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Mean curvature rigidity of horospheres, hyperspheres and hyperplanes

Rabah Souam

Abstract We prove that horospheres, hyperspheres and hyperplanes in a hyperbolic space $\mathbb{H}^n$, $n \geq 3$, admit no perturbations with compact support which increase their mean curvature. This is an extension of the analogous result in the Euclidean spaces, due to M. Gromov, which states that a hyperplane in a Euclidean space $\mathbb{R}^n$ admits no compactly supported perturbations having mean curvature ≥ 0.

Keywords mean curvature, mean convexity, tangency principle.

Mathematics Subject Classification (2010) 53A10, 53C40

The content of the present note is motivated by the following nice result of M. Gromov [1]

Theorem 1. *A hyperplane in a Euclidean space $\mathbb{R}^n$ cannot be perturbed on a compact set so that its mean curvature satisfies $H \geq 0$.*

Gromov [1] showed that Theorem 1 can be derived from the non existence of $\mathbb{Z}^n$-invariant metrics with positive scalar curvature on $\mathbb{R}^n$. He also gave another direct argument to prove it using a symmetrization process.

In what follows, we give another proof of Theorem 1 and an extension to the hyperbolic spaces using a simple argument. More precisely, we prove the following:

Theorem 2. *Let M denote a horosphere, a hypersphere or a hyperplane in a hyperbolic space $\mathbb{H}^n, n \geq 3$ and $H_M \geq 0$ its (constant) mean curvature. Let Σ be a connected properly embedded $\mathcal{C}^2$-hypersurface in $\mathbb{H}^n$ which coincides with M outside a compact subset of $\mathbb{H}^n$. If the mean curvature of Σ is $\geq H_M$, then $\Sigma = M$.*

We recall that a hyperplane in $\mathbb{H}^n$ is a complete totally geodesic hypersurface and a hypersphere is a connected component of a set equidistant from a hyperplane.

The proof uses the tangency principle which goes back to E. Hopf. We recall it with some details in what follows. We first fix some notations and conventions.

Let Σ be an embedded $\mathcal{C}^2$-hypersurface, with a global unit normal ν, in a smooth complete $n-$dimensional Riemannian manifold M. We denote by σ the second fundamental

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form of Σ, which is defined as follows

$$\sigma_p(u,v) = - < \nabla_u \nu, v > \quad \text{for } p \in \Sigma,\ u, v \in T_p\Sigma.$$

The shape operator S of Σ is defined as follows

$$S_p u = -\nabla_u \nu \quad \text{for } p \in \Sigma,\ u \in T_p\Sigma,$$

and the (normalized) mean curvature of Σ is the function

$$H = \frac{1}{n-1} \operatorname{trace} S.$$

Note that these definitions depend on the choice of the unit normal field ν. The mean curvature vector field $\vec{H} := H\nu$ is, instead, independent of the choice of the unit normal field. With our conventions, the mean curvature of a unit sphere in the Euclidean space with respect to its interior unit normal is equal to 1.

Let $p \in \Sigma$ and consider local coordinates $(x_1, \dots, x_n)$ around p in M so that $T_p\Sigma = \mathbb{R}^{n-1} \times \{0\}$ and $\frac{\partial}{\partial x_n}(0) = \nu(p)$. An open neighborhood $\mathcal{U}$ of p in Σ is the graph, in the these coordinates, of a $\mathcal{C}^2$-function u defined on an open neighborhood Ω of the origin in $\mathbb{R}^{n-1} \times \{0\}$. The mean curvature of $\mathcal{U}$, computed with respect to the unit normal field ν, is given by a nonlinear elliptic operator

$$\mathcal{M}(u) = \mathcal{F}(x, u, Du, D^2u)$$

with $\mathcal{F} : (x, z, \xi, \mathbf{s}) \in \Omega \times \mathcal{V} \to \mathcal{F}(\mathrm{x}, \mathrm{z}, \xi, \mathbf{s})$ a smooth function defined in $\Omega \times \mathcal{V}$ where $\mathcal{V}$ is an open subset of $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n}$.

