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GRADIENT ESTIMATES AND MAXIMAL DISSIPATIVITY FOR THE KOLMOGOROV OPERATOR IN Φ_2^4

GIUSEPPE DA PRATO AND ARNAUD DEBUSSCHE

ABSTRACT. We consider the transition semigroup P_t of the Φ_2^4 stochastic quantisation on the torus \mathbb{T}^2 and prove the following new estimate (Theorem 3.9)

$$|DP_t\varphi(x) \cdot h| \leq c t^{-\beta} |h|_{C^{-s}} \|\varphi\|_0 (1 + |x|_{C^{-\alpha}})^\gamma,$$

for some α, β, γ, s positive. Thanks to this estimate, we show that cylindrical functions are a core for the corresponding Kolmogorov equation. Some consequences of this fact are discussed in a final remark.

1. INTRODUCTION

We consider the equation of the Φ_2^4 stochastic quantisation on the torus \mathbb{T}^2 :

$$\begin{cases} dX = (AX - :X^3:)dt + dW, \\ X(0) = x, \end{cases} \quad (1.1)$$

where

$$Ax = \Delta x - x, \quad D(A) = H^2(\mathbb{T}^2).$$

This equation has been the object of several studies. In particular, in [DaDe03-2] existence and uniqueness of a mild solution to (1.1) has been proved for almost all initial datas x with respect to the Gibbs measure ν defined below; later in [MoWe15] well-posedness was proved *for all* x in a negative Besov space. Moreover, in [HaMa16] the strong Feller property of the corresponding transition semigroup $(P_t)_{t \geq 0}$ was proved. This latter result is improved in [TsWe18] where it is proved that, for $t > 0$, P_t maps bounded Borel functions to α -Hölder functions for some $\alpha \in (0, 1)$.

The main result of the present paper (Theorem 3.9 below) goes further in this direction: we prove that P_t maps bounded borelian functions to Lipschitz functions and give an estimate of the gradient of $P_t\varphi$. We use a method similar to the one used in [DaDe03-1] and [DaDe07] as well as similar estimates as in [TsWe18]. In the second part we use this result for showing that cylindrical functions are a core for the infinitesimal generator \mathcal{K} of P_t and present some consequences of this fact.

2. NOTATIONS

We use the classical $L^p = L^p(\mathbb{T}^2)$ spaces for $p \in [1, \infty]$. For $p = 2$, we write $H = L^2$. We use also the L^2 based Sobolev spaces $H^s = H^s(\mathbb{T}^2)$ for $s \in \mathbb{R}$ and the Besov spaces $B_{p,q}^s = B_{p,q}^s(\mathbb{T}^2)$, for $s \in \mathbb{R}$, $p, q \in [1, \infty]$. These are defined in terms of Fourier series and Littlewood-Paley theory (see [BaChDa11]). Note that $H^s = B_{2,2}^s$. When $p = q = \infty$, we write $B_{\infty,\infty}^s = C^s$ and for $s > 0$ non integer these spaces coincide with the classical Hölder spaces.

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Besov spaces are convenient when multiplying two functions when one has negative regularity. Recall that for $p_1, p_2 \in [1, \infty]$ and $\alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha_1 + \alpha_2 > 0$ we have for $\alpha = \alpha_1 \wedge \alpha_2 \wedge (\alpha_1 + \alpha_2)$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\tilde{p} = p_1 \vee p_2$

$$\|uv\|_{B_{p,\tilde{p}}^\alpha} \leq C \|u\|_{B_{p_1,p_1}^{\alpha_1}} \|v\|_{B_{p_2,p_2}^{\alpha_2}}$$

and for any $\kappa > 0$

$$\|uv\|_{B_{p,\tilde{p}}^{\alpha-\kappa}} \leq C \|u\|_{B_{p_1,p_1}^{\alpha_1}} \|v\|_{B_{p_2,p_2}^{\alpha_2}}.$$

We use the eigenprojections Π_N corresponding to the eigenvalues $k^2 \leq N$, $k \in \mathbb{Z}^2$, of A . Define the Gaussian measure

$$\mu = \prod_{k \in \mathbb{Z}^2} N\left(0, \frac{1}{1 + |k|^2}\right), \text{ on } \mathcal{H} = \mathbb{R}^{\mathbb{Z}^2},$$

and for $x_N \in \Pi_N H$

$$:x_N^2: = x_N^2 - \rho_N^2, \quad :x_N^3: = x_N^3 - 3\rho_N^2 x_N,$$

where

$$\rho_N = \frac{1}{2\pi} \left[\sum_{|k| \leq N} \frac{1}{1 + |k|^2} \right]^{1/2}.$$

More generally, the n^{th} Wick power of x_N is defined by $:x_N^n: = \sqrt{n!} \rho_N^n H_n(\frac{1}{\rho_N} x_N)$ where H_n is the n^{th} Hermite polynomial.

As well known, there exists the limit

$$\lim_{N \rightarrow \infty} :(\Pi_N x)^n: =: x^n:, \text{ in } L^r(\mathcal{H}, \mu; B_{p,q}^s), \quad (2.1)$$

for any $r \in [1, \infty)$, $p, q \in [1, \infty]$, $s < 0$.

Then, we define

$$Z_\infty(t) = \int_{-\infty}^t e^{(t-s)A} dW(s).$$

Since $Z_\infty(t)$ has law μ , we know that $:\Pi_N Z_\infty^n:$ converges to $:Z_\infty^n:$ in $L^r(\mathcal{H}, \mu; B_{p,q}^s)$ for any $r \in [1, \infty)$, $p, q \in [1, \infty]$, $s < 0$

We set moreover $X = Y_\infty + Z_\infty = Y + Z$ where

$$Z(t) = \int_0^t e^{(t-s)A} dW(s) = Z_\infty(t) - e^{At} Z_\infty(0),$$

Clearly:

$$:(\Pi_N X)^3: = (\Pi_N Y_\infty)^3 + 3(\Pi_N Y_\infty)^2 \Pi_N Z_\infty + 3(\Pi_N Y_\infty) :(\Pi_N Z_\infty)^2: + :(\Pi_N Z_\infty)^3:.$$

It is therefore natural to consider the following definition for $:X^3:$ in (1.1):

$$:X^3: = Y_\infty^3 + 3Y_\infty^2 Z_\infty + 3Y_\infty :Z_\infty^2: + :Z_\infty^3:.$$

We also have

$$:X^3: = Y^3 + 3Y^2 Z + 3Y :Z^2: + :Z^3:$$

if we set

$$:Z^3: = -(e^{At} Z_\infty(0))^3 + 3(e^{At} Z_\infty(0))^2 Z_\infty - 3(e^{At} Z_\infty(0)) :Z_\infty^2: + :Z_\infty^3:.$$

We will see that Y and Y_∞ has positive regularity whereas Z has the same smoothness as Z_∞ . Using the product rules above, it can be seen that all products above make sense.

Finally, we define $Z_N = \Pi_N Z$ and introduce the following approximation of (1.1):

$$\begin{cases} \frac{dY_N}{dt} = AY_N - (Y_N^3 + 3Y_N^2 Z_N + 3Y_N : Z_N^2 : + : Z_N^3 :), \\ Y_N(0) = x. \end{cases} \quad (2.2)$$

Equivalently, setting $X_N = Y_N + Z_N$, we may write with a slight abuse of notation

$$: X_N^3 : = Y_N^3 + 3Y_N^2 Z_N + 3Y_N : Z_N^2 : + : Z_N^3 :$$

and

$$\begin{cases} dX_N = (AX_N + : X_N^3 :)dt + \Pi_N dW, \\ X_N(0) = x. \end{cases} \quad (2.3)$$

Note that this is not a Galerkin approximation. However, since $: X_N^3 : = X_N^3 - 3\rho_N^2 X_N$ and $\Pi_N dW$ is a finite dimensional white noise, it is classical to prove that X_N exists and is unique. Moreover, it is smooth in space when the initial datum is smooth; in this case Y_N is also spatially smooth and has moreover regularity $3/2^-$ in time. All the computations done in the sections below are rigorous when the initial datum is smooth, for initial data in a negative Besov space, they are obtained thanks to a preliminary smoothing of the initial datum.

