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Sum-of-squares Flight Control Synthesis for Deep-stall Recovery

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In lieu of extensive Monte-Carlo simulations for flight control verification, sum-of-squares programming techniques provide an algebraic approach to the problem of nonlinear control synthesis and analysis. However, their reliance on polynomial models has hitherto limited the applicability to aeronautical control problems. Taking advantage of recently proposed piecewise polynomial models, this paper revisits sum-of-squares techniques for recovery of an aircraft from deep-stall conditions using a realistic yet tractable aerodynamic model. Local stability analysis of classical controllers is presented as well as synthesis of polynomial feedback laws with the objective of enlarging their nonlinear region of attraction. A newly developed synthesis algorithm for backwards-reachability facilitates the design of recovery control laws, ensuring stable recovery by design. The paper’s results motivate future research in aeronautical sum-of-squares applications.

Nomenclature

α	Angle of attack (rad);
α_0	Low-angle of attack boundary ($\alpha_0 = 16.2949^\circ$);
γ_A	Flight-path angle relative to air (rad);
η	Elevator deflection (rad), negative if leading to positive pitch moment;
ϑ	Signed distance ratio ($\vartheta \in \mathbb{R}$);

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λ	Level set ($\lambda \in \mathbb{R}_+$);
ρ	Pseudo-radius ($\rho \in \mathbb{R}_+$);
$\varphi(\cdot)$	Boundary condition function ($\varphi: \mathbb{R}^m \rightarrow \mathbb{R}$);
$d\eta$	Change of elevator deflection (rad/s);
\mathbf{K}	Feedback control law ($\mathbf{K}: \mathbb{R}^n \rightarrow \mathbb{R}^p$);
n	Number of states; system degree;
p	Number of inputs;
P	Positive-definite, polynomial shape function ($P: \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}, P \in \mathbb{R}[\mathbf{x}]$);
q	Pitch rate (rad/s);
V_A	Aircraft speed relative to air (m/s), positive along x_a -axis;
\mathbf{x}^*, η^*	State vector and elevator deflection at trim condition;
$\tilde{\mathbf{x}}, \tilde{\eta}$	Scaled state vector and elevator deflection;
$(\cdot)^{post}$	Domain of high angle of attack;
$(\cdot)^{pre}$	Domain of low angle of attack;
\mathcal{E}	(Quasi)-Ellipsoidal set ($\mathcal{E} = \{\mathbf{x} P(\mathbf{x}) \leq \rho\}$) with shape function P and pseudo-radius ρ ;
$\Sigma[\mathbf{x}]$	Polynomial sum-of-squares cone ($\Sigma[\mathbf{x}] \subset \mathbb{R}[\mathbf{x}]$);
$\mathbb{R}[\mathbf{x}]$	Set of polynomials in \mathbf{x} with real-valued coefficients;

I. Introduction

PREDICTION AND PREVENTION of inflight loss-of-control (LOC-I) commonly requires prior knowledge of the aircraft's dynamics using a reliable and representative aerodynamic model. However, establishing a satisfactory model is rarely straightforward, and ensuring sufficient representation of every aspect of the operational envelope of the true vehicle is extremely challenging. Indeed, dynamics beyond the nominal flight envelope are highly nonlinear and often unstable. Flight control certification for commercial airliners therefore relies today on simple but extensive and cumbersome Monte-Carlo simulations of high-fidelity models [1] in order to analyse the viable subset of the flight envelope, which demands significant investment of time and computational power. Yet, more sophisticated tools based on nonlinear stability theory have

been applied in the literature. Mathematical continuation and bifurcation analysis establishes trim conditions and periodic orbits as well as their stability [2, 3]; however, attraction or reachability of a stable solution cannot be determined quantitatively using the continuation methodology. Reachability analysis, on the other hand, numerically evolves reachable subsets of the state-space over time, identifying possible violations of predefined constraints [4, 5]. An alternative formulation of the reachability problem is the algebraic notion of controlled invariant sets, or *safe sets* [6, 7]. Defined as the largest set such that the aircraft can be kept within the state constraints subject to control input limitations, the safe set determines bounds for prevention and recoverability from LOC-I events. The idea of a safe set as defined by the existence of an admissible control sequences is thus contrasted by the set of converging state trajectories subject to an a priori specified control law, namely, the region of attraction of the closed-loop system.

Determining the region of attraction of a given system up to a desired accuracy is, in general, a non-trivial task [8]. Recently, Lyapunov stability theory and LaSalle’s later extension have been turned into a systematic analysis approach employing sum-of-squares (SOS) programming techniques [9, 10]. Relaxed to semi-definite problems [11], SOS provides global stability proofs [12] as well as provable under-estimates for the region of attraction of systems defined by polynomial dynamics [13]. Those methods for stability analysis can further extended for synthesis of control feedback laws ensuring or enlarging a region of attraction subject to input constraints [14, 15].