Definition. Let Σ_1 and Σ_2 be two embedded hypersurfaces in the Riemannian manifold M oriented by global unit normals ν_1 and ν_2 respectively and let p an interior point of both Σ_1 and Σ_2. We say that $\Sigma_1 \geq \Sigma_2$ near p if the following conditions are satisfied:

(i) $\nu_1(p) = \nu_2(p)$,

(ii) there are local coordinates $(x_1, \dots, x_n)$ around p in M with $T_p\Sigma_1 = T_p\Sigma_2 = \mathbb{R}^{n-1} \times \{0\}$ and $\frac{\partial}{\partial x_n}(0) = \nu_1(p) = \nu_2(p)$, in which Σ_1 and Σ_2 are graphs over an open domain in $\mathbb{R}^{n-1} \times \{0\}$ of functions u_1 and u_2 respectively, satisfying $u_1 \geq u_2$.

Suppose that $\Sigma_1 \geq \Sigma_2$ near p and that their mean curvature functions (computed with respect to the given normals), in local coordinates as above, verify $\mathcal{M}(u_1) \leq \mathcal{M}(u_2)$. The tangency principle asserts that Σ_1 coincides with Σ_2 in a neighborhood of p. The argument goes as follows. Consider the difference function $u = u_2 - u_1$ and set $u_t = u_1 + t(u_2 - u_1)$ for $t \in [0, 1]$. We can write the following :

$$\begin{aligned}\mathcal{M}(u_2) - \mathcal{M}(u_1) = &\left(\int_0^1 \frac{\partial \mathcal{F}}{\partial z}(u_t)dt\right) u + \left(\int_0^1 \frac{\partial \mathcal{F}}{\partial \xi}(u_t)dt\right) . Du \\ &+ \left(\int_0^1 \frac{\partial \mathcal{F}}{\partial \mathbf{s}}(u_t)dt\right) . D^2u.\end{aligned}$$

Here $\left(\int_0^1 \frac{\partial \mathcal{F}}{\partial \xi}(u_t)dt\right) . Du = \sum_i b_i(x)\, u_{x_i}(x)$ with $b_i(x) := \int_0^1 \frac{\partial \mathcal{F}}{\partial \xi_i}(u_t)dt$, and $\left(\int_0^1 \frac{\partial \mathcal{F}}{\partial \mathbf{s}}(u_t)dt\right) . D^2u = \sum_{i,j} a_{ij}(x) u_{x_i x_j}(x)$ with $a_{ij}(x) := \int_0^1 \frac{\partial \mathcal{F}}{\partial \mathbf{s}_{i,j}}(u_t)dt$.

This shows that the $\mathcal{C}^2$-function $u = u_2 - u_1$ satisfies the inequality $Lu \geq 0$ where L is the linear elliptic operator with continuous coefficients defined by

$$Lv = \left(\int_0^1 \frac{\partial \mathcal{F}}{\partial z}(u_t)dt\right) v + \left(\int_0^1 \frac{\partial \mathcal{F}}{\partial \xi}(u_t)dt\right) . Dv + \left(\int_0^1 \frac{\partial \mathcal{F}}{\partial \mathbf{s}}(u_t)dt\right) . D^2 v.$$

By assumption, $u \leq 0$ and u vanishes at the origin. The strong maximum principle (see, for instance, Chapter 10, Addendum 2, Corollary 19 in [2]) then shows that $u \equiv 0$ in a neighborhood of the origin, that is, the hypersurfaces Σ_1 and Σ_2 coincide in a neighborhood of p. We can thus state the tangency principle in the following form:

Tangency Principle. *Let Σ_1 and Σ_2 be two embedded $\mathcal{C}^2$-hypersurfaces in a smooth Riemannian manifold M, oriented by global unit normals. Suppose the mean curvature functions H_1 and H_2 of respectively Σ_1 and Σ_2 verify $H_2 \geq a \geq H_1$ for some real number a. Let p be an interior point of both Σ_1 and Σ_2. If $\Sigma_1 \geq \Sigma_2$ near p, then Σ_1 coincides with Σ_2 in a neighborhood of p.*