It is not difficult to prove that (X_N) converges to the unique solution of (1.1) (see [TsWe18]) in convenient topologies, $C([0, T]; C^{-\alpha})$ for $\alpha > 0$ for instance.

Let us define the Gibbs measure ν on \mathcal{H}

$$\nu(dx) = \mathcal{Z}^{-1} e^{-2U(x)} \mu(dx), \quad (2.4)$$

where

$$U(x) = \frac{1}{4} \int_H : x^4 : (\xi) d\xi \quad (2.5)$$

and

$$\mathcal{Z} := \int_H e^{-2U(x)} \mu(dx).$$

Then

$$e^{-2U(x)} \in L^r(\mathcal{H}, \mu),$$

for any $r \in [1, \infty)$ thanks to the Nelson inequality.

We can write equation (1.1) as

$$\begin{cases} \frac{dY}{dt} = AY - Y^3 - 3Y^2 Z - 3Y : Z^2 : - : Z^3 :, \\ Y(0) = x, \end{cases} \quad (2.6)$$

or, in mild form,

$$Y(t) = e^{At} x - \int_0^t e^{A(t-s)} [Y^3(s) + 3Y^2(s)Z(s) + 3Y(s) : Z^2(s) : + : Z^3(s) :] ds. \quad (2.7)$$

We define the transition semigroups P_t and P_t^N associated to (1.1) and (2.3) respectively and introduce the following modified semigroup:

$$S_t^N \varphi(x) = \mathbb{E} \left(e^{-K \int_0^t V(X_N(s)) ds} \varphi(X_N(t, x)) \right) \quad (2.8)$$

where the potential V is given by

$$V(X_N) = | : X_N^2 : |_{H^{-\alpha}}^p = | X_N^2 - \rho_N^2 |_{H^{-\alpha}}^p$$

for some $K > 0$, $p > 0$, $\alpha > 0$ to be chosen. This quantity is well defined for $\varphi \in \mathcal{B}_b(C^{-\alpha})$, the space of borelian bounded functions on $C^{-\alpha}$.

The following relation formally follows from Duhamel formula since $P_N\varphi$ satisfies the Kolmogorov equation associated to (2.3) and $S_N\varphi$ satisfies the same equation with an extra term due to the potential V :

$$P_t^N \varphi(x) = S_t^N \varphi(x) + K \int_0^t S_{t-s}^N (V P_s \varphi)(x) ds. \quad (2.9)$$

It can be proved rigorously by an approximation argument.

3. ESTIMATES

Let us consider the equation:

$$\begin{cases} \frac{d\eta^{h,N}}{dt} = (A\eta^{h,N} - 3 : X_N^2 : \eta^{h,N}), \\ \eta^{h,N}(0) = h. \end{cases} \quad (3.1)$$

It is classical that $\eta^{h,N} = D_x X_N \cdot h$ is the differential of X_N with respect to the initial data $x \in C^{-\alpha}$ in the direction $h \in C^{-\alpha}$.

In this section, we derive several estimates on η_N and X_N . These hold only on finite time intervals and we restrict to $t \in [0, 1]$.

Lemma 3.1. *Let $\alpha < \frac{1}{3}$, $\epsilon > 0$ such that $\alpha + \epsilon < \frac{1}{3}$ and $p \geq 1$, $s > 0$ such that $\alpha + \epsilon + \frac{2}{3p} + \frac{s}{3} < \frac{1}{3}$ then there exists c depending on α, ϵ, p such that for all $t \in (0, 1]$, $x \in C^{-\alpha}$, $h \in C^{-s}$*

$$|\eta^{h,N}(t)|_{C^{\alpha+\epsilon}} \leq ct^{-(\alpha+\epsilon+s)/2} |h|_{C^{-s}} e^{c \int_0^t |X_N^2(s)|_{H^{-\alpha}}^p ds}$$

and

$$|\eta^{h,N}(t)|_{L^\infty} \leq ct^{-s/2} |h|_{C^{-s}} e^{c \int_0^t |X_N^2(s)|_{H^{-\alpha}}^p ds}.$$

Proof. Write (3.1) in mild form

$$\eta^{h,N}(t) = e^{At} h - 3 \int_0^t e^{A(t-s)} \left(: X_N^2(s) : \eta^{h,N}(s) \right) ds.$$

Below we omit the dependence on N . By the smoothing of the heat semigroup e^{At} , the product rule in Besov spaces and $t \in [0, 1]$, we have

$$\begin{aligned} |\eta^h(t)|_{C^{\alpha+\epsilon}} &\leq ct^{-(\alpha+\epsilon+s)/2} |h|_{C^{-s}} + c \int_0^t |t-r|^{-(\alpha+\epsilon+1/2)} \left| : X^2(r) : \eta^h(r) \right|_{H^{-\alpha-\epsilon/2}} dr \\ &\leq ct^{-(\alpha+\epsilon+s)/2} |h|_{C^{-s}} + c \int_0^t |t-r|^{-(\alpha+\epsilon+1/2)} \left| : X^2(r) : \right|_{H^{-\alpha}} \left| \eta^h(r) \right|_{C^{\alpha+\epsilon}} dr. \end{aligned}$$

Let $\lambda(t) = t^{(\alpha+\epsilon+s)/2} |\eta^h(t)|_{C^{\alpha+\epsilon}}$, then since $t \in [0, 1]$,

$$\lambda(t) \leq c |h|_{C^{-s}} + c \int_0^t |t-r|^{-(\alpha+\epsilon+1/2)} r^{-(\alpha+\epsilon+s)/2} \left| : X^2(r) : \right|_{H^{-\alpha}} \lambda(r) dr.$$

Let $\frac{1}{p} + \frac{1}{q} = 1$ and

$$\frac{(3\alpha + 3\epsilon + 1 + s)q}{2} < 1, \quad i.e. \quad \frac{3\alpha + 3\epsilon + 1 + s}{2} < 1 - \frac{1}{p}.$$

Then

$$\begin{aligned}\lambda^p(t) &\leq c|h|_{C^{-s}}^p + c \left(\int_0^t |t-r|^{-q(\alpha+\epsilon+1/2)} r^{-q(\alpha+\epsilon+s)/2} dr \right)^{p/q} \int_0^t | : X^2(r) : |_{H^{-\alpha}}^p \lambda^p(r) dr \\ &\leq c|h|_{C^{-s}}^p + c \int_0^t | : X^2(r) : |_{H^{-\alpha}}^p \lambda^p(r) dr\end{aligned}$$

The first inequality follows by Gronwall's lemma.