Sum-of-squares techniques have been exploited to analyze the short-period motion of an F/A-18 aircraft model [16]. A special iteration technique, called *\mathcal{V} -s-iteration*, was applied to estimate the region of attraction of the longitudinal motion of the Generic Transport Model [17]. In [18], this technique was employed to validate a revised control law for the F/A-18 “falling-leaf” mode. It is worth noting that this work used a reduced six-state polynomial aircraft model that was derived by sampling the equations of motion rather than the aerodynamic coefficients. Despite SOS techniques being a powerful tool to generate Lyapunov functions for suitable models, few work on SOS for aircraft dynamics has been published since. Simple polynomials are often unsuitable to fit full-envelope aerodynamics accurately, whereas advanced modeling techniques, such as multivariate splines [19], are computationally heavy to analyse using SOS. Simple piecewise-defined models, as proposed by the authors in [20], have the potential of bridging this gap, since they both describe accurately aircraft dynamics in the domains of low and high angles of attack while only slightly increasing the computational load for sum-of-squares programming.

In this article, we synthesize controllers for and verify deep-stall recovery of a small unmanned aircraft using SOS programming and a piecewise polynomial model. The choice of aircraft, a fixed-wing capable of stable deep-stall transition, descent, and recovery [21], allows us to isolate longitudinal dynamics. Continuing our work in [20], we present an extended *\mathcal{V} -s-iteration* for piecewise-defined aircraft dynamics in order to

obtain feedback control laws subject to state and inputs constraints. While we initially rely on common polynomial surfaces to govern the estimate (cf. [13, 14, 17, 18]), we later replace it by a single deep-stall condition. We thus provide a systematic analysis by SOS beyond polynomial aircraft models. The present paper is organised as follows: In Section II, we introduce the aircraft, its piecewise polynomial model and recall the basics of SOS analysis. Section III analyzes stable recovery of a linear-quadratic regulator and synthesizes polynomial controllers subject to an enlarged region of attraction. Section IV concludes the article by a revised formulation of control synthesis specifically for deep-stall recovery.

Note on *polynomial surfaces* For sum-of-squares analysis and control synthesis, we make use of geometric objects defined by polynomial functions on the state-space, i.e., $\mathcal{E} = \{\mathbf{x} \mid P(\mathbf{x}) \leq \rho\}$ for some polynomial P in \mathbf{x} and $\rho > 0$. The most common examples are ellipsoids (in three dimensions) and the related hyper-ellipsoidal surfaces, which are governed by quadratic functions $P = \mathbf{x}^T \mathbf{Q} \mathbf{x}$ with positive-definite matrix \mathbf{Q} . For simplicity, we call both ellipsoids and hyper-ellipsoids *ellipsoidal* surfaces. The concept of (hyper)-ellipsoids can be further extended to positive polynomial surfaces of order larger than two, of which we will refer as *quasi-ellipsoids*.

II. Methodology

We consider an autonomous, 1.65 m-wingspan, unmanned aircraft that, instead of landing gear, is intended to land vertically descending by a deliberate deep-stall manoeuvre. In this manoeuvre, the drag of the wings counteract the gravitational force, leading to a stable trim condition, and the horizontal distance covered during the landing is minimised. For the purpose of stable deep-stall flight, the elevator is designed to exceed the usual range and reach deflections of up to -60° . The parameters of the aircraft used in this study are given by Tab. 1.

Table 1 Parameters of the aircraft.

flight mass	m	1.55	kg
wing span	b	1.66	m
mean chord	c_a	0.174	m
wing area	S	0.277	m ²
air density	ϱ	1.25	kg/m ³
gravitational constant	g	9.81	m/s ²

The aircraft is considered to be laterally stable due to its dihedral wings. We therefore neglect the lateral dynamics for the analysis of stability and, consequently, assume the side-slip angle β to vanish. In deep-stall flight and transition, the aircraft is further unthrottled, i.e., the thrust force is zero ($\mathbf{F} = 0$).

We will refer mainly to the international standard *air-path axis system* (x_a, y_a, z_a) oriented along the

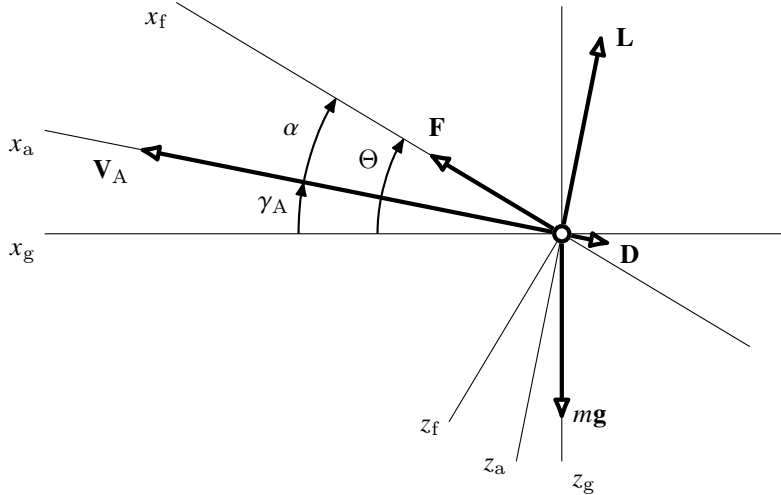


Fig. 1 Longitudinal axes with angles and vectors for $\beta = 0$.

aircraft's velocity vector with respect to air (\mathbf{V}_A) [22]. Lift and drag forces are defined along these axes and denoted \mathbf{L} and \mathbf{D} ; angle of attack α , flight-path angle γ_A , and pitch angle Θ are given by rotations into *body axis system* (x_f, y_f, z_f) as well as *earth-fixed axis system* (x_g, y_g, z_g), defined by the aircraft's fuselage and the ground, and between. (Fig. 1). If not stated otherwise, all variables are in SI-units; angles are however given in degrees where convenient.

