Gamma}$ disconnects $\tilde{\tau}_1$ and $\tilde{\tau}_2$ in $(\mathcal{V}, \mathcal{E})$. For that consider a path in G from $\tilde{\tau}_1$ to $\tilde{\tau}_2$. Let v_a be the last vertex of the path

that belongs to $\{\tilde{\tau}_1\} \cup \tilde{B}$ and v_b the next vertex on the path (which exists since the $\tilde{\tau}_2$ is not in $\{\tilde{\tau}_1\} \cup \tilde{B}$). From Observation 5.5, we see that the edge (v_a, v_b) must belong to $\tilde{\Gamma} = \partial^0(\tilde{\tau}_1 + \tilde{B})$. We have shown that any path in G connecting $\tilde{\tau}_1$ and $\tilde{\tau}_2$ has to contain an edge in $\tilde{\Gamma}$ and the claim is proved.
In particular, the minimum $\tilde{\Gamma}_{\min}$ disconnects $\tilde{\tau}_1$ and $\tilde{\tau}_2$ in $(\mathcal{V}, \mathcal{E})$. As $\tilde{\Gamma}_{\text{LMC}}$ denotes the minimum of Problem 5.2, $\tilde{\Gamma}_{\text{LMC}} \sqsubseteq_{lex} \tilde{\Gamma}_{\min}$ which, together with (5.3) gives us $\tilde{\Gamma}_{\text{LMC}} = \tilde{\Gamma}_{\min}$. We have therefore shown the minimum defined by equation (5.2) coincides with the minimum defined in Problem 5.2. $\square$

5.3 Lexicographic Min Cut in $\mathcal{O}(n\alpha(n))$ time complexity

In this section, we will study an algorithm solving Problem 5.2. As we will only consider the dual graph for this section, we leave behind the dual chain notation: vertices $\tilde{\tau}_1$ and $\tilde{\tau}_2$ are replaced by α_1 and α_2, and the solution to the problem is simply noted Γ_{LMC}.

The following lemma exposes a property about the solution that helps constructing it iteratively.

Lemma 5.8. *Consider Γ_{LMC} solution of Problem 5.2 for the graph $G = (\mathcal{V}, \mathcal{E})$ and $\alpha_1, \alpha_2 \in \mathcal{V}$. Let e_0 be an edge in $\mathcal{V} \times \mathcal{V}$ such that $e_0 < \min\{e \in \mathcal{E}\}$. Then:*

1. *The solution for $(\mathcal{V}, \mathcal{E} \cup \{e_0\})$ is either Γ_{LMC} or $\Gamma_{\text{LMC}} \cup \{e_0\}$.*
2. *$\Gamma_{\text{LMC}} \cup \{e_0\}$ is solution for $(\mathcal{V}, \mathcal{E} \cup \{e_0\})$ if and only if α_1 and α_2 are connected in $(\mathcal{V}, \mathcal{E} \cup \{e_0\} \setminus \Gamma_{\text{LMC}})$.*

Proof. Let's call Γ'_{LMC} the solution for $(\mathcal{V}, \mathcal{E} \cup \{e_0\})$.
Since $\Gamma'_{\text{LMC}} \cap \mathcal{E}$ disconnects α_1, α_2 in $(\mathcal{V}, \mathcal{E})$, one has:

$$\Gamma_{\text{LMC}} \sqsubseteq_{lex} \Gamma'_{\text{LMC}}$$

Since $\Gamma_{\text{LMC}} \cup \{e_0\}$ disconnects $(\mathcal{V}, \mathcal{E} \cup \{e_0\})$, we have:

$$\Gamma'_{\text{LMC}} \sqsubseteq_{lex} \Gamma_{\text{LMC}} \cup \{e_0\}$$

Therefore, $\Gamma_{\text{LMC}} \sqsubseteq_{lex} \Gamma'_{\text{LMC}} \sqsubseteq_{lex} \Gamma_{\text{LMC}} \cup \{e_0\}$.
As $e_0 < \min\{e \in \mathcal{E}\}$, there is no set in $\mathcal{E} \cup \{e_0\}$ strictly between Γ_{LMC} and $\Gamma_{\text{LMC}} \cup \{e_0\}$ for the lexicographic order. It follows that Γ'_{LMC} is either equal to Γ_{LMC} or $\Gamma_{\text{LMC}} \cup \{e_0\}$.
The choice for Γ'_{LMC} is therefore ruled by the property that it should disconnect α_1 and α_2: if α_1 and α_2 are connected in $(\mathcal{V}, \mathcal{E} \cup \{e_0\} \setminus \Gamma_{\text{LMC}})$, Γ_{LMC} does not disconnect $(\mathcal{V}, \mathcal{E} \cup \{e_0\})$ and $\Gamma_{\text{LMC}} \cup \{e_0\}$ has to be the solution for $(\mathcal{V}, \mathcal{E} \cup \{e_0\})$. On the other hand, if α_1 and α_2 are not connected in $(\mathcal{V}, \mathcal{E} \cup \{e_0\} \setminus \Gamma_{\text{LMC}})$, then both Γ_{LMC} and $\Gamma_{\text{LMC}} \cup \{e_0\}$ disconnects $(\mathcal{V}, \mathcal{E} \cup \{e_0\})$, but as $\Gamma_{\text{LMC}} <_{lex} \Gamma_{\text{LMC}} \cup \{e_0\}$, $\Gamma_{\text{LMC}} \cup \{e_0\}$ is not the solution for $(\mathcal{V}, \mathcal{E} \cup \{e_0\})$. $\square$

Building an algorithm from Lemma 5.8 suggest a data structure able to check if vertices α_1 and α_2 are connected in the graph: the disjoint-set data structure, introduced for finding connected components [32], does exactly that. In this structure, each set of elements has a different root value, called representative. Calling the operation **MakeSet** on an element creates a new set containing this element. The **FindSet** operation, given an element of a set, returns the representative of the set. For all elements of the same set, **FindSet** will of course return the same representative. Finally, the structure allows merging two sets, by using the **UnionSet** operation. After this operation, elements of both sets will have the same representative.

We now describe Algorithm 5. The algorithm supposes that the set of edges is sorted in decreasing order according to the lexicographic order.

Algorithm 5: Lexicographic Min Cut

```
Inputs : G = (V, E) and α1, α2 ∈ V, with E = {e_i, i = 1, n} sorted in decreasing order
Output: Γ_LMC
Γ_LMC ← ∅
for v ∈ V do
    MakeSet(v)
end
for e ∈ E in decreasing order do
    e = (v1, v2) ∈ V × V
    r1 ← FindSet(v1), r2 ← FindSet(v2)
    c1 ← FindSet(α1), c2 ← FindSet(α2)
    if {r1, r2} = {c1, c2} then
        Γ_LMC ← Γ_LMC ∪ e
    end
    else
        UnionSet(r1, r2)
    end
end
```

Proposition 5.9. *Algorithm 5 computes the solution of Problem 5.2 for a given graph* $(\mathcal{V}, \mathcal{E})$ *and two vertices* $\alpha_1, \alpha_2 \in \mathcal{V}$*. Assuming the input set of edges* $\mathcal{E}$ *are sorted, the algorithm has* $\mathcal{O}(n\alpha(n))$ *time complexity, where* n *is the cardinal of* $\mathcal{E}$ *and* α *the inverse Ackermann function.*

Proof. We denote by e_i the i^{th} edge along the decreasing order and Γ^i_{LMC} the value of the variable Γ_{LMC} of the algorithm after iteration i.
The algorithm works by incrementally adding edges in decreasing order and tracking the growing connected components of the set associated with α_1 and α_2 in $(\mathcal{V}, \{e \in \mathcal{E}, e \geq e_i\} \setminus \Gamma^i_{\mathrm{LMC}})$, for $i = 1, \ldots, n$.
At the beginning, no edges are inserted, and $\Gamma^0_{\mathrm{LMC}} = \varnothing$ is indeed solution for $(\mathcal{V}, \varnothing)$. With Lemma 5.8, we are guaranteed at each iteration i to find the solution for $(\mathcal{V}, \{e \in \mathcal{E}, e \geq e_i\} \setminus \Gamma^i_{\mathrm{LMC}})$ by only adding to the solution the current edge e_i if α_1 and α_2 are connected in $\{e \in \mathcal{E}, e \geq e_i\} \setminus \Gamma^i_{\mathrm{LMC}}$, which is done in the if-statement. If the edge is not added, we update the connectivity of the graph $(\mathcal{V}, \{e \in \mathcal{E}, e \geq e_i\} \setminus \Gamma^i_{\mathrm{LMC}})$ by merging the two sets represented by r_1 and r_2. At each iteration, Γ^i_{LMC} is solution for $(\mathcal{V}, \{e \in \mathcal{E}, e \geq e_i\})$ and when all edges are processed, Γ^n_{LMC} is solution for $(\mathcal{V}, \mathcal{E})$.

The complexity of the **MakeSet**, **FindSet** and **UnionSet** operations have been shown to be respectively $\mathcal{O}(1)$, $\mathcal{O}(\alpha(v))$ and $\mathcal{O}(\alpha(v))$, where $\alpha(v)$ is the inverse Ackermann function on the cardinal of the vertex set [41]. Assuming sorted edges as input of the algorithm – which can be performed in $\mathcal{O}(n \ln(n))$, the algorithm runs in $\mathcal{O}(n\alpha(n))$ time complexity.

□

Chapter 6

Regular triangulations as lexicographic minimal chains

6.1 Conventions and notations

We recall here some conventions in order to make this section more self contained.

A k-simplex σ being a set of $k+1$ vertices in ambient space $\mathbb{R}^n$, we allow ourself to use set theoretic operators on simplices. For example $\tau \subset \sigma$ means that τ is a face of σ and $\sigma_1 \cup \sigma_2$ is the join of σ_1 and σ_2. $|\sigma|$ denote the underlying space of the simplex σ, i.e. the convex hull of its vertices, which thanks to our generic condition are affinely independent.

Homology coefficients are implicitly in $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, i.e. the integer numbers modulo 2 so that $-1 = 1$ and the vector space of k-chains over K is denoted $C_k(K)$ in place of $C_k(K, \mathbb{Z}_2)$. In this context we can allow ourself to see chains as sets of simplices. For example if $\Gamma \in C_k(K)$ and σ is a k-simplex in K, we allow ourself to write indifferently $\sigma \in \Gamma$ or $\Gamma(\sigma) = 1$. Similarly, if $\Gamma_1, \Gamma_2 \in C_k(K)$, we allow ourself to use indifferently vector or set theoretic operators: $\Gamma_1 + \Gamma_2 = \Gamma_1 - \Gamma_2 = (\Gamma_1 \cup \Gamma_2) \setminus (\Gamma_1 \cap \Gamma_2)$.

If Γ is a chain in $C_k(K)$, we denote by $|\Gamma|$ the *support* of Γ which is the sub-complex of K made of all k-simplices in Γ together with all their faces.

We recall the definition of a lexicographic order on k-chains with coefficients in $\mathbb{Z}_2$ induced by a total order on k-simplices:

Definition 6.1 (Lexicographic Order on chains)**.** *Assume there is a total order $\leq$ on the set of k-simplices of K, defining the* max *on sets of k-simplices. For $\Gamma_1, \Gamma_2 \in \boldsymbol{C_k}(K)$:*

$$\Gamma_1 \sqsubseteq_{lex} \Gamma_2 \underset{def.}{\iff} \begin{cases} \Gamma_1 + \Gamma_2 = 0 \\ \text{or} \\ \max\{\sigma \in \Gamma_1 + \Gamma_2\} \in \Gamma_2 \end{cases}$$

6.2 Main result of the section

We consider both cases of a set $\mathbf{P} = \{(P_1, \mu_1), \ldots, (P_N, \mu_N)\} \subset \mathbb{R}^n \times \mathbb{R}$ of weighted points as well as a the case of a set $\mathbf{P} = \{P_1, \ldots, P_N\} \subset \mathbb{R}^n$ of points, the later case being equivalent to a particular configuration of the first case for which the weights are zero: $\forall i, \mu_i = 0$. In both cases, $\mathcal{CH}(\mathbf{P})$ denote the *convex hull* of the set of points in $\mathbb{R}^n$, that is:

$$\mathcal{CH}(\mathbf{P}) = \mathcal{CH}(\{P_1, \ldots, P_N\})$$

The n-dimensional *full complex over* $\mathbf{P}$, denoted $K_{\mathbf{P}}$, is the simplicial complex made of all possible simplices up to dimension n with vertices in $\mathbf{P}$.

The aim of this section is to prove the following:

Theorem 6.2. *Let $\mathbf{P} = \{(P_1, \mu_1), \ldots, (P_N, \mu_N)\} \subset \mathbb{R}^n \times \mathbb{R}$, with $N \geq n+1$ be in general position and let $K_{\mathbf{P}}$ be the n-dimensional full complex over $\mathbf{P}$. Denote by $\beta_{\mathbf{P}} \in \boldsymbol{C}_{n-1}(K_{\mathbf{P}})$ the $(n-1)$-chain made of simplices belonging to the boundary of $\mathcal{CH}(\mathbf{P})$. If:*

$$\Gamma_{\min} = \min_{\sqsubseteq_{lex}} \{\Gamma \in \boldsymbol{C}_n(K_{\mathbf{P}}), \partial\Gamma = \beta_{\mathbf{P}}\}$$

then the simplicial complex $|\Gamma_{\min}|$ support of $\Gamma_{\min}$ is the regular triangulation of $\mathbf{P}$.

When all the weights are zero: $\forall i, \mu_i = 0$ this gives:

Corollary 6.3. *Let $\mathbf{P} = \{P_1, \ldots, P_N\} \subset \mathbb{R}^n$ with $N \geq n+1$ be in general position and let $K_{\mathbf{P}}$ be the n-dimensional full complex over $\mathbf{P}$.*

Denote by $\beta_{\mathbf{P}} \in \boldsymbol{C}_{n-1}(K_{\mathbf{P}})$ the $(n-1)$-chain made of simplices belonging to the boundary of $\mathcal{CH}(\mathbf{P})$.
Then if:

$$\Gamma_{\min} = \min_{\sqsubseteq_{lex}} \{\Gamma \in \boldsymbol{C}_n(K_{\mathbf{P}}), \partial\Gamma = \beta_{\mathbf{P}}\}$$

then the simplicial complex $|\Gamma_{\min}|$ support of $\Gamma_{\min}$ is the Delaunay triangulation of $\mathbf{P}$.

The relation $\sqsubseteq_{lex}$ among n-chains is the lexicographic order defined according to Definition 6.1 where the total order on n-simplices is given at the end of section 6.4. The general position assumption is a generic condition formalized in condition 6.2 for the general formulation of Theorem 6.2. The definition of regular triangulations is recalled in Definition 6.11 of section 6.5.

Obviously, replacing in Theorem 6.2 the full complex $K_{\mathbf{P}}$ by a complex containing the regular triangulation (or Delaunay in case of zero weights) would give again this triangulation as a minimum.

One option would have been to explain the proof ideas in the particular case of Delaunay triangulations (i.e. zero weights). However, the framework of regular triangulations is required anyway in the proof construction so that eventually the general framework reveals to be more natural for the proof exposition.

Outline of the proof. The global argument of the proof is given in section 6.10. Theorem 6.2 is proven for points with non positive weights which, thanks to observation 6.8, is enough to have it true for any weights. The proof starts with Proposition 6.13, a consequence of the well known equivalence between regular triangulations and convex hulls of lifted points.

Statement of Proposition 6.13 is the same as Theorem 6.2 but for the order along which the minimum is taken. In Proposition 6.13 the minimization is meant for the order $\sqsubseteq_p$ induced by the weighted L^1 norm $\|\cdot\|_{(p)}$ on chains. All the proof consists then in showing that, while the two orders differ, they share the same minimum under the theorem constraints. The bounding weight of a simplex is a generalization, for weighted points, of the radius of the smallest ball enclosing the simplex. It the dominant quantity in the definition of the order on simplices which induces the lexicographic order $\sqsubseteq_{lex}$ on chains.

The proof then proceeds in 3 main steps:

1) For p large enough, the weight $w_p(\sigma)^p$ of any single simplex σ in the $\|\cdot\|_{(p)}$ norm is larger than the sum of all weights of all simplices with smaller bounding weight (bounding radius) than σ (Lemma 6.15). This fact allows to focus on the link of a single bounding weight, the largest bounding weight for which some simplices in the minimum of $\sqsubseteq_{lex}$ and $\sqsubseteq_p$ would differ (Lemma 6.15 and section 6.10).

2) Then we introduce an extension of the classical lifted paraboloid construction that allows to see the simplicial structure of the link of a simplex τ in the Delaunay triangulation as a

convex cone. In this representation, we study the subcomplex of this link corresponding to cofaces of τ with same bounding radius as τ. We show that this subcomplex is isomorphic to the set of bounded and visible faces of a convex polytope. It is therefore a simplicial ball (Lemma 6.23 and section 6.8).

3) Then, by induction on the dimension of convex cones and convex polytopes, one show that this bounded subcomplex of the boundary of a convex polytope visible from the origin can be expressed as a the minimum under boundary constraint (Lemma 6.27 and section 6.9) for another lexicographic order.

This lexicographic order corresponds, on a the subset of the link of τ corresponding to simplices with same bounding weight, to the lexicographic order of corresponding full dimensional simplices in the star of τ (Lemmas 6.28 and 6.29).

6.3 Weighted points and weighted distances

We briefly recall the notions associated to weighted points and distances required to introduce the generalization of Delaunay triangulations, called regular triangulation or sometimes weighted Delaunay. We follow the terminology, notations and conventions of section 4.4 in [11].

We consider a set of weighted points $\mathbf{P} = \{(P_1, \mu_1), \ldots, (P_N, \mu_N)\} \subset \mathbb{R}^n \times \mathbb{R}$ with $N \geq n+1$. A weighted point $(P, 0)$ is seen as the usual point $P \in \mathbb{R}^n$, while, when $\mu > 0$ it is associated to the sphere centered at P with radius $r = \sqrt{\mu}$.

When $\mu < 0$ weighted points may be seen as spheres with imaginary radius $r = \sqrt{\mu}$. We allow ourself to speak about spheres in this later case but we prefer in general to consider weights over radius since we rely on the natural order on real number valued (possibly negative) weights μ: translating this order on complex numbers $r = \sqrt{\mu}$ is not natural when $\mu < 0$.

Weighted distance between weighted points

Definition 6.4 (Section 4.4 in [11]). *Given two weighted points $(P_1, \mu_1), (P_2, \mu_2) \in \mathbb{R}^n \times \mathbb{R}$ their weighted distance is defined as:*

$$\mathrm{D}\left((P_1, \mu_1), (P_2, \mu_2)\right) \underset{def.}{=} (P_1 - P_2)^2 - \mu_1 - \mu_2$$

we say that (P_1, μ_1) and (P_2, μ_2) are orthogonal *if $\mathrm{D}\left((P_1, \mu_1), (P_2, \mu_2)\right) = 0$.*

First generic condition.

Condition 6.1. *We say that $\mathbf{P} = \{(P_1, \mu_1), \ldots, (P_N, \mu_N)\} \subset \mathbb{R}^n \times \mathbb{R}$ satisfies the first generic condition if no $(n+1)$ points $\{P_{i_1}, \ldots, P_{i_{n+1}}\}$ lie on a same $(n-1)$-dimensional affine space.*

Generalization of circumradius and bounding radius. We define the following generalizations to set of weighted points of circumsphere and smallest enclosing ball.

Definition 6.5. *Assume that $\mathbf{P}$ satisfies Condition 6.1. Given a k-simplex $\sigma \subset \mathbf{P}$ with $0 \leq k \leq n$ the generalized circumsphere and smallest enclosing ball of σ are the weighted points $(P_C, \mu_C)(\sigma)$ and $(P_B, \mu_B)(\sigma)$ respectively defined as:*

$$\mu_C(\sigma) \underset{def.}{=} \min \left\{\mu \in \mathbb{R}, \exists P \in \mathbb{R}^n, \forall (P_i, \mu_i) \in \sigma,\ \mathrm{D}\left((P, \mu), (P_i, \mu_i)\right) = 0\right\} \tag{6.1}$$

$$\mu_B(\sigma) \underset{def.}{=} \min \left\{\mu \in \mathbb{R}, \exists P \in \mathbb{R}^n, \forall (P_i, \mu_i) \in \sigma,\ \mathrm{D}\left((P, \mu), (P_i, \mu_i)\right) \leq 0\right\} \tag{6.2}$$

$P_C(\sigma)$ and $P_B(\sigma)$ are respectively the unique points P that realize the minimum in Equations (6.1) and (6.2). The weights $\mu_C(\sigma)$ and $\mu_B(\sigma)$ are called respectively "circumweight" and bounding weight of σ.

The set $\{(P,\mu), \forall (P_i,\mu_i) \in \sigma, \mathrm{D}((P,\mu),(P_i,\mu_i)) = 0\}$, on which the first arg min is taken, is not empty, thanks to the generic condition 6.1.

When $k = n$ there is in fact (generically) a single weighted point (P_C, μ_C) such that $\forall (P_i,\mu_i) \in \sigma, \mathrm{D}((P,\mu),(P_i,\mu_i)) = 0$. If $k < n$ the arg min is required to get what would be the minimal circumsphere in the case of zero weights.

Observation 6.6. *Observe that if the weight of all points in* $\mathbf{P}$ *are assumed non positive, then one has always* $\mu_C(\tau) \geq 0$ *and* $\mu_B(\tau) \geq 0$ *and if moreover* τ *has positive dimension, then* $\mu_C(\tau) > 0$ *and* $\mu_B(\tau) > 0$.

In the particular case with zero weights, corresponding to Delaunay triangulations, then:

$$\mu_C(\tau) = \mathrm{R_C}(\tau)^2 \quad \text{and} \quad \mu_B(\tau) = \mathrm{R_B}(\tau)^2$$

where $\mathrm{R_C}(\tau)$ *is the circumradius of* τ *and* $\mathrm{R_B}(\tau)$ *is the radius of the smallest enclosing ball* $\mathbf{SEB}(\tau)$ *of* τ.

Observation 6.7. *Let* $\psi_\lambda : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}$, *be the transformation that shifts the weight by* λ:

$$\psi_\lambda(P,\mu) = (P, \mu + \lambda)$$

Let $\sigma \in K_{\mathbf{P}}$ *be a simplex, from definitions 6.5 and 6.4 we have:*

$$\begin{aligned} P_C(\psi_\lambda(\sigma)) &= P_C(\sigma) \qquad \text{and} \qquad \mu_C(\psi_\lambda(\sigma)) = \mu_C(\sigma) - \lambda \\ P_B(\psi_\lambda(\sigma)) &= P_B(\sigma) \qquad \text{and} \qquad \mu_B(\psi_\lambda(\sigma)) = \mu_B(\sigma) - \lambda \end{aligned}$$

It follows that a global shift by a constant value λ *results in an opposite shift on the weights of generalized circumcenters. It therefore preserves the relative order between simplices weights* μ_C *and* μ_B:

$$\mu_C(\sigma_1) \leq \mu_C(\sigma_2) \iff \mu_C(\psi_\lambda(\sigma_1)) \leq \mu_C(\psi_\lambda(\sigma_2))$$

and the same relation holds for μ_B. *Since the order* $\leq$ *between simplices (6.9) defined in section 6.4 relies entirely on comparisons on* μ_C *and* μ_B, *this total order is preserved by a global weight translation.*

It follows that:

Observation 6.8. *Both the order between simplices (see observation 6.7) and regular triangulations are invariant under a global translation* $\psi_\lambda(\tau)$ *of the weights. Therefore proving Theorem 6.2 for non positive weights is enough to extend it to any weights.*

We will need the following easy lemma:

Lemma 6.9. *For any* k*-simplex* $\sigma \in K_{\mathbf{P}}$, *one has:*

$$P_B(\sigma) \in |\sigma| \tag{6.3}$$

Proof. If $\sigma = \{(P_0,\mu_0),\ldots,(P_k,\mu_k)\}$, $|\sigma|$ is the convex hull of $\{P_0,\ldots,P_k\}$.

If $P \notin |\sigma|$ the projection of P on $|\sigma|$ decreases the weighted distance from P to all the vertices of σ, which shows that (P,μ) cannot realize the arg min in (6.2). $\square$

Second generic condition.

Condition 6.2. *We say that* $\mathbf{P} = \{(P_1, \mu_1), \ldots, (P_N, \mu_N)\} \subset \mathbb{R}^n \times \mathbb{R}$ *is in* general position *if it satisfies the first generic condition 6.1 and if, for a* k*-simplex* σ *and a* k'*-simplex* σ' *in* $K_{\mathbf{P}}$ *with* $2 \leq k, k' \leq n$*, one has:*

$$\mu_C(\sigma) = \mu_C(\sigma') \Rightarrow \sigma = \sigma' \tag{6.4}$$

Lemma 6.10. *Under generic condition 6.2 on* $\mathbf{P}$*, for any simplex* σ*, there exists a unique inclusion minimal face* $\Theta(\sigma)$ *of* σ *such that* $(P_B, \mu_B)(\sigma) = (P_C, \mu_C)(\Theta(\sigma))$*. Moreover one has* $(P_C, \mu_C)(\Theta(\sigma)) = (P_B, \mu_B)(\Theta(\sigma))$*.*

Figure 6.1 illustrates the possibilities for $\Theta(\sigma)$ in the case $n = 3$ and zero weights.

Proof. In (6.2), denote by $\tau \subset \sigma$ the set of vertex $(P_i, \mu_i) \in \sigma$ for which:

$$\mathrm{D}\left((P, \mu), (P_i, \mu_i)\right) = 0$$

This set cannot be empty as, if all inequalities in (6.2) where strict, a strictly smaller value of μ would still match the inequality, which would contradict with the $\arg\min$ in (6.2). One has then $\tau \neq \varnothing$ and:

$$\forall (P_i, \mu_i) \in \tau, \mathrm{D}\left((P, \mu), (P_i, \mu_i)\right) = 0 \tag{6.5}$$

and of course:

$$\forall (P_i, \mu_i) \in \tau, \mathrm{D}\left((P, \mu), (P_i, \mu_i)\right) \leq 0 \tag{6.6}$$

No other (P, μ) with a smaller value of μ can satisfies (6.5) nor (6.6) as it would again similarly contradict the $\arg\min$ in (6.2). It follows that:

$$(P_B, \mu_B)(\sigma) = (P_B, \mu_B)(\tau) = (P_C, \mu_C)(\tau)$$

□

6.4 Regular triangulation order on simplices

Notation For a n-simplex σ one defines $\Theta_k(\sigma)$, the $(k + \dim(\Theta(\sigma)))$-dimensional face of σ defined by (6.7) and (6.8) below:

$$\Theta_0(\sigma) = \Theta(\sigma) \tag{6.7}$$

where $\Theta(\sigma)$ is defined in Lemma 6.10 and depicted on figure 6.1 in the case of zero weights.

For $k > 0$, $\Theta_k(\sigma)$ is the $(\dim(\Theta_{k-1}(\sigma)) + 1)$-dimensional coface of $\Theta_{k-1}(\sigma)$ with minimal circumradius:

$$\Theta_k(\sigma) = \underset{\substack{\Theta_{k-1}(\sigma) \subset \tau \subset \sigma \\ \dim(\tau) = \dim\left(\Theta_{k-1}(\sigma)\right)+1}}{\arg\min} \mu_C(\tau) \tag{6.8}$$

and:

$$\mu_k(\sigma) = \mu_C(\Theta_k(\sigma))$$

In particular $\mu_0(\sigma) = \mu_B(\sigma)$ and if $k = \dim(\sigma) - \dim(\Theta(\sigma))$ then $\mu_k(\sigma) = \mu_C(\sigma)$.

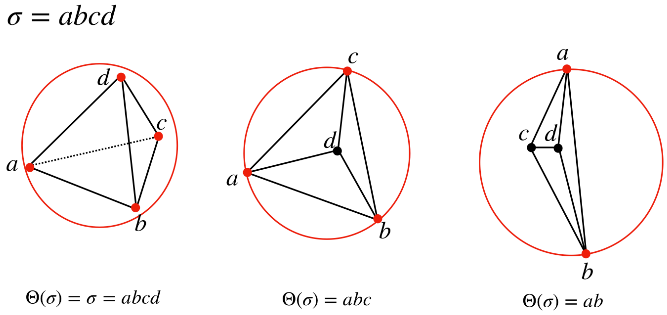

Figure 6.1: Illustration for the definition of $\Theta(\sigma)$ for a tetrahedron $\sigma = abcd$ in the case of zero weights.

Order on n-simplices. Observethat, thanks to generic condition 6.2 and Lemma 6.10, when $\mathbf{P}$ is in general position, one has, for two n-simplices σ_1, σ_2:

$$\mu_B(\sigma_1) = \mu_B(\sigma_2) \Rightarrow \Theta(\sigma_1) = \Theta(\sigma_2)$$

and therefore, if $\mu_B(\sigma_1) = \mu_B(\sigma_2)$, $\mu_k(\sigma_1)$ and $\mu_k(\sigma_2)$ are defined for the same range of values of k.

We define the following relations (recall that $\mu_0(\sigma) = \mu_B(\sigma)$):

$$\sigma_1 < \sigma_2 \underset{\text{def.}}{\iff} \begin{cases} \mu_0(\sigma_1) < \mu_0(\sigma_2) \\ \quad \text{or} \\ \exists k \geq 1,\ \mu_k(\sigma_1) > \mu_k(\sigma_2) \\ \text{and } \forall j, 0 \leq j < k,\ \mu_j(\sigma_1) = \mu_j(\sigma_2) \end{cases} \tag{6.9}$$

and:

$$\sigma_1 \leq \sigma_2 \underset{\text{def.}}{\iff} \begin{cases} \sigma_1 = \sigma_2 \\ \quad \text{or} \\ \sigma_1 < \sigma_2 \end{cases} \tag{6.10}$$

For example, when $n = 2$ and the weights are zero, this order on triangles consists in first comparing the radii of the smallest circles enclosing the triangles $T_i, i = 1, 2$, whose squares are $\mathrm{R_B}(T_i)^2 = \mu_B(T_i) = \mu_0(T_i)$ (Observation 6.6). This is generically enough for acute triangles, but not for obtuse triangles that could generically share the longest edge. In this case the tie is broken by comparing in reverse order the circumradii, whose squares are $\mathrm{R_C}(T_i)^2 = \mu_C(T_i) = \mu_1(T_i)$. As shown in section 7, this order is equivalent to the order $\leq_\infty$ (Lemma 7.6).

Lexicographic order on n-chains. One can check that when $\mathbf{P}$ is in general position, the relation $\leq$ among n-simplices is antisymmetric and a total order.

Following definition 6.1 the order $\leq$ on simplices induces a lexicographic order $\sqsubseteq_{lex}$ on the n-chains of $K_{\mathbf{P}}$. From now on we assume that $\mathbf{P}$ is in general position i.e. it satisfies condition 6.2.

6.5 Regular triangulations of weighted points

We recall now the definition of a regular triangulation over a set of weighted points using a generalization of the empty sphere property of Delaunay triangulations.

Definition 6.11 (Lemma 4.5 in [11]). *A regular triangulation $\mathcal{T}$ of the set of weighted points $\mathbf{P} = \{(P_1, \mu_1), \ldots, (P_N, \mu_N)\} \subset \mathbb{R}^n \times \mathbb{R}$, $N \geq n+1$, is a triangulation of the convex hull of $\{P_1, \ldots, P_N\}$ taking its vertices in $\{P_1, \ldots, P_N\}$ such that for any simplex $\sigma \in \mathcal{T}$, if $(P_C(\sigma), \mu_C(\sigma))$ is the generalized circumsphere of σ, then:*

$$(P_i, \mu_i) \in \mathbf{P} \setminus \sigma \Rightarrow \mathrm{D}\left((P_C(\sigma), \mu_C(\sigma)), (P_i, \mu_i)\right) > 0$$

Under generic condition 6.2 there exists a unique regular triangulation of $\mathbf{P} = \{(P_1, \mu_1), \ldots, (P_N, \mu_N)\}$. However, with variable weights, some points with low weights in $\mathbf{P}$ may not be vertices of the triangulation.

6.6 Lift of weighted points and p-norms

Lift of weighted points. Given a weighted point $(P, \mu) \in \mathbb{R}^n \times \mathbb{R}$ its lift with respect to an implicit origin $O \in \mathbb{R}^n$, denoted $\mathrm{lift}(P, \mu)$, is a point in $\mathbb{R}^n \times \mathbb{R}$ given as:

$$\mathrm{lift}(P, \mu) \underset{\text{def.}}{=} ((P - O),\ (P - O)^2 - \mu)$$

Similarly to Delaunay triangulations, it is a well known fact (and not difficult to check afterward) that simplices of the regular triangulation of $\mathbf{P}$ are in one-to-one correspondence with the lower convex hull of $\mathrm{lift}(\mathbf{P})$:

Proposition 6.12. *A simplex σ is in the regular triangulation of $\mathbf{P}$ if and only if $\mathrm{lift}(\sigma)$ is a simplex on the lower convex hull of $\mathrm{lift}(\mathbf{P})$.*

The lifted paraboloid construction had led some authors to imagine a variational formulation for Delaunay triangulations introduced for the first time in [22]. This idea has been exploited further in order to optimize triangulations in [21, 1]. We follow here the same idea but the variational formulation, while using the same criterion, is applied on the linear space of chains, which can be seen as a superset of the space of triangulations. This point of view allows to revisit Delaunay and regular triangulations in a linear programming setting, as for example in Proposition 6.13.

We define a function on the convex hull of a k-simplex $f_\sigma : |\sigma| \to \mathbb{R}$ where $\sigma = \{(P_0, \mu_0), \ldots, (P_k, \mu_k)\}$ as the difference between the linear interpolation of the height of the lifted vertices and the fonction $x \to (x - O)^2$.

More precisely, for a point $x \in |\sigma|$ with barycentric coordinates $\lambda_i \geq 0$, $\sum_i \lambda_i = 1$ we have $x = \sum_i \lambda_i P_i$ and:

$$f_\sigma : x \mapsto f_\sigma(x) \underset{\text{def.}}{=} \left(\sum_i \lambda_i((P_i - O)^2 - \mu_i)\right) - (x - O)^2 \tag{6.11}$$

A short computation shows that the function f_σ, expressed in terms of barycentric coordinates, is invariant by applying an isometry such as a translation or rotation or symmetry on σ. In particular $f_\sigma(x)$ does not depend on the origin O of the lift.

It follows from Proposition 6.12 that if σ_x is a simplex in the regular triangulation of $\mathbf{P}$ such that $x \in |\sigma_x|$ and if σ' is another simplex with its vertices in $\mathbf{P}$ with $x \in |\sigma_x| \cap |\sigma'|$ then:

$$f_{\sigma_x}(x) \leq f_{\sigma'}(x) \tag{6.12}$$

In the particular case where all weights are non positive, that is $\forall i,\ \mu_i \leq 0$, the convexity of $x \mapsto x^2$ says that the expression of $f_\sigma(x)$ in (6.11) is never negative and in this case (6.12) implies that defining the weight w_p of a n-simplex σ as:

$$w_p(\sigma) \underset{\text{def.}}{=} \|f_\sigma\|_p = \left(\int_{|\sigma|} f_\sigma(x)^p dx\right)^{\frac{1}{p}} \tag{6.13}$$

allows to characterize the regular triangulation as the one induced by the chain Γ_{reg} that, among all chains with boundary $\beta_{\mathbf{P}}$, minimizes:

$$\Gamma \to \|\Gamma\|_{(p)} \underset{\text{def.}}{=} \sum_\sigma |\Gamma(\sigma)| w_p(\sigma)^p \tag{6.14}$$

In this last equation, the notation $|\Gamma(\sigma)|$ instead of $\Gamma(\sigma)$ is there since $\Gamma(\sigma) \in \mathbb{Z}_2$ and the sum is in $\mathbb{R}$: $\Gamma(\sigma) \to |\Gamma(\sigma)|$ convert coefficients in $\mathbb{Z}_2$ into binary real numbers in $\{0,1\}$. The parenthesis around the parameter p indexing the norm in $\|\Gamma\|_{(p)}$ are there to avoid the confusion with a L^p norm (as the one appearing in (6.13)), which is meaningless for coefficients in Z_2, but rather a weighted L^1 norm where the weights $w_p(\sigma)^p$ are parametrized by p.

Formally, we have the following Proposition 6.13, whose proof is given in appendix A.

Proposition 6.13. *Let $\mathbf{P} = \{(P_1,\mu_1),\ldots,(P_N,\mu_N)\} \subset \mathbb{R}^n \times \mathbb{R}$, with $N \geq n+1$ be in general position with with non positive weights: $\forall i,\ \mu_i \leq 0$, and let $K_{\mathbf{P}}$ be the n-dimensional full complex over $\mathbf{P}$.*

Denote by $\beta_{\mathbf{P}} \in \boldsymbol{C}_{n-1}(K_{\mathbf{P}})$ the $(n-1)$-chain made of simplices belonging to the boundary of $\mathcal{CH}(\mathbf{P})$.

Then for any $p \in [1,\infty)$, if:

$$\Gamma_{reg} = \underset{\substack{\Gamma \in \boldsymbol{C}_n(K_{\mathbf{P}}) \\ \partial\Gamma = \beta_{\mathbf{P}}}}{\arg\min} \|\Gamma\|_{(p)}$$

then the simplicial complex $|\Gamma_{reg}|$ support of Γ_{reg} is the regular triangulation of $\mathbf{P}$.

Lemma 6.14. *One has:*

$$\sup_{x \in |\sigma|} f_\sigma(x) = \mu_B(\sigma) \tag{6.15}$$

Proof. Since the expression (6.11) does not depend on the origin O, let us choose this origin to be $O = P_B(\sigma)$. With barycentric coordinates based on the vertices $P_i, i = 0,k$ of σ, i.e. $\lambda_i \geq 0$, $\sum_i \lambda_i = 1$ such that $x = \sum_i \lambda_i P_i$, the expression of f_σ is:

$$f_\sigma(x) = \left(\sum_i \lambda_i \left((P_i - P_B(\sigma))^2 - \mu_i\right)\right) - (x - P_B(\sigma))^2$$

one has $(P_i - P_B(\sigma))^2 - \mu_i - \mu_B(\sigma) = \mathrm{D}\left((P_B(\sigma),\mu_B(\sigma)),(P_i,\mu_i)\right) \leq 0$ so that $(P_i - P_B(\sigma))^2 - \mu_i \leq \mu_B(\sigma)$ and it follows that:

$$\forall x, f_\sigma(x) \leq \mu_B(\sigma) - (x - P_B(\sigma))^2 \tag{6.16}$$

We have from Lemma 6.9 that $P_B(\sigma) \in |\Theta(\sigma)| \subset |\sigma|$ so that, in the expression of $P_B(\sigma)$ as a barycenter of vertices of σ, only coefficients λ_i corresponding to vertices of $\Theta(\sigma) \subset \sigma$ are non zero:

$$P_B(\sigma) = \sum_{(P_i,\mu_i) \in \Theta(\sigma)} \lambda_i P_i$$

One has by definition of (P_C,μ_C):

$$(P_i,\mu_i) \in \Theta(\sigma) \Rightarrow (P_i - P_C(\Theta(\sigma)))^2 - \mu_i = \mu_C(\Theta(\sigma))$$

and since we know from Lemma 6.10 that:

$$(P_B, \mu_B)(\sigma) = (P_B, \mu_B)(\Theta(\sigma)) = (P_C, \mu_C)(\Theta(\sigma))$$

one gets:

$$(P_i, \mu_i) \in \Theta(\sigma) \Rightarrow (P_i - P_B(\sigma))^2 - \mu_i = \mu_B(\sigma)$$

and:

$$f_\sigma(P_B(\sigma)) = \left(\sum_{(P_i,\mu_i)\in\Theta(\sigma)} \lambda_i((P_i - P_B(\sigma))^2 - \mu_i) \right) - 0^2 = \mu_B(\sigma)$$

This with (6.16) ends the proof. $\square$

Since f_σ is continuous, it follows from Lemma 6.14 that:

$$\lim_{p\to\infty} w_p(\sigma) = w_\infty(\sigma) = \|f_\sigma\|_\infty = \sup_{x\in|\sigma|} f_\sigma(x) = \mu_B(\sigma) \tag{6.17}$$

We then have the following:

Lemma 6.15. *Let be $\mathbf{P} = \{(P_1, \mu_1), \ldots, (P_N, \mu_N)\} \subset \mathbb{R}^n \times \mathbb{R}$, with $N \geq n+1$ in general position and with non positive weights: $\forall i$, $\mu_i \leq 0$. Let $K_\mathbf{P}$ be the corresponding n-dimensional full complex.*

For p large enough, the weight $w_p(\sigma)^p$ of any n-simplex $\sigma \in K_\mathbf{P}$, is larger than the sum of alls n-simplices in $K_\mathbf{P}$ with smaller "bounding weight" μ_B. In other words, if $K_\mathbf{P}^{[n]}$ is the set of n-simplices in $K_\mathbf{P}$:

$$\exists p^\star, \forall p \geq p^\star,\ \forall \sigma \in K_\mathbf{P}^{[n]}, \quad w_p(\sigma)^p > \sum_{\tau\in K_\mathbf{P}^{[n]}, \mu_B(\tau)<\mu_B(\sigma)} w_p(\tau)^p$$

Proof. Consider the smallest ratio between two bounding of n-simplices in $K_\mathbf{P}$:

$$\iota = \inf_{\substack{\sigma_1,\sigma_2\in K_\mathbf{P}^{[n]} \\ \mu_B(\sigma_1)<\mu_B(\sigma_2)}} \frac{\mu_B(\sigma_2)}{\mu_B(\sigma_1)}$$

From the finiteness of simplices, one has $\iota > 1$ and since for a finite number of n-simplices $\sigma \in K_\mathbf{P}^{[n]}$ we have from (6.17):

$$\lim_{p\to\infty} w_p(\sigma) = \mu_B(\sigma)$$

It follows that there is some p_0 large enough such that for any $p > p_0$ one has:

$$\forall \sigma_1, \sigma_2 \in K_\mathbf{P}^{[n]},\ \mu_B(\sigma_1) < \mu_B(\sigma_2) \Rightarrow \frac{\mu_B(\sigma_2)}{\mu_B(\sigma_1)} > \frac{1+\iota}{2}$$

Let $\mathcal{N}$ be total number of n-simplices in $K_\mathbf{P}$. Taking:

$$p^\star = \max\left(p_0, \left\lceil \frac{\log \mathcal{N}}{\log \frac{1+\iota}{2}} \right\rceil \right)$$

realizes the statement of the lemma. $\square$

As explained in the proof of the main theorem of section 6.10, Lemma 6.15 allows us to focus on the link of a single simplex τ. However, before that, we need to introduce geometrical constructions that give an explicit representation of this link. These geometrical constructions and associated properties will take next two sections 6.7 and 6.8. Moreover Section 6.9 is required to establish Lemma 6.27 , a key ingredient of the proof of main Theorem.