For the second inequality, we write:

$$\begin{aligned}|\eta^h(t)|_{L^\infty} &\leq ct^{-s/2}|h|_{C^{-s}} + c \int_0^t |t-r|^{-(\alpha+\epsilon+1/2)} \left| : X^2(r) : \eta^h(r) \right|_{H^{-\alpha-\epsilon/2}} dr \\ &\leq ct^{-s/2}|h|_{C^{-s}} + c \int_0^t |t-r|^{-(\alpha+\epsilon+1/2)} \left| : X^2(r) : \right|_{H^{-\alpha}} \left| \eta^h(r) \right|_{C^{\alpha+\epsilon}} dr.\end{aligned}$$

and with p, q as above:

$$\begin{aligned}|\eta^h(t)|_{L^\infty}^p &\leq ct^{-sp/2}|h|_{C^{-s}}^p + c \int_0^t r^{p(\alpha+\epsilon+s)/2} \left| : X^2(r) : \right|_{H^{-\alpha}}^p \left| \eta^h(r) \right|_{C^{\alpha+\epsilon}}^p dr \\ &\leq ct^{-sp/2}|h|_{C^{-s}}^p + c \int_0^t \left| : X^2(r) : \right|_{H^{-\alpha}}^p e^{cp \int_0^t r | : X_N^2(s) : |_{H^{-\alpha}}^2 ds} |h|_{C^{-s}}^p dr \\ &\leq c|h|_{C^{-s}}^p \left(t^{-sp/2} + e^{cp \int_0^t | : X_N^2(s) : |_{H^{-\alpha}}^2 ds} \right).\end{aligned}$$

The conclusion follows. \square

We now estimate moments of Y_N solution of (2.2). We estimate Y_N in $L^r(\mathbb{T}^2)$, $r \geq 2$, using similar arguments as in [TsWe18] and [MoWe15], but in a simplified way since we do not need such refined estimates for large times.

Lemma 3.2. *Let $\alpha \in [0, 1)$ and $r \geq 2$, there exist $c > 0$ depending on α and r such that*

$$\frac{1}{r} \frac{d}{dt} |Y_N|_{L^r}^r + \frac{r-1}{r^2} \left| \nabla((Y_N)^{r/2}) \right|_{L^2}^2 + \frac{1}{2} |Y_N|_{L^{r+2}}^{r+2} \leq c \mathcal{L}_\alpha(Z_N)^{k_r},$$

where $k_r = \frac{r+2}{1-\alpha}$ and

$$\mathcal{L}_\alpha(Z_N) = |Z_N|_{C^{-\alpha}} + | : Z_N^2 : |_{C^{-\alpha}} + | : Z_N^3 : |_{C^{-\alpha}} + 1.$$

Proof. Again we omit the dependence on N . Multiplying equation (2.2) by Y^{r-1} and integrating on \mathbb{T}^2 we obtain:

$$\frac{1}{r} \frac{d}{dt} |Y|_{L^r}^r + \frac{4(r-1)}{r^2} \left| \nabla(Y^{r/2}) \right|_{L^2}^2 + |Y|_{L^{r+2}}^{r+2} = - \int_{\mathbb{T}^2} (3Y^{r+1} Z + 3Y^r : Z^2 : + Y^{r-1} : Z^3 :) d\xi.$$

We estimate each of the three terms in the right hand side. By Proposition A.8 and Proposition A.9 in [TsWe18], we have

$$3 \left| \int_{\mathbb{T}^2} Y^{r+1} Z d\xi \right| \leq 3|Y^{r+1}|_{B_{1,1}^\alpha} |Z|_{C^{-\alpha}} \leq c(|Y^{r+1}|_{L^1}^{1-\alpha} |\nabla Y^{r+1}|_{L^1}^\alpha + |Y^{r+1}|_{L^1}) |Z|_{C^{-\alpha}}.$$

Clearly

$$|Y^{r+1}|_{L^1} = |Y|_{L^{r+1}}^{r+1} \leq c|Y|_{L^{r+2}}^{r+1}.$$

Moreover

$$|\nabla Y^{r+1}|_{L^1} = (r+1)|Y^r \nabla Y|_{L^1} \leq (r+1)|Y^{\frac{r+2}{2}}|_{L^2} |Y^{\frac{r-2}{2}} \nabla Y|_{L^2} = \frac{2(r+1)}{r} |Y|_{L^{r+2}}^{\frac{r+2}{2}} |\nabla Y^{r/2}|_{L^2}.$$

It follows that

$$\begin{aligned} 3 \left| \int_{\mathbb{T}^2} Y^{r+1} Z d\xi \right| &\leq c \left(|Y|_{L^{r+2}}^{(1-\alpha)(r+1)+\alpha(r+2)/2} |\nabla Y^{r/2}|_{L^2}^\alpha + |Y|_{L^{r+2}}^{r+1} \right) |Z|_{C^{-\alpha}} \\ &= c \left(|Y|_{L^{r+2}}^{((1-\alpha/2)r+1)} |\nabla Y^{r/2}|_{L^2}^\alpha + |Y|_{L^{r+2}}^{r+1} \right) |Z|_{C^{-\alpha}} \\ &\leq \frac{1}{6} |Y|_{L^{r+2}}^{r+2} + \frac{r-1}{r^2} |\nabla Y^{r/2}|_{L^2}^2 + c \left(|Z|_{C^{-\alpha}}^{p_1} + |Z|_{C^{-\alpha}}^{p_2} \right) \end{aligned}$$

with

$$\frac{(1-\alpha/2)r+1}{r+2} + \frac{\alpha}{2} + \frac{1}{p_1} = 1$$

and $p_2 = r+2$. Note that since $\alpha \in (0, 1)$, p_1 exists and is equal to $\frac{r+2}{1-\alpha}$. We proceed similarly for the second one and obtain:

$$3 \left| \int_{\mathbb{T}^2} Y^r : Z^2 : d\xi \right| \leq \frac{1}{6} |Y|_{L^{r+2}}^{r+2} + \frac{r-1}{r^2} |\nabla Y^{r/2}|_{L^2}^2 + c \left(| : Z^2 : |_{C^{-\alpha}}^{p_3} + | : Z^2 : |_{C^{-\alpha}}^{p_4} \right)$$

with

$$\frac{\alpha}{2} + \frac{r(1-\alpha/2)}{r+2} + \frac{1}{p_3} = 1$$

and $p_4 = \frac{r+2}{2}$ and for the third one

$$\left| \int_{\mathbb{T}^2} Y^{r-1} : Z^3 : d\xi \right| \leq \frac{1}{6} |Y|_{L^{r+2}}^{r+2} + \frac{r-1}{r^2} |\nabla Y^{r/2}|_{L^2}^2 + c \left(| : Z^3 : |_{C^{-\alpha}}^{p_5} + | : Z^3 : |_{C^{-\alpha}}^{p_6} \right)$$

with

$$\frac{\alpha}{2} + \frac{r(1-\alpha/2)-1}{r+2} + \frac{1}{p_5} = 1$$

and $p_6 = \frac{r+2}{3}$.

Using these three inequalities in the energy equality and noting that $p_1 = \frac{r+2}{1-\alpha} \geq p_i$, $i = 2, \dots, 5$ we obtain

$$\frac{1}{2} \frac{d}{dt} |Y|_{L^r}^r + \frac{r-1}{r^2} \left| \nabla(Y^{r/2}) \right|_{L^2}^2 + \frac{1}{2} |Y|_{L^{r+2}}^{r+2} \leq c \left(|Z|_{C^{-\alpha}} + | : Z^2 : |_{C^{-\alpha}} + | : Z^3 : |_{C^{-\alpha}} + 1 \right)^{p_1}.$$

This gives the result with $k_r = p_1$. □

Lemma 3.3. *Let $\alpha < \frac{1}{3}$, $k \geq 1$ there exists $C_{\alpha,k} > 0$ and $\kappa_{\alpha,k} > 0$ such that*

$$\mathbb{E} \left(\sup_{t \in [0,1]} |Y_N(t)|_{C^{-\alpha}}^k \right) \leq C_{\alpha,k} (|x|_{C^{-\alpha}} + 1)^{\kappa_{\alpha,k}}.$$