A polynomial $g \in \mathbb{R}[\mathbf{x}]$ is a *sum of squares* (SOS) if $g = \sum_i g_i(\mathbf{x})^2$ for some $(g_i)_i \subset \mathbb{R}[\mathbf{x}]$; the set of sum-of-squares polynomials is denoted by $\Sigma[\mathbf{x}]$. It can be proven that $g \in \Sigma[\mathbf{x}]$ if and only if there is a positive semidefinite matrix \mathbf{M} such that $g = \mathbf{z}^T \mathbf{M} \mathbf{z}$, where \mathbf{z} is a vector of monomials in \mathbf{x} [10]. This relation reduces the problem of finding SOS polynomials to semidefinite programming [11], given that the objective is linear in the SOS variables. However, the problem of a single decision variable entering bilinearly into the objective, although bilinear problems are generally \mathcal{NP} -hard, can efficiently be solved as quasi-convex SOS program [10]. Notwithstanding that any SOS polynomial is positive semidefinite, the opposite does not hold.

A. Equations of motion

The longitudinal equations of motion without thrust are given as nonlinear 4-state, ordinary differential equations [17, 23, 24]:

$$\dot{V}_A = -\frac{1}{2}\rho V_A^2 S m^{-1} C_D(\alpha, \eta) - g \sin \gamma_A; \quad (1)$$

$$V_A \dot{\gamma}_A = \frac{1}{2}\rho V_A^2 S m^{-1} C_L(\alpha, \eta) - g \cos \gamma_A; \quad (2)$$

$$I_y \dot{q} = \frac{1}{2}\rho V_A^2 S c_a C_m(\alpha, \eta) - k_{\dot{q}q} q; \quad (3)$$

$$\dot{\Theta} = q; \quad (4)$$

with the pitch angle

$$\Theta = \gamma_A + \alpha \quad (5)$$

and the air speed V_A as norm of the velocity vector relative to air. Then, C_L , C_D , C_m are dimensionless coefficients connected to lift force, drag force, and pitching moment. The elevator deflection η is, by convention, negative when causing a positive pitching moment. The linear damping coefficient $k_{\dot{q}q}$ accounts for non-static aerodynamics (see [21]).

The aerodynamic coefficients of the body, wing, and surfaces have been modeled by piecewise polynomial functions

$$C_{\odot}(\alpha, \dots) = \begin{cases} C_{\odot}^{pre}(\alpha, \dots) & \text{if } \alpha \leq \alpha_0; \\ C_{\odot}^{post}(\alpha, \dots) & \text{else;} \end{cases} \quad (6)$$

with $C_{\odot}^{pre}(\alpha_0, \cdot) \equiv C_{\odot}^{post}(\alpha_0, \cdot)$ and $\alpha_0 = 16.2949^\circ$. Fig. 2 shows the piecewise model and their polynomial segments. Defined as piecewise polynomials, we are able to account for full-envelope characteristics both of the lift and drag coefficients as well as the coefficients in body axes [20]. The resulting models are continuous over the entire domain but not necessarily differentiable in its joint. The pitch-moment coefficient C_m is modeled likewise. The full aircraft model is detailed in [25].

B. Region of attraction estimation

In the following, we develop a region of attraction estimation for piecewise polynomial systems under constrained control inputs using SOS programming and extend this framework to find a control law that enlarges the region of attraction of the controlled system. In order to resolve the resulting bilinear terms, we