6.7 Projection on the bisector of a simplex

In this section we establishes a condition for a simplex τ to belong to the regular triangulation, and when it is the case, we give a characterization of its link in the regular triangulation.

We denote by $\mathbf{bis}_\tau$ the $(n-k)$-dimensional affine space *bisector* of τ formally defined as:

$$\mathbf{bis}_\tau \underset{\text{def.}}{=} \{x \in \mathbb{R}^n, \forall v_1, v_2 \in \tau, \mathrm{D}\left((x,0), v_1\right) = \mathrm{D}\left((x,0), v_2\right)\} \tag{6.18}$$

In the particular case where $\dim(\tau) = 0$, one has $\mathbf{bis}_\tau = \mathbb{R}^n$.

Let $x \mapsto \pi_{\mathbf{bis}_\tau}(x)$ and $x \mapsto d(x, \mathbf{bis}_\tau) = d(x, \pi_{\mathbf{bis}_\tau}(x))$ denote respectively the orthogonal projection on and the minimal distance to $\mathbf{bis}_\tau$. We define a projection $\pi_\tau : \mathbf{P} \to \mathbf{bis}_\tau \times \mathbb{R}$ as follows:

$$\pi_\tau(P, \mu) \underset{\text{def.}}{=} \left(\pi_{\mathbf{bis}_\tau}(P), \mu - d(P, \mathbf{bis}_\tau)^2\right) \tag{6.19}$$

Figure 6.2 illustrates π_τ for ambient dimension 3 and $\dim(\tau) = 1$.

Let $o_\tau = P_C(\tau) \in \mathbf{bis}_\tau$ denote the (generalized) circumcenter of τ. If $(P_i, \mu_i) \in \tau$, then $\mathrm{D}\left((o_\tau, \mu_C(\tau)), (P_i, \mu_i)\right) = 0$.

Since $o_\tau = \pi_{\mathbf{bis}_\tau}(P_i)$ that gives $(P_i - o_\tau)^2 - \mu_i - \mu_C(\tau) = d(P, \mathbf{bis}_\tau)^2 - \mu_i - \mu_C(\tau) = 0$. It follows that if we denote $\mu(\pi_\tau(P_i, \mu_i))$ the weight of $\pi_\tau(P_i, \mu_i)$, one has $\mu(\pi_\tau(P_i, \mu_i)) = \mu_i - d(P, \mathbf{bis}_\tau)^2 = -\mu_C(\tau)$. We have shown therefore that π_τ sends all vertices of τ to a single weighted point:

$$(P_i, \mu_i) \in \tau \Rightarrow \pi_\tau(P_i, \mu_i) = (o_\tau, -\mu_C(\tau)) \tag{6.20}$$

We have then the following lemma:

Lemma 6.16. *Let $\mathbf{P}$ be in general position, $\tau \in K_{\mathbf{P}}$ a k-simplex and $\sigma \in K_{\mathbf{P}}$ a coface of τ. Then σ is in the regular triangulation of $\mathbf{P}$ if and only if $\pi_\tau(\sigma)$ is a coface of the vertex $\{(o_\tau, -\mu_C(\tau))\} = \pi_\tau(\tau)$ in the regular triangulation of $\pi_\tau(\mathbf{P})$.*

Proof of Lemma 6.16. Under generic condition, σ is in the regular triangulation of $\mathbf{P}$ if and only if there is a weighted point $(P_C(\sigma), \mu_C(\sigma))$ such that:

$$\forall (P_i, \mu_i) \in \sigma, \quad \mathrm{D}\left((P_C(\sigma), \mu_C(\sigma)), (P_i, \mu_i)\right) = 0 \tag{6.21}$$

$$\forall (P_i, \mu_i) \in \mathbf{P} \setminus \sigma, \quad \mathrm{D}\left((P_C(\sigma), \mu_C(\sigma)), (P_i, \mu_i)\right) > 0 \tag{6.22}$$

Observe that, since $\tau \subset \sigma$, (6.21) implies:

$$\forall (P_i, \mu_i) \in \tau, \mathrm{D}\left((P_C(\sigma), 0), (P_i, \mu_i)\right) = \mu_C(\sigma)$$

This and the definition (6.18) of bisector show that $P_C(\sigma)$ must be on the bisector of τ:

$$P_C(\sigma) \in \mathbf{bis}_\tau \tag{6.23}$$

We get, for $(P_i, \mu_i) \in \mathbf{P}$:

$$\begin{aligned}
&\mathrm{D}\left((P_C(\sigma), \mu_C(\sigma)), (P_i, \mu_i)\right) \\
&\quad = (P_C(\sigma) - P_i)^2 - \mu_C(\sigma) - \mu_i \\
&\quad = (\pi_{\mathbf{bis}_\tau}(P_i) - P_C(\sigma))^2 + d(P_i, \mathbf{bis}_\tau)^2 - \mu_C(\sigma) - \mu_i \\
&\quad = \mathrm{D}_{\mathbf{bis}_\tau}\left((P_C(\sigma), \mu_C(\sigma)), (\pi_{\mathbf{bis}_\tau}(P_i), \mu_i - d(P_i, \mathbf{bis}_\tau)^2)\right) \\
&\quad = \mathrm{D}_{\mathbf{bis}_\tau}\left((P_C(\sigma), \mu_C(\sigma)), \pi_\tau(P_i, \mu_i)\right)
\end{aligned} \tag{6.24}$$

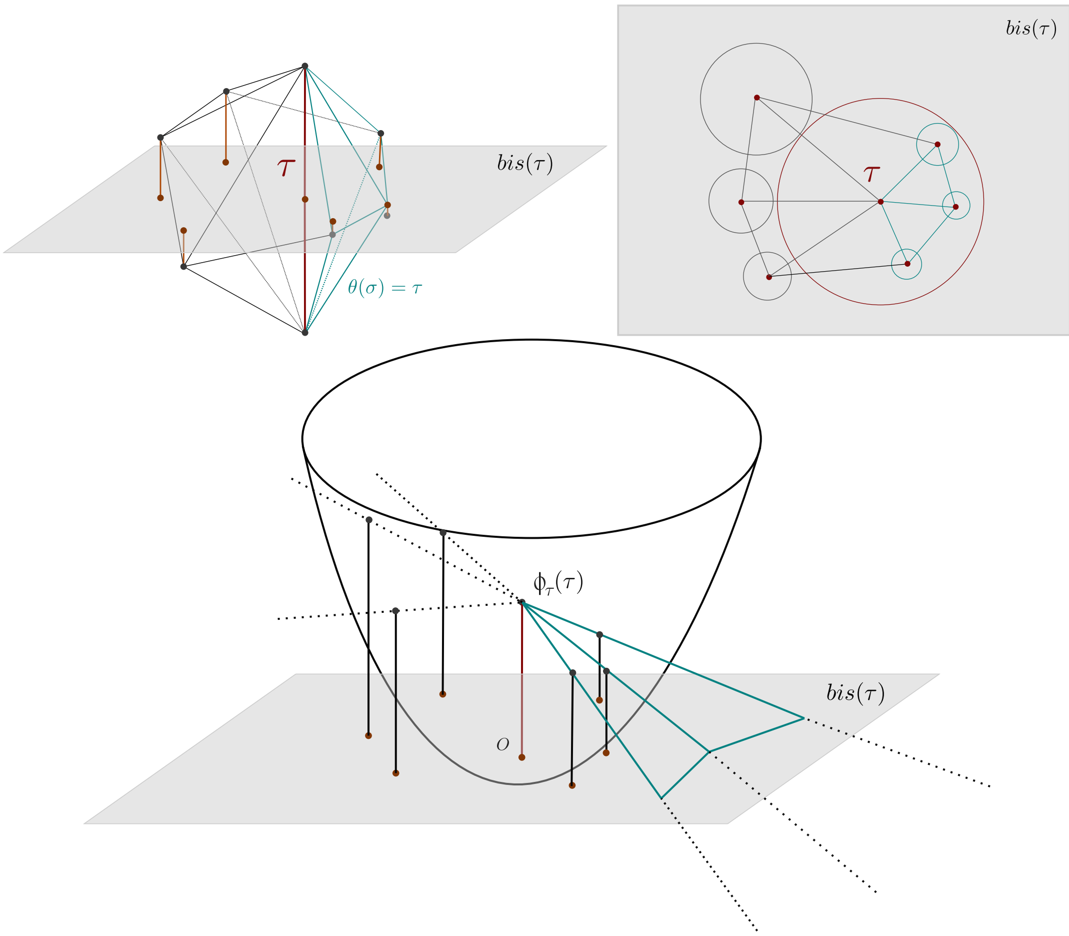

Figure 6.2: Illustration for the definition of $\mathbf{bis}_\tau$ and π_τ (top left and right) and Φ_τ (bottom).

In the last two lines, the weighted distance is denoted $\mathrm{D}_{\mathbf{bis}_\tau}$ instead of D in order to stress that, thanks to (6.23) , it occurs on weighted point of $\mathbf{bis}_\tau \times \mathbb{R}$ rather than $\mathbb{R}^n \times \mathbb{R}$. It results that (6.21) and (6.22) are equivalent to:

$$\begin{array}{rl} \forall (P_i, \mu_i) \in \sigma, & \mathrm{D}_{\mathbf{bis}_\tau}\left((P_C(\sigma), \mu_C(\sigma)), \pi_\tau(P_i, \mu_i)\right) = 0 \\ \forall (P_i, \mu_i) \in \mathbf{P} \setminus \sigma, & \mathrm{D}_{\mathbf{bis}_\tau}\left((P_C(\sigma), \mu_C(\sigma)), \pi_\tau(P_i, \mu_i)\right) > 0 \end{array}$$

which precisely means that $\pi_\tau(\sigma)$ is a coface of the vertex $\pi_\tau(\tau) = \{(o_\tau, -\mu_C(\tau))\}$ in the regular triangulation of $\pi_\tau(\mathbf{P})$. □

Looking at the definition 6.5 in the light of (6.24) in the proof of Lemma 6.16 we get that:

$$\sigma \supset \tau \;\Rightarrow\; \mu_C(\sigma) = \mu_C(\pi_\tau(\sigma)) \text{ and } P_C(\sigma) = P_C(\pi_\tau(\sigma)) \tag{6.25}$$

An immediate consequence of Lemma 6.16 is:

Corollary 6.17. *The projection π_τ preserves the structure of the regular triangulation around τ, more precisely:*

1. *the simplex τ is in the regular triangulation of $\mathbf{P}$ if and only if the vertex $\pi_\tau(\tau)$ is in the regular triangulation of $\pi_\tau(\mathbf{P})$,*
2. *if τ is in the regular triangulation of $\mathbf{P}$, the link of τ in the regular triangulation of $\mathbf{P}$ is isomorphic to the link of vertex $\pi_\tau(\tau) = \{(o_\tau, -\mu_C(\tau))\}$ in the regular triangulation of $\pi_\tau(\mathbf{P})$.*

Proof. The first item results from Lemma 6.16 if we consider the special case $\sigma = \tau$ while the second item result from the same lemma when considering σ as a proper coface of τ. □

6.8 Polytope and shadow associated to a link in a regular triangulation

In this section we are interested in characterizing k-simplices $\tau \in K_{\mathbf{P}}$ such that, following Lemma 6.10, $\tau = \Theta(\sigma)$ where σ is a n-simplex of the regular triangulation. In the case of Delaunay triangulation, one has always $1 \leq k = \dim(\tau)$. In the more general case of a regular triangulation a n-simplex can be "included" in the sphere associated to a single of its vertices v with $\tau = \{v\}$, in which case $k = 0$. We know from Lemma 6.10 that in this case one has $\Theta(\tau) = \tau$. In case of Delaunay this means that the circumsphere and the smallest enclosing ball coincide for the simplex τ. In the general case it says that:

$$(P_B, \mu_B)(\tau) = (P_C, \mu_C)(\tau) \tag{6.26}$$

So we restrict ourself in this section to k-simplices τ that satisfy (6.26) and we study again the link of τ in the regular triangulation of $\mathbf{P}$ or equivalently (Lemma 6.16) the link of the vertex $\pi_\tau(\tau) = \{(o_\tau, -\mu_C(\tau))\}$ in the regular triangulation of $\pi_\tau(\mathbf{P})$.

We know then from Proposition 6.12 that the link of τ in the regular triangulation is isomorphic to the link of $\mathrm{lift}(\pi_\tau(\tau))$ on the boundary of the lower convex hull of $\mathrm{lift}(\pi_\tau(\mathbf{P}))$. We consider the lift with the origin at o_τ, in other words, the image of a vertex $(P, \mu) \in \mathbf{P}$ is:

$$\begin{aligned} \Phi_\tau(P, \mu) &\underset{\text{def.}}{=} \mathrm{lift}\left(\pi_\tau(P, \mu)\right) \\ &= \left(\pi_{\mathbf{bis}_\tau}(P) - o_\tau,\; (\pi_{\mathbf{bis}_\tau}(P) - o_\tau)^2 - \mu + d(P, \mathbf{bis}_\tau)^2\right) \end{aligned}$$

Observe that, from (6.20):

$$\Phi_\tau(\tau) = \{(0, \mu_C(\tau)\}$$

Call P_τ the set of weighted points in $(P_i, \mu_i) \in \mathbf{P}$ such that $\tau \cup (P_i, \mu_i)$ has same bounding weight than τ. In the case of the Delaunay triangulation, P_τ corresponding to the set of points in $\mathbf{P}$ inside the bounding ball of τ. Formally, using the assumption (6.26):

$$P_\tau \underset{\text{def.}}{=} \{(P_i, \mu_i) \in \mathbf{P} \setminus \tau,\ \mathrm{D}\left((P_C(\tau), \mu_C(\tau)), (P_i, \mu_i)\right) < 0\} \tag{6.27}$$

Denote by K_τ the $(n-k-1)$-dimensional simplicial complex made of all up to dimension $(n-k-1)$ simplices over vertices in P_τ.

K_τ can alternatively be defined as the link of τ in the complete n-complex (i.e. containing all possible simplices up to dimension n) over the set of vertices $\tau \cup P_\tau$.

Observe that:

$$\begin{aligned} &\mathrm{D}\left((P_C(\tau), \mu_C(\tau)), (P_i, \mu_i)\right) < 0 \\ \iff &\left(\pi_{\mathbf{bis}_\tau}(P_i) - o_\tau\right)^2 - \mu_i + d\left(P_i, \mathbf{bis}_\tau\right)^2 < \mu_C(\tau) \end{aligned} \tag{6.28}$$

Recall than from observation 6.6, one has, under the assumption of non positive weight in $\mathbf{P}$, that $\mu_C(\tau) \geq 0$.

Denote by Height(lift((P, μ)) the *height* of the lift of a point (P, μ), defined as the last coordinate of the lift, so that:

$$\mathrm{Height}(\Phi_\tau(P, \mu)) = \left(\pi_{\mathbf{bis}_\tau}(P) - o_\tau\right)^2 - \mu + d(P, \mathbf{bis}_\tau)^2 \tag{6.29}$$

Since under our generic conditions we have $\pi_{\mathbf{bis}_\tau}(P) - o_\tau \neq 0$, (6.27), (6.28) and (6.29) imply that:

$$\exists v \in P_\tau \Rightarrow \mathrm{Height}(\Phi_\tau(v, \mu)) > 0 \tag{6.30}$$

And (6.28) and (6.30) can be rephrased as:

Observation 6.18. *A vertex belongs to P_τ if and only if the height of its image by Φ_τ is strictly less than $\mu_C(\tau) > 0$:*

$$v \in P_\tau \iff 0 < \mathrm{Height}(\Phi_\tau(v)) < \mu_C(\tau)$$

It follows from the observation that we are allowed to define the following:

Definition 6.19 (Shadow)**.** *Let v be a vertex in P_τ. The* shadow $\mathrm{Sh}_\tau(v)$ *of v is a point in the $(n-k)$-dimensional Euclidean space $\mathbf{bis}_\tau$ defined as the intersection of the half-line starting at $(0, \mu_C(\tau))$ and going through $\Phi_\tau(v)$ with the space $\mathbf{bis}_\tau$.*

$$\mathrm{Sh}_\tau(P, \mu) \underset{\mathit{def.}}{=} \frac{\mu_C(\tau)}{\mu_C(\tau) - \mathrm{Height}(\Phi_\tau(P, \mu))}\left(\pi_{\mathbf{bis}_\tau}(P) - o_\tau\right)$$

This induces (in general position) an embedding of simplices: the shadow of a simplex $\sigma \in K_\tau$ is a simplex in $\mathbf{bis}_\tau$ whose vertices are the shadows of vertices of σ. This in turn induces "immersions" of the chains defined on K_τ into chains immersed in $\mathbf{bis}_\tau$.

Let Γ_{reg} be the n-chain containing the n-simplices of the regular triangulation of $\mathbf{P}$ and $|\Gamma_{reg}|$ the corresponding simplicial complex. Denote by $\mathrm{X}(\tau) \in \boldsymbol{C}_{n-k-1}(K_\tau)$ the $(n-k-1)$-chain made of simplices $\sigma \in K_\tau$ such that $\tau \cup \sigma \in |\Gamma_{reg}|$:

$$\mathrm{X}(\tau) \underset{\text{def.}}{=} \{\sigma \in K_\tau,\ \dim(\sigma) = n-k-1,\ \tau \cup \sigma \in \Gamma_{reg}\} \tag{6.31}$$

In the following, we call *polytope* a finite intersection of closed half spaces $\cap_i H_i$. The *convex cone* of a polytope at a point p is the intersection of all such H_i whose boundary contain p. It is indeed a convex cone with apex p.

Definition 6.20 (Polytope facet **visible** from the point 0)**.** *We say that a facet f of a polytope is* visible from the point 0*, or* visible *for short, if the closed half-space H containing the polytope and whose boundary is the supporting plane of f does not contains* 0.

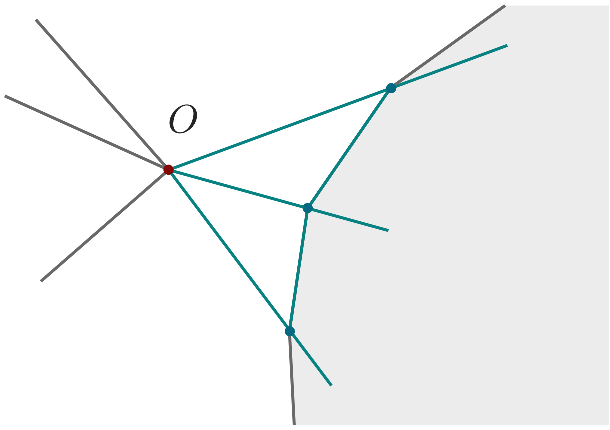

Figure 6.3: Illustration of the shadow polytope of Definition 6.21 corresponding to the example of Figure 6.2.

Definition 6.21 (Shadow Polytope)**.** *The (possibly empty) intersection of the convex cone of* $\Phi_\tau(\mathbf{P})$ *at* $\Phi_\tau(\tau) = (0, \mu_C(\tau))$ *with* $\mathbf{bis}_\tau$ *is called* shadow polytope *of* τ*.*

Figure 6.3 depicts in blue the Shadow polytope corresponding to the example of Figure 6.2. Observe that the shadow polytope is indeed a polytope since for each lower half-space H_j with boundary going through $(0, \mu_C(\tau))$ and contributing to the convex cone of $\Phi_\tau(\mathbf{P})$ at $(0, \mu_C(\tau))$, the intersection $H_j \cap \mathbf{bis}_\tau$ is a $(n-k)$-dimensional half-space in $\mathbf{bis}_\tau$. The shadow polytope is precisely the intersection of all such half-spaces $H_j \cap \mathbf{bis}_\tau$. Since each H_j is a lower half-space and since by observation 6.18 one has $\mu_C(\tau) > 0$, $H_j$does not contains $(0,0)$, which implies that $H_j \cap \mathbf{bis}_\tau$ does not contain the point o_τ in $\mathbf{bis}_\tau$. It follows hat:

Observation 6.22. *All facets of the shadow polytope are visible from* 0*.*

Lemma 6.23. *Let* $\mathbf{P}$ *be in general position with non positive weights. We have the following:*

1. *τ is in the regular triangulation of* $\mathbf{P}$ *if and only if* $\Phi_\tau(\tau) = (0, \mu_C(\tau))$ *is an extremal point of the convex hull of* $\Phi_\tau(\mathbf{P})$*.*

2. *When τ is in the regular triangulation of* $\mathbf{P}$*, its link is isomorphic to the link of the vertex* $\Phi_\tau(\tau) = (0, \mu_C(\tau))$ *in the simplicial complex corresponding to the boundary of the lower convex hull of* $\Phi_\tau(\mathbf{P})$*.*

3. *When τ is in the regular triangulation of* $\mathbf{P}$*,* Sh_τ *induces a simplicial isomorphism between* $|\mathrm{X}(\tau)|$ *and the set of bounded faces of the boundary of the shadow polytope.*

Proof. 1. follows from Corollary 6.17 item 1. together with Proposition 6.12.

2. follows from Corollary 6.17 item 2. together with Proposition 6.12.

For 3. consider a simplex $\sigma \in \mathrm{X}(\tau)$. $\Phi_\tau(\sigma)$ is a simplex in the link of $\Phi_\tau(\tau)$ in the lower convex hull of $\Phi_\tau(\mathbf{P})$ and therefore the convex cone CC_σ with apex $\Phi_\tau(\tau)$ and going through $\Phi_\tau(|\sigma|)$ is on the boundary of the convex cone $CC_\mathbf{P}$ with apex $\Phi_\tau(\tau)$ and going through the convex hull of $\Phi_\tau(\mathbf{P})$.

Therefore $\mathrm{Sh}_\tau(\sigma)$, intersection of CC_σ with $\mathbf{bis}_\tau$ is on the boundary of the shadow polytope, intersection of $CC_\mathbf{P}$ with $\mathbf{bis}_\tau$.

$\mathrm{Sh}_\tau(\sigma)$ is bounded as being the convex hull of the shadow of its vertices. Thanks to observation 6.22, it is a facets of the boundary of the shadow polytope visible from 0. In the reverse direction, a bounded facet of the boundary of the shadow polytope is precisely the shadow of a simplex $\Phi_\tau(\sigma)$ in the link of $\Phi_\tau(\tau)$ in the lower convex hull of $\Phi_\tau(\mathbf{P})$ with all its vertices in P_τ. Therefore $\sigma \in \mathrm{X}(\tau)$. This bijection extends to lower-dimensional faces of $|\mathrm{X}(\tau)|$ and is a simplical map. $\square$

6.9 Convex cone as lexicographic minimal chain

We start this section by giving the two lemmas 6.25 and 6.26, instrumental in the proof of 6.27.

Trace of a chain in a link

Definition 6.24 (Trace of a chain in a link)**.** *Given a k-simplex τ in a simplical complex K and a n-chain on K for $n > k$, we call* trace of Γ on the link of τ *the $(n-k-1)$-chain $\mathrm{Tr}_\tau(\Gamma)$ defined in the link of τ defined by:*

$$\mathrm{Tr}_\tau(\Gamma)(\sigma) = \Gamma(\tau \cup \sigma)$$

Next lemma is used twice in what follows.

Lemma 6.25. *Given a k-simplex τ in a simplical complex K and a n-chain Γ on K for $n > k$, one has:*

$$\partial \,\mathrm{Tr}_\tau(\Gamma) = \mathrm{Tr}_\tau(\partial\Gamma)$$

Proof. We need to prove that for any simplex σ in the link of τ, one has:

$$\partial \,\mathrm{Tr}_\tau(\Gamma)(\sigma) = \mathrm{Tr}_\tau(\partial\Gamma)(\sigma)$$

We have by definition that a $(n-k-2)$-simplex σ is in $\mathrm{Tr}_\tau(\partial\Gamma)(\sigma)$ if and only if $\tau \cap \sigma = \varnothing$ and $\tau \cup \sigma$ is in $\partial\Gamma$. In other words, $\tau \cup \sigma$ has an odd number of n-cofaces in the chain Γ. This in turn means that σ has an odd number of $(n-k)$-cofaces in the trace of Γ in the link of τ, i.e. $\sigma \in \partial \,\mathrm{Tr}_\tau(\Gamma)$. □

Farthest point cannot be in the interior of the visible part of a convex boundary

Lemma 6.26. *Let $C \subset \mathbb{R}^n$ be a polytope and $O \in \mathbb{R}^n \setminus C$. Let $X \subset \partial C$ be a compact set union of facets of C visible from O. If $x \in X$ maximizes the distance to O, then x is in the closure of $\partial C \setminus X$.*

Proof. Assume for a contradiction that x is in the relative interior of X, that is there is some $\rho > 0$ such that $B(x,\rho) \cap X = B(x,\rho) \cap \partial C$. Then all facets containing x are visible from O. if x is not a vertex pf C, it belongs then to the relative interior of a convex face f in ∂C with $\dim f \geq 1$. Then we have a contradiction since the function $y \mapsto d(P,y)$ is convex on f and cannot have an interior local maximum at x. We assume now that x is a vertex of C.

Following for example [31, 40], denote by $\mathrm{Tan}_x C$ and $\mathrm{Nor}_x C$ respectively the Tangent and Normal cone to C at x. In case of a closed polytope they can be expressed as:

$$\mathrm{Tan}_x C = \bigcap_{\rho>0} \{\lambda(c-x), \lambda \geq 0, c \in \mathcal{B}(x,\rho)\}$$

and

$$\mathrm{Nor}_x C = (\mathrm{Tan}_x C)^\perp \underset{\text{def.}}{=} \{u, \forall v \in \mathrm{Tan}_x C, \langle u, v\rangle \leq 0\} \tag{6.32}$$

Since C is a convex polytope, $\mathrm{Tan}_x C$ is a convex closed cone and one has [40]:

$$\mathrm{Tan}_x C = (\mathrm{Nor}_x C)^\perp \underset{\text{def.}}{=} \{v, \forall u \in \mathrm{Nor}_x C, \langle u, v\rangle \leq 0\} \tag{6.33}$$

Each facet F_i of C containing x is supported by a half-space $H_i = \{y, \langle y-x, n_i\rangle \leq 0\}$ and one has:

$$\mathrm{Nor}_x C = \left\{\sum_i \lambda_i n_i, \forall i, \lambda_i \geq 0\right\} \tag{6.34}$$

$$\forall i, \langle O - x, n_i \rangle > 0$$

This with (6.34) gives:

$$\forall u \in \mathrm{Nor}_x C, \langle O - x, u \rangle > 0 \tag{6.35}$$

using (6.33) we get that $O - x$ is in the interior of $-\mathrm{Tan}_x C$ i.e.:

$$x - O \in (\mathrm{Tan}_x C)^\circ \tag{6.36}$$

Since x is a vertex of C, $\mathrm{Tan}_x C$ is the convex hull of $\mathrm{Tan}_x \partial C$, therefore (6.36) implies at there are $t_1, \ldots t_{n+1} \in \mathrm{Tan}_x \partial C$ and $\lambda_1, \ldots \lambda_{n+1} \geq 0$ such that:

$$x - O = \sum_i \lambda_i t_i$$

which gives:

$$0 < \langle x - O, x - O \rangle = \langle x - O, \sum_i \lambda_i t_i \rangle = \sum_i \lambda_i \langle x - O, t_i \rangle$$

Since all the λ_i are not negative, there must be at least one i for which:

$$\langle x - O, t_i \rangle > 0$$

This precisely means that $y \mapsto d(P, y)$ is increasing in the direction t_i in a neighborhood of x in X, a contradiction since x is assumed to be a local maximum of $y \mapsto d(P, y)$ in X. $\square$

lexicographic order on shadows We define now another order on simplices together with its induced order on chains, respectively denoted $\leq_{Sh}$ and $\sqsubseteq_{\mathrm{Sh}}$. We add the $._{Sh}$ suffix because these orders are meant to be applied to shadow simplices and chains.

However, this section is self-contained and does not relies on previous construction. Lemma 6.27 is a result in itself: convex hulls can be defined as minimal lexicographic chains. Applying it to a translation (to match the visibility from 0 requirement) of the lift polytope could give directly a lexicographic order whose minimum is a regular triangulation, at the price of the lost of invariance by isometry of the order on simplices.

We associate to a $(n-1)$-simplex σ in $\mathbb{R}^n$ that does not contain 0 a dimension increasing sequence of faces $\varnothing = \tau_{-1}(\sigma) \subset \tau_0(\sigma) \subset \ldots \subset \tau_{n-1}(\sigma) = \sigma$ with $\dim(\tau_i) = i$. Under a simple generic condition (unicity of the max in (6.37)), it is defined as follows.

$\tau_{-1}(\sigma) = \varnothing$ and $\tau_0(\sigma)$ is the vertex of σ farthest from 0. More generally, define the distance from a flat (an affine space) F to 0 as $d_0(F) = \inf_{p \in F} d(p, 0)$. In other words, $d_0(F)$ is the distance from 0 to the orthogonal projection of 0 on F. If ζ is a non degenerate i-simplex for $i \geq 0$, defines $d_0(\zeta) = d_0(F(\zeta))$ where $F(\zeta)$ is the i-dimensional flat support of ζ.

For $i \geq 0$, $\tau_i(\sigma)$ is the coface of dimension i of $\tau_{i-1}(\sigma)$ whose supporting i-flat is farthest from 0:

$$\tau_i(\sigma) \underset{\text{def.}}{=} \arg \max_{\substack{\zeta \supset \tau_{i-1}(\sigma) \\ \dim(\zeta) = i}} d_0(\zeta) \tag{6.37}$$

For $i = 0, \ldots, n-1$ we set $\delta_i(\sigma) = d_0(\tau_i(\sigma))$ and the comparison $<_{Sh}$ between two $(n-1)$-simplices σ_1 and σ_2 is a lexicographic order on the sequences $(\delta_i(\sigma_1))_{i=0,\ldots n-1}$ and $(\delta_i(\sigma_2))_{i=0,\ldots n-1}$:

$$\sigma_1 <_{Sh} \sigma_2 \underset{\text{def.}}{\iff} \begin{cases} \exists k \geq 0,\ \delta_k(\sigma_1) < \delta_k(\sigma_2) \\ \text{and } \forall j, 0 \leq j < k,\ \delta_j(\sigma_1) = \delta_j(\sigma_2) \end{cases} \tag{6.38}$$

which defines an order relation:

$$\sigma_1 \leq_{Sh} \sigma_2 \underset{\text{def.}}{\iff} \sigma_1 = \sigma_2 \quad \text{or} \quad \sigma_1 <_{Sh} \sigma_2 \tag{6.39}$$

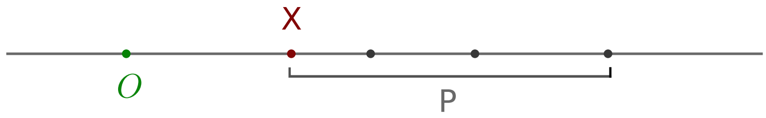

Figure 6.4: Illustration of Lemma 6.27 for $n = 1$.

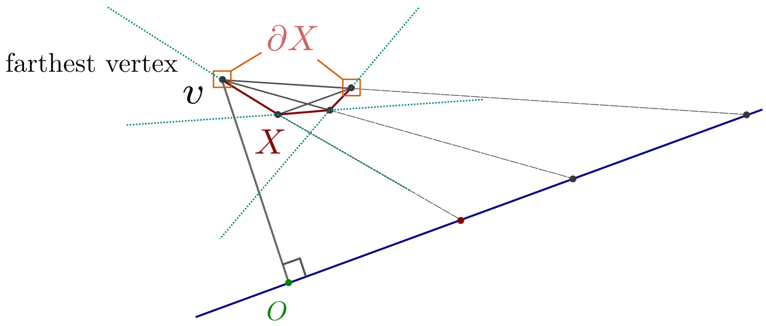

Figure 6.5: Illustration of the recursion in the proof of Lemma 6.27 for $n = 2$.

The following generic condition says that the sequence of $\delta_i(\sigma)$ does not coincide for two different simplices.

Condition 6.3. *Let K be a $(n-1)$-dimensional simplicial complex. For any pair of simplices $\sigma_1, \sigma_2 \in K$:*

$$\dim(\sigma_1) = \dim(\sigma_2) = k \qquad \text{and} \qquad d_0(\sigma_1) = d_0(\sigma_2)$$
$$\Rightarrow \quad \sigma_1 = \sigma_2$$

Observe that, under condition 6.3, $\leq_{Sh}$ is a total order on simplices. As for the lexicographic order $\sqsubseteq_{lex}$, following definition 6.1 the order $\leq_{Sh}$ on simplices induces a lexicographic order $\sqsubseteq_{\mathrm{Sh}}$ on k-chains of K.

Lemma 6.27. *Let P be a set of points in $\mathbb{R}^n$ such that $0 \in \mathbb{R}^n$ is not in the convex hull of P. Let K be the complete $(n-1)$-dimensional simplicial complex over P, i.e. the simplicial complex made of all $(n-1)$-simplices whose vertices are points in P with all their faces. Assume that K satisfies the generic condition 6.3.*

Let X be a $(n-1)$-chain in K whose $(n-1)$-simplices are in convex position, *i.e. contribute to the boundary of the convex hull of P, and are all* visible *from $0 \in \mathbb{R}^n$.*

Then:

$$X = \min_{\sqsubseteq_{\mathrm{Sh}}} \{\Gamma \in \boldsymbol{C}_{n-1}(K), \partial\Gamma = \partial X\} \tag{6.40}$$

where when $n = 1$ the boundary operator in (6.40) is meant as the boundary operator of reduced homology, i.e the linear operator $\tilde{\partial}_0 : \boldsymbol{C}_{n-1}(K) \to \mathbb{Z}_2$ that counts the parity.

Proof. We first claim that the lemma holds for $n = 1$. In this case the fact that 0 is not in the convex hull of P means that the 1-dimensional points in P are either all positive, either all negative. The single simplex in the convex hull boundary visible from 0 is the point in P closest to 0, i.e. the one with the smallest absolute value, which corresponds to the minimum chain with odd parity in the $\sqsubseteq_{\mathrm{Sh}}$ order, which proves the claim.

We assume then the theorem to be true for the dimension $n-1$ and proceed by induction. This recursion is illustrated on figure 6.5 for $n = 2$ and figure 6.6 for $n = 3$.

Consider the minimum:

$$\Gamma_{\min} = \min_{\sqsubseteq_{\mathrm{Sh}}} \{\Gamma \in \boldsymbol{C}_{n-1}(K), \partial\Gamma = \partial X\} \tag{6.41}$$

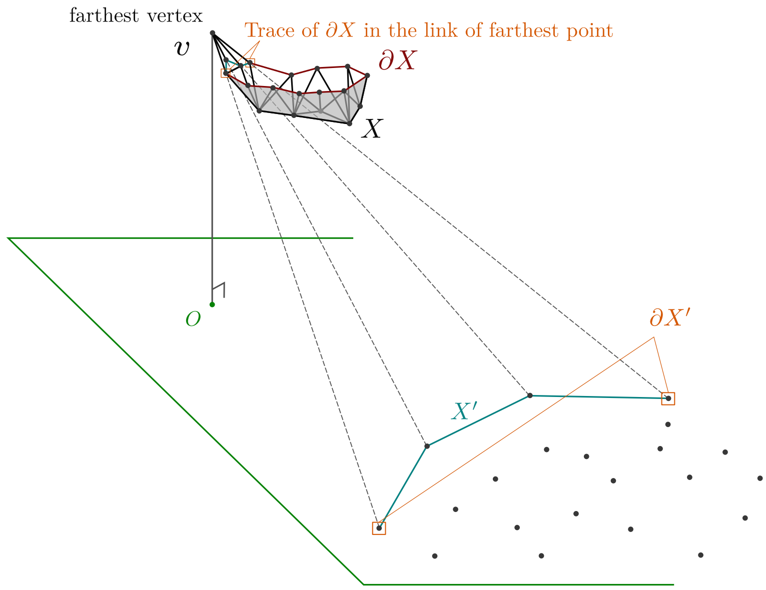

Figure 6.6: Illustration of the recursion in the proof of Lemma 6.27 for $n = 3$.

We need to prove that $\Gamma_{\min} = X$.

Let v be the (unique) vertex in the simplices of ∂X which is farthest from 0. Since v is a vertex in at least one simplex in $\partial X = \partial\Gamma_{\min}$, it must be a vertex in some simplex in $\Gamma_{\min}$.

We claim that v is also the vertex in the simplices of X which is farthest from 0. Indeed, X is made of visible facets of the boundary of $\mathcal{CH}(P)$, the convex hull of P. Thanks to Lemma 6.26, if x is a local maximum in X of the distance to 0 one has $x \in \partial X$ and the claim follows.

Since v is the vertex in X farthest from 0 and since by definition $\Gamma_{\min} \sqsubseteq_{\mathrm{Sh}} X$, we know that $\Gamma_{\min}$ does not contain any vertex farther from the origin than v, therefore v is also the vertex in the simplices of $\Gamma_{\min}$ farthest from 0.

Since $\partial\Gamma_{\min} = \partial X$, Lemma 6.25 implies that:

$$\partial\,\mathrm{Tr}_v(\Gamma_{\min}) = \mathrm{Tr}_v(\partial\Gamma_{\min}) = \mathrm{Tr}_v(\partial X) \tag{6.42}$$

In order to define a lexicographic order on chains on the link of v in K, we use a transformation very similar to the shadow of definition 6.19. Specifically, we consider the hyperplane Π containing 0 and orthogonal to the line $0v$. We associate to any $(n-2)$-simplex $\eta \in \mathrm{Lk}_K(v)$ the $(n-2)$ simplex $\pi_{v\Pi}(\eta)$ conical projection of η on Π with center v. In other words, if u is a vertex of η:

$$\{\pi_{v\Pi}(u)\} = \Pi \cap uv$$

where uv denote the line going through u and v. $\pi_{v\Pi}$ is a conical projection on vertices but it extends to a bijection on simplices and an isomorphism on chains that trivially commutes with the boundary operator.

By definition of the lexicographic order $\sqsubseteq_{\mathrm{Sh}}$, the comparison of two chains whose farthest vertex is v starts by comparing their restrictions to the star of v. Therefore, since v is the farthest vertex in $\Gamma_{\min}$, the restriction of $\Gamma_{\min}$ to the star of v must be minimum under the constraint $\partial\Gamma = \partial X$. The constraint $\partial\Gamma = \partial X$ for the restriction of $\Gamma_{\min}$ to the star of v is equivalent to the constraint given by equation (6.42) or equivalently by:

$$\partial\pi_{v\Pi}\left(\mathrm{Tr}_v(\Gamma_{\min})\right) = \pi_{v\Pi}\left(\mathrm{Tr}_v(\partial X)\right)$$

and the minimization on the restriction of the $(n-1)$-chain $\Gamma_{\min}$ to the star of v can equivalently be expressed as the minimization of the $(n-2)$-chain $\gamma_{\min} = \pi_{v\Pi}\left(\mathrm{Tr}_v(\Gamma_{\min})\right)$ under the constraint $\partial\gamma_{\min} = \pi_{v\Pi}\left(\mathrm{Tr}_v(\partial X)\right)$, we have then:

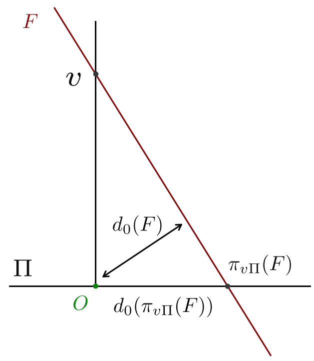

Figure 6.7: Illustration for equation (6.44).

$$
\begin{aligned}
\gamma_{\min} &= \pi_{v\Pi}\left(\mathrm{Tr}_v\left(\Gamma_{\min}\right)\right) \\
&= \pi_{v\Pi}\left(\mathrm{Tr}_v\left(\min_{\sqsubseteq_{\mathrm{Sh}}}\{\Gamma \in \boldsymbol{C}_{n-1}(K), \partial\Gamma = \partial X\}\right)\right) \\
&= \min_{\sqsubseteq_{\mathrm{Sh}}}\{\gamma \in \boldsymbol{C}_{n-2}\left(\pi_{v\Pi}\left(\mathrm{Lk}_K(v)\right)\right), \partial\gamma = \pi_{v\Pi}\left(\mathrm{Tr}_v(\partial X)\right)\} \\
&= \min_{\sqsubseteq_{\mathrm{Sh}}}\{\gamma \in \boldsymbol{C}_{n-2}\left(\pi_{v\Pi}\left(\mathrm{Lk}_K(v)\right)\right), \partial\gamma = \partial\pi_{v\Pi}\left(\mathrm{Tr}_v(X)\right)\}
\end{aligned} \tag{6.43}
$$

In the third equality of (6.43) we have used the fact that the orders on $(n-1)$-simplices in the star of v in K and the order on corresponding $(n-2)$-simplices in the image by $\pi_{v\Pi}$ of the link of v are compatible.