Proof. Since

$$|Y_N|_{L^r}^{r+2} \leq c |Y_N|_{L^{r+2}}^{r+2}$$

it follows from Lemma 3.2 that

$$\frac{d}{dt} |Y_N(t)|_{L^r}^r + c |Y_N(t)|_{L^r}^{r+2} \leq c \mathcal{L}_\alpha(Z_N(t))^{k_r}$$

and by [TsWe18, Lemma 3.8] it follows for $t \in [0, 1]$

$$|Y_N(t)|_{L^r} \leq c \max \left(t^{-r/2}, \sup_{t \in [0,1]} \mathcal{L}_\alpha(Z_N(t))^{\frac{r}{1-\alpha}} \right).$$

We choose $r > \frac{2}{\alpha}$ and use the embedding $L^r \subset C^{-\alpha}$ to deduce

$$|Y_N(t)|_{C^{-\alpha}} \leq c \max \left(t^{-r/2}, \sup_{t \in [0,1]} \mathcal{L}_\alpha(Z_N(t))^{\frac{r}{1-\alpha}} \right). \quad (3.2)$$

It remain to obtain a bound for $|Y_N(t)|_{C^{-\alpha}}$ for small time. Write the mild form for the equation satisfied by Y_N :

$$Y_N(t) = e^{At}x - \int_0^t e^{A(t-s)}(Y_N^3(s) + 3Y_N^2(s)Z_N(s) + 3Y_N(s) : Z_N^2(s) : + : Z_N^3(s) :) ds$$

Let $\alpha < \frac{1}{3}$, $\beta > \alpha$ such that

$$\gamma := \frac{\alpha + \beta}{2} < \frac{1}{3}$$

and define

$$M(t) = \sup_{s \in [0,t]} (|Y_N(s)|_{C^{-\alpha}} + s^\gamma |Y_N(s)|_{C^{-\beta}}).$$

Then similar computations as in the proof of Theorem 3.3 in [TsWe18] give for $t \in [0, 1]$:

$$M(t) \leq c_1 |x|_{C^{-\alpha}} + c_2 t^{1-3\gamma} M(t)^3 + c_3 t^{1-3\gamma} \sup_{t \in [0,1]} \mathcal{L}_\alpha(Z_N(t))^3,$$

for some constants c_1, c_2 which can be chosen larger than 1.

Therefore if we set

$$t^* = \min \left\{ \left(c_3 \sup_{t \in [0,1]} \mathcal{L}_\alpha(Z_N(t))^3 \right)^{-\frac{1}{1-3\gamma}}, (8c_2(c_1|x|_{C^{-\alpha}} + 1)^2)^{-\frac{1}{1-3\gamma}}, 1 \right\}$$

we have

$$\sup_{t \in [0, t^*]} M(t) \leq 2(c_1|x|_{C^{-\alpha}} + 1) \quad (3.3)$$

It follows from (3.2) and (3.3):

$$\begin{aligned} \sup_{t \in [0,1]} |Y_N(t)|_{C^{-\alpha}} &\leq \sup_{t \in [0, t^*]} |Y_N(t)|_{C^{-\alpha}} + \sup_{t \in [t^*, 1]} |Y_N(t)|_{C^{-\alpha}} \\ &\leq 2(c_1|x|_{C^{-\alpha}} + 1) + \max \left((t^*)^{-2/r}, \sup_{t \in [0,1]} \mathcal{L}_\alpha(Z_N(t))^{\frac{rk_r}{r+2}} \right) \end{aligned}$$

The result follows taking the expectation and using finiteness of the moments of

$$\sup_{t \in [0,1]} \mathcal{L}_\alpha(Z_N(t)).$$

□

We set

$$\tilde{Z}_N(t) = e^{At}x + \Pi_N \int_0^t e^{A(t-s)} dW(s) = e^{At}x + Z_N(t),$$

and

$$\tilde{Y}_N(t) = X_N(t) - \tilde{Z}_N(t).$$

Then \tilde{Y}_N satisfies the same equation as Y_N , see (2.2), with Z_N replaced by \tilde{Z}_N and 0 initial datum.

Lemma 3.4. *Let $\alpha \in (0, 1/9)$, $k \geq 1$ then*

$$\mathbb{E} \left(\sup_{t \in [0,1]} |\tilde{Y}|_{L^2}^2 + \int_0^t |\nabla \tilde{Y}_N(s)|_{L^2}^2 ds \right)^k \leq C_{\alpha,k} \left(|x|_{C^{-\alpha}}^{12k/(1-\alpha)} + 1 \right)$$

for any $t > 0$.

Proof. Lemma 3.2 clearly also holds for \tilde{Y}_N provided Z_N is replaced by \tilde{Z}_N . We use it with $r = 2$ and again omit to indicate the dependency on N :

$$\frac{1}{2} \frac{d}{dt} |\tilde{Y}|_{L^2}^2 + \frac{1}{4} |\nabla \tilde{Y}|_{L^2}^2 + \frac{1}{2} |\tilde{Y}|_{L^4}^4 \leq c \mathcal{L}_\alpha(\tilde{Z})^{k_2},$$

c and $k_2 = \frac{4}{1-\alpha}$ are given in Lemma 3.2. We use the product rules in Besov spaces and the smoothing effect of e^{At} to prove:

$$\mathcal{L}_\alpha(\tilde{Z}(s)) \leq C_{\alpha,\epsilon} \left((s^{-(2\alpha+\epsilon)} + 1) |x|_{C^{-\alpha}}^3 + 1 \right) \mathcal{L}_\alpha(Z(s))$$

the result follows by integrating on $[0, T]$ and taking expectation of the k^{th} power. Note that since $\alpha < 1/9$, it is possible to choose $\epsilon > 0$ small enough $(2\alpha + \epsilon)k_2 < 1$. \square

Lemma 3.5. *Let $\alpha \in (0, 1/9)$, $k \geq 1$, there exists $\gamma_{\alpha,k}, C_{\alpha,k}$ such that*

$$\mathbb{E} \left(\sup_{t \in [0,1]} |\tilde{Y}_N(t)|_{H^{\alpha+\epsilon}}^k \right) \leq C_{\alpha,k} (|x|_{C^{-\alpha}} + 1)^{\gamma_{\alpha,k}}.$$

Proof. We write, omitting again N

$$\tilde{Y}(t) = \int_0^t e^{A(t-s)} (\tilde{Y}^3(s) + 3\tilde{Y}^2(s)\tilde{Z}(s) + 3\tilde{Y}(s) : \tilde{Z}^2(s) : + : \tilde{Z}^3(s) :) ds$$

and get

$$|\tilde{Y}(t)|_{H^{\alpha+\epsilon}} \leq c \int_0^t (t-s)^{-(\alpha+\epsilon/2)} \left(|\tilde{Y}^3|_{H^{-\alpha}} + |\tilde{Y}^2 \tilde{Z}|_{H^{-\alpha}} + |\tilde{Y} : \tilde{Z}^2 :|_{H^{-\alpha}} + | : \tilde{Z}^3 :|_{H^{-\alpha}} \right) ds.$$

We have

$$\begin{aligned} |\tilde{Y}^3|_{H^{-\alpha}} &\leq c |\tilde{Y}^2|_{H^{\alpha+\epsilon}} |\tilde{Y}|_{C^{-\alpha}} \leq c |\tilde{Y}|_{B_{4,4}^{\alpha+\epsilon}}^2 |\tilde{Y}|_{C^{-\alpha}} \leq c |\tilde{Y}|_{H^{\alpha+\epsilon+1/2}}^2 |\tilde{Y}|_{C^{-\alpha}} \\ &\leq c \left(|\tilde{Y}|_{L^2}^{2(1/2-\alpha-\epsilon)} |\nabla \tilde{Y}|_{L^2}^{2(1/2+\alpha+\epsilon)} + |\tilde{Y}|_{L^2}^2 \right) |\tilde{Y}|_{C^{-\alpha}}. \end{aligned}$$