We can now prove Theorem 2. We will work with the halfspace model of $\mathbb{H}^n$, that is,

$$\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n,\, x_n > 0\}$$

endowed with the metric

$$ds^2 = \frac{dx_1^2 + \cdots + dx_n^2}{x_n^2}.$$

Let P_0 be a hyperplane in $\mathbb{H}^n$. Without loss of generality, we may assume P_0 is the hyperplane $\{x_1 = 0, x_n > 0\}$. For $t \in \mathbb{R}$, we let P_t denote the hyperplane $\{x_1 = t, x_n > 0\}$.

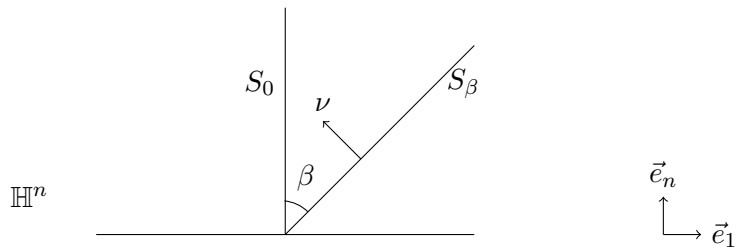
Let Σ be a properly embedded $\mathcal{C}^2$-hypersurface in $\mathbb{H}^n$ which coincides with P_0 outside a compact subset of $\mathbb{H}^n$. Suppose its mean curvature verifies $H_\Sigma \geq 0$. Σ separates $\mathbb{H}^n$ into 2 connected components, a mean convex one V_+ and a mean concave one V_-. We may assume V_+ coincides with the domain $\{x_1 > 0, x_n > 0\}$ outside a compact set; the other case being similar. There is a largest $T \geq 0$ verifying $\Sigma \cap P_T \neq \emptyset$. We orient P_T by its unit normal field pointing in the direction $x_1 > 0$. The hypersurfaces Σ and P_T are both closed subsets of $\mathbb{H}^n$, so their intersection $\Sigma \cap P_T$ is closed in Σ. Let us show it is also open in Σ. Take a point $p \in \Sigma \cap P_T$. Then the normal to Σ at p points in the direction $x_1 > 0$ (since we assumed V_+ coincides with the domain $\{x_1 > 0, x_n > 0\}$ outside a compact set) and $\Sigma \leq P_T$ near p. As $H_\Sigma \geq 0$, the tangency principle shows that Σ coincides with P_T in a neighborhood of p. This shows that $\Sigma \cap P_T$ is open in Σ and since it is also closed, we conclude that $\Sigma = P_T$ and $T = 0$.

Consider now the case of a horosphere which we may suppose, without loss of generality, is the horosphere $\mathcal{H}_1 = \{x_n = 1\}$. We also consider the family of horospheres $\mathcal{H}_t = \{x_n = t\}$, $t > 0$, having the same asymptotic boundary as $\mathcal{H}_1$. We orient the horospheres $\mathcal{H}_t$ by their upward pointing unit normal. For this choice of orientation their mean curvature is equal to one.

Let Σ be a properly embedded $\mathcal{C}^2$-hypersurface in $\mathbb{H}^n$ which coincides with $\mathcal{H}_1$ outside a compact subset of $\mathbb{H}^n$ and having mean curvature ≥ 1. Σ divides $\mathbb{H}^n$ into 2 components. The mean convex one coincides with the domain $\{x_n > 1\}$ outside a compact set since Σ asymptotically coincides with $\mathcal{H}_1$. There is a largest $T \geq 1$ such that $\Sigma \cap \mathcal{H}_T \neq \emptyset$. At a point $p \in \Sigma \cap \mathcal{H}_T$, the unit normals to Σ and the horopshere $\mathcal{H}_T$ coincide and $\Sigma \leq \mathcal{H}_T$ near p. As $H_\Sigma \geq 1$, the tangency principle shows that Σ coincides with $\mathcal{H}_T$ in a neighborhood

of p. So the set $\Sigma \cap \mathcal{H}_T$ is open in Σ and since it is also closed, we conclude that Σ coincides with the horosphere $\mathcal{H}_T$ and that $T = 1$.