Indeed, if F is a k-flat in $\mathbb{R}^n$ going through v, we have (see figure 6.7):

$$
d_0(\pi_{v\Pi}(F)) = d_0(F \cap \Pi) = \frac{d_0(F)\|v-0\|}{\sqrt{(v-0)^2 - d_0(F)^2}} \tag{6.44}
$$

with the convention $d_0(\pi_{v\Pi}(F)) = +\infty$ in the non generic case where $F \cap \Pi = \varnothing$ (the denominator vanishes in this case while since $v \in F$ one has $F \cap \Pi \neq \varnothing \Rightarrow d_0(F) < \|v-0\|$).

As seen on (6.44) $d_0(F) \mapsto d_0(\pi_{v\Pi}(F))$ is an increasing function and the orders are therefore consistent along the induction.

We claim that the minimization problem in the last member of (6.43) satisfies the condition of the theorem for $n' = n-1$ which is assumed true by induction.

Recursion: The hyperplane Π corresponds to $\mathbb{R}^{n'}$ with $n' = n-1$.

- $n' \leftarrow n-1$
- $P' \leftarrow \pi_{v\Pi}(P \setminus \{v\})$
- $K' \leftarrow \pi_{v\Pi}(\mathrm{Lk}_K(v))$
- $X' \leftarrow \pi_{v\Pi}\left(\mathrm{Tr}_v(X)\right)$

Since $(n-1)$-simplices in X are in convex positions, hyperplanes supporting these simplices, in particular the simplices in the star of v, separate all points of P from 0. It follows that the intersection of theses hyperplanes with the horizontal hyperplane, i.e. the images by $\pi_{v\Pi}$ of the hyperplanes, separates $P' = \pi_{v\Pi}(P \setminus \{v\})$ from 0.

It follows that the $(n'-1)$-simplices in $X' = \pi_{v\Pi}\left(\mathrm{Tr}_v(X)\right)$ are in convex position and are visible from 0.

Therefore one can apply our lemma recursively, which gives us, using (6.43):

$$\gamma_{\min} = \pi_{v\Pi}\left(\mathrm{Tr}_v\left(\Gamma_{\min}\right)\right) = \pi_{v\Pi}\left(\mathrm{Tr}_v(X)\right)$$

It follows that the faces in the star of v corresponding to $\mathrm{Tr}_v\left(\Gamma_{\min}\right)$ belong to X. Call Y the $(n-1)$-chain made of these simplices in the star of v. We have both $Y \subset X$ and $Y \subset \Gamma_{\min}$. Since v is the vertex farthest from 0 in both X and in $\Gamma_{\min}$ one has by definition of the lexicographic order:

$$\begin{aligned}
\Gamma_{\min} &= \min_{\sqsubseteq_{lex}}\{\Gamma \in \boldsymbol{C}_{n-1}(K), \partial\Gamma = \partial X\} \\
&= Y + \min_{\sqsubseteq_{lex}}\{\Gamma \in \boldsymbol{C}_{n-1}(K), \partial\Gamma = \partial(X - Y)\}
\end{aligned}$$

So, by considering the new problem $X \leftarrow (X - Y)$ and iterating as long as X is not empty, we get our final result $\Gamma_{\min} = X$. □

6.10 Proof of Theorem 6.2

Equivalence between $\sqsubseteq_{\mathrm{Sh}}$ and $\sqsubseteq_{lex}$ restricted to simplices with same bounding weight In order to establish the connexion between the orders $\sqsubseteq_{lex}$ and $\sqsubseteq_{\mathrm{Sh}}$ we first need the following (K_τ is defined just after (6.27)):

Lemma 6.28. *For $\sigma_1, \sigma_2 \in K_\tau$, one has:*

$$\mu_C(\tau \cup \sigma_1) \geq \mu_C(\tau \cup \sigma_2) \iff d_0(\mathrm{Sh}_\tau(\sigma_1)) \leq d_0(\mathrm{Sh}_\tau(\sigma_2)) \tag{6.45}$$

Proof. Using the definition 6.5, one has:

$$\left(P_C, \mu_C\right)\left(\pi_\tau(\tau) \cup \pi_\tau(\sigma)\right) = \underset{\substack{(P,\mu)\in\mathbb{R}^n\times\mathbb{R} \\ \forall (P_i,\mu_i)\in\pi_\tau(\tau)\cup\pi_\tau(\sigma), \mathrm{D}\left((P,\mu),(P_i,\mu_i)\right)=0}}{\arg\min} \mu \tag{6.46}$$

Since both terms of (6.45) are invariant by a global translation, we can assume without loss of generality and in order to make the computations simpler that $o_\tau = 0$.

In this case, as seen in (6.20) the coordinates of $\pi_\tau(\tau)$ are $(0, -\mu_C(\pi_\tau(\tau)) = (0, -\mu_C(\tau))$ by (6.25). So that $\mathrm{D}\left((P, \mu), \pi_\tau(\tau)\right) = 0$ gives us:

$$\mu = P^2 + \mu_C(\tau) \tag{6.47}$$

It follows that among the weighted points (P, μ) that satisfy $\mathrm{D}\left((P, \mu), \pi_\tau(\tau)\right) = 0$, minimizing μ is equivalent to minimizing P^2 and one can reformulate the characterization (6.46) of $\left(P_C, \mu_C\right)\left(\pi_\tau(\tau \cup \sigma)\right) = \left(P_C, \mu_C\right)\left(\pi_\tau(\tau) \cup \pi_\tau(\sigma)\right)$ as:

$$\left(P_C, \mu_C\right)\left(\pi_\tau(\tau) \cup \pi_\tau(\sigma)\right) = \underset{\substack{(P,\mu)\in\mathbb{R}^n\times\mathbb{R} \\ \forall (P_i,\mu_i)\in\pi_\tau(\tau)\cup\pi_\tau(\sigma), \mathrm{D}\left((P,\mu),(P_i,\mu_i)\right)=0}}{\arg\min} P^2 \tag{6.48}$$

For $(P, \mu) \in \mathbb{R}^{n-k} \times \mathbb{R}$, define the hyperplane $\Pi_{(P,\mu)}$ in $\mathbb{R}^{n-k} \times \mathbb{R}$ as:

$$(X, z) \in \Pi_{(P,\mu)} \underset{\text{def.}}{\iff} z = (\mu - P^2) + 2\langle P, X\rangle \tag{6.49}$$

Observe that:

$$\mathrm{D}\left((P, \mu), (P_i, \mu_i)\right) = 0 \iff \mathrm{lift}(P_i, \mu_i) \in \Pi_{(P,\mu)}$$

So that the definition of $(P_C, \mu_C)(\pi_\tau(\tau) \cup \pi_\tau(\sigma))$ given in (6.48) can be equivalently formulated as $\Pi_{(P_C,\mu_C)}$ being the hyperplane in $\mathbb{R}^{n-k} \times \mathbb{R}$ that minimizes P_C^2 among all hyperplanes containing both $\mathrm{lift}(\pi_\tau(\tau))$ and all the points in $\mathrm{lift}(\pi_\tau(\sigma))$.

But, as seen on (6.49), $2\|P_C\|$ is the slope of the hyperplane $\Pi_{(P_C,\mu_C)}$ so that $\Pi_{(P,\mu)}$ is the hyperplane with minimal slope going through $\Phi_\tau(\tau) = \mathrm{lift}(\pi_\tau(\tau))$ and all the points in $\Phi_\tau(\sigma) = \mathrm{lift}(\pi_\tau(\sigma))$. This slope $2\|P_C\|$ is also the slope of the unique $(\dim(\sigma)+1)$-dimensional affine space F going through $\Phi_\tau(\tau) = \mathrm{lift}(\pi_\tau(\tau)) = (0, \mu_C(\tau))$ and all the points in $\Phi_\tau(\sigma) = \mathrm{lift}(\pi_\tau(\sigma))$. Since $F \cap \mathbb{R}^{n-k} \times \{0\}$ is the affine space supporting $\mathrm{Sh}_\tau(\sigma)$, one has:

$$d_0(\mathrm{Sh}_\tau(\sigma)) = \frac{\mu_C(\tau)}{2\|P_C(\pi_\tau(\tau) \cup \pi_\tau(\sigma))\|}$$

so that, using (6.25) for the second equality:

$$d_0(\mathrm{Sh}_\tau(\sigma)) = \frac{\mu_C(\tau)}{2\sqrt{\mu_C(\pi_\tau(\tau) \cup \pi_\tau(\sigma)) - \mu_C(\tau)}} = \frac{\mu_C(\tau)}{2\sqrt{\mu_C(\tau \cup \sigma) - \mu_C(\tau))}}$$

It follows that the map:

$$\mu_C(\tau \cup \sigma) \to d_0(\mathrm{Sh}_\tau(\sigma))$$

is decreasing. □

For a n-chain Γ denote by $\downarrow_\rho \Gamma$ the chain obtained by removing from Γ all simplices with bounding weight strictly greater than ρ.

$$\downarrow_\rho \Gamma \underset{\text{def.}}{=} \{\sigma \in \Gamma, \mu_B(\sigma) \le \rho\} \tag{6.50}$$

We have then:

Lemma 6.29. *Using the notation and context of section 6.10, for two n-chains $\Gamma_1, \Gamma_2 \in \boldsymbol{C}_n(K_{\mathbf{P}})$ one has:*

$$\begin{aligned} &\downarrow_{\mu_B^{\neq}} \Gamma_1 \quad \sqsubseteq_{lex} \quad \downarrow_{\mu_B^{\neq}} \Gamma_2 \\ \Rightarrow \mathrm{Sh}_{\tau_{\mu_B^{\neq}}}&\left(\mathrm{Tr}_{\tau_{\mu_B^{\neq}}}\left(\downarrow_{\mu_B^{\neq}} \Gamma_1\right)\right) \quad \sqsubseteq_{\mathrm{Sh}} \quad \mathrm{Sh}_{\tau_{\mu_B^{\neq}}}\left(\mathrm{Tr}_{\tau_{\mu_B^{\neq}}}\left(\downarrow_{\mu_B^{\neq}} \Gamma_2\right)\right) \end{aligned}$$

Proof. If we denote by $\underset{\mu_B^{\neq}}{\rightarrow} \Gamma$ the set of simplex in Γ with bounding weight equal to $\mu_B^{\neq}$:

$$\underset{\mu_B^{\neq}}{\rightarrow} \Gamma \underset{\text{def.}}{=} \left\{\sigma \in \Gamma, \mu_B(\sigma) = \mu_B^{\neq}\right\}$$

We claim that:

$$\left(\downarrow_{\mu_B^{\neq}} \Gamma_1 \sqsubseteq_{lex} \downarrow_{\mu_B^{\neq}} \Gamma_2\right) \Rightarrow \left(\underset{\mu_B^{\neq}}{\rightarrow} \Gamma_1 \sqsubseteq_{lex} \underset{\mu_B^{\neq}}{\rightarrow} \Gamma_2\right) \tag{6.51}$$

Indeed, by definition of the lexicographic order, if this did not holds, it would imply $\underset{\mu_B^{\neq}}{\rightarrow} \Gamma_1 \neq \underset{\mu_B^{\neq}}{\rightarrow} \Gamma_2$ and the largest simplex for which $\underset{\mu_B^{\neq}}{\rightarrow} \Gamma_1$ and $\underset{\mu_B^{\neq}}{\rightarrow} \Gamma_2$ differ would be in Γ_1 contradicting $\downarrow_{\mu_B^{\neq}} \Gamma_1 \sqsubseteq_{lex} \downarrow_{\mu_B^{\neq}} \Gamma_2$ which proves the claim (6.51).

Note that, from Lemma 6.10 and generic condition 6.2, all the simplices in $\underset{\mu_B^{\neq}}{\rightarrow} \Gamma_1$ and $\underset{\mu_B^{\neq}}{\rightarrow} \Gamma_2$ are in the star of a single simplex $\tau = \tau_{\mu_B^{\neq}}$ such that $\mu_C(\tau) = \mu_B(\tau) = \mu_B^{\neq}$.

It remains to show that the order $\sqsubseteq_{lex}$ restricted to simplices $\tau \cup \sigma$ with $\mu_B(\tau \cup \sigma) = \mu_B^{\neq}$ corresponds to the order $\sqsubseteq_{\mathrm{Sh}}$ on the shadow of σ.

By definition of $\sqsubseteq_{lex}$, since in (6.9) one has always $\mu_0(\tau \cup \sigma_1) = \mu_0(\tau \cup \sigma_2) = \mu_B^{\neq}$, it goes like this:

$$\sigma_1 < \sigma_2 \underset{\text{def.}}{\iff} \begin{cases} \exists k \geq 1,\ \mu_k(\tau \cup \sigma_1) > \mu_k(\tau \cup \sigma_2) \\ \text{and } \forall j, 0 \leq j < k,\ \mu_j(\tau \cup \sigma_1) = \mu_j(\tau \cup \sigma_2) \end{cases} \tag{6.52}$$

Observe that this expression is similar to (6.38).

For a 0-simplex $\{v\} \in K_{\mathbf{P}}$, the circumweight $\mu_C(\tau \cup \{v\})$ is, according to Lemma 6.28, a decreasing function of the distance $d_0(\mathrm{Sh}_\tau(\eta))$ between its shadow and the origin. It follows that for a $(n-k-1)$-simplex $\sigma \in K_{\mathbf{P}}$, the vertex v for which the circumweight $\mu_C(\tau \cup \{v\})$ is minimal has its shadow $\mathrm{Sh}_\tau(v)$ maximizing the distance to the origin. This minimal circumweight is $\mu_1(\tau \cup \{v\})$ while this maximal distance is $\delta_0(\sigma)$.

More generally, looking at (6.37) and (6.8), Lemma 6.28 allows to check that the simplex $\Theta_k(\sigma)$ of (6.8) in the star of τ in $K_{\mathbf{P}}$ corresponds to the simplex $\tau_{k-1}(\sigma)$ in (6.37) in the link of τ:

$$\Theta_k(\sigma) = \tau \cup \tau_{k-1}(\sigma)$$

So that for $\sigma_1, \sigma_2 \in K_{\mathbf{P}}$ referring to (6.38) and (6.52):

$$\mu_k(\tau \cup \sigma_1) \leq \mu_k(\tau \cup \sigma_2) \iff \delta_{k-1}(\sigma_1) \geq \delta_{k-1}(\sigma_2)$$

It follows that, for $\Gamma_1, \Gamma_2 \in \boldsymbol{C}_n(K_{\mathbf{P}})$ and $\tau = \tau_{\mu_B^{\neq}}$:

$$\underset{\mu_B^{\neq}}{\rightarrow} \Gamma_1 \sqsubseteq_{lex} \underset{\mu_B^{\neq}}{\rightarrow} \Gamma_2 \iff \begin{cases} \mathrm{Sh}_\tau\left(\mathrm{Tr}_{\tau_{\mu_B^{\neq}}}\left(\downarrow_{\mu_B^{\neq}} \Gamma_1\right)\right) \\ \sqsubseteq_{\mathrm{Sh}} \mathrm{Sh}_{\tau_{\mu_B^{\neq}}}\left(\mathrm{Tr}_{\tau_{\mu_B^{\neq}}}\left(\downarrow_{\mu_B^{\neq}} \Gamma_2\right)\right) \end{cases}$$

which, with claim (6.51), ends the proof. □

Proof of main Theorem

Proof of Theorem 6.2. We prove Theorem 6.2 in the case of non positive weights which then extends to any weights thanks to observations 6.8 and 6.7.

As in Proposition 6.13, denote by Γ_{reg} the chain that defines the regular triangulation of $\mathcal{CH}(\mathbf{P})$. As in Theorem 6.2 denote by $\beta_{\mathbf{P}} \in \boldsymbol{C}_{n-1}(K_{\mathbf{P}})$ the $(n-1)$-chain made of simplices belonging to the boundary of $\mathcal{CH}(\mathbf{P})$.

According to Proposition 6.13 Γ_{reg} minimizes $\Gamma \mapsto \|\Gamma\|_{(p)}$ among the chains with boundary β for any $p \geq 1$. In particular Γ_{reg} minimizes $\Gamma \mapsto \|\Gamma\|_{p^\star}$ for the value $p^\star$ of Lemma 6.15.

Proposition 6.13 and Theorem 6.2 consider a minimum with respect to the same boundary condition while their objective differ.

In order to prove Theorem 6.2 we have to show that both minimum agree. For a contradiction, we assume now that they differ, which means that, $\Gamma_{\min} \neq \Gamma_{reg}$, or equivalently that $\Gamma_{\min} + \Gamma_{reg} \neq 0$ where $\Gamma_{\min}$ is the minimal chain of Theorem 6.2.

Consider $\mu_B^{\neq}$ to be the largest bounding weight for which some simplex in $\Gamma_{\min}$ and Γ_{reg} differ:

$$\mu_B^{\neq} \underset{\text{def.}}{=} \max\{\mu_B(\sigma), \sigma \in \Gamma_{\min} + \Gamma_{reg}\}$$

There must be at least one simplex with bounding weight $\mu_B^{\neq}$ in Γ_{reg} as, otherwise, by definition of $\mu_B^{\neq}$ there would be a simplex with radius $\mu_B^{\neq}$ in $\Gamma_{\min}$ and this would give $\Gamma_{reg} \sqsubseteq_{lex} \Gamma_{\min}$ with $\Gamma_{reg} \neq \Gamma_{\min}$ and since $\partial\Gamma_{reg} = \partial\Gamma_{\min} = \beta_{\mathbf{P}}$ this contradicts the definition of $\Gamma_{\min}$.

Similarly, it follows from Lemma 6.15 that if there was no simplex with bounding weight $\mu_B^{\neq}$ in $\Gamma_{\min}$, one would have $\|\Gamma_{\min}\|_{p^\star} < \|\Gamma_{reg}\|_{p^\star}$ and $\partial\Gamma_{reg} = \partial\Gamma_{\min}$: a contradiction with the minimality of Γ_{reg} for norm $\|\cdot\|_{p^\star}$ (Proposition 6.13).

We have shown that if they differ, both Γ_{reg} and $\Gamma_{\min}$ must contain at least one simplex with bounding weight $\mu_B^{\neq}$.

We know from the generic conditions that the set of simplices with bounding weigth $\mu_B^{\neq}$ are all cofaces of some unique dimension minimal simplex $\tau_{\mu_B^{\neq}}$.

If $\dim(\tau_{\mu_B^{\neq}}) = n$ then $\tau_{\mu_B^{\neq}}$ is (from the general position asumption) the unique simplex in $K_{\mathbf{P}}$ whose bounding weight is $\mu_B^{\neq}$. But then Γ_{reg} and $\Gamma_{\min}$ coincide on simplices with bounding weight $\mu_B^{\neq}$, a contradiction with the definition of $\mu_B^{\neq}$.

Assume now that $\dim(\tau_{\mu_B^{\neq}}) = k < n$.

For a n-chain $\Gamma \downarrow_\rho \Gamma$ is defined by (6.50). By definition of $\mu_B^{\neq}$ one has:

$$\downarrow_{\mu_B^{\neq}} \Gamma_{reg} - \downarrow_{\mu_B^{\neq}} \Gamma_{\min} = \Gamma_{reg} - \Gamma_{\min} \tag{6.53}$$

In order to spare our eyes, we allow ourselves to replace for the rest of the section:

$$\downarrow_{\mu_B^{\neq}} \text{ by } \downarrow$$

$$\tau_{\mu_B^{\neq}} \text{ by } \tau$$

It follows from (6.53) that:

$$\partial \left(\downarrow \Gamma_{reg} - \downarrow \Gamma_{\min}\right) = \partial\Gamma_{reg} - \partial\Gamma_{\min} = \beta_{\mathbf{P}} - \beta_{\mathbf{P}} = 0$$

It follows that $\downarrow \Gamma_{reg}$ and $\downarrow \Gamma_{\min}$ have same boundary and Lemma 6.25 says that their trace have also same boundary:

$$\partial \operatorname{Tr}_\tau \left(\downarrow \Gamma_{min}\right) = \partial \operatorname{Tr}_\tau \left(\downarrow \Gamma_{reg}\right) \tag{6.54}$$

Observe that $\operatorname{Tr}_\tau(\downarrow \Gamma_{reg})$ coincide with the definition of $\mathrm{X}(\tau)$ in (6.31). We know then from Lemma 6.23 that the shadows of simplices in $\mathrm{X}(\tau) = \operatorname{Tr}_\tau(\downarrow \Gamma_{reg})$, that is $\operatorname{Sh}_\tau(\mathrm{X}(\tau)) = \operatorname{Sh}_\tau(\operatorname{Tr}_\tau(\downarrow \Gamma_{reg}))$ is a chain in $\operatorname{Sh}_\tau(K_\tau)$ made of the faces of the convex hull of $\operatorname{Sh}_\tau(P_\tau)$ visible from the origin 0, where P_τ and K_τ are as defined in (6.27) and following lines.

In the remaining of this proof we use a lexicographic order $\sqsubseteq_{\mathrm{Sh}}$ on shadows of $(n-k-1)$-chains in K_τ, defined at the beginning of section 6.9. This order is equivalent to the order $\sqsubseteq_{lex}$ on corresponding n-chains restricted to the set of simplices with bounding weight $\mu_B^{\neq}$ (Lemmas 6.28 and 6.29).

This correspondance allows to conclude the proof by applying Lemma 6.27 that says that the chain defined by visible faces of a polytope minimises the lexicographic order $\sqsubseteq_{\mathrm{Sh}}$ among chains with same boundary.

More formally, thanks to Lemma 6.29 we have:

$$\begin{aligned} \downarrow \Gamma_1 \quad &\sqsubseteq_{lex} \quad \downarrow \Gamma_2 \\ \Rightarrow \operatorname{Sh}_\tau\left(\operatorname{Tr}_\tau\left(\downarrow \Gamma_1\right)\right) \quad &\sqsubseteq_{\mathrm{Sh}} \quad \operatorname{Sh}_\tau\left(\operatorname{Tr}_\tau\left(\downarrow \Gamma_2\right)\right) \end{aligned}$$

It follows that $\operatorname{Sh}_\tau(\operatorname{Tr}_\tau(\downarrow \Gamma_{min}))$ is, among all chain in the complex $\operatorname{Sh}_\tau(K_\tau)$, the one that minimises $\sqsubseteq_{\mathrm{Sh}}$ under the constraint (6.54), or equivalently:

$$\partial \operatorname{Sh}_\tau\left(\operatorname{Tr}_\tau\left(\downarrow \Gamma_{min}\right)\right) = \partial \operatorname{Sh}_\tau\left(\operatorname{Tr}_\tau\left(\downarrow \Gamma_{reg}\right)\right)$$

Lemma 6.27 applied with:

$$\begin{aligned} n &\leftarrow n-k \\ X &\leftarrow \operatorname{Sh}_\tau\left(\operatorname{Tr}_\tau\left(\downarrow \Gamma_{reg}\right)\right) = \operatorname{Sh}_\tau\left(\mathrm{X}(\tau)\right) \\ P &\leftarrow \operatorname{Sh}_\tau\left(P_\tau\right) \\ K &\leftarrow \operatorname{Sh}_\tau\left(K_\tau\right) \end{aligned}$$

implies:

$$\mathrm{Sh}_\tau\left(\mathrm{Tr}_\tau\left(\downarrow \Gamma_{min}\right)\right) = \mathrm{Sh}_\tau\left(\mathrm{Tr}_\tau\left(\downarrow \Gamma_{reg}\right)\right)$$

In other words, Γ_{min} and Γ_{reg} coincide on simplices with bounding weight $\mu_B^{\neq}$, a contradiction with the definition of $\mu_B^{\neq}$.

$\square$

Chapter 7

Particular case of 2-chains

We focus now on the case of 2-simplices and 2-chains in Euclidean space $\mathbb{R}^n$, with $n \geq 2$. We consider the Delaunay case, i.e. the weights are zero.

Indeed, when $d = 2$, Lemma 7.3 gives us an explicit expressions of the asymptotic behavior of $w_p(\sigma)$ as $p \to \infty$ that allows a finer analysis.

7.1 $\leq_\infty$, $\sqsubseteq_\infty$ and $\sqsubseteq_{lex}$ preorders

The weights of simplices $\sigma \mapsto w_p(\sigma)$ and the norm of chains $\Gamma \mapsto \|\Gamma_1\|_{(p)}$ have been respectively defined in (6.13) and (6.14) in section 6.6 in the general case. In this section we consider the same definition but we are in the particular case where the weights of points are zero and we focus on the case where the dimension of simplices and chains is 2.

We now introduce the binary relation $\leq_\infty$ on the set of d-simplices of K.

Definition 7.1 (Preorder $\leq_\infty$ on d-simplices of K)**.**

$$\sigma_1 \leq_\infty \sigma_2 \underset{def.}{\iff} \exists p \in [1, \infty[, \forall p' \in [p, \infty[, w_{p'}(\sigma_1) \leq w_{p'}(\sigma_2)$$

It is straightforward to check that $\leq_\infty$ is reflexive and transitive, and therefore a preorder. We show below that, at least for $d = 2$, by assuming a generic condition on the point sample $\mathbf{P}$, it becomes a total order.

We also introduce the binary relation $\sqsubseteq_\infty$ on the set of d-chains of K:

Definition 7.2 ($\sqsubseteq_\infty$ order on chains)**.** *For* $\Gamma_1, \Gamma_2 \in C_d(K)$*:*

$$\Gamma_1 \sqsubseteq_\infty \Gamma_2 \underset{def.}{\iff} \exists p \in [1, \infty), \forall p' \in [p, \infty), \|\Gamma_1\|_{(p')} \leq \|\Gamma_2\|_{(p')}$$

$\sqsubseteq_\infty$ is also a preorder on $C_d(K)$.

When $\leq_\infty$ is a total order on simplices it induces, according to definition 2.1, a total order on chains $\sqsubseteq_{lex}$ on d-chains. We see below that this occurs generically when $d = 2$.

7.2 Limit of Delaunay energy and $\leq_\infty$ order

Limit of Delaunay energy as $p \to \infty$ We say that a triangle $\sigma = abc$ is non-degenerate if it spans a 2-dimensional affine space. For a compact set of points S, $\mathbf{SEB}(S)$ denotes the Smallest Enclosing Ball of S, i.e. the unique closed Euclidean ball with minimal radius containing S. For triangle abc denote $\mathrm{R_C}(abc)$ and $\mathrm{R_B}(abc)$ respectively the circumradius and the radius of the minimal enclosing circle $\mathbf{SEB}(abc)$. Observe that $\mathrm{R_C}(abc)$ and $\mathrm{R_B}(abc)$ coincide if and only if triangle abc is acute or right.

Following lemma gives an explicit expression of the asymptotic behavior of $w_p(\sigma)$ as $p \to \infty$. The proof consists of computations and is reported to Appendix B.

Lemma 7.3. *Consider $\sigma = abc$ a non degenerate triangle. Then we have the following. For any triangle abc there is a function ω_{abc}*

$$\omega_{abc} : [1, \infty) \to \mathbb{R}$$

such that:

$$\lim_{p\to\infty} \omega_{abc}(p) = 1$$

and:

– *if abc is strictly acute:*

$$w_p(abc)^p = \pi \, \frac{1}{p} \, \mathrm{R_B}(abc)^{2+2p} \, \omega_{abc}(p)$$

– *if abc is right:*

$$w_p(abc)^p = \frac{\pi}{2} \, \frac{1}{p} \, \mathrm{R_B}(abc)^{2+2p} \, \omega_{abc}(p)$$

– *if abc is obtuse:*

$$w_p(abc)^p = \frac{1}{h^2} \, \frac{1}{p^2} \, \mathrm{R_B}(abc)^{2+2p} \, \omega_{abc}(p)$$

with:

$$h^2 = \frac{\mathrm{R_C}(abc)^2 - \mathrm{R_B}(abc)^2}{\mathrm{R_B}(abc)^2}$$

As seen before, the binary relation $\leq_\infty$ introduced in definition 7.1 is reflexive and transitive, in other words it is a preorder. We show now that when $d = 2$, and by assuming a generic condition on the point set, it becomes a total order.

We say that a triangle $\sigma = abc$ is non-degenerate if it spans a 2-dimensional affine space.

Since $\lim_{p\to\infty} w_p(\sigma) = w_\infty(\sigma)$, one has:

$$w_\infty(\sigma_1) < w_\infty(\sigma_2) \Rightarrow \sigma_1 \leq_\infty \sigma_2 \tag{7.1}$$

and

$$\sigma_1 \leq_\infty \sigma_2 \Rightarrow w_\infty(\sigma_1) \leq w_\infty(\sigma_2 \tag{7.2}$$

However one can check on the example given below of two obtuse triangles with a same longest edge that the relation $\sigma_1 \leq_\infty \sigma_2$ is finer than $w_\infty(\sigma_1) \leq w_\infty(\sigma_2)$: the converse of (7.2) does not hold.

Recall that a property **P** on sample sets is *generic* if the set of samples with N points, $S \in \mathbb{R}^{nN}$, verifying **P** is a dense open subset of $\mathbb{R}^{nN}$. We may also require its complement to be of zero Lebesgue measure.

For two obtuse triangles abc and abd sharing the same longest edge ab we have:

$$w_\infty(abc) = \mathrm{R_B}(abc)^2 = ((a-b)/2)^2 = \mathrm{R_B}(abd)^2 = w_\infty(abd)$$

Since the property of being obtuse and sharing the same longest edge is stable under infinitesimal perturbations of $\{a, b, c, d\}$, there is no generic property enforcing the binary relation $w_\infty(\sigma_1) \leq w_\infty(\sigma_2)$ to be antisymmetric.

In contrast, as shown by Lemmas 7.4 and 7.6 below, $\leq_\infty$ can be made antisymmetric, and therefore a total order, under generic condition 7.1 below.

Next lemma, easy consequence of Lemma 7.3, describes sufficient geometric implications of $\leq_\infty$.

Lemma 7.4. *If σ_1 and σ_2 are non degenerate triangles, then:*

$$\sigma_1 \leq_\infty \sigma_2 \Rightarrow \begin{cases} \mathrm{R_B}(\sigma_1) < \mathrm{R_B}(\sigma_2) \\ \text{or} \\ \mathrm{R_B}(\sigma_1) = \mathrm{R_B}(\sigma_2)\,\text{and}\,\mathrm{R_C}(\sigma_1) \geq \mathrm{R_C}(\sigma_2) \end{cases}$$

and:

$$\sigma_1 \leq_\infty \sigma_2 \Leftarrow \begin{cases} \mathrm{R_B}(\sigma_1) < \mathrm{R_B}(\sigma_2) \\ \text{or} \\ \mathrm{R_B}(\sigma_1) = \mathrm{R_B}(\sigma_2)\,\text{and}\,\mathrm{R_C}(\sigma_1) > \mathrm{R_C}(\sigma_2) \end{cases}$$

proof of Lemma 7.4. If $\mathrm{R_B}(\sigma_1) \neq \mathrm{R_B}(\sigma_2)$ equation (7.2) gives $\sigma_1 \leq_\infty \sigma_2 \iff \mathrm{R_B}(\sigma_1) < \mathrm{R_B}(\sigma_2)$. Therefore one just has to consider the case when $\mathrm{R_B}(\sigma_1) = \mathrm{R_B}(\sigma_2)$. We assume then $\mathrm{R_B}(\sigma_1) = \mathrm{R_B}(\sigma_2)$: if $\mathrm{R_C}(\sigma_1) < \mathrm{R_C}(\sigma_2)$, both triangles are obtuse and Lemma 7.3 shows that one cannot have $\sigma_1 \leq_\infty \sigma_2$, while if $\mathrm{R_C}(\sigma_1) > \mathrm{R_C}(\sigma_2)$, the same lemma gives $\sigma_1 \leq_\infty \sigma_2$. ☐

Lemma 7.4 suggests the following generic property:

Condition 7.1. *Generic condition:*

1. *Any triangle in K is non-degenerate,*
2. *for any two triangles σ_1 and σ_2 in K, one has:*

$$\mathbf{SEB}(\sigma_1) \neq \mathbf{SEB}(\sigma_2) \Rightarrow \mathrm{R_B}(\sigma_1) \neq \mathrm{R_B}(\sigma_2)$$

3. *for any two triangles σ_1 and σ_2, one has:*

$$\begin{cases} \sigma_1 \neq \sigma_2 \\ \text{and} \\ \mathcal{S} = \mathbf{SEB}(\sigma_1) = \mathbf{SEB}(\sigma_2) \end{cases} \Rightarrow \begin{cases} \mathrm{R_C}(\sigma_1) \neq \mathrm{R_C}(\sigma_2) \\ \text{and} \\ \exists ab = \sigma_1 \cap \sigma_2, \mathcal{S} = \mathbf{SEB}(ab) \end{cases}$$

Lemma 7.5. *When $d = 2$, Condition 7.1 is generic.*

Proof. Since the intersection of a finite number of dense open sets is a dense open set, it is enough, in order to show that Condition 7.1 is generic, to prove that it is the conjonction of a finite set of generic conditions.

The condition for 3 points to span a 2 space, i.e. to not lie on a same line, is clearly generic since it can be expressed as saying that c does not lie on line ab.

For the second condition, we say that a vertex a of σ *contributes* to $\mathbf{SEB}(\sigma)$ if it belongs to the boundary of $\mathbf{SEB}(\sigma)$.
The condition $\mathbf{SEB}(\sigma_1) \neq \mathbf{SEB}(\sigma_2)$ is open and it says that at least either a vertex a_2 of σ_2 which is not a vertex of σ_1 contributes to $\mathbf{SEB}(\sigma_2)$, either a vertex a_1 of σ_1 which is not a vertex of σ_2 contributes to $\mathbf{SEB}(\sigma_1)$. In both cases the set of position of vertex a_2 or a_1 that enforces $\mathrm{R_B}(\sigma_1) \neq \mathrm{R_B}(\sigma_2)$ is open and dense.

For the third condition, let's consider two triangles σ_1, σ_2 verifying $\sigma_1 \neq \sigma_2$ and $\mathcal{S} = \mathbf{SEB}(\sigma_1) = \mathbf{SEB}(\sigma_2)$. We consider the number of points contributing to $\mathcal{S}$. As a smallest enclosing ball of a triangle requires at least two contributing points and having four (or more) points on a sphere is not a generic condition, we have two cases to consider:

– If two points contribute to $\mathcal{S}$, both triangles are obtuse and share this longest edge.

– If three points contribute to $\mathcal{S}$, as $\sigma_1 \neq \sigma_2$, both triangles cannot be acute and therefore one triangle is obtuse and shares its longest edge with the other triangle.

In both case, at least one vertex of the triangles is not on the sphere S and the set of its coordinates satisfying $\mathrm{R_C}(\sigma_1) \neq \mathrm{R_C}(\sigma_2)$ is open and dense. $\square$

Lemma 7.6. *If Condition 7.1 holds, $\leq_\infty$ is a total order on the set of 2-simplices of K with:*

$$\sigma_1 \leq_\infty \sigma_2 \iff \begin{cases} \mathrm{R_B}(\sigma_1) & < \mathrm{R_B}(\sigma_2) \\ \text{or} & \\ \mathrm{R_B}(\sigma_1) & = \mathrm{R_B}(\sigma_2) \quad \text{and} \quad \mathrm{R_C}(\sigma_1) \geq \mathrm{R_C}(\sigma_2) \end{cases} \tag{7.3}$$

Proof. To get the equivalence property from Lemma 7.4, see that under Condition 7.1.2:

$$\mathrm{R_B}(\sigma_1) = \mathrm{R_B}(\sigma_2) \text{ and } \mathrm{R_C}(\sigma_1) = \mathrm{R_C}(\sigma_2) \Rightarrow \mathbf{SEB}(\sigma_1) = \mathbf{SEB}(\sigma_2)$$

which in turn implies $\sigma_1 = \sigma_2$ from Condition 7.1.3.

The preorder $\leq_\infty$ is immediately total using this equivalence property (i.e. $\sigma_1 \leq_\infty \sigma_2$ or $\sigma_2 \leq_\infty \sigma_1$ always holds true). We can now easily show its antisymmetry property from Equation 7.3 rendering $\leq_\infty$ a total order:

$$\begin{aligned} & \sigma_1 \leq_\infty \sigma_2 \text{ and } \sigma_2 \leq_\infty \sigma_1 && (7.4) \\ \Rightarrow \quad & \mathrm{R_B}(\sigma_1) = \mathrm{R_B}(\sigma_2) \text{ and } \mathrm{R_C}(\sigma_1) = \mathrm{R_C}(\sigma_2) && \\ \Rightarrow \quad & \sigma_1 = \sigma_2 && (7.5) \end{aligned}$$

$\square$

In the sequel of the section, we assume generic Condition 7.1. $\leq_\infty$ is a total order and we can define naturally the corresponding increasing and decreasing strict orders $<_\infty$ and $>_\infty$.

7.3 Delaunay $\sqsubseteq_\infty$ and $\sqsubseteq_{lex}$ orders

Thanks to Lemma 7.6, under generic condition 7.1, $\leq_\infty$ is a total order on triangles that induces, according to definition 2.1, a total order $\sqsubseteq_{lex}$ on 2-chains.

We compare now $\sqsubseteq_{lex}$ with the binary relation $\sqsubseteq_\infty$ introduced in Definition 7.2.

$\sqsubseteq_{lex}$ and $\sqsubseteq_\infty$ differ: Even under generic condition 7.1, relations $\sqsubseteq_\infty$ and $\sqsubseteq_{lex}$ are not equivalent in general. Indeed, if $\sigma_1, \sigma_2, \sigma_3$ are obtuse triangles sharing a same longest edge (which happens generically) such that $w_\infty(\sigma_1) = w_\infty(\sigma_2) = w_\infty(\sigma_3)$ with:

$$h_i^2 = \frac{\mathrm{R_C}(\sigma_i)^2 - \mathrm{R_B}(\sigma_i)^2}{\mathrm{R_B}(\sigma_i)^2}$$

such that $h_1 < h_2, h_3$ but

$$\frac{1}{h_1^2} < \frac{1}{h_2^2} + \frac{1}{h_3^2}$$

Lemma 7.3 gives us:

$$\sigma_1 \sqsubseteq_\infty \sigma_2 + \sigma_3 \text{ and } \neg(\sigma_2 + \sigma_3 \sqsubseteq_\infty \sigma_1)$$

while since $h_1 < h_2, h_3 \Rightarrow \sigma_2, \sigma_3 <_\infty \sigma_1$ one has:

$$\sigma_2 + \sigma_3 \sqsubseteq_{lex} \sigma_1$$

However as it will be seen, minima of $\sqsubseteq_\infty$ can coincide with minima of $\sqsubseteq_{lex}$, at least in Čech and Rips complexes.

7.4 When $\sqsubseteq_\infty$ and $\sqsubseteq_{lex}$ have same minima

The aim of this section is to show that for $d = 2$, under generic condition 7.1, $\sqsubseteq_\infty$ and $\sqsubseteq_{lex}$ in Čech and Rips complexes have same minima under boundary or homology constraints.

Čech and Rips complexes Recall that, given a finite set of points $\mathbf{P} \subset \mathbb{R}^n$, the *Čech complex* with parameter λ is the simplicial complex made of all simplices σ such that the radius of $\mathbf{SEB}(\sigma)$ is less or equal to λ. The *proximity graph* $G(\mathbf{P}, \lambda)$ is the graph whose vertices are in bijection with points in $\mathbf{P}$ and with one edge for each pair $\{a, b\} \in \mathbf{P}$ whose length is not greater than λ.

The corresponding *Vietoris-Rips complex* is the flag complex of the proximity graph $G(\mathbf{P}, \lambda)$, i.e the simplicial complex made of all cliques in $G(\mathbf{P}, \lambda)$.

Lemma 7.7. *Let $\mathbf{P} \subset \mathbb{R}^n$ be a finite set of points satisfying generic condition 7.1 and let K be a Čech or Vietoris-Rips complex over $\mathbf{P}$. Let $\mathcal{D} \subset \boldsymbol{C_2}(K)$ be a set of chains in either of these forms:*

$$\mathcal{D} = \{\Gamma \in \boldsymbol{C_2}(K) \mid \exists B \in \boldsymbol{C_3}(K), \Gamma - A = \partial B\} \text{ or,}$$

$$\mathcal{D} = \{\Gamma \in \boldsymbol{C_2}(K) \mid \partial\Gamma = \beta\}$$

for some $A \in \boldsymbol{C_2}(K)$ or for some 1-cycle $\beta \in Z_1(K)$.