Similarly

$$|\tilde{Y}^2 \tilde{Z}|_{H^{-\alpha}} \leq c |\tilde{Y}^2|_{H^{\alpha+\epsilon}} |\tilde{Z}|_{C^{-\alpha}} \leq c \left(|\tilde{Y}|_{L^2}^{2(1/2-\alpha-\epsilon)} |\nabla \tilde{Y}|_{L^2}^{2(1/2+\alpha+\epsilon)} + |\tilde{Y}|_{L^2}^2 \right) |\tilde{Z}|_{C^{-\alpha}}$$

and

$$|\tilde{Y} : \tilde{Z}^2 :|_{H^{-\alpha}} \leq c |\tilde{Y}|_{H^{\alpha+\epsilon}} | : \tilde{Z}^2 :|_{C^{-\alpha}} \leq c \left(|\tilde{Y}|_{L^2}^{1-(\alpha+\epsilon)} |\nabla \tilde{Y}|_{L^2}^{\alpha+\epsilon} + |\tilde{Y}|_{L^2} \right) | : \tilde{Z}^2 :|_{C^{-\alpha}}.$$

Gathering these results gives

$$\begin{aligned} |\tilde{Y}(t)|_{H^{\alpha+\epsilon}} &\leq c \int_0^t (t-s)^{-(\alpha+\epsilon/2)} \left[\left(|\tilde{Y}|_{L^2}^{2(1/2-\alpha-\epsilon)} |\nabla \tilde{Y}|_{L^2}^{2(1/2+\alpha+\epsilon)} + |\tilde{Y}|_{L^2}^2 \right) (|\tilde{Y}|_{C^{-\alpha}} + |\tilde{Z}|_{C^{-\alpha}}) \right. \\ &\quad \left. + \left(|\tilde{Y}|_{L^2}^{1-(\alpha+\epsilon)} |\nabla \tilde{Y}|_{L^2}^{\alpha+\epsilon} + |\tilde{Y}|_{L^2} \right) | : \tilde{Z}^2 :|_{C^{-\alpha}} + | : \tilde{Z}^3 :|_{C^{-\alpha}} \right] ds =: (a) + (b) + (c). \end{aligned}$$

The first term is bounded thanks to Hölder inequality

$$\begin{aligned} (a) &= c \int_0^t (t-s)^{-(\alpha+\epsilon/2)} \left[\left(|\tilde{Y}|_{L^2}^{2(1/2-\alpha-\epsilon)} |\nabla \tilde{Y}|_{L^2}^{2(1/2+\alpha+\epsilon)} + |\tilde{Y}|_{L^2}^2 \right) (|\tilde{Y}|_{C^{-\alpha}} + |\tilde{Z}|_{C^{-\alpha}}) ds \right. \\ &\leq c \left(|\tilde{Y}|_{L^{p_1}(0,t,L^2)}^{2(1/2-\alpha-\epsilon)} (|\nabla \tilde{Y}|_{L^2(0,t,L^2)}^{2(1/2+\alpha+\epsilon)} + |\tilde{Y}|_{L^{p_2}(0,t,L^2)}^2) \right) \left(|Y|_{L^{p_3}(0,t,C^{-\alpha})} + |x|_{C^{-\alpha}} + |Z|_{L^{p_3}(0,t,C^{-\alpha})} \right) \end{aligned}$$

with

$$\frac{2(1/2-\alpha-\epsilon)}{p_1} + (1/2+\alpha+\epsilon) + \alpha + \frac{\epsilon}{2} < 1 + \frac{1}{p_3}$$

and

$$\frac{2}{p_2} + \alpha + \frac{\epsilon}{2} + \frac{1}{p_3} < 1.$$

Since $\alpha < \frac{1}{4}$, ϵ, p_1, p_2, p_3 can be chosen such this is satisfied. Then

$$\begin{aligned} (b) &\leq c \int_0^t (t-s)^{-(\alpha+\epsilon/2)} \left(|\tilde{Y}|_{L^2}^{1-(\alpha+\epsilon)} |\nabla \tilde{Y}|_{L^2}^{\alpha+\epsilon} + |\tilde{Y}|_{L^2} \right) \\ &\times \left[s^{-(\alpha+\epsilon/2)} (|x|_{C^{-\alpha}}^2 + |x|_{C^{-\alpha}} |Z|_{C^{-\alpha}}) + | : Z^2 : |_{C^{-\alpha}} \right] ds \\ &\leq c \left(|\tilde{Y}|_{L^{p_4}(0,t,L^2)}^{1-(\alpha+\epsilon)} |\nabla \tilde{Y}|_{L^2(0,t,L^2)}^{\alpha+\epsilon} + |\tilde{Y}|_{L^{p_5}(0,t,L^2)} \right) \\ &\times (|x|_{C^{-\alpha}}^2 + |x|_{C^{-\alpha}} |Z|_{L^{p_6}(0,t,C^{-\alpha})} + | : Z^2 : |_{L^{p_6}(0,t,C^{-\alpha})}) \end{aligned}$$

with

$$\alpha + \frac{\epsilon}{2} + \frac{1-(\alpha+\epsilon)}{p_4} + \frac{\alpha+\epsilon}{2} + \alpha + \frac{\epsilon}{2} + \frac{1}{p_6} < 1$$

and

$$2(\alpha + \frac{\epsilon}{2}) + \frac{1}{p_5} + \frac{1}{p_6} < 1$$

which again is possible since $\alpha < \frac{1}{4}$. Similarly

$$\begin{aligned} (c) &\leq c \int_0^t (t-s)^{-(\alpha+\epsilon/2)} \left(s^{-(2\alpha+\epsilon)} (|x|_{C^{-\alpha}}^3 + |x|_{C^{-\alpha}}^2 |Z|_{C^{-\alpha}}) \right. \\ &\left. + s^{-(\alpha+\epsilon/2)} |x|_{C^{-\alpha}} | : Z^2 : |_{C^{-\alpha}} + | : Z^3 : |_{C^{-\alpha}} \right) ds \end{aligned}$$

Taking expectation we find thanks to Lemma 3.3, Lemma 3.4, Hölder inequality and finiteness of the moments of $\sup_{t \in [0,1]} |Z|_{C^{-\alpha}}$, $\sup_{t \in [0,1]} | : Z^2 : |_{C^{-\alpha}}$ and $\sup_{t \in [0,1]} | : Z^3 : |_{C^{-\alpha}}$:

$$\mathbb{E} \left(\sup_{t \in [0,1]} |\tilde{Y}(t)|_{H^{\alpha+\epsilon}}^k \right) \leq c (|x|_{C^{-\alpha}}^{\kappa_{\alpha,k}} + 1)$$

for some $\kappa_{\alpha,k}$ which could be written explicitly. □

Lemma 3.6. *Let $\alpha \in (0, 1/9)$, for any $k \geq 1$, there exists $\bar{\gamma}_{\alpha,k}, C_{\alpha,k}$ such that*

$$\mathbb{E} \left(| : X_N(t)^2 : |_{H^{-\alpha}}^k ds \right) \leq C_{\alpha,k} (t^{-\alpha k} + 1) (|x|_{C^{-\alpha}} + 1)^{\bar{\gamma}_{\alpha,k}}.$$

Proof. We have, omitting the dependence in N ,

$$\begin{aligned}
| : X(t)^2 : |_{H^{-\alpha}} &\leq c (|Y^2(t)|_{H^{-\alpha}} + |Y(t)|_{H^{\alpha+\epsilon}} |Z(t)|_{C^{-\alpha}} + | : Z(t)^2 : |_{C^{-\alpha}}) \\
&\leq c (|Y(t)|_{H^{\alpha+\epsilon}} (|Y(t)|_{C^{-\alpha}} + |Z(t)|_{C^{-\alpha}}) + | : Z(t)^2 : |_{C^{-\alpha}}) \\
&\leq c \left((t^{-\alpha} |x|_{C^{-\alpha}} + |\tilde{Y}(t)|_{H^{\alpha+\epsilon}}) (|Y(t)|_{C^{-\alpha}} + |Z(t)|_{C^{-\alpha}}) + | : Z(t)^2 : |_{C^{-\alpha}} \right)
\end{aligned}$$