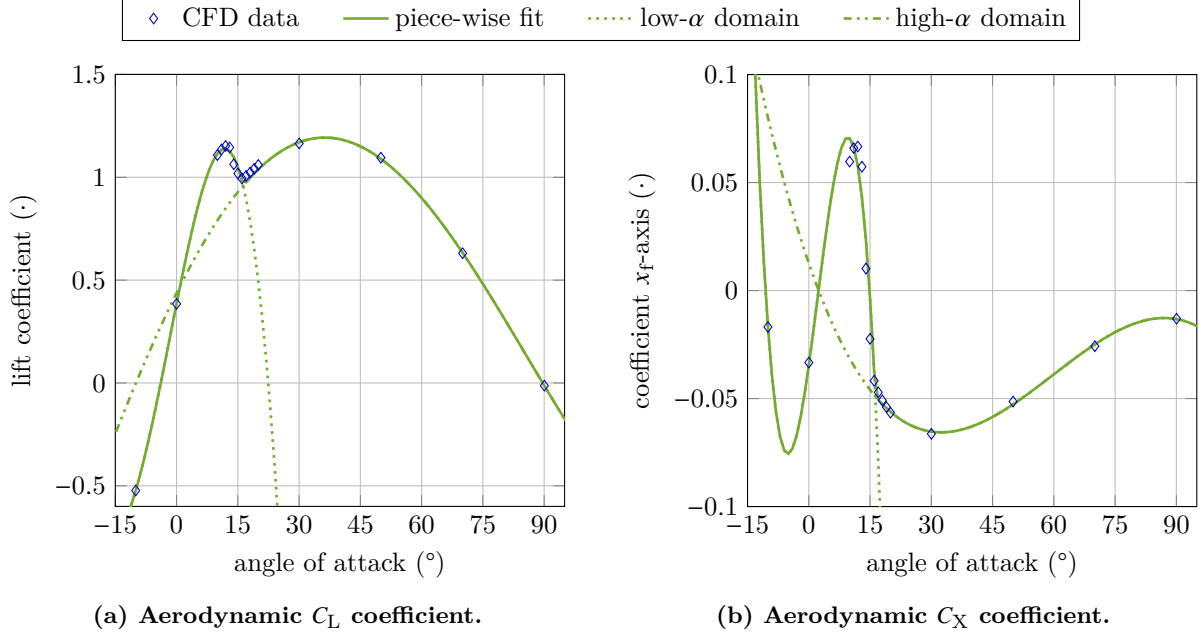


Fig. 2 Piecewise model of aerodynamic coefficients with polynomial segments [21].

will make use of the \mathcal{V} -s-iteration [17].

Let a piecewise polynomial system be defined as

$$\mathbf{f}(\mathbf{x}, \mathbf{u}) = \begin{cases} \mathbf{f}_1(\mathbf{x}, \mathbf{u}) & \text{if } \varphi(\mathbf{x}) \leq 0; \\ \mathbf{f}_2(\mathbf{x}, \mathbf{u}) & \text{else;} \end{cases} \quad (7)$$

with state vector $\mathbf{x} \in \mathcal{X} \subset \mathbb{R}^n$, input vector $\mathbf{u} \in \mathcal{U} \subset \mathbb{R}^p$, submodels $\mathbf{f}_1, \mathbf{f}_2 \in \mathbb{R}[\mathbf{x}, \mathbf{u}]^n$ and boundary $\varphi \in \mathbb{R}[\mathbf{x}]$; assume further $\mathbf{f}(\mathbf{x}^*, \mathbf{u}^*) = 0$. The equilibrium $(\mathbf{x}^*, \mathbf{u}^*)$ is stable if there exists a non-empty set of initial conditions such that the trajectories converge, the *region of attraction* \mathcal{R} , and \mathcal{R} contains \mathbf{x}^* in its interior. Now, $\Omega_\lambda = \{\mathbf{x} | \mathcal{V}(\mathbf{x}) \leq \lambda\}$ is an invariant subset of \mathcal{R} if, for $\mathcal{V}: \mathcal{X} \rightarrow \mathbb{R}$ continuous and positive definite, $\mathcal{V}(0) = 0$, and $\lambda \in \mathbb{R}_+$,

$$\forall \mathbf{x} \in \Omega_\lambda. \quad \nabla \mathcal{V} \mathbf{f}(\mathbf{x}, \mathbf{u}^*) < 0 \quad (8)$$

and Ω_λ is bounded [8]. Moreover, we call Ω_λ *invariant under control* $\mathbf{K}(\cdot)$ if Ω_λ is an invariant set of the closed-loop system $\mathbf{f}_{\mathbf{K}}: \mathbf{x} \mapsto \mathbf{f}(\mathbf{x}, \mathbf{K}(\mathbf{x}))$ for some control law $\mathbf{u} = \mathbf{K}(\mathbf{x})$ and $\mathbf{K}(\mathbf{x}) \in \mathcal{U}$ for all $\mathbf{x} \in \Omega_\lambda$.

As Lyapunov functions are non-unique, alternative $\mathcal{V}(\cdot)$ give rise to different estimates of the region of attraction. For comparison of the size of an invariant subset, the \mathcal{V} -s-iteration introduces a surface $\mathcal{E}_\rho = \{\mathbf{x} | P(\mathbf{x}) \leq \rho\}$ with $P \in \mathbb{R}[\mathbf{x}]$ positive (quadratic) chosen as parameter of the estimation [13]. The

estimation of a maximal invariant subset of the region of attraction is then subject to the optimisation problem

$$\max_{\substack{\mathcal{V}, \epsilon \in \mathbb{R}[\mathbf{x}] \\ \lambda > 0}} \rho > 0 \quad \text{s.t. (8) and } \mathcal{E}_\rho \subset \Omega_\lambda.$$

as well as $\mathcal{V}(\mathbf{x}) > 0$ if $\mathbf{x} \neq 0$ and $\mathcal{V}(0) = 0$.