If $\Gamma \in \boldsymbol{C}_2(K)$ is a minimum in $\mathcal{D}$ for one of the orders $\sqsubseteq_\infty$ or $\sqsubseteq_{lex}$, then Γ cannot contain two obtuse triangles sharing the same longest edge.

Proof. Consider two obtuse triangles $abc, abd \in K$ with same longest edge ab.

If K is Cech complex with parameter λ we must have $\|b - a\| \leq \lambda/2$. Since the ball with diameter ab contains the points a, b, c, d, the triangles acd and bcd as well as the tetrahedron $abcd$ belongs to K.

Now if K is a Vietoris-Rips complex with parameter λ we must have $\|b-a\| \leq \lambda$ and, again, acd and bcd as well as the tetrahedron $abcd$ belongs to K.

It follows that, in both cases, $abc+abd$ is homologous to and has same boundary as $acd+bcd$ in K. Therefore, if some chain Γ contains abc and abd then, the chain $\Gamma' = \Gamma+abc+abd+acd+bcd$ is homologous and has same boundary as Γ. One checks easily that, while $\Gamma' \neq \Gamma$, one has $\Gamma' \sqsubseteq_\infty \Gamma$ and $\Gamma' \sqsubseteq_{lex} \Gamma$ and Γ cannot be minimum in $\mathcal{D}$ for one of the orders $\sqsubseteq_\infty$ or $\sqsubseteq_{lex}$. □

The formal proof of next lemma is reported to Appendix C. However, it is intuitively plausible since Lemma 7.7 forbids the situation described in section 7.3 where $\sqsubseteq_\infty$ and $\sqsubseteq_{lex}$ differ.

Lemma 7.8. *Let $\mathbf{P} \subset \mathbb{R}^n$ be a finite set of points satisfying generic condition 7.1 and let K be a Čech or Vietoris-Rips complex over $\mathbf{P}$. Let $\mathcal{D} \subset \boldsymbol{C_2}(K)$ be a set of chains in either of these forms:*

$$\mathcal{D} = \{\Gamma \in \boldsymbol{C_2}(K) \mid \exists B \in \boldsymbol{C_3}(K), \Gamma - A = \partial B\} \text{ or,}$$

$$\mathcal{D} = \{\Gamma \in \boldsymbol{C_2}(K) \mid \partial\Gamma = \beta\}$$

for some $A \in \boldsymbol{C_2}(K)$ or for some 1-cycle $\beta \in Z_1(K)$.

Then:

$$\min_{\sqsubseteq_\infty} \mathcal{D} = \min_{\sqsubseteq_{lex}} \mathcal{D}$$

Lemma 7.8 applies to 2-chains in Euclidean n-space. In the particular case where $n = 2$ it allows to give an elementary proof of a specialization of Theorem 6.2 or Corollary 6.3:

Proposition 7.9. *Let $\mathbf{P} = \{P_1, \ldots, P_N\} \subset \mathbb{R}^2$ with $N \geq 3$ be in general position and let $K_{\mathbf{P}}$ be the 2-dimensional full complex over $\mathbf{P}$. Denote by $\beta_{\mathbf{P}} \in \boldsymbol{C}_1(K_{\mathbf{P}})$ the 1-chain made of edges belonging to the boundary of $\mathcal{CH}(\mathbf{P})$.*
Then if:

$$\Gamma_{\min} = \min_{\sqsubseteq_{lex}}\{\Gamma \in \boldsymbol{C}_2(K_{\mathbf{P}}), \partial\Gamma = \beta_{\mathbf{P}}\}$$

then the simplicial complex $|\Gamma_{\min}|$ support of $\Gamma_{\min}$ is the Delaunay triangulation of $\mathbf{P}$.

Proof. Looking at Definition 7.2, we see that if $\Gamma_1 \leq_\infty \Gamma_2$ there is $p(\Gamma_1, \Gamma_2)$ large enough such that for any $p' \geq p(\Gamma_1, \Gamma_2)$, $\|\Gamma_1\|_{(p')} \leq \|\Gamma_2\|_{(p')}$. Since there are only a finite number of chains Γ in a given finite simplicial complex, one can define for a given simplicial complex $\tilde{p} = \max_{\Gamma_1 \sqsubseteq_\infty \Gamma_2} p(\Gamma_1, \Gamma_2)$ such that:

$$\Gamma_1 \sqsubseteq_\infty \Gamma_2 \iff \|\Gamma_1\|_{(\tilde{p})} \leq \|\Gamma_2\|_{(\tilde{p})}$$

Consider now Proposition 6.13 in the case of zero weights and $n = 2$. It implies that the support of the chain:

$$\min_{\sqsubseteq_\infty}\{\Gamma \in \boldsymbol{C}_2(K_{\mathbf{P}}), \partial\Gamma = \beta_{\mathbf{P}}\}$$

is the Delaunay triangulation of $\mathbf{P}$. Applying Lemma 7.8 complete the proof. $\square$

Chapter 8

Application to point cloud triangulations

Sections 3, 4 and 5 were motivated by the study of polynomial algorithms in a specialization of the NP-hard OCHP. However, the order on simplices was not specified and one can wonder if choosing such an ordering makes the specialization of OCHP too restrictive for it to be useful. In this section, in the light of the total order defined in Section 6 (Equation 6.9) and its equivalent order for a 2D Delaunay triangulation in Section 7 (Lemma 7.6), we give a concrete example where this restriction makes sense and provides a simple and elegant application to the problem of point cloud triangulation. Whereas algorithms of Sections 3, 4 and 5 only dealt with abstract simplicial complexes, we now consider a bijection between vertices and a set of points in Euclidean space, allowing to compute geometric quantities on simplices.

8.1 Related works

Many methods have been proposed to answer the problem of surface reconstruction in specific acquisition contexts [35, 36, 37]: [8] classifies a large number of these methods according to the assumptions and information used in addition to geometry. In the family of purely geometric reconstruction based on a Delaunay triangulation, one very early contribution is the sculpting algorithm by Boissonnat [9]. The crust algorithm by Amenta et al. [2, 3] and an algorithm based on natural neighbors by Boissonnat et al. [10] were the first algorithms to guarantee a triangulation of the manifold under sampling conditions. However, these general approaches usually have difficulties far from these sampling conditions, in applications where point clouds are noisy or under-sampled. This difficulty can be circumvented by providing additional information on the nature of the surface [23, 33]. Our contribution lies in this category of approaches. We provide some topological information of the surface: a boundary for the open surface reconstruction and an interior region and exterior region for the closed surface reconstruction.

8.2 Simplicial ordering

We use the total order on 2-simplices defined in Lemma 7.6 for this section. Indeed, the 2D Delaunay triangulation has some well-known optimality properties, such as maximizing the minimal angle, and we can hope that using the lexicographic order induced by this total order to minimize 2-chains in dimension 3 will keep some of those properties. In fact, Section 9 is devoted to showing that, for a Čech or Vietoris-Rips complex, under strict conditions linking the point set sampling, the parameter of the complex and the reach of the underlying manifold of Euclidean space, the minimal lexicographic chain using the described simplex order is a triangulation of the sampled manifold. Experimental results (Figure 8.1) show that this

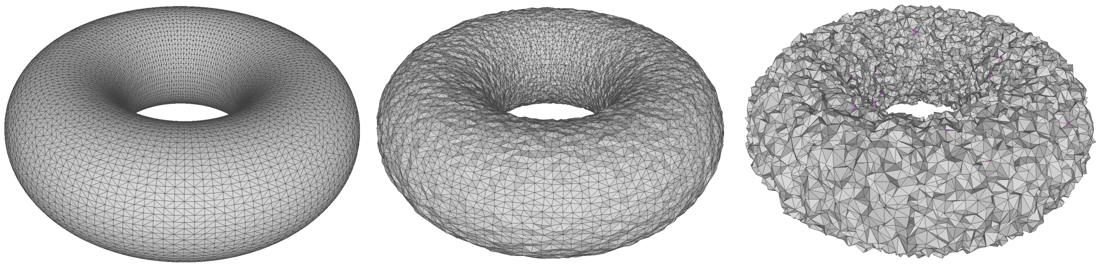

Figure 8.1: Watertight reconstructions under different perturbations. Under small perturbations (first two images from the left), the reconstruction is a triangulation of the sampled manifold. A few non-manifold configurations appear however under larger perturbations (Rightmost image).

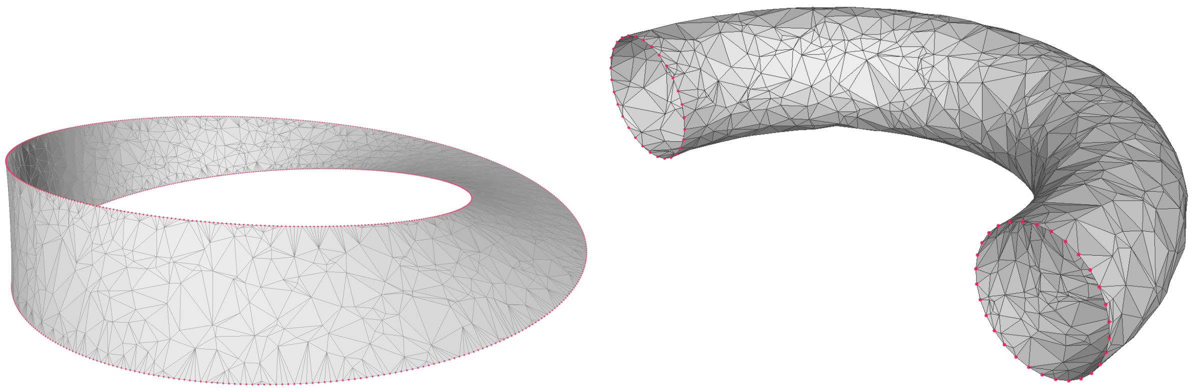

Figure 8.2: Open surface triangulations under imposed boundaries (red cycles).

property remains true relatively far from these theoretical conditions.

8.3 Open surface triangulation

Using the Phat library [7], we generate a Čech complex of the point cloud and the points of a provided cycle, with a sufficient parameter to capture the topology of the object [17, 6]. After constructing the 2-boundary, we apply the boundary reduction algorithm, slightly modified to calculate as well the transformation matrix V. We then apply Algorithm 4, and in the case the cycle is a boundary, we get a chain bounded by the provided cycle. We then apply Algorithm 3 to minimize the chain under imposed boundary. Figure 8.2 shows results of this method.

8.4 Closed surface triangulation

Using Algorithm 5 requires a strongly connected 3-pseudomanifold: we therefore use a 3D Delaunay triangulation, for its efficiency and non-parametric nature, using the CGAL library [34], and complete it into a 3-sphere by connecting, for any triangle on the convex hull of the Delaunay triangulation, its dual edge to an "infinite" dual vertex.

The choice of α_1 and α_2 will define the location of the closed separating surface. We can guide the algorithm by interactively adding multiple α_1 and α_2 regions as depicted in Figure 8.3. Algorithm 5 requires to be slightly modified to take as input multiple α_1, α_2: after creating all sets with MakeSet, we need to combine all α_1 sets together, and all α_2 sets together. The algorithm remains unchanged for the rest.

Experimentally, sorting triangles does not require exact predicates: the $\mathrm{R_B}$ and $\mathrm{R_C}$ quantities can simply be calculated in fixed precision. The quasilinear complexity of Algorithm 5 makes

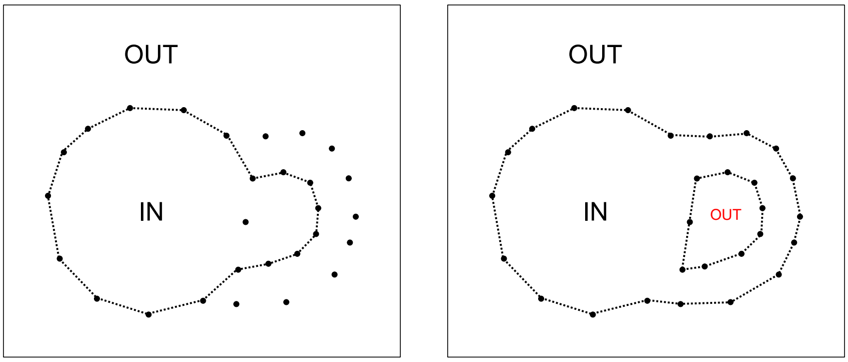

Figure 8.3: Providing additional topological information can improve the result of the reconstruction.

it competitive in large point cloud applications (Figure 8.4).

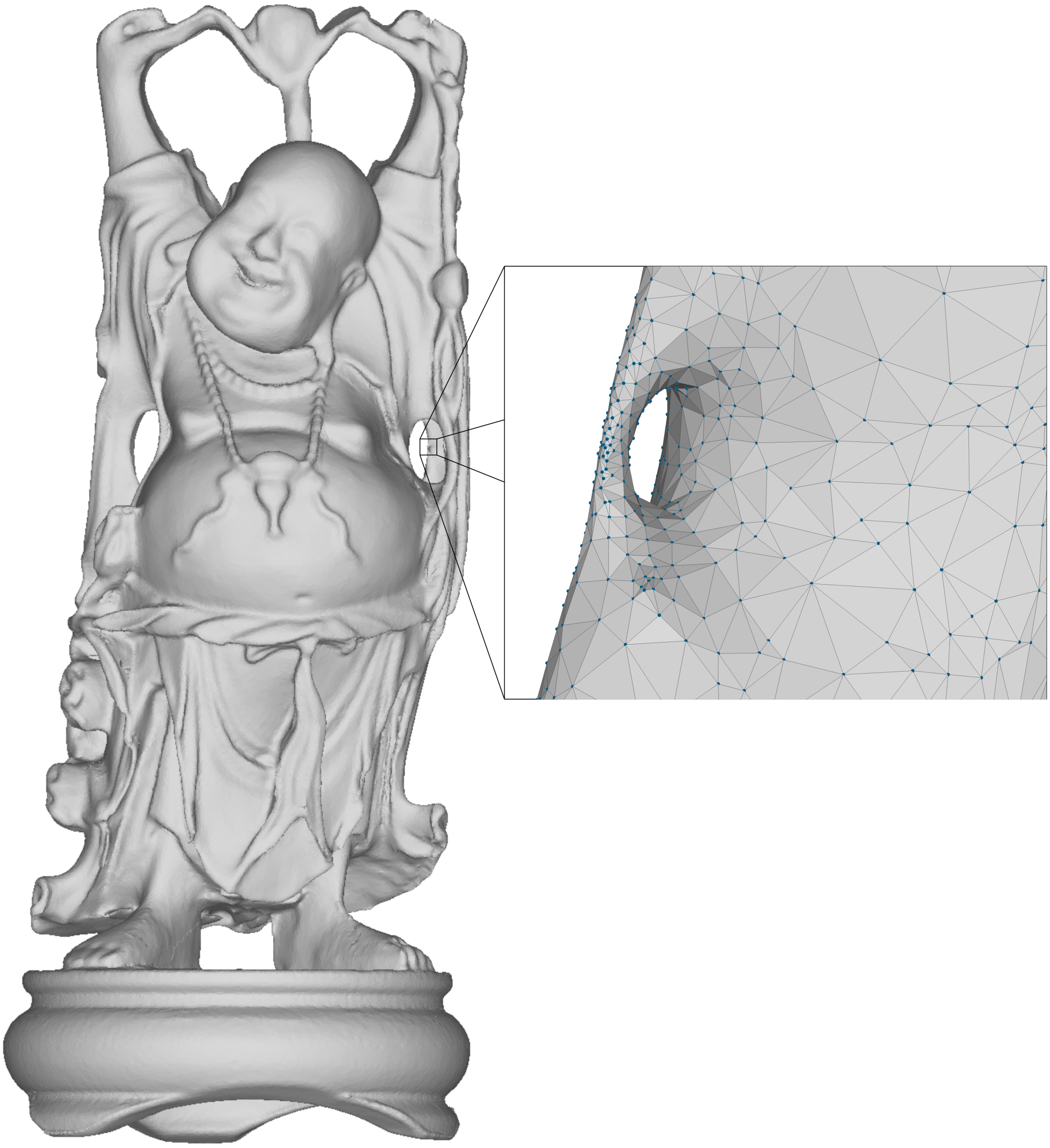

Figure 8.4: Closed surface triangulation of 440K points in 7.33 seconds.

Chapter 9

Triangulation of smooth 2-submanifolds of Euclidean space by minimal chains over point samples

9.1 Main result of the section

Fondamental class When $\mathcal{M}$ is a connected compact d-manifold (orientability is not required here since the coefficients field is $\mathbb{Z}_2$), It is well known that $H_d(\mathcal{M})$ is one dimensional and is generated by the so called *fundamental class* of $\mathcal{M}$, and, since the field is $\mathbb{Z}_2$, has a single non zero element: $H_d(\mathcal{M}) \simeq \mathbb{Z}_2$. In both contexts of simplicial and singular homology, this fundamental class can be built from a triangulation of $\mathcal{M}$.

In the following, we consider:

Embedded manifold: For $n > 2$, we denote by $\mathcal{M}$ a connected, compact and C^2 smooth embedded 2-manifold $\mathcal{M} \subset \mathbb{R}^n$ with reach $\mathcal{R} > 0$.

Sampling set S: $S \subset \mathcal{M}$ a finite "sampling set" such that for any point $x \in \mathcal{M}$ there is at least one point in S at distance less than ϵ from x, and such that any two points in S are at least η apart, with $0 < \eta \leq \epsilon/2$.

Simplicial complex Let K be the λ-Chech complex of S (in fact the 3-skeleton of the λ-Chech complex is enough since it defines $H_2(K)$). It is known [38] that for $\epsilon < \sqrt{3/5}\mathcal{R}$, K has the homotopy type of $\mathcal{M}$.

Main result

We show that for some constant $C_3 > 0$ (independent on the ambient dimension n), for any $n > 2$, and any such $\mathcal{M}$ and S such that $\epsilon < C_3 \left(\frac{\eta}{\epsilon}\right)^{10} \mathcal{R}$, the support of the chain $\Gamma_{\min} \in C_2(K, \mathbb{Z}/2\mathbb{Z})$ minimizing the lexicographic order $\sqsubseteq_{lex}$ in the homology class of the fundamental class of $\mathcal{M}$ is a triangulation of $\mathcal{M}$. More precisely, the restriction to $|\Gamma_{\min}|$ of the projection on $\mathcal{M}$ is an homeomorphism.

While this is not yet formally proven at this stage, it seems very likely that this minimum coincides with the tangential Delaunay complex [13] when all stars are consistent.

We denote by $\pi_{\mathcal{M}}$ the orthogonal projection on $\mathcal{M}$, in other words the map from $\left(\mathcal{M}^{\oplus\mathcal{R}}\right)^{\circ} \to \mathcal{M}$ that associates to each point its closest point on $\mathcal{M}$.

Theorem 9.1. *There are constants C_1, C_1, C_3 such that: If $\mathcal{M}$ is a C^2 manifold embedded in $\mathbb{R}^n$ with reach $\mathcal{R}$, S a (ϵ, η)-sampling of $\mathcal{M}$ and $K = \check{\mathrm{C}}(S, \lambda)$ the Čech complex on S with parameter λ, such that:*

1. *If:*

$$C_1\epsilon < \lambda < C_2\mathcal{R} \tag{9.1}$$

then one has $\beta_2(K) = 1$.

2. *Let* $\mathcal{T}$ *be the lexicographic minimal representative of the unique homology class in* $H_2(K)$*, i.e. the lexicographic minimal* 2*-cycle which is not a boundary:*

$$\mathcal{T} \ = \ \min_{\sqsubseteq_{lex}} \operatorname{Ker}(\partial_2) \setminus \operatorname{im}(\partial_3) \tag{9.2}$$

If

$$\frac{\epsilon}{\mathcal{R}} < C_3\left(\frac{\eta}{\epsilon}\right)^{10} \tag{9.3}$$

then the restriction of $\pi_{\mathcal{M}}$ *to* $|\mathcal{T}|$ *is an homeomorphism on* $\mathcal{M}$*. In particular* $(|\mathcal{T}|, \pi_{\mathcal{M}})$ *is a triangulation of* $\mathcal{M}$*.*

Remark 9.2. *Observe that::*

- *Thanks to Lemma 7.8, it follows that* $\min_{\sqsubseteq_{lex}}$ *could be replaced by* $\sqsubseteq_\infty$ *and therefore, for a any complex* K *satisfying the condition of the theorem, there exists* p *large enough such that:*

$$\mathcal{T} \ = \ \underset{\Gamma\in\operatorname{Ker}(\partial_2)\setminus\operatorname{im}(\partial_3)}{\arg\min} \ \|\Gamma\|_{(p)} \tag{9.4}$$

- *when the theorem holds for a range of Čech complex parameter* $\lambda, \sqrt{2}\lambda$ *there is a Rips-Vietoris complex* R *such that* $\check{\mathrm{C}}(S,\lambda) \subset R \subset \check{\mathrm{C}}(S,\sqrt{2}\lambda)$*. Under stronger sampling conditions [6] this inclusion induces isomorphisms on Homology groups. In this situation one could replace Čech complex by Rips-Vietoris complex in the theorem (one may need Lemma 9.17 for a formal proof, to guarantee that taking larger simplices cannot give a lexicographic smaller representative).*

The sequel of the section is devoted to the proof of Theorem 9.1.

9.2 Properties of lexicographic minimal homologous 2-chains

In this section $\mathcal{T}$ is (implicitly) assumed to be a 2-cycle in a Čech complex $\check{\mathrm{C}}(S,\lambda)$ over a finite set $S \subset \mathbb{R}^n$ which is a lexicographic minimum in its homology class. So for that for $K = \check{\mathrm{C}}(S,\lambda)$ and some cycle $A \in \boldsymbol{C_2}(K)$:

$$\mathcal{T} = \min_{\sqsubseteq_{lex}} \{\Gamma \in \boldsymbol{C_2}(K) \mid \exists B \in \boldsymbol{C_3}(K), \Gamma - A = \partial B\}$$

We establishes some properties on $\mathcal{T}$ that does not require any geometric assumption on the set S, in particular S is not assumed to be the sampling of anything at this stage.

Lemma 9.3. *If* $abc \in \mathcal{T}$ *is an acute triangle, then* $\mathbf{SEB}(abc)$*, the Smallest Enclosing Ball of* abc*, has no point of* S *in its interior:*

$$\mathbf{SEB}(abc)^\circ \cap S = \varnothing$$

Proof. For a contradiction, assume that $\mathcal{T}(abc) \neq 0$ and $\mathbf{SEB}(\sigma)^\circ \cap S \neq \emptyset$. Take $d \in \mathbf{SEB}(\sigma)^\circ \cap S$, then $\mathrm{R_B}(abcd) = \mathrm{R_B}(abc) \leq \lambda$ and therefore, $abd, bcd, cad, abcd \in K$.

One has $\mathrm{R_B}(abd), \mathrm{R_B}(bcd), \mathrm{R_B}(cad) < \mathrm{R_B}(abc)$ and then $abd + bcd + cad \sqsubset_{lex} abc$. Consider:

$$\mathcal{T}' = \mathcal{T} + abc + abd + bcd + cad = \mathcal{T} + \partial(abcd)$$

Then, $\mathcal{T}'$ is homologous to $\mathcal{T}$ and is strictly lower for $\sqsubset_{lex}$:

$$\mathcal{T}' \sqsubset_{lex} \mathcal{T}$$

A contradiction with the definition of $\mathcal{T}$. □

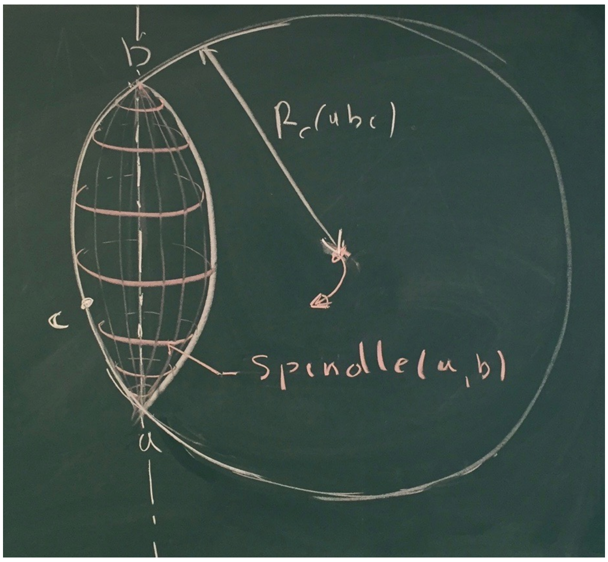

Figure 9.1: Spindle of edge ab with radius $\mathrm{R_C}(abc)$.

Definition 9.4. *Given two points $a, b \in \mathbb{R}^n$ and a radius $r \geq \frac{1}{2}\|a-b\|$, we call* spindle *of edge ab with radius r the intersection of all balls with radius r containing ab (equivalently with a and b on their boundary).*

$$\begin{aligned}
\mathrm{SP}_r(ab) \underset{def.}{=} & \bigcap_{\substack{\mathcal{B}(o,\rho)\supset\{a,b\} \\ \rho\leq r}} \mathcal{B}(o,\rho) = \bigcap_{\mathcal{B}(o,r)\supset\{a,b\}} \mathcal{B}(o,\rho) \\
= & \bigcap_{\substack{(a-o)^2\leq r^2 \\ (b-o)^2\leq r^2}} \mathcal{B}(o,r) = \bigcap_{\substack{(a-o)^2=r^2 \\ (b-o)^2=r^2}} \mathcal{B}(o,r)
\end{aligned}$$

Similarly, if abc is obtuse with longest edge ab we call the the Spindle of edge ab in triangle abc the Spindle of edge ab with radius $\mathrm{R_C}(abc)$, equivalently:

$$\mathrm{SP}_{abc}(ab) \underset{def.}{=} \mathrm{SP}_{\mathrm{R_C}(abc)}(ab) = \{x \in \mathbb{R}^n,\ \angle axb \ \geq\ \angle acb\,\}$$

If $r = \frac{1}{2}\|a-b\|$ then $\mathrm{SP}_r(ab) = \mathbf{SEB}(ab)$.

One check easily that:

$$c \in \mathrm{SP}_{abc}(ab)$$

From the definition we have also that for $r, r' \geq \frac{1}{2}\|a-b\|$

$$\mathrm{SP}_r(ab) \subset \mathrm{SP}_{r'}(ab) \iff r \geq r'$$

It follows that, if abc is obtuse with longest edge ab:

$$c \in \mathrm{SP}_r(ab) \iff \mathrm{R_C}(abc) \geq r \tag{9.5}$$

Lemma 9.5. *If $abc \in \mathcal{T}$ is an obtuse triangle with ab its longest edge, then the spindle of edge ab with radius $\mathrm{R_C}(abc)$ has no point of S in its interior.*

Proof. For a contradiction, assume there is $abc \in \mathcal{T}$ and $\mathrm{SP}_{abc}(ab)^\circ \cap S \neq \emptyset$. Take $d \in \mathrm{SP}_{abc}(ab)^\circ \cap S$, then $\mathrm{R_B}(abcd) = \mathrm{R_B}(abc) = \mathrm{R_B}(ab) \leq \lambda$, therefore, $abd, bcd, cad, abcd \in K$.

One has

$$\mathrm{R_B}(bcd), \mathrm{R_B}(cad) < \mathrm{R_B}(abc) = \mathrm{R_B}(abd)$$

and

$$\mathrm{R_C}(abc) < \mathrm{R_C}(abd)$$

and it follows $abd, bcd, cad < abc$. Consider:

$$\mathcal{T}' = \mathcal{T} + abc + abd + bcd + cad = \mathcal{T} + \partial(abcd)$$

Then, $\mathcal{T}'$ is homologous to $\mathcal{T}$ and is strictly lower for $\sqsubset_{lex}$:

$$\mathcal{T}' \sqsubset_{lex} \mathcal{T}$$

A contradiction with the definition of $\mathcal{T}$. $\square$

Lemma 9.6. *If triangles $\sigma_1, \sigma_2 \in \mathcal{T}$ then:*

$$\mathrm{R_B}(\sigma_1) = \mathrm{R_B}(\sigma_2) \Rightarrow \sigma_1 = \sigma_2 \tag{9.6}$$

and if $\sigma \in \mathcal{T}$ then either σ is is acute and is the unique triangle in K with same $\mathrm{R_B}(\sigma) = \mathrm{R_C}(\sigma)$, either σ is obtuse, in which case it is the unique triangle with maximal circumradius among triangles in K with same bounding radius $\mathrm{R_B}(\sigma)$.

Proof. In the context of condition 7.1, $\mathrm{R_B}(\sigma_1) = \mathrm{R_B}(\sigma_2) \Rightarrow \mathbf{SEB}(\sigma_1) = \mathbf{SEB}(\sigma_2)$, and, in order to prove (9.6) it is enough to consider the case when $\exists ab = \sigma_1 \cap \sigma_2, \mathbf{SEB}(\sigma_1) = \mathbf{SEB}(\sigma_2) = \mathbf{SEB}(ab)$ and $\mathrm{R_B}(\sigma_2) = \mathrm{R_B}(\sigma_1) \leq \mathrm{R_C}(\sigma_1) < \mathrm{R_C}(\sigma_2)$.

If $\mathrm{R_B}(\sigma_1) = \mathrm{R_C}(\sigma_1)$, σ_1 is acute and we get then a contradiction with Lemma 9.3 since if $abc = \sigma_1$ and $abd = \sigma_2$ then $\mathrm{R_C}(abc) < \mathrm{R_C}(abd)$ implies that $d \in \mathbf{SEB}(abc)^\circ$.

Now if $\mathrm{R_B}(\sigma_1) < \mathrm{R_C}(\sigma_1)$, σ_1 is obtuse and we get then a contradiction with Lemma 9.5 since if $abc = \sigma_1$ and $abd = \sigma_2$ then $\mathrm{R_C}(abc) < \mathrm{R_C}(abd)$ implies that $d \in \mathrm{SP}_{abc}(ab)^\circ$.

The second part of the Lemma follows similarly from Lemmas 9.3 when σ is acute and from Lemma 9.5 when σ is obtuse. $\square$

Observe that we get immediately from Lemma 9.5 and the genericity condition that in $\mathcal{T}$ one can not have two obtuse triangles sharing their longest edge. More precisely we have the following:

Lemma 9.7. *If $abc \in \mathcal{T}$ is an obtuse triangle with ab its longest edge, then ab is shared by at least another triangle abd in $\mathcal{T}$ such that:*

- *either abd is acute and $\mathrm{R_C}(abc) \leq \mathrm{R_B}(abd)$,*
- *either abd is obtuse with longest edge ad or bd and $\mathrm{R_C}(abc) \leq \mathrm{R_C}(abd)$.*

Moreover any triangle abd' sharing ab obeys to the same alternative.

Proof. Since $\partial\mathcal{T} = 0$ one has in particular $\partial\mathcal{T}(ab) = 0$:

$$0 = \partial\mathcal{T}(ab) = \sum_{q \in \mathrm{Lk}_{|\mathcal{T}|}(ab)} \mathcal{T}(abq)$$

Since $\Gamma(abc) = 1$, there exists at least some triangle abd with $d \neq c$ and $\Gamma(abd) = 1$. If abd is acute, then we have from Lemma 9.3 that $c \notin \mathbf{SEB}(abd)$ and therefore $\mathrm{R_C}(abc) \leq \mathrm{R_C}(abd)$. Assume now that abd is obtuse. ab can not be the longest edge of abd: indeed if it was we get from the generic condition that either $\mathrm{R_C}(abc) > \mathrm{R_C}(abd)$ either $\mathrm{R_C}(abc) < \mathrm{R_C}(abd)$, that is, according to (9.5) that either $c \in \mathrm{SP}_{abd}(ab)$, either $d \in \mathrm{SP}_{abc}(ab)$, and both cases contradicts Lemma 9.5.

Without loss of generality, we can assume therefore that ad is the longest edge of abd. It remains to prove that in this case one has $\mathrm{R_C}(abc) \leq \mathrm{R_C}(abd)$.

For a contradiction, assume that $\mathrm{R_C}(abc) > \mathrm{R_C}(abd)$. It follows from (9.5) that

$$c \in \mathrm{SP}_{\mathrm{R_C}(abd)}(ab) \tag{9.7}$$

Observe that, from the definition of spindle one has in general:

$$\{a, b\} \subset \mathcal{B}(o, r) \iff \mathrm{SP}_r(ab) \subset \mathcal{B}(o, r)$$

and, if for any set $X \subset \mathrm{SP}_r(ad)$

$$\mathrm{SP}_r(ad) \subset \mathcal{B}(o, r) \Rightarrow X \subset \mathcal{B}(o, r)$$

Since $\{a, b\} \subset \mathrm{SP}_{abd}(ad) = \mathrm{SP}_{\mathrm{R_C}(abd)}(ad)$ we get that:

$$\begin{aligned}
\mathrm{SP}_{\mathrm{R_C}(abd)}(ab) &= \bigcap_{\{a,b\}\subset\mathcal{B}(o,\mathrm{R_C}(abd))} \mathcal{B}(o, \mathrm{R_C}(abd)) \\
&\subset \bigcap_{\mathrm{SP}_{\mathrm{R_C}(abd)}(ad)\subset\mathcal{B}(o,\mathrm{R_C}(abd))} \mathcal{B}(o, \mathrm{R_C}(abd)) \\
&= \bigcap_{\{a,d\}\subset\mathcal{B}(o,\mathrm{R_C}(abd))} \mathcal{B}(o, \mathrm{R_C}(abd)) \\
&= \mathrm{SP}_{\mathrm{R_C}(abd)}(ad)
\end{aligned}$$

This with (9.7) gives:

$$c \in \mathrm{SP}_{\mathrm{R_C}(abd)}(ad)$$

a contradiction with Lemma 9.5. □

It follows from Lemma 9.7 that:

Corollary 9.8. *Any obtuse triangle $T_0 \in \mathcal{T}$ belongs to at least one sequence $T_0, \ldots, T_m \in \mathcal{T}$ of triangles with increasing bounding radius and circumradius, such that $T_0, \ldots, T_{m-1}$ are obtuse, T_m is acute and for $i = 0, \ldots, m-1$, the longest edge of T_i is an edge of T_{i+1} and if $i < m-1$ it is not the longest edge of T_{i+1}. In particular, this sequence is increasing for $\leq_\infty$.*

Lemma 9.9 below give us an explicit bound on the length of the increasing sequences of Corollary 9.8.

Lemma 9.9. *In a sequence of increasing triangles defined in Corollary 9.8, one has:*

$$m \leq \frac{\mathrm{R_B}(T_m)^2}{\eta^2} - 1 \tag{9.8}$$

Proof. In the sequence, if $0 \leq i \leq m-1$ triangle T_i share its longest edge with an edge of triangle T_{i+1}. Denote the vertices of triangle T_i as $a_i b_i c_i = T_i$ where $a_i b_i$ is the longest edge of T_i if $0 \leq i \leq m-1$. Since, for $i \leq m-1$, T_i is obtuse one has for all $i = 0, \ldots, m-1$:

$$(b_i - a_i)^2 > (c_i - a_i)^2 + (c_i - b_i)^2 \tag{9.9}$$

Denotes by l_i the length of edge $a_i b_i$ for $0 \leq i \leq m-1$. Since all edges have length greater or equal than η, one has $(c_i - a_i)^2, (c_i - b_i)^2 \geq \eta^2$ and (9.9) gives in particular:

$$l_0^2 > \eta^2 + \eta^2 = 2\eta^2$$

and since for $i \geq 1$ one of $(c_i - a_i)^2, (c_i - b_i)^2$ is l_{i-1}^2 (9.9) gives for all $i = 1, \ldots, m-1$:

$$l_i^2 > l_{i-1}^2 + \eta^2$$

The two last equations gives us:

$$l_{m-1}^2 > (m+1)\eta^2$$

Since $l_{m-1} \leq \mathrm{R_B}(T_m)$, this proves the lemma. □

9.3 Where the reach comes into play

Geometry of submanifolds with positive reach We recall here a few geometric lemma about general set with positive reach as well as on C^2 submanifold of euclidean space.

Lemma 9.10 (Lemma 5 in [5]). *Let $A \subset \mathbb{R}^d$ be a compact set and $B(z, \alpha)$ a closed ball with center z and radius α. If $0 \le \alpha < \mathrm{Reach}\,(A)$ and $A \cap B(z,\alpha) \neq \emptyset$ then* $\mathrm{Reach}\,(A) \le \mathrm{Reach}\,(A \cap B(z,\alpha))$.

Theorem 9.11 (Theorem 1 in [14]). *If $A \subset \mathbb{R}^d$ is a closed set, then*

$$\mathrm{Reach}\,(A) = \sup\left\{ r > 0,\ \forall a, b \in A,\ |a-b| < 2r \Rightarrow d_A(a,b) \le 2r \arcsin \frac{|a-b|}{2r} \right\},$$

where the sup *over the empty set is* 0.

and its corollary:

Corollary 9.12 (Corollary 1 in [14]). *Let $A \subset \mathbb{R}^d$ be a closed set with positive reach* $\mathrm{Reach}\,(A) > 0$*. Then, for any $r <$ $\mathrm{Reach}\,(A)$ and any $x \in \mathbb{R}^d$, if $B(x,r)$ is the closed ball centered at x with radius r, then $A \cap B(x,r)$ is geodesically convex in A.*

Lemma 9.13 (Corollary 3 in [14]). *if $p, q \in \mathcal{M}$, then*

$$\sin \frac{\angle(T_p\mathcal{M}, T_q\mathcal{M})}{2} \le \frac{\|p-q\|}{2\mathcal{R}}$$

Lemma 9.14 (Theorem 4.8(7) in [31]). *if $p, q \in \mathcal{M}$, then*

$$d(q, T_p\mathcal{M}) \le \frac{\|p-q\|^2}{2\mathcal{R}}$$

Local expression of $\mathcal{M}$ as the graph of a function In this section we assume, in adition to the assumption made on $\mathcal{T}$ ay the begiing of the previous section, that $\mathcal{T}$ meets the conditions of Theorem 9.1, in particular S is an (ϵ, η)-sampling of a compact, connected submanifold $\mathcal{M}$ of $\mathbb{R}^n$ with reach $\mathcal{R}$.

Let $m \in M$ and Π_m the plane tangent to M at m. Denote by $\pi_m : \mathbb{R}^n \to \Pi_m$ the orthogonal projection on Π_m. Denote by $\mathcal{D}(m, \mathcal{R}/4) \subset \Pi_m$ the disk centered at m of radius $\mathcal{R}/4$ and $\mathcal{C}(m, \mathcal{R}/2, \mathcal{R}/4)$ the set of points in $\mathcal{B}(m, \mathcal{R}/2)$ that project on $\mathcal{D}(m, \mathcal{R}/4)$:

$$\begin{aligned}
\mathcal{D}(m, \mathcal{R}/4) &= \{q \in\subset \Pi_m, \|q-m\| \le \mathcal{R}/4\} \\
\mathcal{C}(m, \mathcal{R}/2, \mathcal{R}/4) &= \{q \in \mathcal{B}(m, \mathcal{R}/2), \pi_m(q) \in \mathcal{D}(m, \mathcal{R}/4)\} \\
&= \mathcal{B}(m, \mathcal{R}/2) \cap \pi_m^{-1}(\mathcal{D}(m, \mathcal{R}/4))
\end{aligned}$$

Lemma 9.15. *$\mathcal{M} \cap \mathcal{C}(m, \mathcal{R}/2, \mathcal{R}/4)$ is the graph of a map:*

$$\phi : \mathcal{D}(m, \mathcal{R}/4) \to \mathbb{R}^{n-2}$$

In other words, with the identification $\mathbb{R}^n = \mathbb{R}^2 \times \mathbb{R}^{n-2}$, if we consider an orthogonal frame of $\mathbb{R}^n$ centered at m and aligned with Π_m, we get:

$$\mathcal{M} \cap \mathcal{C}(m, \mathcal{R}/2, \mathcal{R}/4) = \{(u, \phi(u)), u \in \mathcal{D}(m, \mathcal{R}/4)\}$$

With, for any $u \in \mathcal{D}(m, \mathcal{R}/4)$

$$\|\phi(u)\| < \frac{1}{\mathcal{R}} \|u-m\|^2 \tag{9.10}$$

$$\left\| \frac{\partial \phi}{\partial u}(u) \right\| < \frac{2}{\mathcal{R}} \|u-m\| \tag{9.11}$$

Proof of Lemma 9.15 is given in Appendix D.

From now on we assume the following bound on Cech parameter λ: $\lambda < \mathcal{R}/10$. (This bound may certainly be relaxed and the constants in lemmas below improved).

Bounding the triangles circumradii

Lemma 9.16. *If $abc \in \mathcal{T}$ is an acute triangle, then* $\mathrm{R_B}(abc) < 2\epsilon$

The idea of the proof, given in Appendix D, is that otherwise there would be at least one sampling point not too far from the triangle circumcenter contradicting Lemma 9.3.

This bound extends in fact to all triangles in $\mathcal{T}$, more precisely:

Lemma 9.17. *If $abc \in \mathcal{T}$, then*

$$\frac{1}{\sqrt{3}}\eta < \mathrm{R_C}(abc) < 2\epsilon$$

Proof. If abc is acute, then $\mathrm{R_C}(abc) = \mathrm{R_B}(abc)$ and the result follows from Lemma 9.16. If abc is obtuse this follows from Corollary 9.8 (and Lemma 9.16). The lower bound follows from the fact that edge lengths are lower bounded by η. □

Controlling triangle planes and circumcenters For a triangle abc, we denote by $h_{\min}(abc)$ its smallest height.