The result follows by Lemma 3.3 and Lemma 3.5. \square

3.1. Estimates on S_t^N . We have

$$S_t^N \varphi(x) = \mathbb{E} \left(e^{-K \int_0^t V(X_N(s)) ds} \varphi(X_N(t, x)) \right)$$

where the potential V is given by

$$V(x) = | : x^2 : |_{H^{-\alpha}}^p.$$

Lemma 3.7. *Let $\alpha \in (0, \frac{1}{9})$, $s > 0$, $p \geq 1$ such that $\alpha + \frac{1}{3p} + \frac{s}{3} < \frac{1}{3}$ and $\alpha(2p-1) + s < 2$. Then there exists γ such that for any $\varphi \in \mathcal{B}_b(C^{-\alpha})$ we have for $t \in [0, 1]$, $x \in C^{-\alpha}$, $h \in H$*

$$|DS_t^N \varphi(x) \cdot h| \leq c(t^{-(1+s+2\alpha p)/2} + 1) |h|_{C^{-s}} \|\varphi\|_0 (1 + |x|_{C^{-\alpha}})^\gamma$$

and

$$|DS_t^N (V\varphi)(x) \cdot h| \leq c(t^{-(1+s+2\alpha p)/2} + 1) |h|_{C^{-s}} \|\varphi\|_0 (1 + |x|_{C^{-\alpha}})^\gamma.$$

Remark 3.8. $V\varphi$ is not bounded but it is clear from Lemma 3.6 that S_t^N can be extended to borelian functions with polynomial growth.

Proof. We only prove the second inequality, the first one is similar.

We omit to mention the dependence of N . From [DaZa97] we know that $S_t^N \varphi$ is differentiable in any direction $h \in H$ and we have the following modified Bismut–Elworthy–Li formula:

$$\begin{aligned}
DS_t(V\varphi)(x) \cdot h &= \frac{1}{t} \mathbb{E} \left(e^{-K \int_0^t V(X(s, x)) ds} V(X(t, x)) \varphi(X(t, x)) \int_0^t (\eta^h(s), dW(s)) \right) \\
&\quad + 2K \mathbb{E} \left(e^{-K \int_0^t V(X(s, x)) ds} V(X(t, x)) \varphi(X(t, x)) \int_0^t (1 - \frac{s}{t}) V'(X(s, x)) \cdot \eta^h(s) ds \right) \\
&= (a) + (b).
\end{aligned}$$

Clearly

$$\begin{aligned}
(a) &\leq \frac{1}{t} \|\varphi\|_0 \left[\mathbb{E} (V(X(t, x)))^2 \right]^{1/2} \\
&\quad \times \left[\mathbb{E} \left(e^{-2K \int_0^t V(X(s, x)) ds} \left| \int_0^t (\eta^h(s), dW(s)) \right|^2 \right) \right]^{1/2}.
\end{aligned}$$

By Lemma 3.6 we have for K large enough

$$\left[\mathbb{E} (V(X(t, x)))^2 \right]^{1/2} \leq c(t^{-\alpha p} + 1)(1 + |x|_{C^{-\alpha}})^{\bar{\gamma}_{\alpha, 2p/2}}.$$

To estimate the second factor, we proceed as in the proof of [DaDe03-2, Lemma 4.1]. Writing Itô's formula for

$$d \left| e^{-K \int_0^t V(X(s, x)) ds} \int_0^t (\eta^h(s), dW(s)) \right|^2$$

we deduce for K large enough

$$\begin{aligned}
& \mathbb{E} \left(e^{-2K \int_0^t V(X(s,x)ds} \left| \int_0^t (\eta^h(s), dW(s)) \right|^2 \right)^{1/2} \\
& \leq \mathbb{E} \left(\int_0^t e^{-2K \int_0^s V(X(r,x)dr} \left| \eta^h(s) \right|^2 ds \right)^{1/2} \\
& \leq c \mathbb{E} \left(\int_0^t \sigma^{-s} ds \right)^{1/2} |h|_{C^{-s}}
\end{aligned}$$

thanks to Lemma 3.1 and the embedding $L^\infty \subset L^2$. In Lemma 3.1, we choose $\epsilon > 0$ such that $\alpha + \epsilon + \frac{1}{3p} + \frac{s}{3} < \frac{1}{3}$.

It follows that

$$(a) \leq c(t^{-(1+s)/2} + 1)(t^{-\alpha p} + 1) |h|_{C^{-s}} \|\varphi\|_0 (1 + |x|_{C^{-\alpha}})^{\gamma_{\alpha, 2p/2}}.$$

Similarly, the other term is bounded by

$$(b) \leq c \|\varphi\|_0 \left[\mathbb{E} (V(X(t,x)))^2 \right]^{1/2} \mathbb{E} \left(e^{-2K \int_0^t V(X(s,x)ds} \left| \int_0^t V'(X(s,x)) \cdot \eta^h(s) ds \right|^2 \right)^{1/2}$$

We have

$$\begin{aligned}
|V'(X(s,x)) \cdot \eta^h(s)| &= 2p(| : X^2(s) : |_{H^{-\alpha}}^{p-2} |(: X^2(s) :, X(s)\eta^h(s))|_{H^{-\alpha}} \\
&\leq 2p(| : X^2(s) : |_{H^{-\alpha}}^{p-1} |X(s)\eta^h(s)|_{H^{-\alpha}} \\
&\leq 2p(| : X^2(s) : |_{H^{-\alpha}}^{p-1} |X(s)|_{H^{-\alpha}} |\eta^h(s)|_{C^{\alpha+\epsilon}}.
\end{aligned}$$

By Lemma 3.1, Minkowski inequality, Lemma 3.3 and Lemma 3.5, we thus can write for K large enough:

$$\begin{aligned}
& E \left(e^{-2K \int_0^t V(X(s,x)ds} \left| \int_0^t V'(X(s,x)) \cdot \eta^h(s) ds \right|^2 \right)^{1/2} \\
& \leq c \mathbb{E} \left(\left(\int_0^t | : X^2(s) : |_{H^{-\alpha}}^{p-1} |X(s)|_{H^{-\alpha}} s^{-(\alpha+\epsilon+s)/2} ds \right)^2 \right)^{1/2} |h|_{C^{-s}} \\
& \leq c \int_0^t \mathbb{E} \left(\left(| : X^2(s) : |_{H^{-\alpha}}^{p-1} |X(s)|_{H^{-\alpha}} \right)^2 \right)^{1/2} s^{-(\alpha+\epsilon+s)/2} ds |h|_{C^{-s}} \\
& \leq c \left(\int_0^t (1 + s^{-(\alpha+\epsilon+s+2\alpha(p-1))/2}) ds \right) (1 + |x|_{C^{-\alpha}})^{(\bar{\gamma}_{\alpha, 2p/2} + \bar{\gamma}_{\alpha, 4(p-1)} + \kappa_{\alpha, 4})/4} |h|_{C^{-s}}.
\end{aligned}$$

We deduce for $\epsilon > 0$ small enough so that $\alpha + \epsilon + s + 2\alpha(p-1) < 2$:

$$(b) \leq c \|\varphi\|_0 (1 + t^{-\alpha p}) (1 + |x|_{C^{-\alpha}})^{(\bar{\gamma}_{\alpha, 4(p-1)} + \kappa_{\alpha, 4})/4} |h|_{C^{-s}}$$