1. Piecewise region of attraction

Recall that Ω_λ bounded is an invariant subset of \mathcal{R} under control \mathbf{K} if $\mathcal{V}(\cdot)$ is a continuous and positive definite function and $\nabla \mathcal{V} \mathbf{f}_{\mathbf{K}i}(x) < 0$ for any $\mathbf{x} \in \Omega_\lambda$ and $i \in \{1, 2\}$ such that $\mathbf{f}_{\mathbf{K}i}$ is active. For $\mathcal{V}_{1,2} \in \mathbb{R}[\mathbf{x}]$, we have the sufficient SOS constraint [20] (see also [13, Lemma 10])

$$\mathcal{V}_{1,2}(\mathbf{x}) - l_a \in \Sigma[\mathbf{x}] \tag{9}$$

$$-\nabla \mathcal{V}_1 \mathbf{f}_{\mathbf{K}1} - l_b + (\mathcal{V}_1 - \lambda) s_{1,\lambda} + \varphi s_{1,\varphi} \in \Sigma[\mathbf{x}] \tag{10}$$

$$-\nabla \mathcal{V}_2 \mathbf{f}_{\mathbf{K}2} - l_b + (\mathcal{V}_2 - \lambda) s_{2,\lambda} + \varphi s_{2,\varphi} \in \Sigma[\mathbf{x}], \tag{11}$$

where $s_{i,\lambda}, s_{i,\varphi} \in \Sigma[\mathbf{x}]$, $i \in \{1, 2\}$ relax negativity of $\nabla \mathcal{V}_i \mathbf{f}_i$ to the respective partitions defined by Ω_λ and $\varphi(\cdot)$ and $l_{a,b} \in \mathbb{R}[\mathbf{x}]$ are positive definite terms [13], e.g., $l_a = l_b = \epsilon \mathbf{x}^T \mathbf{x}$ with some small $\epsilon > 0$. As $\mathcal{V}(\cdot)$ is defined piecewise itself—i.e., $\mathcal{V}(\mathbf{x}) = \mathcal{V}_1(\mathbf{x})$ if $\varphi(\mathbf{x}) \leq 0$; $\mathcal{V}(\mathbf{x}) = \mathcal{V}_2(\mathbf{x})$ else;—, we ensure continuity along the boundary $\varphi \equiv 0$ by the additional SOS constraint

$$-\mathcal{V}_1 + \mathcal{V}_2 + \varphi s_{\varphi,1} - \varphi s_{-\varphi,1} \in \Sigma[\mathbf{x}] \tag{12}$$

$$-\mathcal{V}_2 + \mathcal{V}_1 + \varphi s_{\varphi,2} - \varphi s_{-\varphi,2} \in \Sigma[\mathbf{x}], \tag{13}$$

where $s_{\varphi,i}, s_{-\varphi,i} \in \Sigma[\mathbf{x}]$ with $i \in \{1, 2\}$ enforce that both $\mathcal{V}_1(\mathbf{x}) \leq \mathcal{V}_2(\mathbf{x})$ and $\mathcal{V}_2(\mathbf{x}) \leq \mathcal{V}_1(\mathbf{x})$ if $\varphi(\mathbf{x}) = 0$.

2. Invariant sets under control

The definition (7) and subsequent constraints (9)–(11), as is easy to see, apply equally to closed-loop controlled systems $\mathbf{f}_{\mathbf{K}}$ where $\mathbf{K}(\cdot)$ is linear or polynomial in \mathbf{x} . This alone, however, is insufficient for invariance under control as we have defined it above. We might now assume \mathcal{U} to be defined as conjunction $\mathcal{U} = \{\mathbf{u} \mid p_1(\mathbf{u}) \leq 0, \dots, p_m(\mathbf{u}) \leq 0\}$ with $(p_j)_{1 \leq j \leq m} \subset \mathbb{R}[\mathbf{u}]$; if furthermore $\mathbf{K} \in \mathbb{R}[\mathbf{x}]^P$, we can state a necessary

SOS constraint for invariance of Ω_λ under control \mathbf{K} as

$$-(p_j \circ \mathbf{K}) + (\mathcal{V} - \lambda) s_{j,\lambda} \in \Sigma[\mathbf{x}], \quad (14)$$

where $(p_j \circ \mathbf{K})(\mathbf{x}) = p_j(\mathbf{K}(\mathbf{x}))$, with $s_{j,\lambda} \in \Sigma[\mathbf{x}]$ for all $1 \leq j \leq m$. The constraints (9)–(11) and (14), in fact, hold independently of each other and can be evaluated separately. As the subset-relation “ \subseteq ” on the level sets $\Omega_{(\cdot)}$ constitutes a total order, $\Omega_{\lambda_{\mathbf{K}}}$ with $\lambda_{\mathbf{K}} = \min\{\lambda, \lambda'\}$ is an invariant subset of the region of attraction of $\mathbf{f}_{\mathbf{K}}(\cdot)$ under control \mathbf{K} .

3. \mathcal{V} -s-iteration

The ellipsoid \mathcal{E}_ρ can be fitted inside the invariant Ω_λ if

$$-(\mathcal{V}_i - \lambda) + (P - \rho) s_{i,P} \in \Sigma[\mathbf{x}] \quad (15)$$

with $s_{i,P} \in \Sigma[\mathbf{x}]$ for $i \in \{1, 2\}$.