Lemma 9.18. *If $abc \in \mathcal{T}$ is acute then its smallest height $h_{\min}(abc)$ is lower bounded by* $\mathrm{R_C}(abc)$*. If $abc \in \mathcal{T}$ is obtuse then its smallest height $h_{\min}(abc)$ is lower bounded by:*

$$h_{\min}(abc) \geq \mathrm{R_C}(abc)\left(\frac{\eta}{\mathrm{R_C}(abc)}\right)^2 \geq \frac{1}{2}\epsilon\left(\frac{\eta}{\epsilon}\right)^2 = \frac{\eta^2}{2\epsilon} \tag{9.12}$$

In any case, the sinus of the smallest angle $\theta_{\min}$ of triangle abc is lower bounded by:

$$\sin\theta_{\min} > \frac{1}{4}\frac{\eta}{\epsilon}$$

Proof of Lemma 9.18 is given in Appendix D. Denote by $\check{h}_{\min}$ the minimal altitude (height) of all triangles:

$$\check{h}_{\min} \underset{\text{def.}}{=} \min_{T\in K} h_{\min}(T)$$

Lemma 8.11 in [11] gives an upper bound on the angle between the supporting plane $\Pi(T)$ of a triangle $T \in \mathcal{T}$ and and Π_m the plane tangent to $\mathcal{M}$ at m. Expressed in our setting it gives:

Lemma 9.19 (Lemma 8.11 in [11])**.** *if $T \in K$ and m is a vertex of T, then*

$$\sin\angle(\Pi(T), T_m\mathcal{M}) \leq \frac{2L^2}{\check{h}_{\min}\mathcal{R}}$$

Where L is an upper bound on the triangles edge lengths, so that in our case, thanks to Lemma 9.17 $L < 4\epsilon$ and gives, using (10.21)

$$\sin\angle(\Pi(T), T_m\mathcal{M}) \leq \frac{32\epsilon^2}{\check{h}_{\min}\mathcal{R}} \leq 64\frac{\epsilon}{\mathcal{R}}\left(\frac{\epsilon}{\eta}\right)^2 \tag{9.13}$$

Next Lemma synthesizes the geometric bounds constrained by the manifold reach that are used in subsequent proofs.

Lemma 9.20. *Consider $T \in \mathcal{T}$ and m a vertex of T. Denote by $\Pi(T)$ the supporting plane of T and Π_m the plane tangent to $\mathcal{M}$ at m, then nearby point clouds are near these plane, more precisely for any given constant $C > 1$:*

$$S \cap \mathcal{B}(m, C\epsilon) \subset \Pi_m^{\oplus\beta_1} \tag{9.14}$$

with:

$$\beta_1 = C^2 \frac{\epsilon}{\mathcal{R}} \epsilon = O\left(\frac{\epsilon}{\mathcal{R}} \epsilon\right)$$

and:

$$S \cap \mathcal{B}(m, C\epsilon) \subset \Pi(T)^{\oplus \beta_2} \tag{9.15}$$

with:

$$\beta_2 = C^2 \left(1 + 64 \left(\frac{\epsilon}{\eta}\right)^2\right) \frac{\epsilon}{\mathcal{R}} \epsilon = O\left(\frac{\epsilon}{\mathcal{R}} \left(\frac{\epsilon}{\eta}\right)^2 \epsilon\right)$$

Also the angle between Π_m and $\Pi(T)$ is upper bounded:

$$\angle \Pi_m, \Pi(T) < \beta' \tag{9.16}$$

And for any triangle $abc \in \mathcal{T}$ such that $\{a, b, c\} \subset \mathcal{B}(m, C\epsilon)$ on has:

$$\angle \Pi(abc), \Pi(T) < 3\beta' \tag{9.17}$$

with:

$$\beta' = 64C \frac{\epsilon}{\mathcal{R}} \left(\frac{\epsilon}{\eta}\right)^2 = O\left(\frac{\epsilon}{\mathcal{R}} \left(\frac{\epsilon}{\eta}\right)^2\right)$$

Proof of Lemma 9.20 is given in Appendix D.

Lemma 9.21. *For any constant $C > 0$, $m \in \mathcal{M}$, and $T \in \mathcal{T}$ such that $T \subset S \cap \mathcal{B}(m, C\epsilon)$, denote by $\Pi(T)$ the supporting plane of T. The distance between the circumcenter of T and the plane tangent to $\mathcal{M}$ at m is bounded by β with:*

$$\beta = 8C \frac{\epsilon}{\mathcal{R}} \left(\frac{\epsilon}{\eta}\right)^2 \epsilon = O\left(\frac{\epsilon}{\mathcal{R}} \left(\frac{\epsilon}{\eta}\right)^2 \epsilon\right)$$

Proof. From 9.20 the angle between the plane Π_m tangent to $\mathcal{M}$ at m and the plane $\Pi(T)$ is $\beta' = 4C \frac{\epsilon}{\mathcal{R}} \left(\frac{\epsilon}{\eta}\right)^2 = O\left(\frac{\epsilon}{\mathcal{R}} \left(\frac{\epsilon}{\eta}\right)^2\right)$. Since the distance from m to the circumcenter o_T of T is upper bounded by 2ϵ from Lemma 9.17, the distance between the circumcenter of T and the plane tangent to $\mathcal{M}$ at m is bounded by $2\epsilon \sin \beta'$ which proves the lemma. □

Big O notation We use the notation $O(f(\frac{\epsilon}{\mathcal{R}}))$ and $\Omega(f(\frac{\epsilon}{\mathcal{R}}))$ implicitly means respectively $O(f(\frac{\epsilon}{\mathcal{R}}))_{\frac{\epsilon}{\mathcal{R}} \to 0}$ and $\Omega(f(\frac{\epsilon}{\mathcal{R}}))_{\frac{\epsilon}{\mathcal{R}} \to 0}$. More explicitly:

$$g(\frac{\epsilon}{\mathcal{R}}) \in O(f(\frac{\epsilon}{\mathcal{R}})) \quad \text{if and only if:}$$
$$\exists C, n_0 > 0, \forall n \geq n_0, \qquad \tfrac{\epsilon}{\mathcal{R}} < \tfrac{1}{n} \quad \Rightarrow \quad g(\frac{\epsilon}{\mathcal{R}}) < Cf(\frac{\epsilon}{\mathcal{R}})$$

and

$$g(\frac{\epsilon}{\mathcal{R}}) \in \Omega(f(\frac{\epsilon}{\mathcal{R}})) \quad \text{if and only if:}$$
$$\exists C, n_0 > 0, \forall n \geq n_0, \quad \tfrac{\epsilon}{\mathcal{R}} < \tfrac{1}{n} \quad \Rightarrow \quad g(\frac{\epsilon}{\mathcal{R}}) > Cf(\frac{\epsilon}{\mathcal{R}})$$

Functions considered below f and g are in fact of the form:

$$C \left(\frac{\epsilon}{\mathcal{R}}\right)^k \left(\frac{\epsilon}{\eta}\right)^l \epsilon^m$$

C is a true constant: it does not depends on the ambiant dimension or on S or $\mathcal{M}$ of course) and each of the three other factors can be seen as independent parameters. One could make things simpler taking $\mathcal{R} = 1$ and maybe $\epsilon = 2\eta$ since this is always possible (ε-nets) and the one important parameter for the proof is $\frac{\epsilon}{\mathcal{R}}$ for which one allow ourselves to fix an upper bound taken as small as required to make the proof work without fighting for optimizing the constants.

9.4 Triangles in minimal chains have almost empty circumspheres

For a ball $\mathcal{B}(o,R)$ we denote by $\mathrm{P}_{\mathcal{B}(o,R)}(s)$ the power of point s with respect to $\mathcal{B}(o,R)$:

$$\mathrm{P}_{\mathcal{B}(o,R)}(s) = (s-o)^2 - R^2$$

For a triangle T we denote by $\mathrm{P}_T(s)$ the power of point s with respect to the circumscribing ball of T:

$$\mathrm{P}_T(s) = \mathrm{P}_{\mathcal{B}(\mathrm{C_C}(T),\mathrm{R_C}(T))}(s) = (s - \mathrm{C_C}(T))^2 - \mathrm{R_C}(T)^2$$

Definition 9.22 (almost empty triangle)**.** *For $\alpha \geq 0$, we say that triangle T is* α-almost empty *if:*

$$\forall s \in S, \mathrm{P}_T(s) \geq -\alpha$$

From Lemma 9.3 we know that acute triangles in $\mathcal{T}$ are 0-almost empty.

The aim of this section is to prove that triangles in $\mathcal{T}$ are α-almost empty for small α:

Lemma 9.23. *If $T \in \mathcal{T}$, then T is α-almost empty with:*

$$\alpha = O\left(\left(\frac{\epsilon}{\mathcal{R}}\right)^2 \left(\frac{\epsilon}{\eta}\right)^{10} \epsilon^2\right)$$

The proof of Lemma 9.23 is in Appendix E. It proceeds by induction, as in the proof of Lemma 9.17, along the directed acyclic graph given by Corollary 9.8.

9.5 Proximity graph G_δ of triangle circumcenters

Connected components of G_δ are small In this section, for a given $\delta > 0$, we study the proximity graph G_δ of circumcenters of triangles in $\mathcal{T}$. For $T_1, T_2 \in \mathcal{T}$, $\mathrm{C_C}(T_1)$ and $\mathrm{C_C}(T_2)$ are connected if $\|\mathrm{C_C}(T_2) - \mathrm{C_C}(T_1)\| \leq \delta$.

Let $T_0 \in \mathcal{T}$. Lemma 9.24 below says that for δ small enough, the connected component of $\mathrm{C_C}(T_0)$ in the proximity graph is included in $\mathcal{B}(\mathrm{C_C}(T_0), \rho)$ for $\rho < \frac{1}{8}\frac{\eta^2}{\epsilon}$.

Lemma 9.24 below is central in the proof of the theorem. The main difficulty is to prove that, locally, the projection on a local tangent plane is injective. More precisely, one have to show that the projection of two triangles on a common nearby tangent plane have disjoint interiors. Thanks to Lemma 9.24 this is done separately for pairs of triangles in separate connected components of G_δ (section 9.5.1) and for pairs of triangles in the same connected component (section 9.6).

In section 9.5.1, for triangles in separate connected components we want δ to be significantly larger than the bound given in Lemma 9.21 on the distance between circumcenters and tangent planes in order to guarantee a lower bound on the distance between projections on nearby tangent plane of corresponding circumcenters. For triangles in the same connected component, we need in section 9.6 to upper bound the distances between circumcenters of projections on tangent plane of triangles by $\frac{1}{8}\frac{\eta^2}{\epsilon}$, which motivate the value chosen for ρ in Lemma 9.24.

Lemma 9.24. *Given $T_0 \in \mathcal{T}$, for:*

$$\delta = \left(\frac{\epsilon}{\mathcal{R}}\right)^{1/2} \epsilon$$

the connected component of $\mathrm{C_C}(T_0)$ in the proximity graph G_δ is included in $\mathcal{B}(\mathrm{C_C}(T_0), \rho)$ with

$$\rho \in O\left(\left(\frac{\epsilon}{\mathcal{R}}\right)^{1/2} \left(\frac{\epsilon}{\eta}\right)^6\right) \epsilon$$

Proof of Lemma 9.24. Let m be a vertex of T_0.

From Lemma 9.20, if $\rho < \epsilon$, $S \cap S \cap \mathcal{B}(\mathrm{C_C}(T_0), \rho + 2\epsilon)$ is at distance $\beta_1 \in O\left(\frac{\epsilon}{\mathcal{R}}\epsilon\right)$ from plane Π_m. From the sampling conditions, balls centered on S with radius $\eta/2$ are disjoint. The intersection of these balls with plane Π_m are disks with radius lower bounded by:

$$\sqrt{(\eta/2)^2 - \beta_1^2}$$

which is arbitrarily close to $\eta/2$, for $\frac{\epsilon}{\mathcal{R}}$ small enough. These disks are of course disjoint as well which allows us to use a packing argument: the number of disjoint disks of radius say r included in a disk of radius R is upper bonded by he ratio of the disks area $\frac{R^2}{r^2}$. For a set X, $\sharp(X)$ denotes the cardinal of X. The packing argument gives us:

$$\sharp(S \cap \mathcal{B}(\mathrm{C_C}(T_0), \rho + 2\epsilon)) < C\left(\frac{\rho + 2\epsilon}{\eta/2}\right)^2$$

Where C is a small constant, say $C < 6^{\frac{1}{3}}$.

Since triangles circumradii are bounded by 2ϵ (Lemma 9.17), if $T \in \mathcal{T}$ is such that $\mathrm{C_C}(T) \in \mathcal{B}(\mathrm{C_C}(T_0), \rho)$ then the vertices of T are in $\mathcal{B}(\mathrm{C_C}(T_0), \rho + 2\epsilon)$. Since N points define less than $\frac{1}{6}N^3$ triangles:

$$\sharp\{T \in \mathcal{T}, \mathrm{C_C}(T) \in \mathcal{B}(\mathrm{C_C}(T_0), \rho)\} < \left(\frac{\rho + 2\epsilon}{\eta/2}\right)^6$$

It follows that for $\rho < \epsilon$ if:

$$\delta\left(\frac{\rho + 2\epsilon}{\eta/2}\right)^6 < \rho - \delta \tag{9.18}$$

then any path in the connected component of $\mathrm{C_C}(T_0)$ in G_δ cannot get out of $\mathcal{B}(\mathrm{C_C}(T_0), \rho)$ since it has exhausted the maximal number of triangles with circumcenters in $\mathcal{B}(\mathrm{C_C}(T_0), \rho)$ after a length of $\rho - \delta$.

It follows that in this case the cardinal of the connected component of $\mathrm{C_C}(T_0)$ in G_δ is upper bounded by:

$$\left(\frac{\rho + 2\epsilon}{\eta/2}\right)^6$$

Since $\rho < \epsilon$, (9.18) is satisfied if:

$$\delta < \frac{\rho}{2^7}\left(\frac{\eta}{3\epsilon}\right)^6 < \frac{1}{2^7 3^6}\left(\frac{\eta}{\epsilon}\right)^6 \epsilon \tag{9.19}$$

Then taking as in the lemma $\delta = \left(\frac{\epsilon}{\mathcal{R}}\right)^{1/2}\epsilon$ we get that for $\frac{\epsilon}{\mathcal{R}}$ small enough (9.19) is satisfied and:

$$\rho \leq \delta\left(\frac{\rho + 2\epsilon}{\eta/2}\right)^6 < 2^6 3^6 \left(\frac{\epsilon}{\eta}\right)^6 \left(\frac{\epsilon}{\mathcal{R}}\right)^{1/2} \epsilon \in O\left(\left(\frac{\epsilon}{\mathcal{R}}\right)^{1/2}\left(\frac{\epsilon}{\eta}\right)^6\right)\epsilon$$

□

9.5.1 Triangles in separate connected components of G_δ do not overlap

Definition 9.25. *If $A, B \subset \mathbb{R}^2$, we denote by $S(A, B)$ the* separation distance *of A, B defined by:*

$$S(A, B) = \sup_{\|y\|=1} \inf_{\substack{a \in A \\ b \in B}} \langle y, b - a \rangle$$

Since $\langle y, b-a\rangle = \langle -y, a-b\rangle$ one has:

$$S(A,B) = S(B,A)$$

We denote by $|A|$ the convex hull of A. Since

$$\forall y, \inf_{a\in A}\langle y,a\rangle = \inf_{a'\in|A|}\langle y,a'\rangle$$

one has:

$$S(A,B) = S(|A|,B) = S(A,|B|) = S(|A|,|B|)$$

if $|A|\cap|B| = \emptyset$:

$$S(A,B) = \inf_{\substack{a\in|A|\\ b\in|B|}} \|a-b\|$$

If A and B are compact sets, then:

$$S(A,B) > 0 \iff |A|\cap|B| = \emptyset$$

We define $\mathrm{width}(A)$ for a convex set A as:

Definition 9.26.

$$\mathrm{width}(A) = -S(A,A) = \inf_{\|y\|=1}\ \sup_{a,a'\in A}\langle y, a'-a\rangle$$

One has

$$S(A,B) = S(|A|,|B|) = \sup_{\|y\|=1}\ \inf_{\substack{a\in|A|\\ b\in|B|}}\langle y,b-a\rangle \leq \sup_{\|y\|=1}\ \inf_{a,b\in|A|\cap|B|}\langle y,b-a\rangle$$

and therefore: If $|A|\cap|B|\neq\emptyset$:

$$S(A,B) \leq -\,\mathrm{width}(|A|\cap|B|) \tag{9.20}$$

Given a triangle T, $\mathrm{V}(T)$ denotes the set of its 3 vertices.

Lemma 9.27. *For any constant $C>0$, if $\frac{\epsilon}{\mathcal{R}}$ is small enough, if $T,T'\in\mathcal{T}$ such that $\mathrm{V}(T),\mathrm{V}(T')\subset S_{m,C\epsilon}$, if:*

$$\|\mathrm{C_C}(T) - \mathrm{C_C}(T')\| > \delta = \left(\frac{\epsilon}{\mathcal{R}}\right)^{1/2}\epsilon$$

then

$$S(\pi_m(T),\pi_m(T')) > -\beta$$

with:

$$\beta \in O\left(\left(\frac{\epsilon}{\mathcal{R}}\right)^{3/2}\left(\frac{\epsilon}{\eta}\right)^{10}\epsilon\right)$$

Proof. Denote s_i for $i=1,2,3$ the vertices of T and Denote s'_j for $j=1,2,3$ the vertices of T'. According to Lemma 9.23 one has:

$$\forall i, \mathrm{P}_{T'}(s_i) \geq -\alpha \quad \text{and} \quad \forall j, \mathrm{P}_T(s'_j) > -\alpha$$

with

$$\alpha = O\left(\left(\frac{\epsilon}{\mathcal{R}}\right)^{2}\left(\frac{\epsilon}{\eta}\right)^{10}\epsilon^2\right) \tag{9.21}$$

Since by definition:

$$\forall j, \mathrm{P}_T(s_i) = 0 \quad \text{and} \quad \forall j, \mathrm{P}_{T'}(s'_j) = 0$$

It follows:

$$\begin{aligned}\forall i,j,-2\alpha &\leq \left(\mathrm{P}_T(s'_j)-\mathrm{P}_{T'}(s'_j)\right)-\left(\mathrm{P}_T(s_i)-\mathrm{P}_{T'}(s_i)\right)\\ &= \left((\mathrm{C_C}(T)-s'_j)^2-\mathrm{R_C}(T)^2\right)-\left((\mathrm{C_C}(T')-s'_j)^2-\mathrm{R_C}(T')^2\right)\\ &- \left((\mathrm{C_C}(T)-s_i)^2-\mathrm{R_C}(T)^2\right)-\left((\mathrm{C_C}(T')-s_i)^2-\mathrm{R_C}(T')^2\right)\\ &= 2\langle \mathrm{C_C}(T)-\mathrm{C_C}(T'), s_i-s'_j\rangle\end{aligned}$$

This gives:

$$\langle \mathrm{C_C}(T)-\mathrm{C_C}(T'), s_i-s'_j\rangle \geq -\alpha \tag{9.22}$$

If we denote π_m the projection of vectors in $\mathbb{R}^n$ on tangent plane to $\mathcal{M}$ at m and $\pi_m^\perp$ the projection on the orthogonal $(n-2)$-linear space in such a way that for any vector $w \in \mathbb{R}^n$, $\pi_m(w)+\pi_m^\perp(w)=w$. One has:

$$\begin{aligned}\langle \mathrm{C_C}(T)-\mathrm{C_C}(T'), s_i-s'_j\rangle &= \langle \pi_m\left(\mathrm{C_C}(T)-\mathrm{C_C}(T')\right), \pi_m\left(s_i-s'_j\right)\rangle\\ &+ \langle \pi_m^\perp\left(\mathrm{C_C}(T)-\mathrm{C_C}(T')\right), \pi_m^\perp\left(s_i-s'_j\right)\rangle\end{aligned}$$

From Lemma 9.21 one has

$$\left\|\pi_m^\perp\left(\mathrm{C_C}(T)-\mathrm{C_C}(T')\right)\right\| \in O\left(\frac{\epsilon}{\mathcal{R}}\left(\frac{\epsilon}{\eta}\right)^2\epsilon\right)$$

and from Lemma 9.20 one has:

$$\left\|\pi_m^\perp\left(s_i-s'_j\right)\right\| \in O\left(\frac{\epsilon}{\mathcal{R}}\epsilon\right)$$

which makes:

$$\left|\langle \pi_m^\perp\left(\mathrm{C_C}(T)-\mathrm{C_C}(T')\right), \pi_m^\perp\left(s_i-s'_j\right)\rangle\right| \in O\left(\left(\frac{\epsilon}{\mathcal{R}}\right)^2\left(\frac{\epsilon}{\eta}\right)^2\epsilon^2\right)$$

Since $\left(\left(\frac{\epsilon}{\mathcal{R}}\right)^2\left(\frac{\epsilon}{\eta}\right)^2\epsilon^2\right) \in O\left(\left(\frac{\epsilon}{\mathcal{R}}\right)^2\left(\frac{\epsilon}{\eta}\right)^{10}\epsilon^2\right)$ We therefore get from (9.22), with α still obeying the same asymptotic behavior given in (9.21) above:

$$\langle \pi_m\left(\mathrm{C_C}(T)-\mathrm{C_C}(T')\right), \pi_m\left(s_i-s'_j\right)\rangle \geq -\alpha \tag{9.23}$$

From Lemma 9.21 we know that taking $\frac{\epsilon}{\mathcal{R}}$ small enough forces $\left\|\pi_m^\perp\left(\mathrm{C_C}(T)-\mathrm{C_C}(T')\right)\right\|$ to be arbitrary small with respect to $\|\mathrm{C_C}(T)-\mathrm{C_C}(T')\| > \delta = \left(\frac{\epsilon}{\mathcal{R}}\right)^{1/2}\epsilon$, which makes:

$$\left\|\pi_m\left(\mathrm{C_C}(T)-\mathrm{C_C}(T')\right)\right\| \in \Omega\left(\left(\frac{\epsilon}{\mathcal{R}}\right)^{1/2}\epsilon\right) \tag{9.24}$$

If we define:

$$y = \frac{\pi_m\left(\mathrm{C_C}(T)-\mathrm{C_C}(T')\right)}{\left\|\pi_m\left(\mathrm{C_C}(T)-\mathrm{C_C}(T')\right)\right\|}$$

(9.23) and (9.24) gives us:

$$\forall i,j, \langle y, \pi_m(s_i)-\pi_m(s'_j)\rangle \geq -\beta$$

with

$$\beta \in O\left(\left(\frac{\epsilon}{\mathcal{R}}\right)^{3/2}\left(\frac{\epsilon}{\eta}\right)^{10}\epsilon\right)$$

which gives,

$$S(\pi_m(T),\pi_m(T')) > -\beta$$

□

Recall that for a triangle $T = abc$, $\mathrm{V}(T) = \{a, b, c\}$, denotes the set of vertices of T, that $|T|$ denotes the convex hull of T and, if A is a set, $d(v, A) = \inf_{a \in A} d(v, a)$.

Lemma 9.28. *For $\frac{\epsilon}{\mathcal{R}}$ small enough, let $T = s_1, s_2, s_3 \in \mathcal{T}$ and $s \in \mathcal{M}$, such that $\forall i, s \neq s_i$ with $s, s_1, s_2, s_3 \in S_{m,\mathcal{R}/2,C\epsilon}$ then the distance from $\pi_m(s)$ to $\pi_m(|T|)$ is lower bounded by some "constant" distance from T:*

$$d(\pi_m(s), \pi_m(|T|)) > \frac{1}{64}\frac{\eta^2}{\epsilon} \in O(1)$$

Proof. According to Lemma 9.23 , T is α-almost empty with:

$$\alpha = O\left(\left(\frac{\epsilon}{\mathcal{R}}\right)^2 \left(\frac{\epsilon}{\eta}\right)^{10} \epsilon^2\right) \tag{9.25}$$

We show first that $d(s, |T|) \geq \frac{1}{8}\frac{\eta^2}{\epsilon}$. We consider the maps π_T and $\pi_T^{\perp}$ similar to the one introduced in the proof of previous Lemma, with $\forall w \in \mathbb{R}^n, \pi_T(w) + \pi_T^{\perp}(w) = w$ where π_T is now the projection on the supporting plane Π_T of triangle T and vector $\pi_T^{\perp}(w)$ in the orthogonal linear $n-2$-space. π_T denote also the affine projection on Π_T when applied to points.

One has $(s_i - s)^2 = \pi_T(s_i - s)^2 + \pi_T^{\perp}(s_i - s)^2$.

By sampling conditions, $(s_i - s)^2 \geq \eta^2$ and by Lemma 9.20, $\pi_T^{\perp}(s_i - s)^2 > 4\beta_2^2$ with $\beta_2 = O\left(\frac{\epsilon}{\mathcal{R}}\left(\frac{\epsilon}{\eta}\right)^2 \epsilon\right)$. It follows that for $\frac{\epsilon}{\mathcal{R}}$ small enough:

$$\forall i, \pi_T(s_i - s)^2 = (s_i - \pi_T(s))^2 > \eta^2/2 \tag{9.26}$$

Since $|T|$ is compact there is a projection of s on $|T|$ denoted p such that $p \in |T|$ and $d(s, p) = d(s, |T|)$.

If $\pi_T(s) \in |T|$, then $p = \pi_T(s)$. If not, if $d(s, |T|)^2 < \eta^2/2$ then $d(\pi_T(s), |T|)^2 < \eta^2/2$ while, after equation (9.26) if p was vertex s_i of T one would have $d(\pi_T(s), |T|)^2 = d(\pi_T(s), s_i)^2 > \eta^2/2$. Therefore, if $\pi_T(s) \notin |T|$ then p is on an edge, say, w.l.o.g., s_1s_2. Then, similarly to the case $\pi_T(s) \in |T|$, we can split $s_i - s$ in a component on the vectorial line supporting s_1s_2 and its orthogonal space. In both cases we have:

$$\forall i, (s_i - p)^2 > \eta^2/2 \tag{9.27}$$

Therefore on has:

$$d(s, |T|) = d\left(s, |T| \setminus \bigcup_i \mathcal{B}\left(s_i, \frac{\eta}{2}\right)\right)$$

The map $x \mapsto \mathrm{P}_T(x)$ is convex and therefore its maximum on $|T| \setminus \bigcup_i \mathcal{B}\left(s_i, \frac{\eta}{2}\right)$ is reached on the boundary and more precisely on convex corners of the boundary. The boundary of $|T| \setminus \bigcup_i \mathcal{B}\left(s_i, \frac{\eta}{2}\right)$ is a union of parts of edges of T and intersections of balls boundaries with $|T|$. Convex corners appears at the intersection of edges and balls. Considers edge s_1s_2. The two convex corners appears at the intersection of spheres with radius $\eta/2$ centered at s_1 or s_2.

Let us denote by $o = \mathrm{C_C}(T)$ the circumcenter of T, by $m = (s_1s_2)/2$ the middle of edge s_1s_2, and by c a corner. With $h = \|m - o\|$ and $d = \|m - s_1\| = \|m - s_2\| \geq \eta/2$, one has:

$$(o - c)^2 = h^2 + (d - \eta/2)^2$$

With $R = \mathrm{R_C}(T) = h^2 + d^2$ we have:

$$\mathrm{P}_T(c) = (o - c)^2 - R^2 = -d\,\eta + (\eta/2)^2$$

And since $d \geq \eta/2$ we get $\mathrm{P}_T(c) \leq -\eta^2/2 + (\eta/2)^2 = -(\eta/2)^2$ Since $x \mapsto \mathrm{P}_T(x)$ reaches its maximum on a corner such as c one has:

$$\forall x \in |T| \setminus \bigcup_i \mathcal{B}\left(s_i, \frac{\eta}{2}\right), \mathrm{P}_T(x) \leq -(\eta/2)^2$$

and this gives:

$$|T| \setminus \bigcup_i \mathcal{B}\left(s_i, \frac{\eta}{2}\right) \subset \mathcal{B}\left(o, \sqrt{R^2 - (\eta/2)^2}\right)$$

Since by Lemma 9.23, T is α-almost empty with $\alpha = O\left(\left(\frac{\epsilon}{\mathcal{R}}\right)^2 \left(\frac{\epsilon}{\eta}\right)^{10} \epsilon^2\right)$ we have that $s \notin \mathcal{B}\left(o, \sqrt{R^2 - \alpha}\right)$ and therefore:

$$\begin{aligned} d(s, |T|) &= d\left(s, |T| \setminus \bigcup_i \mathcal{B}\left(s_i, \frac{\eta}{2}\right)\right) \\ &\geq \sqrt{R^2 - \alpha} - \sqrt{R^2 - (\eta/2)^2} \\ &\geq \left((\eta/2)^2 - \alpha\right)/(2R) \end{aligned}$$

Since $R \leq 2\epsilon$ and, for $\frac{\epsilon}{\mathcal{R}}$ small enough, $\alpha < (\eta/2)^2/2$, we get:

$$d(s, |T|) \geq \frac{1}{32}\left(\frac{\eta}{\epsilon}\right)^2 \epsilon$$

Now, from Lemma 9.20, $d(s, \pi_T(s)) < \beta_2$ and, if d_h denote the Hausdorff distance, one get easily that $d_h(|T|, \pi_T(|T|)) < \beta_2$ with $\beta_2 = O\left(\frac{\epsilon}{\mathcal{R}}\left(\frac{\epsilon}{\eta}\right)^2 \epsilon\right)$. It follows that for $\frac{\epsilon}{\mathcal{R}}$ small enough, $\beta_2 < \frac{1}{128}\left(\frac{\eta}{\epsilon}\right)^2 \epsilon$ which gives:

$$d(\pi_m(s), \pi_m(|T|)) \geq d(s, |T|) - d(s, \pi_T(s)) - d_h(|T|, \pi_T(|T|)) > \frac{1}{64}\frac{\eta^2}{\epsilon} \in O(1)$$

$\square$

Lemma 9.29. *For $\frac{\epsilon}{\mathcal{R}}$ small enough, let $T \in \mathcal{T}$, $s \in \mathcal{M}$ such that $\mathrm{V}(T) \subset S_{m,\mathcal{R}/2,C\epsilon}$. Then any height of triangle $\pi_m(T)$ is larger than $\frac{\eta^2}{4\epsilon}$. Any angle of triangle $\pi_m(T)$ has his sinus larger than $\frac{\eta}{8\epsilon}$.*

Proof. Lemma 9.18 gives similar lower bounds for T. Since by Lemma 9.20 the distance between T and the plane Π_m is $O\left(\frac{\epsilon}{\mathcal{R}}\epsilon\right)$, for $\frac{\epsilon}{\mathcal{R}}$ small enough this distance become arbitrarily small with respect to the lower bound η on edge lengths. It results that the transformation $T \to \pi_m(T)$ is as close as we want to an isometry. And the lower bounds on angle and heights can be made arbitrarily close to the one for T. It follows in particular that small enough $\frac{\epsilon}{\mathcal{R}}$ allows to guarantee lower bounds for $\pi_m(T)$ twice smaller than the lower bounds on T. $\square$

Lemma 9.30. *For $\frac{\epsilon}{\mathcal{R}}$ small enough, for $s \in \mathcal{M}$, let $T, T' \in \mathcal{T}$ such that $\mathrm{V}(T), \mathrm{V}(T') \subset S_{m,\mathcal{R}/2,C\epsilon}$, if $\pi_m(T)^\circ \cap \pi_m(T')^\circ \neq \emptyset$ one has:*

$$\mathrm{width}\left(\pi_m(T) \cap \pi_m(T')\right) > 2^{-10}\left(\frac{\eta}{\epsilon}\right)^3 \epsilon$$

Proof. if $\pi_m(T)^\circ \cap \pi_m(T')^\circ \neq \emptyset$ then $\pi_m(T) \cap \pi_m(T')$ is a convex polygon with non empty interior. There is a disk with positive maximal radius included in $\pi_m(T) \cap \pi_m(T')$. Generically this disk is tangent to three sides of the polygon (for non generic configurations it is enough to consider arbitrarily close generic configurations).

Consider the first case where this three sides belong to the same triangle, say $\pi_m(T) = abc$. Since the radius of the incircle of a triangle is $\rho(abc) = 2S/(ab + bc + ac)$ where S is the area of abc, we have that if h is the smallest height of abc, then $\rho(abc) \geq h/3$. Since $\mathrm{width}\left(\pi_m(T) \cap \pi_m(T')\right) > 2\rho(abc)$, thanks to Lemma 9.29 we get the result in this case with a lower bound of $\frac{2}{3}\frac{\eta^2}{8\epsilon} > 2^{-10}\left(\frac{\eta}{\epsilon}\right)^3 \epsilon$.

Now consider the case when only two of the three tangent edges to the maximal disk, say ab and ac, belong to triangle T.

If a is a also a vertex of $\pi_m(T') = ab'c'$, this means that the maximal disk has is third tangent point on $b'c'$. call b'' (respectively c'') the intersection of $b'c'$ with ab (respectively ac). Then the triangle $ab''c''$ is a subset of $\pi_m(T) \cap \pi_m(T')$. It follows that width $(\pi_m(T) \cap \pi_m(T'))$ is lower bounded by the smallest height of $ab''c''$. But from Lemma 9.29 we have that the sinus of the angle $\angle b''ac'' = \angle bac'$ is lower bounded by $\frac{\eta}{8\epsilon}$. Since the height $d(a,(b''c'')) = d(a,(b',c'))$ is lower bounded (Lemma 9.29 again) by $\frac{\eta^2}{8\epsilon}$, we have that $a, b'', ac'' \geq \frac{\eta^2}{8\epsilon}$. We get therefore that the smallest height of $ab''c''$ is lower bounded by $\frac{\eta^2}{8\epsilon}\frac{\eta}{8\epsilon}$.

It remains to consider the case where a is not a vertex of $\pi_m(T')$. It follows that, thanks to Lemma 9.28, $d(\pi_m(a), \pi_m(|T'|)) > 2^{-7}\left(\frac{\eta}{\epsilon}\right)^2 \epsilon$. Denote by t_{ab} (respectively t_{ac}) the point where the maximal disk is tangent to ab (respectively ac). Since $t_{ab}, t_{ac} \in \pi_m(|T'|)$ we get $at_{ab} = at_{ac} > 2^{-7}\left(\frac{\eta}{\epsilon}\right)^2 \epsilon$. The radius ρ of the maximal disk is then $\rho = at_{ab}\tan\frac{\angle bac}{2}$.

Since, from Lemma 9.29 one has $\tan\frac{\angle bac}{2} \geq \frac{1}{2}\sin\angle bac > \frac{\eta}{16\epsilon}$, we get:

$$\rho > 2^{-7}\left(\frac{\eta}{\epsilon}\right)^2 \epsilon \frac{\eta}{16\epsilon} = 2^{-11}\left(\frac{\eta}{\epsilon}\right)^3 \epsilon$$

One has width $(\pi_m(T) \cap \pi_m(T')) \geq 2\rho$ and we get the lower bound.

□

Lemma 9.31. *For $\frac{\epsilon}{\mathcal{R}}$ small enough, let $T, T' \in \mathcal{T}$, $s \in \mathcal{M}$ such that $\mathrm{V}(T), \mathrm{V}(T') \subset S_{m,\mathcal{R}/2,C\epsilon}$. If T and T' are not in the same connected component of G_δ for $\delta = \left(\frac{\epsilon}{\mathcal{R}}\right)^{1/2}\epsilon$ one has:*

$$\pi_m(T)^\circ \cap \pi_m(T')^\circ = \emptyset$$

Proof. For a contradiction assume $\pi_m(T)^\circ \cap \pi_m(T')^\circ \neq \emptyset$. From Lemma 9.30 we have then that:

$$\text{width}\left(\pi_m(T) \cap \pi_m(T')\right) > 2^{-10}\left(\frac{\eta}{\epsilon}\right)^3 \epsilon$$

and from Lemma 9.27, if T and T' are not in the same connected component of G_δ this means that $\|\mathrm{C_C}(T) - \mathrm{C_C}(T')\| > \delta$ and Lemma 9.27 gives:

$$S(\pi_m(T), \pi_m(T')) > -\beta$$

with:

$$\beta \in O\left(\left(\frac{\epsilon}{\mathcal{R}}\right)^{3/2}\left(\frac{\epsilon}{\eta}\right)^{10}\epsilon\right)$$

which, for $\frac{\epsilon}{\mathcal{R}}$ small enough is smaller than $2^{-10}\left(\frac{\eta}{\epsilon}\right)^3\epsilon$ and then:

$$S(\pi_m(T), \pi_m(T')) > -\,\text{width}\left(\pi_m(T) \cap \pi_m(T')\right)$$

which contradicts (9.20). □

9.6 Triangles in same connected component of G_δ

9.6.1 Partition of a connected components CC of G_δ into dual graph connected components $(CC_j)_{j=1,k}$

All along this section 9.6 we assume that $\delta = \left(\frac{\epsilon}{\mathcal{R}}\right)^{1/2}\epsilon$, as assumed in Lemmas 9.24 and 9.31.

Each connected component of G_δ corresponds to a set $CC \subset \mathcal{T}$. We consider $CC = \{T_1, \ldots, T_m\}$ a such connected connected component of triangles in G_δ. We consider the *dual* *graph* in CC, i.e. the graph whose vertices are triangles in CC and two vertices are connected by an edge in the graph if the corresponding triangles share an edge. Connected components $(CC_j)_{j=1,k}$ in this dual graph defines again a partition of $CC = \bigcup_{j=1,k} CC_j$.

Recall that, for given $m \in \mathcal{M}$ and C we say that two triangles T, T' with vertices in $S_{m,\mathcal{R}/2,C\epsilon}$ overlap if the interior of their projections on plane Π_m are not disjoint, i.e. $\pi_m(T)^\circ \cap \pi_m(T')^\circ \neq \emptyset$.

In order to prove that, locally, no two triangle's projection on Π_m overlap we consider the three alternatives. Lemma 9.31 above says that triangles in separate connected components of G_δ do not overlap. Then Lemma 9.39 in section 9.6.4 below states the same property for two triangles in a same connected component CC_i of the dual graph over a connected component CC of G_δ. Lastly, Lemma 9.40 in section 9.6.5 states the same property for two triangles in a same connected component CC of G_δ but in separate connected components CC_i, CC_j of the dual graph in CC.

9.6.2 A pentagram as counter example

In some sense the problem of proving the manifoldness of the support of the minimal chain is reduced to the simple case of quasicocyclic points. However the simplicity of the configuration do not allow to conclude directly and the minimality condition has to be used further in the argument as is shown by the following counter example: Denote by $V_0, \ldots, V_4$ the 5 vertices of a pentagon with $V_k = e^{\frac{k}{5}2\pi i}, k = 0, \ldots, 4$.
Consider the chain

$$\tau = V_0V_1V_3 + V_1V_2V_4 + V_2V_3V_0 + V_3V_4V_1 + V_4V_0V_2$$

Observes that $\partial\tau = V_0V_1 + V_1V_2 + V_2V_3 + V_3V_4 + V_4V_0$

The support of τ is not a valid triangulation of the pentagon, it does not collapse either on a triangulation since it a Moebius strip. Remark: this example does not work with $\mathbb{R}$ as field coefficients since it is not orientable (find an orientable counter-example ?).

9.6.3 Some properties of triangles in a same connected component of G_δ

Let V be the set of vertices of all triangles in a given connected component CC of G_δ. Since circumradii are upper bounded by 2ϵ and since all circumcenters in a connected component of G_δ lie in a ball of radius $\rho \in O\left(\left(\frac{\epsilon}{\mathcal{R}}\right)^{1/2}\left(\frac{\epsilon}{\eta}\right)^6\right)\epsilon$, by Lemma 9.24.

Moreover we assume that for some $m \in \mathcal{M}$ and C one has:

$$V \subset S_{m,\mathcal{R}/2,C\epsilon}$$

Lemma 9.32. *Let $V \subset S_{m,\mathcal{R}/2,C\epsilon}$ be the set of vertices of all triangles in a connected component of G_δ, for $\delta = \left(\frac{\epsilon}{\mathcal{R}}\right)^{1/2}\epsilon$. There is a circle $C_0 \subset \Pi_m$ of radius in $[\eta/2, 2\epsilon]$ such that any vertex in $v \in V$ as well as its projection $\pi_m(s)$ on Π_m are at distance less than $O\left(\left(\frac{\epsilon}{\mathcal{R}}\right)^{1/2}\left(\frac{\epsilon}{\eta}\right)^6\epsilon\right)$ from C_0. In other words, if π_{C0} denote the projection on (i.e. the closest point on) circle C_0, for any $v \in V$:*

$$\begin{aligned}
\|v - \pi_{C0}(v)\| &\in O\left(\left(\frac{\epsilon}{\mathcal{R}}\right)^{1/2}\left(\frac{\epsilon}{\eta}\right)^6\epsilon\right) \\
\|\pi_m(v) - \pi_{C0}(v)\| &\in O\left(\left(\frac{\epsilon}{\mathcal{R}}\right)^{1/2}\left(\frac{\epsilon}{\eta}\right)^6\epsilon\right)
\end{aligned}$$

Proof. Call $T_0, \ldots, T_{m-1}$ the m triangles in the connected component. From Lemma 9.24, all circumradius lies in a ball of radius

$$\rho \in O\left(\left(\frac{\epsilon}{\mathcal{R}}\right)^{1/2}\left(\frac{\epsilon}{\eta}\right)^6\right)\epsilon$$

It follows that any circumcenter c_i of a triangle T_i lies at distance less than 2ρ from c_0. For $i \neq 0$ considers a vertex s_i of T_i. One has:

$$\|s_i - c_0\| \leq \|s_i - c_i\| + \|c_i - c_0\| \leq \|s_i - c_i\| + 2\rho \leq R_i + 2\rho$$

Where $R_i = \mathrm{R_C}(T_i)$.