The conclusion follows when gathering the estimates on (a) and (b). □

3.2. **Estimate on P_t^N .** Recall that

$$P_t^N \varphi(x) = S_t^N \varphi(x) + K \int_0^t S_{t-s}^N (V P_s^N \varphi) ds.$$

Theorem 3.9. *Let $s \in (0, 1)$ and $\alpha \in (0, 1/9)$ such that $\frac{2\alpha}{1-s} < 1 - s - 3\alpha$. Let $p \geq 1$ such that $\frac{2\alpha}{1-s} < \frac{1}{p} < 1 - s - 3\alpha$, then there exists $\gamma > 0$, and $c > 0$ depending on s, α and p such that for any $\varphi \in \mathcal{B}_b(C^{-\alpha})$ we have for $t \in [0, 1]$, $x \in C^{-\alpha}$, $h \in H$:*

$$|DP_t^N \varphi(x) \cdot h| \leq c(t^{-(1+s+2\alpha p)/2} + 1) |h|_{C^{-s}} \|\varphi\|_0 (1 + |x|_{C^{-\alpha}})^\gamma.$$

Remark 3.10. *It is easy to find α, s satisfying the assumptions of Theorem 3.9. For instance, choose any $s \in (0, 1)$, $\delta \in (0, \frac{1-s}{2})$ and $\alpha < \min\{\frac{\delta(1-s)}{1+3\delta}; s\}$. Then*

$$\frac{\alpha}{\delta} < 1 - s - 3\alpha.$$

Moreover

$$\frac{\alpha}{\delta} > \frac{2\alpha}{1-s}.$$

It follows that one may choose p such that $\frac{2\alpha}{1-s} < \frac{\alpha}{\delta} < \frac{1}{p} < 1 - s - 3\alpha$ and Theorem 3.9 implies

$$|DP_t^N \varphi(x) \cdot h| \leq c(t^{-(1+s+\delta)/2} + 1) |h|_{C^{-s}} \|\varphi\|_0 (1 + |x|_{C^{-\alpha}})^\gamma.$$

Remark 3.11. *Since H is dense in C^{-s} , this result shows that $DP_t^N \varphi(x)$ belongs to the dual of C^{-s} and the bound extends to $h \in C^{-s}$.*

Proof. Since for $t \geq 1$, $\|P_t \varphi\|_0 \leq \|\varphi\|_0$, it suffices to prove the result for $t \in [0, 1]$ and use the Markov property.

Since $\frac{2\alpha}{1-s} > \frac{2\alpha}{2-s+\alpha}$, we have $\frac{2\alpha}{2-s+\alpha} < \frac{1}{p} < 1 - s - 3\alpha$ and this is equivalent to the assumptions of Lemma 3.7, therefore we may write:

$$\begin{aligned} |DP_t^N \varphi(x) \cdot h| &\leq |DS_t^N \varphi(x) \cdot h| + K \int_0^t |DS_{t-s}^N (V P_s \varphi)(x)| ds \\ &\leq c(t^{-(1+s+2\alpha p)/2} + 1) |h|_{C^{-s}} \|\varphi\|_0 (1 + |x|_{C^{-\alpha}})^\gamma \\ &\quad + K \int_0^t c(t-s)^{-(1+s+2\alpha p)/2} |h|_{C^{-s}} \|\varphi\|_0 (1 + |x|_{C^{-\alpha}})^\gamma ds. \end{aligned}$$

This implies the result since $1 + s + 2\alpha p < 2$. □

Letting $N \rightarrow \infty$ in Theorem 3.9, we deduce:

Corollary 3.12. *Let $s \in (0, 1)$ and $\alpha \in (0, 1/9)$ such that $\frac{2\alpha}{1-s} < 1 - s - 3\alpha$. Let $p \geq 1$ such that $\frac{2\alpha}{1-s} < \frac{1}{p} < 1 - s - 3\alpha$, then there exists $\gamma > 0$, and $c > 0$ depending on s, α and p such that for any $\varphi \in \mathcal{B}_b(H)$ we have for $t \in [0, 1]$, $x, y \in C^{-\alpha}$:*

$$|P_t \varphi(x) - P_t \varphi(y)| \leq c(t^{-(1+s+2\alpha p)/2} + 1) |x - y|_{C^{-s}} \|\varphi\|_0 (1 + |x|_{C^{-\alpha}} + |y|_{C^{-\alpha}})^\gamma.$$

4. THE KOLMOGOROV OPERATOR

Given $N \in \mathbb{N}$, we denote by $\mathcal{F}_N C_b^\infty$ the set of all functions $\varphi : H \rightarrow \mathbb{R}$ of the form

$$\varphi(x) = g(x_j, |j| \leq N), \tag{4.1}$$

where $g : \mathbb{R}^{(2N+1)^2} \rightarrow \mathbb{R}$, $(\xi_j, |j| \leq N) \rightarrow g(\xi_j, |j| \leq N)$ is of class C_b^∞ . Moreover, we set

$$\mathcal{F}C_b^\infty = \bigcap_{N=1}^{\infty} \mathcal{F}_N C_b^\infty.$$

Note that if $\varphi \in \mathcal{F}_N C_b^\infty$ is of the form (4.1) we have

$$D\varphi(x) = \sum_{|k| \leq N} D_k g(\xi_j, |j| \leq N) e_k \quad (4.2)$$

and

$$D^2\varphi(x) = \sum_{|k| \leq N, |h| \leq N} D_h D_k g(\xi_j, |j| \leq N) e_h \otimes e_k. \quad (4.3)$$

Let us introduce the Kolmogorov operator on $\mathcal{F}C_b^\infty$, setting

$$\mathcal{K}\varphi := \frac{1}{2} \text{Tr} [D^2\varphi] + \langle Ax - :x^3:, D\varphi \rangle, \quad \varphi \in \mathcal{F}C_b^\infty. \quad (4.4)$$

This definition is meaningful because if φ is given by (4.2) we have

$$\langle :x^3:, D\varphi \rangle = \sum_{|k| \leq N} \langle D_k g(\xi_j, |j| \leq N) e_k, :x^3: \rangle.$$

We set

$$\rho(x) = Z^{-1} e^{-2U(x)},$$

so that

$$D \log \rho(x) = -2DU(x) = -2 :x^3: . \quad (4.5)$$

We also consider the approximate measure ν_N in H

$$\nu_N(dx) = Z_N^{-1} e^{-U_N(x)} \mu_N(dx), \quad (4.6)$$

where

$$U_N(x) = \frac{1}{4} \int_H :x_N^4: (\xi) d\xi \quad (4.7)$$

and

$$Z_N := \int_H e^{-U_N(x)} \mu_N(dx).$$

Note that

$$\mathcal{K}(\varphi^2) = 2\varphi\mathcal{K}\varphi + |D\varphi|^2, \quad \forall \varphi \in \mathcal{F}_N C_b^\infty. \quad (4.8)$$

Integrating with respect to ν over \mathcal{H} , yields

$$\int_{\mathcal{H}} \mathcal{K}\varphi \varphi d\nu = -\frac{1}{2} \int_{\mathcal{H}} |D\varphi|^2 d\nu, \quad \forall \varphi \in \mathcal{F}C_b^\infty. \quad (4.9)$$

We also introduce an approximating operator \mathcal{K}_N on $\mathcal{F}C_b^\infty$ setting, for $\varphi(x) = g(x_j, |j| \leq M)$

$$\begin{aligned} \mathcal{K}_N \varphi &:= \frac{1}{2} \text{Tr} [\Pi_N D^2\varphi] + \langle Ax - :x_N^3:, D\varphi \rangle \\ &= \frac{1}{2} \sum_{|h| \leq N \wedge M} D_h^2 g(\xi_j, |j| \leq M) - \sum_{|h| \leq M} (\alpha_h - \rho_N) x_h D_h g(\xi_j, |j| \leq M) \\ &\quad - \sum_{|h| \leq M} \sum_{|j_1|, |j_2|, |j_3| \leq M} x_{j_1} x_{j_2} x_{j_3} \delta_{h, j_1+j_2+j_3} D_h g(\xi_j, |j| \leq M), \end{aligned} \quad (4.10)$$

where $\alpha_h = (1 + |h|^2)$.