Note that we require $\mathcal{V}_{1,2}$ to be of some fixed degree. However, some constraints involve bilinear terms of the form $\mathcal{V}_i s_{(\cdot)}$ once $\mathcal{V}_{1,2}$ become decision variables. The \mathcal{V} -s-iteration uses a bisection approach of iteratively-alternating steps; a detailed discussion of the basic \mathcal{V} -s-iteration has been given by Chakraborty et al. [17]. We extend here the approach in order to incorporate control input constraints for a $\mathbf{K}(\cdot)$ given a priori and, later on, synthesize an optimal control feedback:

- 1a) Find λ° maximal such that (10)–(11) hold for $\mathcal{V}_{1,2}$ fixed;
- 1b) Find λ^* maximal such that (14) holds for $\mathcal{V}_{1,2}$ fixed;
- 2) Find ρ° maximal such that (15) holds for $\mathcal{V}_{1,2}$ and $\lambda^\dagger = \min\{\lambda^\circ, \lambda^*\}$ fixed;
- 3) Find $\mathcal{V}_1, \mathcal{V}_2 \in \mathbb{R}[\mathbf{x}]$ of fixed degree such that (9)–(15) hold for ρ° and λ^\dagger fixed.

The purpose of \mathcal{E}_ρ here is twofold: first, to quantify the size of the provable invariant subset Ω_λ for each iteration; and second, to prevent the last step from yielding a smaller estimate than hitherto achieved.

C. Control synthesis

Until now, we have considered the control law to be given and fixed. Yet, we can further adapt our approach to find a suitable \mathbf{K} in the attempt to enlarge the (estimated) region of attraction within the bounds imposed by the control input constrains. As the SOS constraints must be linear in the prospective control function, we require the control system to be in the *companion form* affine in \mathbf{u} , viz. $\dot{\mathbf{x}} = \mathbf{f}_{\mathbf{x}}(\mathbf{x}) + \mathbf{f}_{\mathbf{u}}(\mathbf{x}) \mathbf{u}$ with $\mathbf{f}_{\mathbf{x}} \in \mathbb{R}[\mathbf{x}]^n, \mathbf{f}_{\mathbf{u}} \in \mathbb{R}[\mathbf{x}]^{n \times p}$, such that $\mathbf{f}_{\mathbf{K}}(\mathbf{x}) = \mathbf{f}_{\mathbf{x}}(\mathbf{x}) + \mathbf{f}_{\mathbf{u}}(\mathbf{x}) \mathbf{K}(\mathbf{x})$. The input constraints $(p_j)_j$, too, must be

linear in \mathbf{u} , $p_j(\mathbf{u}) = \mathbf{p}_j^T \mathbf{u}$ with $\mathbf{p}_j \in \mathbb{R}^p$, and \mathbf{K} guarantees control-invariance of $\Omega_{\lambda'}$ if

$$-\mathbf{p}_j^T \mathbf{K} + (\mathcal{V} - \lambda') s_{j,\lambda'} \in \Sigma[\mathbf{x}] \quad (16)$$

with $s_{j,\lambda'} \in \Sigma[\mathbf{x}]$ for all j .

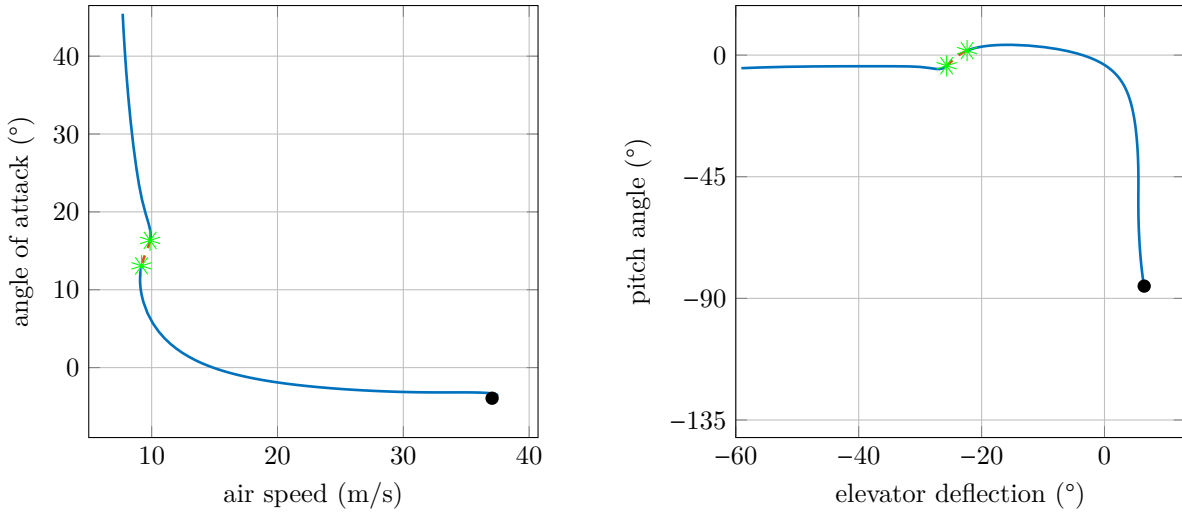
Then, in order to circumvent bilinearities, we execute again steps 1) and 2) for \mathbf{K} fixed and incorporate a supplementary second-to-last step

3a) Find $\mathbf{K} \in \mathbb{R}[\mathbf{x}]$ of fixed degree such that (10)–(11) and (16) hold for $\mathcal{V}_{1,2}$ as well as ρ° and λ^\dagger fixed.