We consider now the case of hyperspheres. Let us denote, this time, the hyperplane $\{x_1 = 0, x_n > 0\}$ by S_0. For $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, we let S_β be the intersection with $\mathbb{H}^n$ of the hyperplane P_β in $\mathbb{R}^n$ of equation $\cos\beta\, x_1 = \sin\beta\, x_n$. Note that β is the oriented angle between P_β and P_0. S_β is equidistant from S_0, the distance between them being equal to $\log \frac{1+|\sin\beta|}{\cos\beta}$. We orient S_β by the normals pointing in the same direction as the vector $-\cos\beta\, \vec{e}_1 + \sin\beta\, \vec{e}_n$, where $\vec{e}_1, \ldots, \vec{e}_n$ is the canonical basis in $\mathbb{R}^n$. For this choice of orientation the mean curvature of S_β is given by $H_\beta = \sin\beta$.



Given a hypersphere in $\mathbb{H}^n$, it is congruent to S_{β_0} for some $\beta_0 \in (0, \frac{\pi}{2})$. Let Σ be a properly embedded $\mathcal{C}^2$-hypersurface in $\mathbb{H}^n$ which coincides with S_{β_0} outside a compact set and having mean curvature $H_\Sigma \geq \sin\beta_0$. Σ separates $\mathbb{H}^n$ in 2 components and the mean convex one coincides, outside a compact set, with the mean convex domain bounded by S_{β_0}. There is a smallest $\beta \in (-\frac{\pi}{2}, \beta_0]$ so that $\Sigma \cap S_\beta \neq \emptyset$. At a point $p \in \Sigma \cap S_\beta$, the unit normals to S_β and Σ coincide, $\Sigma \leq S_\beta$ near p and $H_\Sigma \geq \sin\beta_0 \geq \sin\beta = H_\beta$. By the tangency principle, we know that Σ and S_β coincide in a neighborhood of p. This shows that the intersection $\Sigma \cap S_\beta$ is open in Σ. As it is also closed, we conclude that $\Sigma = S_\beta$ and also that $\beta = \beta_0$.

Remarks.

1. To prove Theorem 1, one uses, as in the case of a hyperplane in $\mathbb{H}^n$, the family of hyperplanes parallel to the given deformed hyperplane.

2. The case of a hyperplane in $\mathbb{H}^n$ can also be treated using the family of hyperspheres equidistant to it.

3. The argument above can also be utilized to obtain analogous rigidity results for hypersurfaces in other ambient manifolds. Consider, for instance, a product manifold $M \times \mathbb{R}$ where M is a connected non compact orientable manifold without boundary. Suppose $M \times \mathbb{R}$ is equipped with a Riemannian metric so that $M_t = M \times \{t\}$ has constant mean curvature for each $t \geq 0$. This happens, for instance, for a warped product metric, that is, a metric of the form $f(t)ds_M^2 + dt^2$, where ds_M^2 is a complete metric on M and f a smooth positive function on $[0, +\infty)$. Call H_t the mean curvature of M_t computed with respect to the unit normal field pointing into the domain $M \times [t, +\infty)$ and suppose the function $t \in [0, +\infty) \to H_t$ is non increasing. Then proceeding as above one shows that, for each $t_0 \geq 0$, if Σ is a connected properly embedded hypersurface which coincides with M_{t_0} outside a compact subset of $M \times \mathbb{R}$ and has mean curvature $H_\Sigma \geq H_{t_0}$, then $\Sigma = M_{t_0}$.

References

[1] M. Gromov, *Mean Curvature in the Light of Scalar Curvature.* arXiv:1812.09731.
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