Similarly as above, we have

$$\int_{\mathcal{H}} \mathcal{K}_N \varphi \varphi \, d\nu_N = -\frac{1}{2} \int_{\mathcal{H}} |D\varphi|^2 d\nu_N, \quad \forall \varphi \in \mathcal{F}_N C_b^\infty. \quad (4.11)$$

4.1. m -dissipativity of \mathcal{K} . Fix $N_0 \in \mathbb{N}$ and let $f \in \mathcal{F}_{N_0} C_b^\infty$. For $N \in \mathbb{N}$ we consider the solution φ_N of the approximating equation

$$\lambda \varphi_N - \frac{1}{2} \operatorname{Tr} [D^2 \Pi_N \varphi_N] + \langle Ax_N - :x_N^3:, D\varphi_N \rangle = f. \quad (4.12)$$

Then, taking into account (4.11) we find the estimate

$$\int_H |D\varphi_N|^2 d\nu_N \leq \frac{2}{\lambda} \int_H f^2 d\nu_N. \quad (4.13)$$

Theorem 4.1. \mathcal{K} is extendible to an m -dissipative operator in $L^1(\mathcal{H}, \nu)$ whose extension we denote by $\mathcal{K}^{(1)}$. Moreover, $\mathcal{F} C_b^\infty$ is a core for $\mathcal{K}^{(1)}$.

Proof. We have only to show

$$\lim_{N \rightarrow \infty} \int_{\mathcal{H}} \langle :x_N^3: - :x^3:, D\varphi_N \rangle d\nu = 0 \quad (4.14)$$

and then to apply the Lumer–Phillips theorem in some Besov spaces.

We take α, s and p satisfying the assumptions of Theorem 3.9 and write

$$|DP_t^N f(x) \cdot h| \leq c t^{-(1+s+\alpha p)/2} + 1) |h|_{C^{-s}} \|f\|_0 (1 + |x|_{C^{-\alpha}})^\gamma, \quad t \geq 0,$$

for some $\alpha < \frac{1}{8}$, $\epsilon > 0$ and c, γ depending on s, α, ϵ . Moreover, we can choose $1 + \alpha + \epsilon + s < 2$.

We write

$$\varphi_N(x) = \int_0^\infty e^{-\lambda t} P_t^N f(x) ds$$

and deduce

$$|D\varphi_N(x) \cdot h| \leq c |h|_{C^{-s}} \|f\|_0 (1 + |x|_{C^{-\alpha}})^\gamma.$$

It then suffices to write

$$\left| \int_{\mathcal{H}} \langle :x_N^3: - :x^3: \rangle D\varphi_N \right| d\nu \leq c \int_{\mathcal{H}} | :x_N^3: - :x^3: |_{C^{-s}} \|f\|_0 (1 + |x|_{C^{-\alpha}})^\gamma d\nu$$

By the Hölder inequality this converges to 0. □

Remark 4.2. Arguing as in [DaDe07] one can show that the closure $\mathcal{K}^{(p)}$ of \mathcal{K} in $L^p(\mathcal{H}, \nu)$, $p \geq 1$ is m -dissipative. Moreover the gradient operator

$$D : \mathcal{F} C_b^1 \subset L^p(\mathcal{H}, \nu) \rightarrow L^p(\mathcal{H}, \nu; H)$$

is closable. We denote by \overline{D}_p its closure and by $W^{1,p}(\mathcal{H}, \nu)$ its domain.

Finally,

$$D(\mathcal{K}^{(p)}) \subset W^{1,2}(\mathcal{H}, \nu)$$

and it results

$$\int_{\mathcal{H}} \mathcal{K}^{(2)} \varphi \varphi \, d\nu = -\frac{1}{2} \int_{\mathcal{H}} |D\varphi|^2 d\nu, \quad \forall \varphi \in D(\mathcal{K}^{(2)}). \quad (4.15)$$

Moreover since, as easily checked

$$\int_{\mathcal{H}} \mathcal{K}^{(2)} \varphi \psi \, d\nu = -\frac{1}{2} \int_{\mathcal{H}} \langle D\varphi, D\psi \rangle d\nu, \quad \forall \varphi, \psi \in \mathcal{F} C_b^2(H),$$

it results

$$\int_{\mathcal{H}} \mathcal{K}^{(2)} \varphi \psi d\nu = -\frac{1}{2} \int_{\mathcal{H}} \langle D\varphi, D\psi \rangle d\nu, \quad \forall \varphi, \psi \in D(\mathcal{K}^{(2)}), \quad (4.16)$$

so that $\mathcal{K}^{(2)}$ is *symmetric*. Finally, let us recall the classical integration by parts formula for the Gaussian measure μ

$$\int_{\mathcal{H}} \langle D\varphi, z \rangle \psi d\mu = - \int_{\mathcal{H}} \langle D\psi, z \rangle \varphi d\mu + \int_{\mathcal{H}} W_z \psi \varphi d\mu, \quad h \in \mathbb{Z}^2, \psi, \varphi \in \mathcal{FC}_b^1(\mathcal{H}), \quad (4.17)$$

where

$$W_z(x) = \langle Q^{-1/2}x, z \rangle = \sum_{k \in \mathbb{Z}^2} \sqrt{1 + |k|^2} z_k x_k, \quad \phi \in \mathcal{H}$$

and Q is the covariance of μ . Since $\nu(dx) = \rho(x)\mu(dx)$, setting $\psi = \rho$ in (4.17) yields

$$\int_{\mathcal{H}} \langle D\varphi, z \rangle \rho d\mu = - \int_{\mathcal{H}} \langle D\rho, z \rangle \varphi d\mu + \int_{\mathcal{H}} W_z \rho \varphi d\mu, \quad h \in \mathbb{Z}^2, \varphi \in \mathcal{FC}_b^1(\mathcal{H}).$$

which is equivalent to

$$\int_{\mathcal{H}} \langle D\varphi, z \rangle d\nu = - \int_{\mathcal{H}} \langle D \log \rho, z \rangle \varphi d\nu + \int_{\mathcal{H}} W_z \varphi d\nu, \quad h \in \mathbb{Z}^2, \varphi \in \mathcal{FC}_b^1(\mathcal{H}).$$

Since $\langle D \log \rho, z \rangle = -\langle : x^3 :, z \rangle$ we have

$$\int_{\mathcal{H}} \langle D\varphi, z \rangle d\nu = \int_{\mathcal{H}} \langle : x^3 :, z \rangle \varphi d\nu + \int_{\mathcal{H}} W_z \varphi d\nu, \quad h \in \mathbb{Z}^2, \varphi \in \mathcal{FC}_b^1(\mathcal{H}).$$

Therefore, for any $p > 1$ and any $z \in \mathbb{Z}^2$ there is $C_p > 0$ such that

$$\left| \int_{\mathcal{H}} \langle D\varphi, z \rangle d\nu \right| \leq C_p \|\varphi\|_{L^p(\mathcal{H}, \nu)} |z|. \quad (4.18)$$

This inequality is useful to develop a geometrical theory for the measure ν , see [DaDe17], [DaDe16] and [DaLuTu18].

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