The old and new last step is once more computed for \mathbf{K} fixed. The thus augmented iteration is performed by Algorithm 1.

D. Preliminary stability analysis

In [21], we have applied bifurcation theory as well as optimization techniques in order to derive the linear pitch-damping model in Eq. (3). We have thus identified an optimal coefficient $C_{mq} \approx 1.96$. In consequence, Fig. 3 shows the location and stability of longitudinal trim conditions (note that the choice of C_{mq} does not affect the location of stationary solutions as $q = 0$ is a necessary conditions for trim), parametrized by the elevator deflection η . The black dot in Fig. 3 indicates the largest deflection, $\eta = 6.5^\circ$, for which the aircraft enters a steep, nose-down descent; the minimal elevator deflection is -60° . Shortly after stall, the aircraft encounters an unstable regime of stationary solutions with a family of limit cycles (*Hopf bifurcation*).



(a) Angle of attack α over air speed V .

(b) Pitch angle Θ over elevator deflection η .

Fig. 3 Trim conditions of longitudinal motion with unstable regimes dashed [21].

Algorithm 1 Extended \mathcal{V} -s-iteration for control synthesis under state and input constraints.

1: **for** $N = 1$ to N_{\max}

2: **if** $N > 1$ **then**

3: find $\mathbf{K}_N \in \mathbb{R}[\mathbf{x}]$ s.t. for all $j \in [1, m]$, $i \in \{1, 2\}$,

$$\begin{aligned} & -\mathbf{p}_j^T \mathbf{K}_N + (\mathcal{V}_{N,i} - \lambda^\dagger) s_{ij,\lambda'} \in \Sigma[\mathbf{x}] \\ & -\nabla \mathcal{V}_{N,i} \mathbf{f}_{\mathbf{K}_N i} - l_b + (\mathcal{V}_{N,i} - \lambda^\dagger) s_{i,\lambda} + \varphi s_{i,\varphi} \in \Sigma[\mathbf{x}] \end{aligned}$$

4: find $\mathcal{V}_{N,1}, \mathcal{V}_{N,2} \in \mathbb{R}[\mathbf{x}]$ s.t. $s_{\varphi,i}, s_{-\varphi,i} \in \Sigma[\mathbf{x}]$ and for all $j \in [1, m]$, $i \in \{1, 2\}$,

$$\begin{aligned} & \mathcal{V}_{N,i}(\mathbf{x}) - l_a \in \Sigma[\mathbf{x}] \\ & -(\mathcal{V}_{N,i} - \lambda^\dagger) + (P - \rho^\circ) s_{i,P} \in \Sigma[\mathbf{x}] \\ & -\nabla \mathcal{V}_{N,i} \mathbf{f}_{\mathbf{K}_N i} - l_b + (\mathcal{V}_{N,i} - \lambda^\dagger) s_{i,\lambda} + \varphi s_{i,\varphi} \in \Sigma[\mathbf{x}] \\ & -\mathbf{p}_j^T \mathbf{K}_N + (\mathcal{V}_{N,i} - \lambda^\dagger) s_{ij,\lambda'} \in \Sigma[\mathbf{x}] \\ & -\mathcal{V}_{N,i} + \mathcal{V}_{N,3-i} + \varphi s_{\varphi,i} - \varphi s_{-\varphi,i} \in \Sigma[\mathbf{x}] \end{aligned}$$

5: **end**

6: **for** $i \in \{1, 2\}$

7: find $\lambda_i^\circ := \max_{\lambda \geq 0} \lambda$ s.t. $s_{i,\lambda}, s_{i,\varphi} \in \Sigma[x]$ and

$$-\nabla \mathcal{V}_{N,i} \mathbf{f}_{\mathbf{K}_N i} - l_b + (\mathcal{V}_{N,i} - \lambda) s_{i,\lambda} + \varphi s_{i,\varphi} \in \Sigma[\mathbf{x}]$$

8: find $\lambda_i^* := \max_{\lambda' \geq 0} \lambda'$ s.t. $s_{ij,\lambda'} \in \Sigma[\mathbf{x}]$ and for all $j \in [1, m]$,

$$-\mathbf{p}_j^T \mathbf{K}_N + (\mathcal{V}_{N,i} - \lambda') s_{ij,\lambda'} \in \Sigma[\mathbf{x}]$$

9: **end**

10: $\lambda^\dagger := \min\{\lambda_1^\circ, \lambda_2^\circ, \lambda_1^*, \lambda_2^*\}$