But on another side from Lemma 9.23 T_0 is α-almost empty with:

$$\alpha \in O\left(\left(\frac{\epsilon}{\mathcal{R}}\right)^2 \left(\frac{\epsilon}{\eta}\right)^{10} \epsilon^2\right)$$

Therefore $\mathrm{P}_{T_0}(s) \geq -\alpha$,

$$\begin{aligned}
\|s_i - c_0\| &= \sqrt{R_0^2 + \mathrm{P}_{T_0}(s_i)} \\
&= R_0\sqrt{1 + \frac{\mathrm{P}_{T_0}(s_i)}{R_0^2}} \\
&\geq R_0\left(1 - \frac{\alpha}{R_0^2}\right) \\
&\geq R_0 - \alpha'
\end{aligned}$$

With: $\alpha' \in O\left((\frac{\epsilon}{\mathcal{R}})^2 \left(\frac{\epsilon}{\eta}\right)^{10} \epsilon\right)$. The two last inequalities gives:

$$R_0 - \alpha' \leq \|s_i - c_0\| \leq R_i + 2\rho$$

It follows that:

$$R_0 - R_i \leq 2\rho + \alpha' \in O\left(\left(\frac{\epsilon}{\mathcal{R}}\right)^{1/2} \left(\frac{\epsilon}{\eta}\right)^6\right)\epsilon$$

Triangle T_0 has been chosen arbitrarily, swapping T_0 and T_i and combining the two inequalities gives:

$$|R_0 - R_i| \leq 2\rho + \alpha' \in O\left(\left(\frac{\epsilon}{\mathcal{R}}\right)^{1/2} \left(\frac{\epsilon}{\eta}\right)^6\right)\epsilon$$

So that we have proven that any point in V lies at distance

$$O\left(\left(\frac{\epsilon}{\mathcal{R}}\right)^{1/2} \left(\frac{\epsilon}{\eta}\right)^6\right)\epsilon$$

from the sphere of centre c_0 and radius R_0. From Lemma 9.21, we know that:

$$\|c_0 - \pi_m(c_0)\| \leq \beta$$

with:

$$\beta = O\left(\frac{\epsilon}{\mathcal{R}} \left(\frac{\epsilon}{\eta}\right)^2 \epsilon\right)$$

From Lemma 9.20:

$$s_i \in \Pi_{T_0}^{\oplus\beta_2}$$

with:

$$\beta_2 = O\left(\frac{\epsilon}{\mathcal{R}} \left(\frac{\epsilon}{\eta}\right)^2 \epsilon\right)$$

The statement of the Lemma follows with the circle $C_0 \subset \Pi_m$ with center $\pi_m(c_0)$ and radius R_0. $\square$

In the remaining of this section, following the notation introduced in the proof of Lemma 9.32, we consider a connected component $CC = \{T_0, \ldots, T_{m-1}\}$ of G_δ whose triangles are sharing a set of vertex $V = \cup_i T_i$ (in this notation we see a triangle as a set of three points $T = \{a, b, c\}$). We denote by C_0 the circle introduced in Lemma 9.32 which is the circumcircle of T_0. For any $s \in V$ we denote by $\pi_{C_0}(s)$ its projection on C_0, i.e. the closest point of s on C_0.

Inequalities on $\pi_{C_0}(V)$ In this section we states inequalities satisfied by vertices in $\pi_{C_0}(V)$, as $\pi_{C_0}(V)$ is used below as an approximation (Lemma 9.32) of V and $\pi_m(V)$.

Lemma 9.33. *For $\frac{\epsilon}{\mathcal{R}}$ small enough the vertices in $\pi_{C_0}(V)$ satisfies the following properties.*

(a) For any $a, b \in V$, $a \neq b$:

$$\|\pi_{C_0}(a) - \pi_{C_0}(b)\| \geq \frac{\eta}{2}$$

(b) For $a, b, c \in V$ such that either $\pi_{C_0}(ab)$ is a diameter of C_0, either $\pi_{C_0}(ab)$ is not a diameter of C_0 but $\pi_{C_0}(c)$ is on the smallest of the two arcs of circle of $C_0 \setminus \{\pi_{C_0}(a), \pi_{C_0}(b)\}$, then one has:

$$\|\pi_{C_0}(a) - \pi_{C_0}(c)\|, \|\pi_{C_0}(b) - \pi_{C_0}(c)\| < \|\pi_{C_0}(a) - \pi_{C_0}(b)\| - \frac{1}{144}\left(\frac{\eta}{\epsilon}\right)^3 \epsilon$$

(c) If $a, b, c \in V$ and bc is not the longest edge of abc, then $\pi_{C_0}(bc)$ is not a diameter of C_0 and $\pi_{C_0}(a)$ is on the longest of the two arcs of circle of $C_0 \setminus \{\pi_{C_0}(b), \pi_{C_0}(c)\}$and:

$$\|\pi_{C_0}(b) - \pi_{C_0}(c)\| \leq 2R_0 - \frac{1}{4096}\left(\frac{\eta}{\epsilon}\right)^3 \epsilon$$

(d) If $a, b, c \in V$ and bc is not the longest edge of abc, then if $d \in V$ is such that $\pi_{C_0}(d)$ is not on the same arc of circle of $C_0 \setminus \{\pi_{C_0}(b), \pi_{C_0}(c)\}$ than $\pi_{C_0}(a)$, then:

$$\|\pi_{C_0}(c) - \pi_{C_0}(d)\|, \|\pi_{C_0}(b) - \pi_{C_0}(d)\| \leq \|\pi_{C_0}(b) - \pi_{C_0}(c)\| - \frac{1}{144}\left(\frac{\eta}{\epsilon}\right)^3 \epsilon$$

Proof. One has by sampling hypothesis that $\|a - b\| \geq \eta$. By Lemma 9.32 we have $\|\pi_{C_0}(a) - \pi_{C_0}(b)\| \geq \frac{\eta}{2}$ for $\frac{\epsilon}{\mathcal{R}}$ small enough and (a) is proved.

Under the assumptions of (b) one has $(\pi_{C_0}(a) - \pi_{C_0}(c))^2 < (\pi_{C_0}(a) - \pi_{C_0}(b))^2 - (\pi_{C_0}(b) - \pi_{C_0}(c))^2$. From (a) we know that $(\pi_{C_0}(b) - \pi_{C_0}(c))^2 \leq \frac{\eta^2}{4}$ and from Lemma 9.17 we know that $\|a-b\| < 2\epsilon$, which gives for $\frac{\epsilon}{\mathcal{R}}$ small enough by Lemma 9.32 $\|\pi_{C_0}(a) - \pi_{C_0}(b)\| < 3\epsilon$. We get:

$$(\pi_{C_0}(a) - \pi_{C_0}(c))^2 < (\pi_{C_0}(a) - \pi_{C_0}(b))^2 \left(1 - \frac{\eta^2}{36\epsilon^2}\right)$$

and then:

$$\begin{aligned} \|\pi_{C_0}(a) - \pi_{C_0}(c)\| &< \|\pi_{C_0}(a) - \pi_{C_0}(b)\| \sqrt{1 - \frac{\eta^2}{36\epsilon^2}} \\ &\leq \|\pi_{C_0}(a) - \pi_{C_0}(b)\| \left(1 - \frac{\eta^2}{72\epsilon^2}\right) \end{aligned}$$

And since $\|\pi_{C_0}(a) - \pi_{C_0}(b)\| > \eta/2$ one gets (b).

In the context of (c), without loss of generality assume that ab is the longest edge of abc. If $\pi_{C_0}(bc)$ was a diameter of C_0, or if it was not but with a on the shortest of the two arcs of circle of $C_0 \setminus \{\pi_{C_0}(b), \pi_{C_0}(c)\}$, one could apply (b) to get $\|\pi_{C_0}(a) - \pi_{C_0}(b)\| < \|\pi_{C_0}(c) - \pi_{C_0}(b)\| - \frac{1}{144}\left(\frac{\eta}{\epsilon}\right)^3 \epsilon$ which, for $\frac{\epsilon}{\mathcal{R}}$ small enough, contradicts the fact that $ab > bc$.

On another hand a short computation shows that if $\|\pi_{C_0}(b) - \pi_{C_0}(c)\| > 2R_0 - \frac{1}{4096}\left(\frac{\eta}{\epsilon}\right)^3 \epsilon$ then he distance between $\pi_{C_0}(c)$ and the point $\widehat{\pi_{C_0}(b)}$ opposite to $\pi_{C_0}(b)$ on circle C_0 is less than $\eta/8$. It follows from (a) that $\pi_{C_0}(a)$ is at distance at least $3\eta/8$ from $\widehat{\pi_{C_0}(b)}$ and, for $\frac{\epsilon}{\mathcal{R}}$ small enough, this would contradicts $ab > bc$.

In the context of (d), by (c), $\pi_{C_0}(bc)$ is not a diameter of C_0 and $\pi_{C_0}(d)$ is on the shortest arc of circle of $C_0 \setminus \{\pi_{C_0}(b), \pi_{C_0}(c)\}$. Then applying (b) gives the required inequality for (d). $\square$

Properties on $\pi_m(V)$

Lemma 9.34. *For $\frac{\epsilon}{\mathcal{R}}$ small enough the vertices in $\pi_v(V)$ and triangles in CC satisfy the following properties.*

(a) For a given oriented frame in Π_m, if $a, b, c \in V$, $\pi_m(abc)$ and $\pi_{C_0}(abc)$ have consistent orientation (and are not flat).

(b) Vertices of $\pi_m(V)$ are in strict convex position.

(c) For $T_1, T_2 \in CC$ one has

$$\pi_m(T_1)^\circ \cap \pi_m(T_2)^\circ = \emptyset \iff \pi_{C_0}(T_1)^\circ \cap \pi_{C_0}(T_2)^\circ = \emptyset$$

Proof. Points in $\pi_{C_0}(V)$ are on circle C_0 with radius $R_0 \leq 2\epsilon$ and their mutual distances are lower bounded by $\eta/2$ by Lemma 9.33(a). Therefore for any triples of points $v_1, v_2, v_3 \in \pi_{C_0}(V)$, $\frac{\eta^3}{16\epsilon}$ is a lower bound on the norm of determinant $|\det(\pi_{C_0}(v_2) - \pi_{C_0}(v_1), \pi_{C_0}(v_3) - \pi_{C_0}(v_1))|$. By Lemma 9.32 we get that for $\frac{\epsilon}{\mathcal{R}}$ small enough then
$\det(\pi_{C_0}(v_2) - \pi_{C_0}(v_1), \pi_{C_0}(v_3) - \pi_{C_0}(v_1))$ and
$\det(\pi_m(v_2) - \pi_m(v_1), \pi_m(v_3) - \pi_m(v_1))$ are non zero and have same sign. This gives (a). But properties (b) and(c) relie entirely on the signe of such determinants: since $\pi_{C_0}(V)$ is on a circle it is obviously in convex position and so is $\pi_m(V)$ which gives (b). Similarly the equality of determinant signs implies (c). □

9.6.4 Triangles in a same connected component CC_i of the dual graph

By Lemma 9.34, if we chose an orientation of plane Π_m this defines an orientation of circle C_0 and a cyclic corresponding cyclic order on he vertices $\pi_{C_0}(V)$ and $\pi_m(V)$ which From Lemma 9.34(b) we know that vertices in $\pi_m(V)$ are in convex position.

Denote by V_{CC_i} the set of vertices of triangles in CC_i. We call *convex boundary edge* of CC_i an edge connecting two vertices of V_{CC_i} which are successive in the cyclic order, equivalently an edge of $\mathcal{CH}(\pi_m(V))$, where $\mathcal{CH}(X)$ denotes the convex hull of X.

Lemma 9.35. *If an edge ab of a triangle in CC_i is shared by a triangle not in CC_i then ab is a convex boundary edge.*

Proof. If ab is shared by a triangle not in CC_i then, by definition of CC_i it is shared by a triangle not in CC. For a contradiction assume that there are vertices c and d on opposite side of ab in triangles of CC_i. By Lemma 9.34, since the vertices $\pi_m(V_{CC_i})$ are in convex position and $\mathcal{CH}(\pi_m(V_{CC_i} \setminus [\pi_m(a), \pi_m(b)]$ has two connected components containing respectively $\pi_m(c)$ and $\pi_m(d)$. A path γ in the union of closed triangle (convex hulls of) from c to d, such that $\gamma((0,1))$ remains in the interior of the union, of triangles in CC_i will project on Π_m on a path $\pi_m \circ \gamma$ that will cross the relative interior of $\pi_m(ab)$. But since there is a triangle not in CC that shares edge ab we get a contradiction with Lemma 9.31. □

Observes that since $\mathcal{T}$ is minimum, any 2-chain σ with vertices in V must be the minimum chain with boundary $\partial\sigma$. Indeed the restriction K_V of K to V contains all possible tetrahedra because V is contained in a ball of radius smaller than λ and has trivial homology. It follows that $\partial\sigma = \partial\sigma'$ if and only there is a 3-chain $\alpha \in C_3(K_V)$ such that $\sigma - \sigma' = \partial\alpha$.

Also one has a direct consequence of Lemma 9.35:

Lemma 9.36. *If σ is the chain corresponding to CC_i, the edges in $\partial\sigma$ are precisely the convex boundary edges of CC_i.*

Proof. If an edge is in $\partial\sigma$ is must be shared by a triangle not in CC_i and therefore , from Lemma 9.35 it is a convex boundary edge. The set of convex boundary edges is a topological circle. Since $\partial\sigma$ is a cycle (i.e. $\partial\partial\sigma = 0$) and $\partial\sigma$ cannot be 0 (if it was σ could be removed from $\mathcal{T}$ and contradict the minimality of $\mathcal{T}$) it must cover the whole circle. $\square$

Lemma 9.37. *(for $\frac{\epsilon}{\mathcal{R}}$ small enough), If ab is the longest edge of all triangles in CC_i it can not be shared by two folded triangles of CC_i.*

Proof. Let ab be the longest edge of all triangles in CC_i and assume for a contradiction that $abc, abd \in CC_i$ are folded along ab.Without loss of generality, we assume that a, c, d, b are ordered according to the cyclic order. By assumption we have $bc < ab$. We are in the situation of Lemma 9.33(c). From Lemma 9.32 we get that for $\frac{\epsilon}{\mathcal{R}}$ small enough $O\left((\frac{\epsilon}{\mathcal{R}})^{1/2}\left(\frac{\epsilon}{\eta}\right)^6 \epsilon\right)$ will be smaller than $\frac{1}{4}\frac{1}{256}\left(\frac{\eta}{\epsilon}\right)^3\epsilon$ and therefore we get by triangle inequality $dc, bd < bc$ and bcd is obtuse with longest edge bc. It results that $bcd <_\infty abd$. Similarly, one has $adc <_\infty abc$. But $(abc + abd) - (adc + bdc) = \partial abcd$ with $abcd \in K$ since $\lambda > 3\epsilon$ and $adc + bdc \sqsubseteq_\infty abc + abd$ contradicts the minimality of $\mathcal{T}$. $\square$

Lemma 9.38. *If bc is the not the longest edge of a triangle $abc \in CC_i$ and is not a convex boundary edge of V_{CC_i} , then the vertex $v \in V_{CC_i} \setminus \{b, c\}$ that minimizes abv for $\leq_\infty$ order is obtuse with largest edge bc and $\pi_{C_0}(v)$ is on the smallest of the two arcs of circle of $C_0 \setminus \{\pi_{C_0}(b), \pi_{C_0}(c)\}$,*

Proof. Takes $v_1 \in V_{CC_i} \setminus \{b, c\}$ such that $\pi_{C_0}(v_1)$ is on the smallest of the two arcs of circle of $C_0 \setminus \{\pi_{C_0}(b), \pi_{C_0}(c)\}$. Then $\pi_{C_0}(b)\pi_{C_0}(c)\pi_{C_0}(v_1)$ is obtuse with longest edge $\pi_{C_0}(b), \pi_{C_0}(c)$ and, according to Lemma 9.33(c):

$$\mathrm{R_B}\left(\pi_{C_0}(b)\pi_{C_0}(c)\pi_{C_0}(v_1)\right) = \frac{1}{2}\left\|\pi_{C_0}(b) - \pi_{C_0}(c)\right\| < R_0 - \frac{1}{8192}\left(\frac{\eta}{\epsilon}\right)^3\epsilon \tag{9.28}$$

where R_0 is the radius of C_0.

Consider now $v_2 \in V_{CC_i} \setminus \{b, c\}$ such that $\pi_{C_0}(v_2)$ is not on the smallest of the two arcs of circle of $C_0 \setminus \{\pi_{C_0}(b), \pi_{C_0}(c)\}$.

If $\pi_{C_0}(b)\pi_{C_0}(c)\pi_{C_0}(v_2)$ is acute then

$$\mathrm{R_B}\left(\pi_{C_0}(b)\pi_{C_0}(c)\pi_{C_0}(v_2)\right) = R_0 \tag{9.29}$$

We know from Lemma 9.32 that the distance between points in V_{CC_i} and their projection on C_0 is in $O\left((\frac{\epsilon}{\mathcal{R}})^{1/2}\left(\frac{\epsilon}{\eta}\right)^6 \epsilon\right)$ and therefore, for for $\frac{\epsilon}{\mathcal{R}}$ small enough, applying triangle inequality on (9.28) and (9.29) gives $\mathrm{R_B}(bcv_1) < \mathrm{R_B}(bcv_2)$ and therefore $bcv_1 \leq_\infty bcv_2$.

If $\pi_{C_0}(b)\pi_{C_0}(c)\pi_{C_0}(v_2)$ is obtuse then $\pi_{C_0}(b)\pi_{C_0}(c)$ cannot be its longest edge, since then $\pi_{C_0}(v_2)$ would be on the smallest of the two arcs of circle of $C_0 \setminus \{\pi_{C_0}(b), \pi_{C_0}(c)\}$. Assume now without loss of generality that $\pi_{C_0}(a)\pi_{C_0}(v_2)$ is the longest edge. Applying Lemma 9.33(b) we get again, for $\frac{\epsilon}{\mathcal{R}}$ small enough, $\mathrm{R_B}(bcv_1) < \mathrm{R_B}(bcv_2)$ and $bcv_1 \leq_\infty bcv_2$.

$\square$

Lemma 9.39. *Assume that for some $m \in \mathcal{M}$ and $C > 0$ one has $V \subset S_{m,\mathcal{R}/2,C\epsilon}$. The algorithm 6 below computes the chain σ with support CC_i. In the result of the algorithm no pair of triangles in CC_i overlap:*

$$\forall T, T' \in CC_i,\ T \neq T', \quad \pi_m(T)^\circ \cap \pi_m(T')^\circ = \emptyset$$

Of course the algorithm is purely conceptual, since one assume an oracle giving us the initialization, (which would be easy if one already had the solution).

proof of Lemma 9.39. Since there are no possible ambiguities from the context, we denote by $\sigma = CC_i$ both the set of triangles and the corresponding chain. Similarly $\tilde{\tau}$ designate both the

Let ab be the longest edge of CC_i and V_{CC_i} the set of vertices of triangles in CC_i.

$\tilde{\tau} \leftarrow \tau_{init}$ = triangles in CC_i sharing ab
while $(\partial\tilde{\tau} \nsubseteq \partial\sigma)$
    $cd \leftarrow$ longest edge in $\partial\tilde{\tau} \setminus \partial\sigma$
    $e \in V_{CC_i}$ such that $\forall v \in V_{CC_i}$, $cde \leq_\infty cdv$
    $\tilde{\tau} \leftarrow \tilde{\tau} + cde$

variable of algorithm 6 seen as a set of triangles and the corresponding chain. From Lemma 9.36 we know that $\partial\sigma$ is the set of convex boundary edges of V_{CC_i}.

Observes that from Lemma 9.37, if ab is a convex boundary edge it is shared by exactly one triangle in CC_i and if not it is shared by exactly two opposite triangles in CC_i. In the first case the initialization τ_{init} has cardinal $\sharp\tau_{init} = 1$ and in the second case to $\sharp\tau_{init} = 2$ Denote $\tau_0 = \tau_{init}$ and τ_i the value of variable $\tilde{\tau}$ at iteration i.

We want to prove that at each iteration $\tau_i \sqsubseteq_{lex} \sigma$ and that the algorithm stop at iteration m with $\tau_m \leq \sigma$ and $\partial\tau_m = \partial\sigma$ which gives $\tau_m = \sigma$ by minimality of σ.

We claim by induction that $\partial\tau_i$ is the set of convex boundary edge on the set V_{τ_i}. Indeed it is true at initialization by Lemma 9.37. At each iteration a new vertex is inserted in $\partial\tau_i$ to produce $\partial\tau_{i+1}$, and from Lemma 9.38 we get that it is inserted along the cyclic order defined by C_0. Also, at each step, the longest edge in $\partial\tau_i$ is replaced by two strictly shorter ones (Lemma 9.33(b) together with Lemma 9.32 shows that this is true for $\frac{\epsilon}{\mathcal{R}}$ small enough). Since, after initialization, triangles added at each step of the algorithm are obtuse by Lemma 9.38 with longest edge being the one from which they have been built, we get that triangles added along the execution of the algorithm are decreasing along $\leq_\infty$ ordering.

We claim by induction that we have $\tau_i \sqsubseteq_{lex} \sigma$. Indeed, at initialization one has $\tau_{init} \subset \sigma$ and therefore $\tau_{init} \sqsubseteq_{lex} \sigma$.

At step i, if $\tau_i \subset \sigma$, since the edge cd selected by the algorithm is in $\partial\tau_i \setminus \partial\sigma$ there must be at least one triangle $\hat{T}$ in $\sigma \setminus \tau_i$ with cd as boundary. Two things may happen. Either $\hat{T} = cde$ is the triangle picked by the algorithm and one get inductively $\tau_{i+1} \subset \sigma$ and obviously $\tau_{i+1} \sqsubseteq_{lex} \sigma$ in which case we reproduce the same situation. Either $\hat{T} \neq cde$ which, by the choice of vertex e means $cde <_\infty \hat{T}$ and since the triangles are added along the algorithm in decreasing order the chain τ_i will remain strictly smaller in lexicographic order than σ at each iteration.

If $\partial\tilde{\tau} \subseteq \partial\sigma$, the program stop and we claim that $\tilde{\tau} = \sigma$. Indeed, since $\partial\tilde{\tau} \subseteq \partial\sigma$ and from Lemma 9.36, edges in $\partial\tilde{\tau}$ are convex boundary edges and as in the proof of Lemma 9.36, $\partial\tilde{\tau}$ is a non zero cycle in topological circle and therefore $\partial\tilde{\tau} = \partial\sigma$. Since by induction hypothesis $\tilde{\tau} \sqsubseteq_{lex} \sigma$ and since σ is minimal under the constraint $\partial x = \partial\sigma$ we get $\tilde{\tau} = \sigma$.

The fact that no pair of triangles in $\tilde{\tau} = \sigma$ overlap follows Lemma 9.34(c) and the fact that at each step of the algorithm:

$$\pi_{C_0}(\cup\tilde{\tau}) = \pi_{C_0}\left(\bigcup_{T\in\tilde{\tau}} \pi_{C_0}(T)\right)$$

is a convex polygon with vertices on C_0 and at each step a triangle T is added to $\tilde{\tau}$ its projection interior $\pi_{C_0}(T)$ is disjoint with $\pi_{C_0}(\cup\tilde{\tau})$. $\square$

9.6.5 Triangles in a same connected component CC of G_δ but in separate connected components of the dual graph

We consider now the case of triangles in a same connected component CC of G_δ but in separate connected components CC_i, CC_j of the dual graph in CC.

Lemma 9.40. *For given $C > 0$ and for $\frac{\epsilon}{\mathcal{R}}$ small enough, let be CC is a connected component of G_δ and V the set of vertex of triangles in CC be such that for some $m \in \mathcal{M}$ one has $V \subset S_{m,\mathcal{R}/2,C\epsilon}$. If CC_i and CC_j are two distinct connected components of the dual graph in CC, for $T_1 \in CC_i$ and $T_2 \in CC_j$ one has:*

$$\pi_m(T_1)^\circ \cap \pi_m(T_2)^\circ = \emptyset$$

Proof. From Lemma 9.34(c) we know that for two triangles $T_1 \in CC_i$ and $T_2 \in CC_j$ one has

$$\pi_m(T_1)^\circ \cap \pi_m(T_2)^\circ = \emptyset \iff \pi_{C_0}(T_1)^\circ \cap \pi_{C_0}(T_2)^\circ = \emptyset$$

If $\pi_m(T_1)^\circ \cap \pi_m(T_2)^\circ \neq \emptyset$, We proceed as in the proof of Lemma 9.35. Since triangles in CC_i and CC_j do not share any edges and since both $\pi_m(\cup_{T\in CC_i} T)$ and $\pi_m(\cup_{T\in CC_j} T)$ are convex polygons (Lemma 9.34(b)), There must be an edge $\pi_m(ab)$ boundary of $\pi_m(\cup_{T\in CC_i} T)$ whose relative interior intersects the interior of $\pi_m(\cup_{T\in CC_j} T)$ or the reverse.

Indeed, if no boundary edge of $\pi_m(\cup_{T\in CC_i} T)$ has its relative interior intersecting the interior of $\pi_m(\cup_{T\in CC_j} T)$, then if p is a point in the relative interior of an edge $\pi_m(ab)$ boundary of $\pi_m(\cup_{T\in CC_i} T)$ it cannot be on the boundary of $\pi_m(\cup_{T\in CC_j} T)$ (we know that $\pi_m(\cup_{T\in CC_i} T)$ and $\pi_m(\cup_{T\in CC_j} T)$ have no boundary edges in common) and is therefore outside $\pi_m(\cup_{T\in CC_j} T)$. Then if $q \in \pi_m(T_1)^\circ \cap \pi_m(T_2)^\circ$ the segment $[pq]$ should cut the boundary of $\pi_m(\cup_{T\in CC_j} T)$. But since there is a triangle not in CC that shares edge ab we get a contradiction with Lemma 9.31. □

9.7 Proof of Theorem 9.1

As a consequence of Lemmas 9.31, 9.39 and 9.40 we get:

Lemma 9.41. *For given C and for $\frac{\epsilon}{\mathcal{R}}$ small enough, if $T_1, T_2 \in \mathcal{T}$ have vertices in $S_{m,\mathcal{R}/2,C\epsilon}$, then :*

$$\pi_m(T_1)^\circ \cap \pi_m(T_2)^\circ = \emptyset$$

From 9.17 we have $L < 4\epsilon$. So that taking $C > 4 \times 2.8$ in Lemma 9.41, with our sampling conditions and Lemmas 9.17 and 9.18 we can check that the conditions of Theorem 4 in [12] are met.

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Chapter 10

Appendices

A Proof of Proposition 6.13

Proof. Note that Γ_{reg} is unique under the assumed general position.

Since, in the regular triangulation, all $(n-1)$-simplices that are not on the boundary of $\mathcal{CH}(\mathbf{P})$ are shared by exactly two n-simplices, while only those in $\beta_{\mathbf{P}}$ have a single n-coface, we have:

$$\partial\Gamma_{reg} = \beta_{\mathbf{P}}$$

We claim now that :

$$\partial\Gamma = \beta_{\mathbf{P}} \Rightarrow \|\Gamma_{reg}\|_{(p)} \leq \|\Gamma\|_{(p)} \tag{10.1}$$

Indeed, (6.13) and (6.14) gives:

$$\|\Gamma\|_{(p)}^p = \sum_{\sigma\in\Gamma} \int_{|\sigma|} \delta_\sigma(x)^p dx = \int_{\mathcal{CH}(\mathbf{P})} \sum_{\substack{\sigma\in\Gamma \\ |\sigma|\ni x}} \delta_\sigma(x)^p dx$$

We get:

$$\|\Gamma\|_{(p)}^p = \int_{\mathcal{CH}(\mathbf{P})\setminus|K_{\mathbf{P}}^{n-1}|} \sum_{\substack{\sigma\in\Gamma \\ |\sigma|\ni x}} \delta_\sigma(x)^p dx \tag{10.2}$$

From the equivalence between regular triangulations and convex hull on lifted points we know that if $\sigma_{reg} \in \Gamma_{reg}$, then for any n-simplex in $\sigma \in K$:

$$x \in |\sigma| \cap |\sigma_{reg}| \Rightarrow \delta_{\sigma_{reg}}(x) \leq \delta_\sigma(x) \tag{10.3}$$

According to Lemma 10.1, in (10.2), there is an odd number of, and therefore at least one, simplex $\sigma \in \Gamma$ satisfying $|\sigma| \ni x$ in the condition on the sum. Therefore (10.3) gives:

$$x \in |\sigma_{reg}| \Rightarrow \delta_{\sigma_{reg}}(x)^p \leq \sum_{\substack{\sigma\in\Gamma \\ |\sigma|\ni x}} \delta_\sigma(x)^p \tag{10.4}$$

And, since, from definition of triangulation, for $x \in \mathcal{CH}(\mathbf{P}) \setminus |K_{\mathbf{P}}^{n-1}|$ there is exactly one simplex σ_{reg} such that $|\sigma_{reg}| \ni x$, (10.4) can be rewritten as:

$$\sum_{\substack{\sigma\in\Gamma_{reg} \\ |\sigma|\ni x}} \delta_\sigma(x)^p \leq \sum_{\substack{\sigma\in\Gamma \\ |\sigma|\ni x}} \delta_\sigma(x)^p \tag{10.5}$$

which, together with (10.2) gives the claim (10.1).

Now, if some n-simplex $\sigma \in \Gamma$ with $|\sigma| \ni x$, for some $x \in \mathcal{CH}(\mathbf{P}) \setminus |K_{\mathbf{P}}^{n-1}|$, is not Delaunay, then

$$\sum_{\substack{\sigma\in\Gamma_{reg} \\ |\sigma|\ni x}} \delta_\sigma(x)^p = \delta_{\sigma_{reg}}(x)^p < \sum_{|\sigma|\ni x} \delta_\sigma(x)^p$$

and since the function is continuous, this implies:

$$\|\Gamma\|_{(p)} > \|\Gamma_{reg}\|_{(p)}$$

$\square$

Lemma 10.1. *Given $\mathbf{P} = \{P_1, \dots, P_N\} \subset \mathbb{R}^n$, with $N \geq n+1$, in general position, denote by $\beta_{\mathbf{P}} \in C_{n-1}(K_{\mathbf{P}})$ the $(n-1)$-chain made of simplices belonging to the boundary of $\mathcal{CH}(\mathbf{P})$.*
Let $\Gamma \in C_n(K_{\mathbf{P}})$ be such that:

$$\partial\Gamma = \beta_{\mathbf{P}}$$

If $x \in \mathcal{CH}(\mathbf{P}) \setminus |K_{\mathbf{P}}^{n-1}|$ then there is an odd number of n-simplices $\sigma \in \Gamma$ such that $x \in |\sigma|$.

Proof. We claim that since $x \in \mathcal{CH}(\mathbf{P}) \setminus |K_{\mathbf{P}}^{n-1}|$ there is $x_\star \in \mathbb{R}^n \setminus \mathcal{CH}(\mathbf{P})$ such that:

$$[x_\star\, x] \cap |K_{\mathbf{P}}^{n-2}| = \varnothing \tag{10.6}$$

where $[x_\star\, x]$ denote the line segment in $\mathbb{R}^n$ between $x_\star$ and x.

Indeed, we consider moving a point x_t from x_0 to some x_1, picking x_0, x_1 far away enough to have $[x_0\, x_1] \cap \mathcal{CH}(\mathbf{P}) = \varnothing$ and in such a way that $[x_0\, x_1]$ belongs to none of the affine hyperplanes spanned by $(n-1)$-simplices in $K_{\mathbf{P}}$), which occurs generically. Then, the negation of condition (10.6) with $x_\star = x_t$, occurs only as isolated values of t. We can pick a value $t^\star$ for which is does not occur: set $x_\star = x_{t^\star}$ and the claim is proved. | Next, we navigate a point $y(t) = (1-t)x_\star + tx$ along segment $[x_\star\, x]$. This segment intersect transversally the $(n-1)$-faces $|\tau|$, for $\tau \in K_{\mathbf{P}}$. At each intersection point we can keep track of the change in the number of covering n-simplices, where, by *covering simplices* we name the n-simplices $\sigma \in \overline{\Gamma}$ such that $x \in |\sigma|$.

We know that this number is zero at $x_\star$ since $x_\star \notin \mathcal{CH}(\mathbf{P})$. Since $\mathcal{CH}(\mathbf{P}) \cap [x_\star\, x]$ is convex there is a single intersection point $y(t_b)$ between $[x_\star\, x]$ ans the boundary of $\mathcal{CH}(\mathbf{P})$.

This point $y(t_b) \in (x_\star, x)$ hits a $(n-1)$-simplex $\tau_b \in |\beta_{\mathbf{P}}|$, face of the convex hull boundary. Since $\partial\Gamma = \beta_{\mathbf{P}}$, we know that τ_b is shared by an odd number n_b of n-simplices in $|\Gamma|$. By definition of the convex hull, and since $\mathbf{P}$ is in general position, for each n-simplex σ coface of τ_b, $|\sigma|$ is on the inner side of the convex hull supporting half plane. It follows that the number of covering simplices become the odd number n_b just after the first crossing.

Then when crossing any other $(n-1)$-simplex $\tau_i \notin |\beta_{\mathbf{P}}|$, at some point $y(t_i)$, the condition $\partial\Gamma = \beta_{\mathbf{P}}$ requires the number n_i of n-simplices in $|\Gamma|$ coface of τ_i to be even. When crossing $|\tau_i|$, along $[x_\star\, x]$, point $y(t_i)$ exits k^-, and enters k^+ n-simplices in $|\Gamma|$, with $k^- + k^+ = n_i$. The current number of covering n-simplices value is incremented by $k^+ - k^-$. Since n_i is even and:

$$k^+ - k^- = k^+ + k^- - 2k^- = n_i - 2k^-$$

$k^+ - k^-$ is even and the number of covering simplices remains odd all along the path $[x_\star\, x]$. $\square$

B Proof of Lemma 7.3

We need a preliminary definition and lemma.

Definition 10.2 (Upper set measure D_ϕ)**.** *Consider a compact set $\mathcal{D} \subset \mathbb{R}^d$. For a continuous function $\phi : \mathcal{D} \to [0,1]$ we denote by D_ϕ the* upper set measure *of ϕ, the map $D_\phi : [0,1] \to \mathbb{R}^+$ defined as:*

$$D_\phi(t) =_{def} \mu_L\left(\{u \in \mathcal{D}, \phi(u) \geq 1-t\}\right)$$

where μ_L denotes the Lebesgues measure.

It follows from the definition that the upper set measure $t \mapsto D_\phi(t)$ is non decreasing.

For next Lemma, recall that, by Rademacher Theorem, Lipschitz function are differentiable almost everywhere.

Lemma 10.3. *Consider a compact set $\mathcal{D} \subset \mathbb{R}^d$ and a continuous function $\phi : \mathcal{D} \to [0,1]$. Assumes the upper set measure D_ϕ of ϕ is Lipschitz and denote by D'_ϕ its derivative defined a.e.. Then one has:*

$$\int_{\mathcal{D}} \phi(u)^p = \int_0^1 D'_\phi(t)(1-t)^p dt$$

Moreover if there is $\beta > 0$ such that D'_ϕ is defined and continuous on $(0, \beta)$ with:

$$\forall t \in (0, \beta), D'_\phi(t) > 0$$

Then, for any given $\alpha > 0$, (with $\alpha \leq 1$):

$$\lim_{p \to \infty} \frac{\int_0^\alpha D'_\phi(t)(1-t)^p dt}{\int_{\mathcal{D}} \phi(u)^p} = 1$$

Proof. Since, in the interval $[0,1]$, the derivative of the map $s \mapsto s^p$ is upper bounded by p, one has:

$$\forall a, b \in [0,1], |a^p - b^p| \leq p|b-a|$$

In particular, for a positive integer k:

$$\forall a \in [0,1], \left| a^p - \left(\frac{1}{k} \lfloor ka \rfloor \right)^p \right| \leq \frac{p}{k}$$

For a compact set $\mathcal{D} \subset \mathbb{R}^d$ and a continuous function $\phi : \mathcal{D} \to [0,1]$ one has therefore:

$$\left| \int_{\mathcal{D}} \phi(u)^p - \int_{\mathcal{D}} \left(\frac{1}{k} \lfloor k\phi(u) \rfloor \right)^p \right| \leq \frac{p}{k} \mu_L(\mathcal{D})$$

Since:

$$\int_{\mathcal{D}} \left(\frac{1}{k} \lfloor k\phi(u) \rfloor \right)^p = \sum_{i=0}^{k-1} \left(D_\phi\left(\frac{i+1}{k}\right) - D_\phi\left(\frac{i}{k}\right) \right) \left(1 - \frac{i}{k}\right)^p$$

Using the Riemann integrability of $t \mapsto p(1-t)^{p-1} D_\phi(t)$, we get:

$$\begin{aligned}
\int_{\mathcal{D}} \phi(u)^p &= \lim_{k \to \infty} \sum_{i=0}^{k-1} \left(D_\phi\left(\frac{i+1}{k}\right) - D_\phi\left(\frac{i}{k}\right) \right) \left(1 - \frac{i}{k}\right)^p \\
&= \lim_{k \to \infty} \sum_{i=1}^{k-1} \left(\left(1 - \frac{i-1}{k}\right)^p - \left(1 - \frac{i}{k}\right)^p \right) D_\phi\left(\frac{i}{k}\right) \\
&= \int_0^1 -D_\phi(t) p (1-t)^{p-1} dt \\
&= [-D_\phi(t)(1-t)^p]_0^1 + \int_0^1 D'_\phi(t)(1-t)^p dt \\
&= \int_0^1 D'_\phi(t)(1-t)^p dt
\end{aligned} \tag{10.7}$$

For the second part of the lemma, one has:

$$\frac{\int_{\mathcal{D}} \phi(u)^p}{\int_0^\alpha D'_\phi(t)(1-t)^p dt} = \frac{\int_0^1 D'_\phi(t)(1-t)^p dt}{\int_0^\alpha D'_\phi(t)(1-t)^p dt} = 1 + \frac{\int_\alpha^1 D'_\phi(t)(1-t)^p dt}{\int_0^\alpha D'_\phi(t)(1-t)^p dt}$$

Since D_ϕ is increasing and K-Lipschitz for some constant K, one has $D'_\phi(t) \in [0, K]$ whenever it is defined which gives:

$$\int_\alpha^1 D'_\phi(t)(1-t)^p dt \leq K(1-\alpha)^p$$

and:

$$\int_0^\alpha D'_\phi(t)(1-t)^p dt \geq \int_0^{\alpha/2} D'_\phi(t)(1-t)^p dt \geq A(1-\alpha/2)^p$$

with:

$$A = \int_0^{\alpha/2} D'_\phi(t)dt$$

Since there is $\beta > 0$ such that D'_ϕ is defined and continuous on $(0, \beta)$ with $\forall t \in (0, \beta), D'_\phi(t) > 0$ we have $A > 0$ and:

$$\left| \frac{\int_{\mathcal{D}} \phi(u)^p}{\int_0^\alpha D'_\phi(t)(1-t)^p dt} - 1 \right| \leq \frac{K(1-\alpha)^p}{A(1-\alpha/2)^p} = \frac{K}{A}\left(\frac{1-\alpha}{1-\alpha/2}\right)^p$$

□

Proof of Lemma 7.3. Applying a scale factor λ to a triangle abc would multiply both its area and the function $x \mapsto L_{x_0}(x) - (x-x_0)^2$ by a ratio λ^2. Therefore $w_p(abc)^p = \int_{abc} \left(L_{x_0}(x) - (x-x_0)^2\right)^p dx$ would by multiplied by λ^{2+2p}. Therefore, without loss of generality, and for the clarity of the equations, one assume in the sequel of the proof that:

$$\mathrm{R_B}(abc) = 1$$

Consider then $\sigma = abc$ with $\mathrm{R_B}(abc) = 1$. We introduce the coordinates system:

$$\begin{aligned} \xi_{abc} : \quad \Pi_{abc} \quad &\to \mathbb{R}^2 \\ q \quad &\mapsto (x_q, y_q) = \xi_{abc}(q) \end{aligned}$$

defined by an orthogonal and normed frame of the supporting plane Π_{abc} of abc centered at circumcenter of $\mathrm{C_C}(abc)$ and such that $x_a = x_b = h$ and $x_c > h$. In the case abc is not strictly acute, ab is assumed (w.l.o.g.) to be the longest edge of triangle abc. One consider the map: $\phi_\sigma : \mathbb{R}^2 \to \mathbb{R}$:

$$\phi_\sigma(x, y) = L_{(0,0)}((x, y)) - ((x, y) - (0, 0))^2 = \mathrm{R_C}(abc)^2 - x^2 - y^2$$

Motivated by:

$$w_p(\sigma)^p = \int_{|\sigma|} \phi_\sigma(x, y)^p dxdy$$

In this frame, if abc is strictly acute, then $\mathrm{R_C}(abc) = 1$, $h < 0$ and $(0, 0)$ is in the interior of abc and therefore there is $\alpha > 0$ small enough such that:

$$\forall t \in [0, \alpha], \phi_\sigma^{-1}([1-t, 1]) = \{(x, y), x^2 + y^2 \leq t\}$$

which gives for the upper set measure D_{ϕ_σ}:

$$\forall t \in [0, \alpha], D_{\phi_\sigma}(t) = \pi t$$

and then, if D'_{ϕ_σ} denotes the derivative of D_{ϕ_σ}:

$$\sigma \text{ strictly acute triangle} \Rightarrow \forall t \in [0, \alpha], D'_{\phi_\sigma}(t) = \pi \tag{10.8}$$

Now if abc is right, one has $\mathrm{R_C}(abc) = 1$ and $h = 0$. There is $\alpha > 0$ small enough such that:

$$\forall t \in [0, \alpha], \phi_\sigma^{-1}([1-t, 1]) = \{(x, y), x^2 + y^2 \leq t \text{ and } x \geq 0\}$$

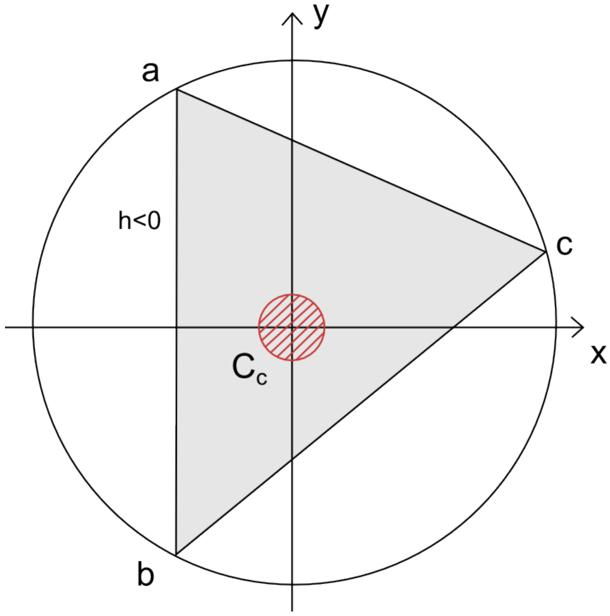

Figure 10.1: Acute triangle case

which gives for the upper set measure D_{ϕ_σ}:

$$\forall t \in [0, \alpha], D_{\phi_\sigma}(t) = \frac{\pi}{2} t$$

and then:

$$\sigma \text{ right triangle} \Rightarrow \forall t \in [0, \alpha], D'_{\phi_\sigma}(t) = \frac{\pi}{2} \tag{10.9}$$

In any case (acute, right or obtuse), the map D_{ϕ_σ} is Lipschitz with Lipschitz constant π since for $0 \leq t < t + u \leq 1$:

$$0 \leq D_{\phi_\sigma}(t+u) - D_{\phi_\sigma}(t) \leq \mu_L \left((x, y) \in \mathbb{R}^2, h^2 + t \leq x^2 + y^2 \leq h^2 + t + u \} \right) \leq \pi u$$

Therefore, Lemma 10.3 apply and one can estimate $w_p(\sigma)^p$.

If abc is acute, then 10.8 gives:

$$\lim_{p \to \infty} \frac{\int_0^\alpha \pi (1-t)^p dt}{\int_{\mathcal{D}} \phi(u)^p} = 1$$

Since:

$$\int_0^\alpha (1-t)^p dt = \frac{1 - (1-\alpha)^p}{p+1}$$

we get:

$$\sigma \text{ strictly acute triangle} \Rightarrow \lim_{p \to \infty} \frac{w_p(\sigma)^p}{\pi / p} = 1 \tag{10.10}$$

Similarly, using equation 10.9 we get:

$$\sigma \text{ right triangle} \Rightarrow \lim_{p \to \infty} \frac{w_p(\sigma)^p}{\pi / (2p)} = 1 \tag{10.11}$$

Now, if abc is strictly obtuse one has $h > 0$, $\mathrm{R_C}(abc) > 1$ and $h^2 = \mathrm{R_C}(abc)^2 - 1$. There is $\alpha > 0$ small enough such that :

$$\forall t \in [0, \alpha], \phi_\sigma^{-1}([1-t, 1]) = \{(x, y), x^2 + y^2 \leq h^2 + t \text{ and } x \geq h\}$$

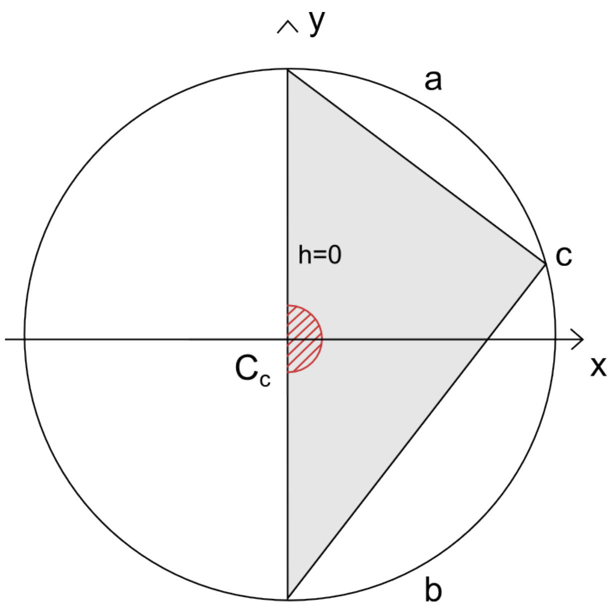

Figure 10.2: Right triangle case

which gives:

$$\forall t \in [0, \alpha], D_{\phi_\sigma}(t) = (h^2 + t)\left(\frac{\pi}{2} - \arcsin\sqrt{\frac{h^2}{h^2+t}}\right) - \sqrt{h^2 t}$$

and then:

$$\sigma \text{ strictly obuse triangle} \Rightarrow \forall t \in [0, \alpha], D'_{\phi_\sigma}(t) = \arccos\sqrt{\frac{h^2}{h^2+t}} \tag{10.12}$$

This gives:

$$\lim_{p\to\infty} \frac{\int_0^\alpha (1-t)^p \arccos\sqrt{\frac{h^2}{h^2+t}}dt}{w_p(\sigma)^p} = 1$$

Using a change of variable $u = \frac{\sqrt{t}}{h}$, $t = h^2u^2$, $dt = 2h^2u\,du$ and $\arccos\sqrt{\frac{h^2}{h^2+t}} = \arctan u$ we get:

$$\int_0^\alpha (1-t)^p \arccos\sqrt{\frac{h^2}{h^2+t}}dt = 2h^2\int_0^{\sqrt{\alpha}/h} (1-h^2u^2)^p u \arctan u\,du$$

using $\lim_{u\to 0}\frac{\arctan u}{u} = 1$ and applying an integration by part to compute $\int_0^{\sqrt{\alpha}/h}(1-h^2u^2)^p u^2\,du$ a short computation gives us:

$$\sigma \text{ strictly obtuse triangle} \Rightarrow \lim_{p\to\infty}\frac{w_p(\sigma)^p}{1/(p^2h^2)} = 1 \tag{10.13}$$

□

C Proof of Lemma 7.8

Proof of Lemma 7.8. Let $\mathcal{D}' \subset \mathcal{D}$ be the set of 2-chains in $\mathcal{D}$ that does not contain any pair of obtuse triangles sharing the same longest edge. Thanks to Lemma 7.7, one has:

$$\begin{aligned}\min_{\sqsubseteq_\infty}\mathcal{D} &= \min_{\sqsubseteq_\infty}\mathcal{D}'\\ \min_{\sqsubseteq_{lex}}\mathcal{D} &= \min_{\sqsubseteq_{lex}}\mathcal{D}'\end{aligned}$$

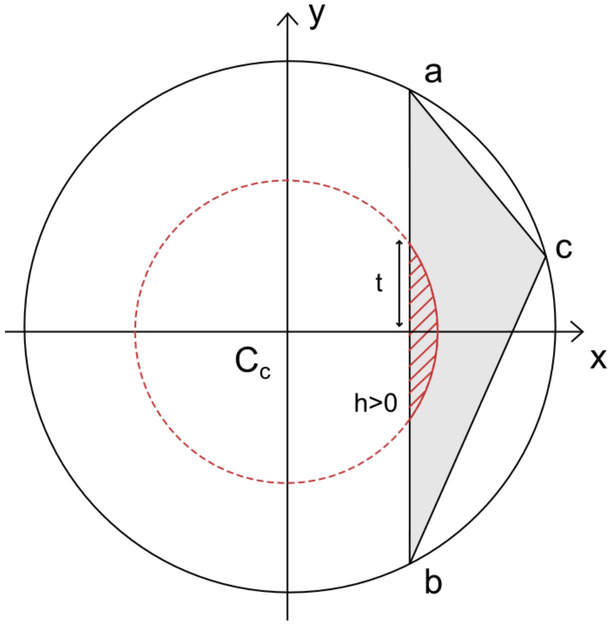

Figure 10.3: Obtuse triangle case

We claim now that the orders $\sqsubseteq_\infty$ and $\sqsubseteq_{lex}$ coincide when restricted to $\mathcal{D}'$. This would give $\min_{\sqsubseteq_\infty} \mathcal{D}' = \min_{\sqsubseteq_{lex}} \mathcal{D}'$ and end the proof.

In order to prove this claim, consider a chain $\Gamma \in \mathcal{D}'$. Under generic condition 7.1 and using the definition of $\mathcal{D}'$, no pair of simplices in Γ have the same bounding radius:

$$\forall \sigma_1, \sigma_2 \in \Gamma,\ \sigma_1 \neq \sigma_2 \iff \mathrm{R_B}(\sigma_1) \neq \mathrm{R_B}(\sigma_2)$$

It follows that the order $\leq_\infty$ coincide with the order on bounding radius:

$$\forall \sigma_1, \sigma_2 \in \Gamma,\ \sigma_1 \leq_\infty \sigma_2 \iff \mathrm{R_B}(\sigma_1) \leq \mathrm{R_B}(\sigma_2)$$

Now consider $\Gamma_1, \Gamma_2 \in \mathcal{D}'$ with $\Gamma_1 \neq \Gamma_2$ and $\Gamma_1 \sqsubseteq_{lex} \Gamma_2$.
According to definition 2.1, this means that:

$$\sigma_{\max} = \max_{\leq_\infty}\{\sigma \in \Gamma_1 + \Gamma_2\} \in \Gamma_2 \setminus \Gamma_1$$

If we denote by Γ_+ the chain made of all simplices in Γ_2 (or equivalently in Γ_1) greater than $\sigma_{\max}$ we can write:

$$\begin{aligned} \Gamma_1 &= \Gamma_+ + \Gamma_{1-} \\ \Gamma_2 &= \Gamma_+ + \Gamma_{2-} \end{aligned}$$

where all triangles in Γ_{1-} or Γ_{2-} are strictly smaller than any triangles in Γ_+.
According to Definition 7.2, one has:

$$\Gamma_1 \sqsubseteq_\infty \Gamma_2 \underset{\text{def.}}{\iff} \exists p \in [1, \infty), \forall p' \in [p, \infty), \|\Gamma_1\|_{p'} \leq \|\Gamma_2\|_{p'}$$

and for all $q \in [1, \infty)$:

$$\begin{aligned} \|\Gamma_1\|_q \leq \|\Gamma_2\|_q &\iff \|\Gamma_1\|_q^q \leq \|\Gamma_2\|_q^q \\ &\iff \sum_{\sigma \in \Gamma_1} w_q(\sigma)^q \leq \sum_{\sigma \in \Gamma_2} w_q(\sigma)^q \\ &\iff \sum_{\sigma \in \Gamma_{1-}} w_q(\sigma)^q \leq \sum_{\sigma \in \Gamma_{2-}} w_q(\sigma)^q \end{aligned} \tag{10.14}$$

If $\Gamma_{1-} = 0$ we have then trivially $\Gamma_1 \sqsubseteq_\infty \Gamma_2$.
Now if $\Gamma_{1-} \neq 0$ denote by $\sigma_{1-,\max}$ the maximal simplex in Γ_{1-} and by n_1 the number of simplices in Γ_{1-}.
Since $\sigma_{1-,\max} <_\infty \sigma_{\max}$ we must consider two possible situations:

1. $\mathrm{R_B}(\sigma_{1-,\max}) < \mathrm{R_B}(\sigma_{\max})$,
2. $\mathrm{R_B}(\sigma_{1-,\max}) = \mathrm{R_B}(\sigma_{\max})$ and $\mathrm{R_C}(\sigma_{1-,\max}) > \mathrm{R_C}(\sigma_{\max})$.

Using the notation of Lemma 7.3, we introduce

$$\omega_1(p) = \max_{\sigma\in\Gamma_{1-}} \omega_\sigma(p) \quad \text{with} \quad \lim_{p\to\infty} \omega_1(p) = 1 \tag{10.15}$$

Case 1. Assume for this case:

$$\mathbf{R_B(\boldsymbol{\sigma}_{1-,\max}) < R_B(\boldsymbol{\sigma}_{\max})}$$

At fixed $\mathrm{R_B}(\sigma_{1-,max})$, independently of the nature of the triangle, we can bound $w_p(\sigma_{1-,max})$ by the acute triangle expression for p large enough:

$$w_p(\sigma_{1-,max})^p \leq \frac{\pi}{p}\,\mathrm{R_B}(\sigma_{1-,max})^{2+2p}\omega_1(p)$$

We have therefore:

$$\sum_{\sigma\in\Gamma_{1-}} w_p(\sigma)^p \leq \frac{n_1\pi}{p}\,\mathrm{R_B}(\sigma_{1-,\max})^{2+2p}\omega_1(p) \tag{10.16}$$

We can as well lower bound $w_p(\sigma_{\max})^p$. There exists a constant C such that, for p large enough:

$$\frac{C}{p^2}\,\mathrm{R_B}(\sigma_{\max})^{2+2p}\omega_{\sigma_{\max}}(p) \leq w_p(\sigma_{\max})^p$$

We have therefore, for p large enough:

$$\frac{C}{p^2}\,\mathrm{R_B}(\sigma_{\max})^{2+2p}\omega_{\sigma_{\max}}(p) \leq \sum_{\sigma\in\Gamma_{2-}} w_p(\sigma)^p \tag{10.17}$$

Since $\mathrm{R_B}(\sigma_{1-,\max}) < \mathrm{R_B}(\sigma_{\max})$ and $\lim_{p\to\infty}\omega_{\sigma_{\max}}(p)/\omega_1(p) = 1$, there exist a p large enough to hold true Equations (10.16) and (10.17) , which gives us:

$$\exists p \in [1,\infty), \forall p' \in [p,\infty), \sum_{\sigma\in\Gamma_{1-}} w_{p'}(\sigma)^{p'} \leq \sum_{\sigma\in\Gamma_{2-}} w_{p'}(\sigma)^{p'}$$

and we get $\Gamma_1 \sqsubseteq_\infty \Gamma_2$ with (10.14).

Case 2. Assume for this case:

$$\mathbf{R_B(\boldsymbol{\sigma}_{1-,\max}) = R_B(\boldsymbol{\sigma}_{\max})} \text{ and } \mathbf{R_C(\boldsymbol{\sigma}_{1-,\max}) > R_C(\boldsymbol{\sigma}_{\max})}$$

Two generic triangle configurations can verify this case: both triangles are obtuse or σ_{max} is strictly acute and $\sigma_{1-,max}$ is obtuse. If both $\sigma_{1-,max}$ and σ_{max} are obtuse, we have:

$$h^2(\sigma_{1-,max}) \geq h^2(\sigma_{max})$$

with:
$$h^2(abc) = \frac{\mathrm{R_C}(abc)^2 - \mathrm{R_B}(abc)^2}{\mathrm{R_B}(abc)^2}$$

Therefore, using Lemma 7.3, $w_p(\sigma_{1-,max})^p \leq w_p(\sigma_{max})^p$. The same is true for σ_{max} strictly acute and $\sigma_{1-,max}$ obtuse, as for a p large enough:

$$\frac{1}{h^2}\,\frac{1}{p^2}\;\mathrm{R_B}(\sigma_{1-,max})^{2+2p}\,\omega_{\sigma_{1-,max}}(p) \leq \pi\,\frac{1}{p}\;\mathrm{R_B}(\sigma_{max})^{2+2p}\,\omega_{\sigma_{max}}(p)$$

In both cases,

$$\begin{aligned} \sum_{\sigma\in\Gamma_{1-}} w_p(\sigma)^p &\leq n_1 w_p(\sigma_{1-,max})^p \frac{\omega_1(p)}{\omega_{\sigma_{\max}}(p)} \\ w_p(\sigma_{max})^p &\leq \sum_{\sigma\in\Gamma_{2-}} w_p(\sigma)^p \end{aligned}$$

and therefore $\Gamma_1 \sqsubseteq_\infty \Gamma_2$ with (10.14).

We have shown that for $\Gamma_1, \Gamma_2 \in \mathcal{D}'$,

$$\Gamma_1 \sqsubseteq_{lex} \Gamma_2 \Rightarrow \Gamma_1 \sqsubseteq_\infty \Gamma_2$$

Since $\sqsubseteq_{lex}$ is a total order, if $\Gamma_1 \neq \Gamma_2$:

$$\neg\,(\Gamma_1 \sqsubseteq_{lex} \Gamma_2) \Rightarrow \Gamma_2 \sqsubseteq_{lex} \Gamma_1 \Rightarrow \Gamma_2 \sqsubseteq_\infty \Gamma_1 \Rightarrow \neg\,(\Gamma_1 \sqsubseteq_\infty \Gamma_2)$$

whose contraposition gives us the reverse inclusion:

$$\Gamma_1 \sqsubseteq_\infty \Gamma_2 \Rightarrow \Gamma_1 \sqsubseteq_{lex} \Gamma_2$$

□

D Proof of Lemmas of section 9.3

Proof of Lemma 9.15. First, we claim that the restriction to $\mathcal{M}\cap\mathcal{B}(m,\mathcal{R}/2)\cap\pi_m^{-1}(\mathcal{D}(m,\mathcal{R}/4))$ of the projection π_m is surjective onto $\mathcal{D}(m,\mathcal{R}/4)$.

Indeed consider a point q on the relative boundary of $\mathcal{M}\cap\mathcal{C}(m,\mathcal{R}/2,\mathcal{R}/4)$ in $\mathcal{M}$. q belongs to the boundary $\partial\,(\mathcal{C}(m,\mathcal{R}/2,\mathcal{R}/4))$ of $\mathcal{C}(m,\mathcal{R}/2,\mathcal{R}/4))$. But since $q \in \mathcal{B}(m,\mathcal{R}/2)$, we know from lemma 9.14 that:
$$d(q, T_m\mathcal{M}) \leq \mathcal{R}/8$$

Since:
$$\partial\mathcal{B}(m,\mathcal{R}/2)\cap\pi_m^{-1}(\mathcal{D}(m,\mathcal{R}/4)\cap T_m\mathcal{M}^{\oplus\mathcal{R}/8} = \varnothing$$

where $T_m\mathcal{M}^{\oplus\mathcal{R}/8}$ is the $\mathcal{R}/8$ offset of the tangent plane $T_m\mathcal{M}$. It follows that $q\in\partial\,\mathcal{C}(m,\mathcal{R}/2,\mathcal{R}/4)\cap T_m\mathcal{M}^{\oplus\mathcal{R}/8}$ cannot be in $\partial\mathcal{B}(m,\mathcal{R}/2)$, therefore one has:

$$\pi_m(q) \in \partial\mathcal{D}(m,\mathcal{R}/4)$$

Now if $q' \in \mathcal{M}\cap\mathcal{C}(m,\mathcal{R}/2,\mathcal{R}/4)$, Lemma 9.13 gives:

$$\sin\frac{\angle(T_{q'}\mathcal{M}, T_m\mathcal{M})}{2} \leq \frac{1}{4}$$

So that $\angle(T_{q'}\mathcal{M}, T_m\mathcal{M}) \leq 2\arcsin\frac{1}{4}$. This angle is always below $\pi/2$, which implies that the restriction to $\mathcal{M}\cap\mathcal{C}(m,\mathcal{R}/2,\mathcal{R}/4)$ of the projection π_m has a full rank derivative and is therefore an open map.

So that the boundary of $\pi_m(\mathcal{M}\cap\mathcal{C}(m,\mathcal{R}/2,\mathcal{R}/4))$ can only be on $\partial\mathcal{D}(m,\mathcal{R}/4))$. It follows that since m is in the image of $\mathcal{M}\cap\mathcal{C}(m,\mathcal{R}/2,\mathcal{R}/4)$ this image has to be the whole set $\mathcal{D}(m,\mathcal{R}/4)$. Put differently, the image is a relatively open and compact, non empty subset of the connected set $\mathcal{D}(m,\mathcal{R}/4)$: it must be the whole set $\mathcal{D}(m,\mathcal{R}/4)$ and the first claim is proven.

Second we claim that the restriction to $\mathcal{M}\cap\mathcal{C}(m,\mathcal{R}/2,\mathcal{R}/4)$ of the projection π_m is injective. Indeed assume for a contradiction that $q,q'\in\mathcal{M}\cap\mathcal{C}(m,\mathcal{R}/2,\mathcal{R}/4)$ with $\pi(q)=\pi(q')$. Thanks to Lemma 9.10, the reach of $\mathcal{M}\cap\mathcal{B}(m,\mathcal{R}/2)$ is not larger than $\mathcal{R}$. Thanks to Corollary 9.12, the shortest path between q and q' in $\mathcal{M}\cap\mathcal{B}(m,\mathcal{R}/2)$ remains in any ball of radius $r<\mathcal{R}$ containing q and q'. It follows that this shortest path γ remains in the spindle $\mathrm{SP}_r(q,q')$:

$$\gamma\subset\mathcal{M}\cap\mathrm{SP}_r(q,q') \tag{10.18}$$

However, in a neighborhood of q the intersection of the tangent cone (i.e. the set of tangent vectors as defined in Definition 4.3 in [31]) to $\mathcal{M}$ at q, i.e. $T_q\mathcal{M}$, and the tangent cone to $\mathrm{SP}_r(q,q')$ at q, i.e. a cone centered on the direction qq', orthogonal to $T_q\mathcal{M}$, with apex half-angle $\theta=\arcsin\frac{2r}{\|q'-q\|}<\pi/2$, contains only 0 so that q is an isolated point in $\mathcal{M}\cap\mathrm{SP}_r(q,q')$, a contradiction with (10.18).

Since the restriction to $\mathcal{M}\cap\mathcal{C}(m,\mathcal{R}/2,\mathcal{R}/4)$ of the projection π_m is a bijection, one can define the map ϕ of the lemma. Then Lemmas 9.13 and 9.14 give (9.10) and (9.11) after a short computation. ∎

Proof of Lemma 9.16. Let $o=\mathrm{C_C}(abc)$ be the circumcenter of abc.
We claim that there exists a point $o'\in\mathcal{M}$ such that:

$$\|o'-o\|\leq 5\,\mathrm{R_B}(abc)^2/\mathcal{R} \tag{10.19}$$

Indeed since $\mathrm{R_B}(abc)<\lambda<\mathcal{R}/10$, and

$$\|a-b\|,\|a-c\|\leq 2\,\mathrm{R_B}(abc)<\mathcal{R}/5$$

we have $b,c\in\mathcal{M}\cap\mathcal{C}(a,\mathcal{R}/2,\mathcal{R}/4)$ and therefore one can apply Lemma 9.15 for $m=a$.

Consider an orthogonal frame of $\mathbb{R}^n$ centered at a and aligned with Π_a, i.e. for which the two first basis vectors are in the linear space associated to Π_a. any x of $\mathbb{R}^n$ is the sum of a point in tangent plane Π_a and a vector in the $n-2$ dimensional linear space $\Pi_a^\perp$ orthogonal to Π_a. Denotes by $\mathfrak{F}_a:\mathbb{R}^2\times\mathbb{R}^{n-2}\to\mathbb{R}^n$ the map that assign to the $(u,v)\in\mathbb{R}^2\times\mathbb{R}^{n-2}=\mathbb{R}^n$ the corresponding point on the orthogonal frame.

One has $a=\mathfrak{F}_a(0,0)$ and there are $u_b,u_c\in\mathcal{D}(0,\mathcal{R}/2)$ such that $b=\mathfrak{F}_a(u_b,\phi(u_b))$ and $c=\mathfrak{F}_a(u_c,\phi(u_c))$ with $\|\phi(u_b)\|\leq\|u_b\|^2/\mathcal{R}\leq 4\,\mathrm{R_B}(abc)^2/\mathcal{R}$.
Similarly $\|\phi(u_c)\|\leq 4\,\mathrm{R_B}(abc)^2/\mathcal{R}$. Since abc is acute, o belongs to its convex hull: $o\in|abc|$ and therefore we have $o=\mathfrak{F}_a(u_o,v_o)$ with $u_o\in\mathcal{D}(0,\mathcal{R}/2)$ and $\|v_o\|\leq 4\,\mathrm{R_B}(abc)^2/\mathcal{R}$.
Take $o'=\mathfrak{F}_a(u_o,\phi(u_o))\in\mathcal{M}$. Since $\|u_o\|\leq\|a-o\|=\mathrm{R_C}(abc)=\mathrm{R_B}(abc)$, we have by Lemma 9.15 $\|\phi(u_o)\|<\mathrm{R_B}(abc)^2/\mathcal{R}$ and:

$$\|o'-o\|=\|\phi(u_o)-v_o\|\leq\|\phi(u_o)\|+\|v_o\|\leq 5\,\mathrm{R_B}(abc)^2/\mathcal{R}$$

which proves the claim.

Since, thanks to lemma 9.3, $\mathcal{B}(o,\mathrm{R_B}(abc))^\circ\cap S=\emptyset$, we have

$$\mathcal{B}\left(o',\mathrm{R_B}(abc)-5\,\mathrm{R_B}(abc)^2/\mathcal{R}\right)^\circ\cap S=\emptyset$$

And since $\mathrm{R_B}(abc) < \lambda < \mathcal{R}/10$, one has $5\,\mathrm{R_B}(abc)^2/\mathcal{R} < \frac{1}{2}\,\mathrm{R_B}(abc)$ which gives

$$\mathcal{B}\left(o', \frac{1}{2}\,\mathrm{R_B}(abc)\right) \cap S = \emptyset$$

But from the sampling condition we know that $\mathcal{B}(o', \epsilon) \cap S \neq \emptyset$ and therefore $\frac{1}{2}\,\mathrm{R_B}(abc) < \epsilon$. □

Proof of Lemma 9.18. A triangle abc is obtuse if and only, when inscribed in its circumcircle, it does not cover the centre of the circle. It follows that abc is obtuse if and only if $h_{\min}(abc) < \mathrm{R_C}(abc)$. If abc is acute one has then: $h_{\min}(abc) \geq \mathrm{R_C}(abc) \geq \frac{1}{\sqrt{3}}\eta$

If abc is obtuse with longest edge ab, its smallest height is the height above ab. Considers triangle abc inscribed in its circumcircle with radius $R = \mathrm{R_C}(abc)$ and center o, with angle $\angle aob$ positive. Taking: $\theta = \angle\frac{aob}{2}$ One has:

$$h_{\min}(abc) = R\,(1 - \cos\theta) \geq R\frac{\theta^2}{4} \tag{10.20}$$

and:

$$\theta = \angle aoc + \angle cob$$

with

$$\begin{aligned}
\angle aoc &= 2\arcsin\frac{ac}{2} \geq 2\arcsin\frac{\eta}{2R} \geq \frac{\eta}{R} \\
\angle cob &= 2\arcsin\frac{cb}{2} \geq 2\arcsin\frac{\eta}{2R} \geq \frac{\eta}{R}
\end{aligned}$$

This makes:

$$\theta \geq 2\frac{\eta}{R}$$

and with (10.20):

$$h_{\min}(abc) \geq R\left(\frac{\eta}{R}\right)^2 \tag{10.21}$$

Since by Lemma 9.17 $R = \mathrm{R_C}(abc) < 2\epsilon$ we get the lower bound for $h_{\min}(abc)$. For the smallest angle, using the well known equality for general triangles:

$$\frac{\sin\angle cab}{\|b-c\|} = \frac{\sin\angle abc}{\|c-a\|} = \frac{\sin\angle bca}{\|a-b\|}$$

Since any edge length lower bonded by η and upper bounded by 2ϵ, one has that the ratio between the smallest sin and the largest one is bounded by $\frac{2\epsilon}{\eta}$ and since the largest angle cannot be smaller than $\pi/3$, we get that the smallest angle is lower bounded by:

$$\sin\theta_{\min} > \frac{\eta}{2\epsilon}\sin(\pi/3) > \frac{\eta}{4\epsilon}$$

□

Proof of Lemma 9.20. (9.14) is direct consequence of Lemma 9.14 with $\|p - q\| \leq C\epsilon$ (we loose a factor 2 but we don't care). Similarly, (9.16) is a direct consequence of Lemma 9.19.

For (9.15), the rotation angle θ between $\Pi(T)$ and Π_m may increases the distance to the plane by $C\epsilon\sin\theta$. since $\sin\theta \leq \theta \leq \beta'$ one gets $\beta_2 < \beta_1 + C\epsilon\beta'$.

For (9.17), one has:

$$\angle\,\Pi(abc), \Pi(T) \leq \angle\,\Pi(abc), \Pi_a + \angle\,\Pi_a, \Pi_m + \angle\,\Pi_m, \Pi(T) \leq 2\beta' + \angle\,\Pi_a, \Pi_m$$

And by Lemma 9.13, one has $\angle\,\Pi_a, \Pi_m \leq \frac{C\epsilon}{2\mathcal{R}} < \beta'$

□

E Proof of Lemma 9.23

The proof of Lemma 9.23 proceeds by induction, as in the proof of Lemma 9.17, along the directed acyclic graph given by Corollary 9.8.

Before giving the proof we need some preliminary lemmas.

Lemma 10.4. *Let $abc \in \mathcal{T}$ be an obtuse triangle with longest edge ab. If $o = \mathrm{C_C}(T)$ and $o' = a + b - o$ is the point symmetric to o with respect to $(a+b)/2$, then have:*

$$\forall s \in S, \max\left(\mathrm{P}_{\mathcal{B}(o,\mathrm{R_C}(T))}(s), \mathrm{P}_{\mathcal{B}(o',\mathrm{R_C}(T))}(s)\right) \geq -\alpha_{Sp}$$

with:

$$\alpha_{Sp} = O\left(\left(\frac{\epsilon}{\mathcal{R}}\right)^2 \left(\frac{\epsilon}{\eta}\right)^{10} \epsilon^2\right)$$

Proof. In the condition of the theorem, Consider the hyperplane H orthogonal to ab going through s. Let us denote $m = (a+b)/2$ and $h = \|a-b\|/2$ the center and radius of the open ball $\mathcal{B}^\circ(m,h)$ with diameter ab and by $R = \mathrm{R_C}(abc)$ the circumradius of triangle abc. Since

$$\mathcal{B}(o,R)^\circ \cap \mathcal{B}(o',R)^\circ \subset \mathcal{B}(m,h)^\circ$$

We get:

$$s \notin \mathcal{B}(m,h)^\circ \Rightarrow \max\left(\mathrm{P}_{\mathcal{B}(o,R)}(s), \mathrm{P}_{\mathcal{B}(o',R)}(s)\right) \geq 0$$

Therefore the theorem holds whenever $s \notin S \cap \mathcal{B}(m,h)^\circ$. We assume now that $s \in S \cap \mathcal{B}(m,h)^\circ$.

We denote o_H the orthogonal projection of o on H, s_{ab} the orthogonal projection of s on line ab, and s_{abc} the orthogonal projection of s on plane $\Pi(abc)$, Observe that, since $s \in \mathcal{B}(m,h)^\circ$, s_{ab} belongs to the open segment (a,b). From Lemma 9.5, we have:

$$\|s - s_{ab}\| \geq \sqrt{R^2 - (s_{ab} - m)^2} - \sqrt{R^2 - h^2} \tag{10.22}$$

Also, since $a, b, s \in S$, one has from the sampling conditions, $\min(\|a-s\|, \|b-s\|) \geq \eta$. Since for $\epsilon/\mathcal{R}$ small enough one has $2\epsilon > R \geq h$ and $s \in \mathcal{B}(m,h)^\circ$ we get, as a rough estimate, $\min(\|a - s_{ab}\|, \|b - s_{ab}\|) \geq 1/8(\eta/\epsilon)^2\epsilon$ and therefore:

$$h - \|s_{ab} - m\| \geq 1/8(\eta/\epsilon)^2\epsilon$$

From this and (10.22) we get:

$$\|s - s_{ab}\| \geq \frac{h}{2R}\frac{1}{8}(\eta/\epsilon)^2\epsilon \geq \frac{1}{64}(\eta/\epsilon)^3\epsilon \tag{10.23}$$

Since s_{ab} and s_{abc} are in plane $\Pi(abc)$ and the vector $(s - s_{abc})$ is orthogonal to it, one has:

$$(s_{ab} - s_{abc})^2 + (s - s_{abc})^2 = (s - s_{ab})^2 \tag{10.24}$$

From Lemma 9.20 we have also $s \in \Pi(T)^{\oplus\beta}$, with $\beta = O\left(\frac{\epsilon}{\mathcal{R}}\left(\frac{\epsilon}{\eta}\right)^2\epsilon\right)$ and therefore

$$(s - s_{abc})^2 \leq \beta^2 = O\left(\left(\frac{\epsilon}{\mathcal{R}}\right)^2 \left(\frac{\epsilon}{\eta}\right)^4 \epsilon^2\right) \tag{10.25}$$

This with (10.23) and (10.24) gives, for $\epsilon/\mathcal{R}$ small enough:

$$(s_{ab} - s_{abc})^2 > \frac{1}{128}(\eta/\epsilon)^3\epsilon \tag{10.26}$$

Since abc is obtuse with longest edge ab, one has $o \neq m$ and then $s_{ab} - o_H = m - o \neq 0$. Since s_{ab},s_{abc} and o_H all lie on the same line $H \cap \Pi(abc)$ and since, from (10.26) $s_{ab} - s_{abc} \neq 0$ we get:

$$\langle s_{ab} - o_H, s_{abc} - s_{ab} \rangle \neq 0$$

If $\langle s_{ab} - o_H, s_{abc} - s_{ab} \rangle > 0$ we prove that $\mathrm{P}_{\mathcal{B}(o,R)}(s) \geq -\alpha$. The proof that $\mathrm{P}_{\mathcal{B}(o',R)}(s) \geq -\alpha$ when $\langle s_{ab} - o_H, s_{abc} - s_{ab} \rangle < 0$ goes similarly. So assume now that $\langle s_{ab} - o_H, s_{abc} - s_{ab} \rangle > 0$.

In this case, one has:

$$\|s_{abc} - o_H\| = \|s_{abc} - s_{ab}\| + \|s_{ab} - o_H\|$$

This with $(s-o)^2 = (s - s_{abc})^2 + (s_{abc} - o_H)^2 + (o_H - o)^2$ gives:

$$(s-o)^2 = (s - s_{abc})^2 + (\|s_{abc} - s_{ab}\| + \|s_{ab} - o_H\|)^2 + (o_H - o)^2$$

And in particular we have:

$$(s-o)^2 \geq (\|s_{abc} - s_{ab}\| + \|s_{ab} - o_H\|)^2 + (o_H - o)^2 \tag{10.27}$$

Also $(s - s_{ab})^2 = (s - s_{abc})^2 + (s_{abc} - s_{ab})^2$ and then:

$$\|s_{abc} - s_{ab}\| = \|s - s_{ab}\| \sqrt{1 - \frac{(s - s_{abc})^2}{(s - s_{ab})^2}} \geq \|s - s_{ab}\| \left(1 - \frac{(s - s_{abc})^2}{(s - s_{ab})^2}\right)$$

and from (10.25) and (10.23) we get:

$$\frac{(s - s_{abc})^2}{(s - s_{ab})^2} = O\left(\left(\frac{\epsilon}{\eta}\right)^{10} \left(\frac{\epsilon}{\mathcal{R}}\right)^2\right)$$

Therefore $\|s_{abc} - s_{ab}\| \geq \|s - s_{ab}\| - O\left(\left(\frac{\epsilon}{\eta}\right)^{10} \left(\frac{\epsilon}{\mathcal{R}}\right)^2 \epsilon\right)$ and equation (10.27) gives:

$$(s-o)^2 \geq (\|s - s_{ab}\| + \|s_{ab} - o_H\|)^2 + (o_H - o)^2 - O\left(\left(\frac{\epsilon}{\eta}\right)^{10} \left(\frac{\epsilon}{\mathcal{R}}\right)^2 \epsilon^2\right)$$

Using equation (10.22) and the fact that $\|s_{ab} - o_H\| = \|m - o\| = \sqrt{R^2 - h^2}$ this gives:

$$(s-o)^2 \geq R^2 - (s_{ab} - m)^2 + (o_H - o)^2 - O\left(\left(\frac{\epsilon}{\eta}\right)^{10} \left(\frac{\epsilon}{\mathcal{R}}\right)^2 \epsilon^2\right)$$

But since $s_{ab} - m = o_H - o$ we get:

$$\mathrm{P}_{\mathcal{B}(o,R)}(s) = (s-o)^2 - R^2 \geq -O\left(\left(\frac{\epsilon}{\eta}\right)^{10} \left(\frac{\epsilon}{\mathcal{R}}\right)^2 \epsilon^2\right) = -\alpha$$

And this concludes the proof of the lemma. □

Definition 10.5 (Unfolded adjacent triangles)**.** *Let $abc, abd \in \mathcal{T}$ be adjacent triangles in $\mathcal{T}$. The circle centered at $(a+b)/2$, orthogonal to ab in the 3-affine space spanned by $\{a,b,c,d\}$ and going through d cuts the plane $\Pi(abc)$ at two points. Since the angle between $\pi(abc)$ and $\Pi(abd)$ is small for ϵ/R small enough (Lemma 9.20), one of these two ponts is strictly closer than the other from d. It is called the* unfolded point of d in plane $\Pi(abc)$ along ab. *If d' is the unfolded point of d in plane $\Pi(abc)$ along ab, triangle abd' is called the* unfolded triangle of abd in plane $\Pi(abc)$ along ab.

Lemma 10.6. *For $abc, abd \in \mathcal{T}$, if abd is α-almost empty and abd' is the unfolded triangle of abd in plane $\Pi(abc)$ along ab then abd' is $(\alpha+\alpha')$-almost empty, with:*

$$\alpha' = O\left(\left(\frac{\epsilon}{\mathcal{R}}\right)^2 \left(\frac{\epsilon}{\eta}\right)^4 \epsilon^2\right)$$

Proof. Denote by $m = (a+b)/2$ the middle of segment ab. Denote by $\overrightarrow{d_{ab}} \in \mathbb{R}^n$ the unit vector in direction $\overrightarrow{ab}$ and by $\overrightarrow{e_c} \in \mathbb{R}^n$ the unit vector in plane $\Pi(abc)$ orthogonal to ab and such that $\langle c-a, e_c\rangle > 0$. Vector $\overrightarrow{n}$ is in the linear space spanned by $\{a,b,c,d\}$ and orthogonal to plane abc. More explicitly takes:

$$\begin{array}{rcl} \overrightarrow{d_{ab}} & = & (b-a)/\|b-a\| \\ \overrightarrow{e_c} & = & (c-a) - \langle c-a, \overrightarrow{d_{ab}}\rangle \overrightarrow{d_{ab}} \\ \overrightarrow{n} & = & (d-a) - \langle d-a, \overrightarrow{d_{ab}}\rangle \overrightarrow{d_{ab}} - \langle d-a, \overrightarrow{e_c}\rangle \overrightarrow{e_c} \end{array}$$

$(m, \overrightarrow{e_c}, \overrightarrow{n})$ is a frame for the 2-plane in linear space spanned by $\{a,b,c,d\}$ and orthogonal to ab.

For a point $t \in R^n$, we denote its "coordinates" $(x_t, z_t, Y_t) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^{d-2}$. Where $x_t = \langle t-m, \overrightarrow{e_n}\rangle$, $z_t = \langle t-m, \overrightarrow{n}\rangle$, and the vector $Y_t \in \mathbb{R}^{d-2}$ contains the coordinates in some orthogonal frame of the $d-2$ linear space orthogonal to $\overrightarrow{e_c}$ and $\overrightarrow{n}$ of the component of vector $t-o$ in this linear space: $t-o-\langle t-o, \overrightarrow{e_c}\rangle\overrightarrow{e_c} - \langle t-o, \overrightarrow{n}\rangle\overrightarrow{n}$. The capital Y of the third coordinate Y_t are there to emphasizes the fact that it represents $d-2$ scalar coordinates.

Denote by $\omega = \mathrm{C_C}(abd)$ and $\omega' = \mathrm{C_C}(abd')$ the respective circumcenters of abd and abd'. Take $s \in S \cap \mathcal{B}(v, C\epsilon)$.

$$(s-\omega)^2 - (s-\omega')^2 = \langle (s-\omega)+(s-\omega'), (s-\omega)-(s-\omega')\rangle = \langle \omega'-\omega, 2s-\omega'-\omega\rangle$$

Denote by θ the angle between $\pi(abc)$ and $\Pi(abd)$.

One has: $x_\omega = \|\omega-m\|\cos\theta$, $z_\omega = \|\omega-m\|\sin\theta$ and $x_{\omega'} = \|\omega-m\|$, $z_{\omega'} = 0$, and $Y_\omega = Y_{\omega'} = 0$. We get:

$$\begin{array}{rcl} (s-\omega)^2 - (s-\omega')^2 & = & \langle (s-\omega)+(s-\omega'), (s-\omega)-(s-\omega')\rangle \\ & = & \langle \omega'-\omega, 2s-\omega'-\omega\rangle \\ & = & (\|\omega-m\|(1-\cos\theta))\,(2x_s - \|\omega-m\|(1+\cos\theta)) \\ & + & (-\|\omega-m\|\sin\theta)\,(2z_s - \|\omega-m\|\sin\theta) \end{array}$$

According to Lemma 9.20 one has

$$\theta \le O\left(\frac{\epsilon}{\mathcal{R}}\left(\frac{\epsilon}{\eta}\right)^2\right)$$

, This gives $\sin\theta \le O\left(\frac{\epsilon}{\mathcal{R}}\left(\frac{\epsilon}{\eta}\right)^2\right)$, $1-\cos\theta \le O\left(\left(\frac{\epsilon}{\mathcal{R}}\right)^2\left(\frac{\epsilon}{\eta}\right)^4\right)$.

$$|z_s| \le O\left(\frac{\epsilon}{\mathcal{R}}\left(\frac{\epsilon}{\eta}\right)^2\epsilon\right)$$

and $\|\omega-m\|, |x_s| \le O(\epsilon)$.

We get as required:

$$\begin{array}{rcl} |\mathrm{P}_{abd'}(s) - \mathrm{P}_{abd}(s)| & = & \left|(s-\omega)^2 - (s-\omega')^2\right| \\ & \le & \alpha' = \mathrm{C}\left(\left(\frac{\epsilon}{\mathcal{R}}\right)^2\left(\frac{\epsilon}{\eta}\right)^4\epsilon^2\right) \end{array}$$

□

Proof of Lemma 9.23. If $T \in \mathcal{T}$ is acute, we have seen, from Lemma 9.3 that T is 0-almost empty. If $T = abc$ is obtuse with longest edge ab, abc has a successor abd in the sequence defined in Corollary 9.8. Denote by $m(abc)$ the length of the sequence defined in Corollary 9.8 from triangle abc to the last, and therefore acute, triangle in the sequence. One has $m(abc) = m(abd) + 1$ and, according to Lemma 9.9 one has $m(abc) \leq O\left((\epsilon/\eta)^2)\right)$.

We proceed by a finite induction along the sequence. If abd is successor of abc, we will show that

$$\forall s \in S, \mathrm{P}_{abc}(s) \geq -\max\left(\alpha_{Sp}, \alpha(abd')\right)$$

Where $\alpha_{Sp} \leq O\left(\left(\frac{\epsilon}{\mathcal{R}}\right)^2 \left(\frac{\epsilon}{\eta}\right)^{10} \epsilon^2\right)$ is given by Lemma 10.4 And

$$\alpha(abd') = \alpha(abd) + \alpha'$$

with $\alpha' = O\left(\left(\frac{\epsilon}{\mathcal{R}}\right)^2 \left(\frac{\epsilon}{\eta}\right)^4 \epsilon^2\right)$ given by Lemma .10.6.

Since the last triangle T_{last} in the sequence, i.e. $m(T_{\text{last}}) = 0$, is acute, and therefore 0-almost empty.

Our induction hypothesis is that $\alpha(abd)$-almost empty with:

$$\alpha(abd) = O\left(\left(\frac{\epsilon}{\mathcal{R}}\right)^2 \left(\frac{\epsilon}{\eta}\right)^{10} \epsilon^2\right)$$

This is indeed correct for the acute triangle T_{last}, last of the sequence.

Considers $\overrightarrow{e_c} \in \mathbb{R}^n$ a vector in plane $\Pi(abc)$, orthogonal to ab in direction c (since $abc \in \mathcal{T}$ it is not degenerate). More explicitly takes:

$$\begin{aligned} \overrightarrow{d_{ab}} &= (b-a)/\|b-a\| \\ \overrightarrow{e_c} &= (c-a) - \langle (c-a), \overrightarrow{d_{ab}} \rangle \overrightarrow{d_{ab}} \end{aligned}$$

Consider the closed half space $H+$ and $H-$ defined by:

$$\begin{aligned} H^+ &= \{x \in \mathbb{R}^n, \langle x-a, \overrightarrow{e_c} \rangle \geq 0\} \\ H^- &= \{x \in \mathbb{R}^n, \langle x-a, \overrightarrow{e_c} \rangle \leq 0\} \end{aligned}$$

Since $H^+ \cup H^- = \mathbb{R}^n$ we split he proof in proving successively that

$$\forall s \in S \cap H^+, \mathrm{P}_{abc}(s) \geq -\alpha_{Sp}$$

and

$$\forall s \in S \cap H^-, \mathrm{P}_{abc}(s) \geq -(\alpha(abd) + \alpha')$$

In fact, in the computation below one has to consider particular cases, when abd is acute and abc nearly co-circular with abd', that follows induction proof.

Consider $o' = a + b - o$ as in Lemma 10.4. Since the hyper-plane $H^+ \cap H^-$ is the radical plane of $\mathcal{B}(o, \mathrm{R_C}(T))$ and $\mathcal{B}(o', \mathrm{R_C}(T))$, and since abc is obtuse with longest edge ab, one has $o \in H^- \setminus H^+$ and $o' \in H^+ \setminus H^-$.

It follows that we have $\forall s \in H^+, \mathrm{P}_{\mathcal{B}(o', \mathrm{R_C}(T))}(s) < \mathrm{P}_{\mathcal{B}(o, \mathrm{R_C}(T))}(s) = \mathrm{P}_{abc}(s)$. This together with Lemma 10.4 gives:

$$\forall s \in S \cap H^+, \mathrm{P}_{abc}(s) \geq -\alpha_{Sp}$$

Now for $s \in S \cap H^-$ we uses the induction hypothesis that abd is $\alpha(abd)$-almost empty and, from Lemma 10.6 we have that, if d' is the unfolded of d in plane $\Pi(abc)$ along ab, abd' is α''-almost empty for:

$$\alpha'' = \alpha(abd) + \alpha' \leq O\left(\left(\frac{\epsilon}{\mathcal{R}}\right)^2 \left(\frac{\epsilon}{\eta}\right)^{10} \epsilon^2\right)$$

Denote by $\omega = \mathrm{C_C}\left(abd'\right)$ and $\rho = \mathrm{R_C}(abd') = \mathrm{R_C}(abd)$ respectively the center and radius of the circumscribing circle of abd'. α''-almost empty writes as:

$$\forall s \in S, (s-\omega)^2 - \rho^2 \geq \alpha'' \tag{10.28}$$

Denote by $m = (a+b)/2$ and $h = ab/2$ respectively the middle of ab and its half length. Since abc is obtuse with longest edge ab we have:

$$(m-c)^2 < h^2 \tag{10.29}$$

Denote by $o = \mathrm{C_C}\left(abc\right)$ and $R = \mathrm{R_C}(abc)$ the respectively the center and radius of the circumscribing circle of abc. One has (abc obtuse and Lemma 9.7):

$$\rho > R \geq h \tag{10.30}$$

In plane $\Pi(abc)$ we consider the orthonormal frame $(m, \overrightarrow{e_c}, \overrightarrow{d_{ab}})$ and denote by $(x_o, y_o) = (x_o, 0)$, (x_c, y_c) and $(x_\omega, y_\omega) = (x_\omega, 0)$ the respective coordinates of o, c and ω in this frame.

Observes that the radical hyper-plane of spheres $\mathcal{B}(o, R)$ and $\mathcal{B}(\omega, \rho)$ is orthogonal to the line $o\omega$ and goes through m. Then, if $x_\omega < x_o$ one has

$$\forall s \in S \cap H^-, \mathrm{P}_{abc}(s) = \mathrm{P}_{\mathcal{B}(o,R)}(s) \geq \mathrm{P}_{\mathcal{B}(\omega,\rho)}(s) \geq -\alpha''$$

So that we have proven the induction when $x_\omega < x_o$.

We examine now the case when $x_\omega \geq x_o$. As we see below this may arise only if abd is acute. Since abc is obtuse with longest edge ab we have $x_o < 0$. Since $c \in S$, (10.28) gives:

$$(x_c - x_\omega)^2 + y_c^2 - \rho^2 \geq -\alpha''$$

that is:

$$x_c^2 + y_c^2 \geq \rho^2 + 2x_c x_\omega - \alpha''$$

And with equation (10.29) this gives:

$$h^2 > \rho^2 + 2x_c x_\omega - \alpha''$$

and with equation (10.30) we get:

$$2x_c x_\omega < \alpha''$$

and since $x_c > 0$ and, from the lower bound on triangles heights $|x_c| > \frac{\eta^2}{8\epsilon}$, one has $x_c > \frac{\eta^2}{8\epsilon}$:

$$x_\omega < \frac{8\epsilon}{\eta^2}\alpha'' \tag{10.31}$$

On another hand, from (10.30) we have $\rho^2 = x_\omega^2 + h^2 > x_o^2 + h^2 = R^2$ which gives $x_\omega^2 > x_o^2$ and, since $x_o < 0$ and $x_\omega > x_o$ one has:

$$0 \leq -x_o < x_\omega < \frac{8\epsilon}{\eta^2}\alpha'' \tag{10.32}$$

We claim that for $\epsilon/\mathcal{R}$ small enough inequations (10.32) can not hold when abd is obtuse: indeed, if abd is obtuse, its longest edge is ad or bd (Lemma 9.7). W.l.o.g. assume it is ad. Since the angle $\angle abd > \pi/2$, we have $ad^2 > ab^2 + bd^2 \geq (2h)^2 + \eta^2$. Since $ad \leq 2\rho$ we get $\eta^2/4 < \rho^2 - h^2 = x_\omega^2$ and for $\epsilon/\mathcal{R}$ small enough we have $\left(\frac{8\epsilon}{\eta^2}\alpha''\right)^2 < \eta^2/4$ which proves the claim.

It follows that it just remains to consider the case when abd, and therefore abd', is acute. abd is 0-almost empty and, from Lemma 10.6 abd' is α''-almost empty with

$$\alpha(abd') \leq O\left(\left(\frac{\epsilon}{\mathcal{R}}\right)^2 \left(\frac{\epsilon}{\eta}\right)^4 \epsilon^2\right)$$

From (10.32) we get that:

$$\|o-\omega\| < \frac{16\epsilon}{\eta^2}\alpha(abd') \tag{10.33}$$

Since, for $\epsilon/\mathcal{R}$ small enough we have $R \le \rho < 2\epsilon$.

If $\|o-s\| > 2\epsilon$ then $\mathrm{P}_{abc}(s) = (s-o)^2 - R^2 > 0$.

Now if $\|\omega - s\| > 3\epsilon$, then $\|o-s\| \ge \|\omega - s\| - \|o-\omega\| > 3\epsilon - \frac{16\epsilon}{\eta^2}\alpha(abd')$ and for $\epsilon/\mathcal{R}$ small enough $3\epsilon - \frac{16\epsilon}{\eta^2}\alpha(abd') > 2\epsilon$ which gives again $\|o-s\| > 2\epsilon$ and $\mathrm{P}_{abc}(s) > 0$.

Assume now that $\|o-s\| \le 2\epsilon$ and $\|\omega - s\| \le 3\epsilon$. (10.33) gives:

$$|(o-s)^2 - (\omega-s)^2| = |\langle o-\omega, (o-s)+(\omega-s)\rangle| \le \|o-\omega\|\|(o-s)+(\omega-s)\| \le (\frac{16\epsilon}{\eta^2}\alpha(abd'))(5\epsilon)$$

Which gives in particular:

$$\mathrm{P}_{abc}(s) - \mathrm{P}_{abd'}(s) - (\rho^2 - R^2) = (o-s)^2 - (\omega-s)^2 \ge -\alpha^\star$$

with

$$\alpha^\star \le O\left(\left(\frac{\epsilon}{\mathcal{R}}\right)^2 \left(\frac{\epsilon}{\eta}\right)^6 \epsilon^2\right)$$

Which gives, since $\rho^2 - R^2 > 0$, $\mathrm{P}_{abc}(s) \ge \mathrm{P}_{abd'}(s) - \alpha^\star$. Since abd is acute, by Lemma 10.6, abd' is α'-almost empty with $\alpha' = O\left((\frac{\epsilon}{\mathcal{R}})^2 \left(\frac{\epsilon}{\eta}\right)^4 \epsilon^2\right)$, therefore, one has $\forall s \in S, \mathrm{P}_{abc}(s) \ge -\alpha' - \alpha^\star$ which proves the induction in this case and concludes the proof of the lemma. $\square$

F An alternative to Whitney's Lemma

The proof of Theorem **??** makes use of [12] for its final global argument.[12] relies heavily on a Whitney Lemma which we consider to be the simplest argument there.

However we produced an alternative argument to Whitney Lemma that perform a recursion on the link of spherical complexes. Since this argument, beside Whitney lemma, may have its own merit, we give it in this section.

F.1 Background

For a k-simplex σ we denote by $|\sigma|$ the compact d-manifold simplex $\{x = (x_0, \dots, x_k) \in \mathbb{R}^d, \sum x_i \le 1; \forall i,\, x_i \ge 0\}$ and by σ° the open manifold relative interior of $|\sigma|$.

For a simplicial complex K, K_d designates the set of d-dimensional simplex in K. K is said *full dimensional* if each simplex has at least one dimension d coface.

The topological space $|K|$ is the support of K:

$$|K| = \bigcup_{\sigma \in K} |\sigma|$$

$\operatorname{Lk}\sigma$ and $\operatorname{St}\sigma$ denotes respectively the link and the star of the simplex σ in K. $\operatorname{St}^\circ\sigma$ denotes the corresponding open topological space:

$$\operatorname{St}^\circ \sigma = \bigcup_{\tau \in \operatorname{St}\sigma} \tau^\circ$$

$\sigma \star \tau$ denotes the join of σ and τ

Recall that a map f between topological spaces is said *open* if it maps open sets to open sets: Uopen $\Rightarrow f(U)$ open. We say that f is *open at x* if the restriction of f to some neighborhood of x is open. It is easy to check that $f : E \to F$ is open iff. it is open at any $x \in E$.

We recall here the invariance of domain theorem (Brouwer, Leray ??) used in the proof of Theorem 10.12.

Theorem 10.7 (Invariance of Domain Theorem, L. E. J. Brouwer 1911 [15])**.** *If M and N are d-manifolds without boundary and $\phi : M \to N$ is injective and continuous, then ϕ is open.*

For a set $E \subset \mathbb{R}^n$, $\mathrm{span}_{\mathrm{L}}(E)$ and $\mathrm{span}_{\mathrm{A}}(E)$ respectively denotes the linear span and the affine span of E.

F.2 Technical lemmas

Lemmas in this section are used in the proof of Theorem 10.12.

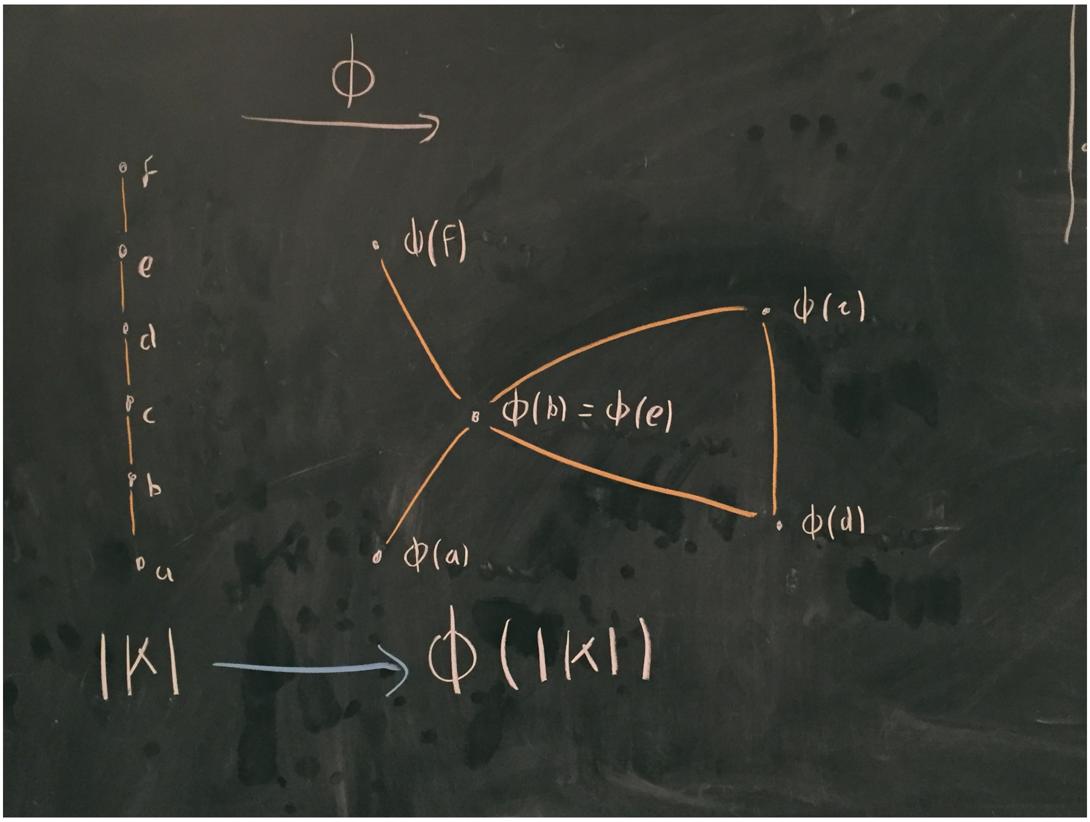

Figure 10.4: ϕ does not meet all conditions of Lemmas 10.8 and 10.9.

Recall that a topological space E is said *Hausdorff* if:

$$\forall x_1, x_2 \in E, x_1 \neq x_2 \Rightarrow \exists \text{ open sets } V_1 \ni x_1, V_2 \ni x_2, V_1 \cap V_2 = \emptyset$$

Metric spaces are obviously Hausdorff. We denote by $\phi_{|W}$ the restriction of ϕ to W.

Lemma 10.8. *Let $\phi : E \to F$ be a continuous and open map. If E is Hausdorff and $W \subset E$ is a dense open subset such that $\phi_{|W}$ is injective, ϕ is injective.*

Proof. For a contradiction assume $x_1 \neq x_2$ and $\phi(x_1) = \phi(x_2)$. Since E is Hausdorff, there are V_{x_1} and V_{x_2} respective open neighborhood of x_1 and x_2 such that $V_{x_1} \cap V_{x_2} = \emptyset$. Since ϕ is open, $\phi(V_{x_1})$ and $\phi(V_{x_2})$ are open sets and so is $U = \phi^{-1}(\phi(V_{x_1}) \cap \phi(V_{x_2}))$. We claim that $\phi(U) = \phi(U \cap V_{x_1}) = \phi(U \cap V_{x_2})$. Indeed, if $y \in \phi(V_{x_1}) \cap \phi(V_{x_2})$ there are $u_1 \in V_{x_1}$ and $u_2 \in V_{x_2}$ such that $\phi(u_1) = \phi(u_2) = y$ and therefore $u_1 \in U \cap V_{x_1}$ which gives us $y \in \phi(U \cap V_{x_1})$. We have then $\phi(U) = \phi(V_{x_1}) \cap \phi(V_{x_2}) \subset \phi(U \cap V_{x_1}) \subset \phi(U)$ and this proves the claim.

Since $x_1 \in U \cap V_{x_1}$, $U \cap V_{x_1}$ is a non empty open set and since W is dense in E, $W_1 = W \cap U \cap V_{x_1}$ is an open set dense in $U \cap V_{x_1}$. The same holds for $W_2 = W \cap U \cap V_{x_2}$. Since ϕ is continuous and open, $\phi(W_1)$ and $\phi(W_2)$ are open sets dense in $\phi(U)$. The intersection of two dense open sets is a dense open set. Therefore $\phi(W_1) \cap \phi(W_2)$ is dense in $\phi(U)$ and in particular non empty. But if $y \in \phi(W_1) \cap \phi(W_2)$, there are $w_1 \in W_1 \subset W$ and $w_2 \in W_2 \subset W$ such that $w_1, w_2 \in W$, $\phi(w_1) = \phi(w_2) = y$ and, since $W_1 \cap W_2 = \emptyset$, $w_1 \neq w_2$, a contradiction. J'ai l'impression qu'on pourrait raccourcir cette preuve. □

Lemma 10.9. *Let $\phi : E \to F$ be a continuous map . If E is Hausdorff and $W \subset E$ is a dense open subset such that $\phi_{|W}$ is injective and open and $\phi_{|E\setminus W}$ is injective, ϕ is injective.*

Proof. For a contradiction, assume $x_1 \neq x_2$ and $\phi(x_1) = \phi(x_2)$. Since $\phi_{|W}$ and $\phi_{|E\setminus W}$ are injective one can assume w.l.o.g. that $x_1 \in W$ and $x_2 \in E \setminus W$. Since E is Hausdorff, there are V_{x_1} and V_{x_2} respective open neighborhood of x_1 and x_2 such that $V_{x_1} \cap V_{x_2} = \emptyset$. $V_{x_1} \cap W \ni x_1$ and is open and since $\phi_{|W}$ is open, $\phi(V_{x_1} \cap W)$ is open. Since $\phi(x_2) = \phi(x_1) \in \phi(V_{x_1} \cap W)$ and ϕ continuous, there exists an open neighborhood $U_2 \subset V_{x_2}$ of x_2 such that $\phi(U_2) \subset \phi(V_{x_1} \cap W)$. Since W is dense, $W \cap U_2 \neq \emptyset$. If $y_2 \in W \cap U_2$, $\phi(y_2) \in \phi(V_{x_1} \cap W)$ and there exits $y_1 \in V_{x_1} \cap W$ such that $\phi(y_1) = \phi(y_2)$ and, since $U_2 \cap V_{x_1} = \emptyset$, $y_1 \neq y_2$, a contradiction since $y_1, y_2 \in W$ □

Definition 10.10 (Convex preserving). *We say that a map from $\mathbb{R}^n$ to $\mathbb{R}^m$ is* convex preserving *if the image of the convex hull is the convex hull of the image.*

If a map is injective and convex preserving, it sends simplex to simplex of the same dimension and images of joins are joins of images.
Homographies are convex preserving. We consider in the proof of the theorem below an homography h. We denote by $\operatorname{dom} h \subset \mathbb{R}^n$, its domain, i.e. the set of points where the denominator does not vanish.
One can check that $h_{|C}$, the restriction of h to a compact convex set $C \subset \operatorname{domain}(h)$ is convex preserving.

Lemma 10.11. *Let $|\tau| \subset \mathbb{R}^n$ be a k-simplex and τ° its relative interrior. For $x \in \tau^\circ$ and $z_1, z_2 \in \mathbb{R}^n \setminus \operatorname{span}_{\mathrm{A}}(\tau^\circ)$, if there is $\lambda > 0$, such that:*

$$(z_2 - x) - \lambda(z_1 - x) \in \operatorname{span}_{\mathrm{L}}(\tau^\circ)$$

then:

$$\{z_1\} \star |\tau| \;\cap\; \{z_2\} \star |\tau| \;\setminus\; |\tau| \;\neq \emptyset$$

Proof. If $t \in \operatorname{span}_{\mathrm{L}}(\tau^\circ)$ is such that:

$$t = (z_2 - x) - \lambda(z_1 - x)$$

One has:

$$z_2 = x + \lambda * (z_1 - x) + t$$

Consider, for $s \in [0, 1]$:

$$g_2(s) = (1 - s)x + sz_2$$

For $0 < s \leq 1$, $g_2(s) \in \{z_2\} \star |\tau| \;\setminus\; |\tau|$.
Also a short computation gives:

$$g_2(s) = (1 - \lambda s)\left(x + \frac{s}{1 - \lambda s} t\right) + \lambda s z_1$$

For some $\hat{s} > 0$ small enough, one has $\lambda \hat{s} \leq 1$ and:

$$x + \frac{\hat{s}}{1 - \lambda \hat{s}} t \in \tau^\circ$$

and therefore:

$$g_2(\hat{s}) \in \{z_1\} \star |\tau| \;\cap\; \{z_2\} \star |\tau| \;\setminus\; |\tau|$$

$\square$

F.3 Case $\mathcal{M} = \mathbb{S}^d$

Notice that the theorem below is not true for infinite simplicial complexes as one can see if K is a 1 dimensional simplicial complex whose support $|K| = \mathbb{R}$ and edges are $\{[j, j+1], j \in \mathbb{Z}\}$ and $\phi : |K| \to \mathbb{S}^1$, such that $\phi(j) = e^{i \arctan j}$.

Theorem 10.12. *Let $d \geq 0$ and K be a finite, d-dimensional simplicial complex. For a piecewise linear map $\phi : |K| \to \mathbb{R}^{d+1} \setminus \{0\}$ define $\mathring{\phi} : |K| \to \mathbb{S}^d$ as :*

$$\mathring{\phi}(x) = \frac{\phi(x)}{\|\phi(x)\|}$$

If K and $\mathring{\phi}$ satisfy:

1. *Any simplex $\tau \in K$ with $\dim \tau \leq d-1$ has at least two d-dimensional cofaces for $d \geq 1$, and K has at least two 0-dimensional simplices for $d = 0$.*
2. *For any $\sigma \in K_d$, the restriction of $\mathring{\phi}$ to $|\sigma|$ is injective.*
3. *The restriction of $\mathring{\phi}$ to*
$$\bigcup_{\sigma \in K_d} \sigma^\circ$$
is injective.

Then $\mathring{\phi}$ is a homeomorphism between $|K|$ and $\mathbb{S}^d$.

Remark: Condition 2. could be expressed in a more specific form: *0 and the image by ϕ of the $d+1$ vertices of any d simplex in K span a $(d+1)$-dimensional affine space.* We keep the general formulation since we expect the theorem to extend to more general situations.

Proof of Theorem 10.12. The theorem is true for $d = 0$: since K has at least two vertices and ϕ is injective (condition 3.), K is made of exactly two vertices and $\mathring{\phi}$ is a bijection onto $\mathbb{S}^0$. The proof for $d \geq 1$ goes by induction on the dimension. We consider the case $d \geq 1$ and we assume the theorem to be true for any dimension $d' < d$.

Barycentric subdivisions of K leads to complex K_{sub} with natural homeomorphisms $h : |K_{\text{sub}}| \to |K|$ such that K_{sub} and $\phi_{\text{sub}} = \phi \circ h$ satisfy the hypothesis of the theorem since ϕ does. It is possible to apply barycentric subdivisions until we have that if $x, y \in |\sigma|$ for $\sigma \in K_{\text{sub}}$, $\langle \mathring{\phi}_{\text{sub}}(x), \mathring{\phi}_{\text{sub}}(y) \rangle > 0$.

We therefore, for simplicity, assume without loss of generality that K and ϕ meet themselves this condition:

$$x \in |\sigma| \Rightarrow \forall y \in |\sigma|, \langle \mathring{\phi}(x), \mathring{\phi}(y) \rangle > 0 \tag{10.34}$$

It is enough to prove that $\mathring{\phi}$ is open. Indeed, by Lemma 10.8 $\mathring{\phi}$ is then one to one on its image and, since it is open, the $\mathring{\phi}^{-1}$ is continuous. But since $|K|$ is compact (as finite simplicial complex) and open, $\mathring{\phi}(|K|)$ is a compact and open subset of $\mathbb{S}^d$. Since $\mathbb{S}^d$ is connected, $\mathring{\phi}(|K|) = \mathbb{S}^d$.

We introduce the open set $W_k \subset |K|$:

$$W_k = \bigcup_{\substack{\tau \in K \\ \dim \tau \geq k}} \tau^\circ$$

One has

$$\bigcup_{\sigma \in K_d} \sigma^\circ = W_d \subset W_{d-1} \subset \ldots \subset W_0 = |K|$$

Since $\phi_{|W_d}$ is injective (condition 2.), it follows from the *invariance of domain theorem* that $\mathring{\phi}$ is open on W_d. We prove now by induction on decreasing k that $\mathring{\phi}_{|W_k}$ is open and injective. For $k < d$, we assume that this is true for any k' such that $k < k' \leq d$.

For $x \in W_k \setminus W_{k+1}$, there is a unique simplex $\tau_x \in K$ such that $x \in \tau_x^\circ$. One has $\dim \tau_x = k$.

Since $\mathrm{St}^\circ \tau_x \setminus \tau_x^\circ \subset W_{k+1}$ is open and $\mathring{\phi}_{|W_{k+1}}$ is open and injective, $\mathring{\phi}_{|\,\mathrm{St}^\circ \tau_x \setminus \tau_x^\circ}$ is open and injective. Then one can apply Lemma 10.9 with $E \leftarrow \mathrm{St}^\circ \tau_x$ and $W \leftarrow \mathrm{St}^\circ \tau_x \setminus \tau_x^\circ$. It follows that $\mathring{\phi}_{|\,\mathrm{St}^\circ \tau_x}$ is injective. It remains to prove that it is open at x.

Denote by Π_x the plane tangent to $\mathbb{S}^d$ at $\mathring{\phi}(x)$ and by $|\,\mathrm{St}\,\tau_x| = \cup_{\sigma \in \mathrm{St}\,\tau_x} |\sigma|$ the closure of $\mathrm{St}^\circ \tau_x$.

Let $\psi_x : |\,\mathrm{St}\,\tau_x| \to \Pi_x$ defined as:

$$\psi_x(y) = \frac{\mathring{\phi}(y)}{\langle \mathring{\phi}(x), \mathring{\phi}(y) \rangle} - \mathring{\phi}(x) = \frac{\phi(y)}{\langle \mathring{\phi}(x), \phi(y) \rangle} - \mathring{\phi}(x)$$

It is well defined by 10.34. The restriction of ψ_x to some simplex in $\mathrm{St}\,\tau_x$ is an homography restricted in a convex set included in its domain by 10.34. It is therefore convex preserving.

Observes also that since the restriction of:

$$z \to \frac{z}{\langle \mathring{\phi}(x), z \rangle} - \mathring{\phi}(x)$$

to $\mathring{\phi}(|\,\mathrm{St}\,\tau_x|)$ is an homeomorphism on its image, properties 2. and 3. of $\mathring{\phi}$ are inherited by ψ_x:

- For any $\sigma \in \mathrm{St}\,\tau_x$, the restriction of ψ_x to $|\sigma|$ is injective and convex preserving.
- The restriction of ψ_x to $W_{k+1} \cap |\,\mathrm{St}\,\tau_x|$ is injective.

In the identification $\Pi_x = \mathbb{R}^d$, without loss of generality one can assume that $\psi_x(x) = 0$ and $\psi_x(\tau_x^\circ)$ spans $\mathbb{R}^k \times \{0\}$.
Define $\pi_{\tau^\perp} : \mathbb{R}^d \to \mathbb{R}^{d-k}$ projection on the $(d-k)^{\text{th}}$ last coordinates.

Lemma 10.13. *$\phi' : \mathrm{Lk}\,\tau_x \to \mathbb{R}^{d-k}$, defined by $\phi' = \pi_{\tau^\perp} \circ \psi_{x\,|\,\mathrm{Lk}\,\tau_x}$ satisfies the theorem for $d' = d - k - 1$*

Lemma 10.13 allows us to recursively apply the theorem to get that $|\,\mathrm{Lk}\,\tau_x|$ is homeomorphic to $\mathbb{S}^{d'}$ and therefore $\mathrm{St}^\circ \tau_x$ is a manifold.
Since $\mathring{\phi}_{|\,\mathrm{St}^\circ \tau_x}$ is continuous injective from a d-manifold to a d-manifold, the invariance of domain theorem says that it is open, in particular $\mathring{\phi}$ is open at x. Since this is true for any $x \in W_k \setminus W_{k+1}$, it follows that $\mathring{\phi}_{|W_k}$ is open. We can apply Lemma 10.8 with $E \leftarrow W_k$ and $W \leftarrow W_{k+1}$ (or $W \leftarrow W_d$) to get that $\mathring{\phi}_{|W_k}$ is injective and open.

One has in particular at the end of the recursion that $\mathring{\phi}_{|W_0} = \mathring{\phi}$ is open and injective and we are done. □

Proof of Lemma 10.13. Observe first that $\mathrm{Lk}\,\tau_x$ is a d'-dimensional simplicial complex, with $d' = d - k - 1$ such that any simplex has at least 2 d'-dimensional cofaces for $d' > 0$ and $\mathrm{Lk}\,\tau_x$ has has least two vertex if $d' = 0$. It follows that condition 1. of the theorem is met by ϕ'.

We prove now that $\forall y \in |\,\mathrm{Lk}\,\tau_x|, \phi'(y) \in \mathbb{R}^d \setminus \{0\}$. Assume that there is $y \in |\,\mathrm{Lk}\,\tau_x|$ with $\phi'(y) = 0$, that is $\psi_x(y) \in \mathrm{span}_{\mathrm{A}}(\psi_x(\tau_x^\circ))$ where $\mathrm{span}_{\mathrm{A}}(\psi_x(\tau_x^\circ))$ is the affine span of $\psi_x(\tau_x^\circ)$ (it corresponds also to the linear span since $0 = \psi_x(x) \in \psi_x(\tau_x^\circ))$. There is a simplex $\sigma' \in \mathrm{Lk}\,\tau_x$ such that $y \in \sigma'^\circ$ and:

$$\psi_x(y) \in \mathrm{span}_{\mathrm{A}}(\psi_x(\tau_x^\circ)) \cap \psi_x(\sigma'^\circ)$$

which gives $\dim \operatorname{span}_{\mathrm{A}} [\psi_x(\tau_x^\circ) \star \psi_x(\sigma'\circ)] \leq \dim \psi_x(\tau_x^\circ) + \dim \psi_x(\sigma'^\circ)$. The restriction of ψ_x to $\tau_x^\circ \star \sigma'^\circ$ is convex preserving and injective. It follows that $\psi_x(\tau_x^\circ) \star \psi_x(\sigma'^\circ) = \psi_x(\tau_x^\circ \star \sigma'^\circ)$ has dimension $\dim \tau_x + \dim \sigma' + 1$, a contradiction.

For the proof of the lemma it remains to check that ϕ' satisfies conditions 2. and 3. of the theorem. Assume that $\mathring{\phi}'(y_1) = \mathring{\phi}'(y_2)$. This is equivalent to the existence a positive real number $\lambda > 0$ such that $\phi'(y_1) = \lambda\phi'(y_2)$. In other words $\psi_x(y_1) - \lambda\psi_x(y_2) \in \operatorname{span}_{\mathrm{A}}(\tau_x) = \operatorname{span}_{\mathrm{L}}(\tau_x)$. We can then apply Lemma 10.11 and we have:

$$(\{\psi_x(y_1)\} \star |\tau_x| \cap \{\psi_x(y_2)\} \star |\tau_x|) \setminus |\tau_x| \neq \emptyset \tag{10.35}$$

If, for some $\sigma' \in \operatorname{St} \tau_x$, $y_1, y_2 \in |\sigma'|$ and $y_1 \neq y_2$, (10.35) contradicts the fact that the restriction of ψ_x to $|\tau_x \star \sigma'|$ is injective. This proves that ϕ' satisfies condition 2. of the theorem.
Now If $y_1 \in {\sigma'_1}^\circ$ and $y_2 \in {\sigma'_2}^\circ$ with $\sigma'_1, \sigma'_2 \in \operatorname{St} \tau_x$, $\sigma'_1 \neq \sigma'_2$ and $\dim \sigma'_1 = \sigma'_2 = d'$, then (10.35) contradicts the fact that the restriction of ψ_x to $W_{k+1} \cap |\operatorname{St} \tau_x|$ is injective. This proves that ϕ' satisfies condition 3. of the theorem. □

F.4 Local injectivity

Theorem 10.14 is an easy consequence of Theorem 10.12.

For a vertex v, we denote by $\overline{\operatorname{St} v}$ the closed star of v, i.e. the open star $\operatorname{St}^\circ v$ of v augmented by all the faces of simplices in $\operatorname{St}^\circ v$. Note that $\operatorname{Lk} v \subset \overline{\operatorname{St} v}$.

Theorem 10.14. *Let $d \geq 1$ and K be a finite, d-dimensional simplicial complex such that for any vertex $v \in K_0$ there exists a piecewise linear map $\pi : \overline{\operatorname{St} v} \to \mathbb{R}^d$ such that:*

1. *Any simplex in $\tau \in K$ with $\dim \tau \leq d-1$ has at least two d-dimensional cofaces in K.*
2. *For any $\sigma \in \overline{\operatorname{St} v}$, the restriction of π to $|\sigma|$ is injective.*
3. *The restriction of π to*
$$\bigcup_{\sigma \in K_d \cap \operatorname{St} v} \sigma^\circ$$
is injective.

Then $\operatorname{St}^\circ v$ is homeomorphic to an open ball in $\mathbb{R}^d$ and π is injective and its restriction to $\operatorname{St}^\circ v$ is open. In particular $|K|$ is a d-manifold.

Proof. Denote by $\pi_{\operatorname{Lk} v}$ the restriction of π to $\operatorname{Lk} v$. $\pi_{\operatorname{Lk} v}$ satisfies the conditions of Theorem 10.12. It follows that $\pi_{\operatorname{Lk} v}$ is injective and open and that $|\operatorname{Lk} v|$ is homeomorphic to $\mathbb{S}^{d-1}$. We get then that since $\overline{\operatorname{St} v}$ is the cone on $\operatorname{Lk} v$, $|\overline{\operatorname{St} v}|$ and $\operatorname{St}^\circ v$ are homeomorphic respectively to the closed and an open ball in $\mathbb{R}^d$. Since $|K|$, as any realization of simplicial complex, is Hausdorff and since any point $x \in |K|$ belongs to the open star $\operatorname{St}^\circ v$ of some vertex v, it has a neighborhood homeomorphic t a neihgborhood in $\mathbb{R}^d$. It follows hat $|K|$ is a d-manifold. It remain to prove that π is injective and open. If $\pi(x_1) = \pi(x_2)$ and x_1 and x_2 are in a same simplex, then condition 2. of the theorem implies $x_1 = x_2$. Assume now for a contradiction that $\pi(x_1) = \pi(x_2)$ and that x_1 and x_2 does not belong to a same simplex. We know then that $x_1 \neq v$ and $x_2 \neq v$ since if for example $x_1 = v$ then there would be a d-dimensional simplex containing both x_2 and, necessarily, $v = x_1$. Without loss of generality we can assume that $\pi(v) = 0$ and $\pi(x_1) = \pi(x_2) \neq 0$. We have, for some $\lambda_1, \lambda_2 > 2$, $x_i = (1-\lambda_i)v + \lambda_i \tilde{x_i}$ with $\tilde{x_i} \in |\operatorname{Lk} v|$ and $\pi(x_i) = \lambda_i \pi(\tilde{x_i})$ for $i = 1, 2$. Then $\pi(x_1) = \pi(x_2) \Rightarrow \lambda_1 \pi(\tilde{x_1}) = \lambda_2 \pi(\tilde{x_2})$ and, using the notation of Theorem 10.12 this gives $\mathring{\pi}(\tilde{x_1}) = \mathring{\pi}(\tilde{x_2})$ and since $\mathring{\pi}$ is one to one we get $\tilde{x_1} = \tilde{x_2}$, a contradiction with the fact that x_1 and x_2 does not belong to a same simplex. We have proven that π is injective and since it is continuous and $\operatorname{St}^\circ v$ is a d-manifold, Invariance of Domain (Theorem 10.7) applies and the restriction of π to $\operatorname{St}^\circ v$ is open. □

We come back now to the context of Theorem Theorem **??**.

Lemma 10.15. *If $|\mathcal{T}|$ satisfies the conditions of Theorem* **??**, *for $m \in \mathcal{M}$, the restriction of π_m to:*

$$U = \bigcup_{v \in S_{m-4,\mathcal{R}/2,C\epsilon}} \mathrm{St}^{\circ}_{\mathcal{T}}\, v$$

is injective and open. In particular $|\mathcal{T}|$ is a 2-manifold.

Proof. We first check that for $\frac{\epsilon}{\mathcal{R}}$ small enough, $\mathcal{T}$ and the projections on local tangent plane satisfy the conditions of Theorem 10.14. Observes that , from the bound 2ϵ on the circumradius of triangles in $\mathcal{T}$, U is a subset of the set of triangles having their vertices in $S_{m,\mathcal{R}/2,C\epsilon}$. Condition 1. of Theorem 10.14 follows from the fact that $\partial\mathcal{T} = 0$. Indeed $\mathcal{T}$ is full dimensional and if a $d-1$ simplex τ would have a single d-dimensional coface, then τ would belong to $\partial\mathcal{T}$. Condition 2. follows from Lemma 9.20. Indeed the upper bound on the angle between a triangle and Π_m enforces the restriction of π_m to any simplex to be injective. Condition 3. follows from Lemma 9.41. We have shown that $|\mathcal{T}|$ is a 2-manifold. Since by Theorem 10.14 the restriction of π_m to any open star of vertices in $S_{m-4,\mathcal{R}/2,C\epsilon}$ is open it follows that it is locally open and therefore open. But then we can apply Lemma 10.8 to π_m with W being the union of interiors of triangles. Indeed Lemma 9.41 gives that the restriction of π_m to W is injective and Lemma 10.8 gives that the restriction of π_m to U is injective. $\square$