11: **for** $i \in \{1, 2\}$

12: find $\rho_i := \max_{\rho \geq 0} \rho$ s.t. $s_{i,P} \in \Sigma[x]$ and

$$-(\mathcal{V}_{N,i} - \lambda^\dagger) + (P - \rho) s_{i,P} \in \Sigma[\mathbf{x}]$$

13: **end**

14: $\rho^\circ = \min\{\rho_1, \rho_2\}$

15: **end**

III. Region of Attraction at Trim Point

Where classical control synthesis relies upon linearized models, the region of attraction estimation provides knowledge about the limitations of the chosen control implementation. Unlike the safe set [see, e.g., 6, 7], that provides an exploratory study in order to estimate the abilities of the aircraft to be controlled, we must study a region of attraction in the context of a given controller and the respective trim condition [20]. For the latter we choose a low-inclination gliding descent trim at $\eta_{\text{glide}}^* = -5^\circ$ (see Appendix A).

In this section, we investigate the capability of different controllers to stably recover the aircraft from a deep-stall trim condition. We first consider a linear quadratic regulator, which could have been derived by classical control techniques. Later on however, we apply the SOS tools in order to derive a polynomial control law that improves stability and recovery of the vehicle. For both analysis and control synthesis, we scale the state vector $\mathbf{x}^T = [V_A \quad \gamma_A \quad q \quad \alpha]$ by $\text{diag}(10 \text{ m/s}, 45^\circ, 150^\circ/\text{s}, 45^\circ)^{-1}$ and the input η , by $(80^\circ)^{-1}$, in order to normalize states and inputs. The scaled variables are henceforth denoted by $\tilde{\mathbf{x}}, \tilde{\eta}$, etc. The viable (unscaled) elevator inputs are given to $\mathcal{U} = [-60^\circ; +20^\circ]$ and represent the physical limits of the aircraft elevator. We further approximate the non-polynomial functions (sine, cosine, inverse) by Taylor series expansions and truncate high-order polynomial terms (Appendix A) in order to facilitate the resulting SOS problems [17].

The aircraft longitudinal motion is commonly divided into *short-period* dynamics involving pitch rate and angle of attack as well as the long-period *phugoid* oscillation of airspeed and flight-path angle and often discussed separately, as-if uncoupled. When discussing the region of attraction, we take into account the full, coupled 4-state model of Eqs. (1)–(4) but display the estimates as projections into either phugoid V_A - γ_A plane or short-period α - q plane. Details for all SOS computations are given in Appendix B.

A. Analysis of the Linear Quadratic Regulator

A further but more advanced element of the classical linear toolbox is the renown *linear quadratic regulator* (LQR). Here, we minimize the quadratic cost function $\tilde{J} = \int_0^\infty \tilde{\mathbf{x}}(t)^T \tilde{\mathbf{Q}} \tilde{\mathbf{x}}(t) + \tilde{R} \tilde{\eta}(t)^2 dt$ taking into account the linearized dynamics $\tilde{\mathbf{f}}(\tilde{\mathbf{x}}^* + \delta\tilde{\mathbf{x}}, \tilde{\eta}^* + \delta\tilde{\eta}) \approx \tilde{\mathbf{A}}\delta\tilde{\mathbf{x}} + \tilde{\mathbf{b}}\delta\tilde{\eta}$ in order to find a linear feedback. We obtain the optimal cost-to-go for an initial condition $\tilde{\mathbf{x}}_0 = \tilde{\mathbf{x}}(0)$ as $\tilde{J}_{\text{opt}} = \tilde{\mathbf{x}}_0^T \tilde{\mathbf{S}} \tilde{\mathbf{x}}_0$, where $\tilde{\mathbf{S}}$ denotes the solution to the general Riccati equation with $(\tilde{\mathbf{A}}, \tilde{\mathbf{b}}, \tilde{\mathbf{Q}}, \tilde{R})$ and the LQR feedback is given as $\tilde{\eta} = \tilde{\mathbf{K}}_{\text{LQR}} = -[\tilde{R}^{-1} \tilde{\mathbf{b}}^T \tilde{\mathbf{S}}] \tilde{\mathbf{x}}$. For weights $\tilde{\mathbf{Q}} = \mathbb{I}_{4 \times 4}$, $\tilde{R} = 10$, the LQR feedback is synthesized to

$$\tilde{\mathbf{K}}_{\text{LQR}} = -0.1163\tilde{V}_A + 0.3881\tilde{\gamma}_A + 0.2412\tilde{q} + 0.0007\tilde{\alpha}. \quad (17)$$

The control-invariant estimate of the region of attraction for $\tilde{\mathbf{K}}_{\text{LQR}}$ is presented in Fig. 4 with the ellipsoidal shape \mathcal{E} governed by $\tilde{P} = 4\tilde{V}_A^2 + 4\tilde{\gamma}_A^2 + \tilde{q}^2 + \tilde{\alpha}^2$. Again, \mathcal{E} and the control inputs returned by the LQR

feedback are illustrated in Fig. 4, too.

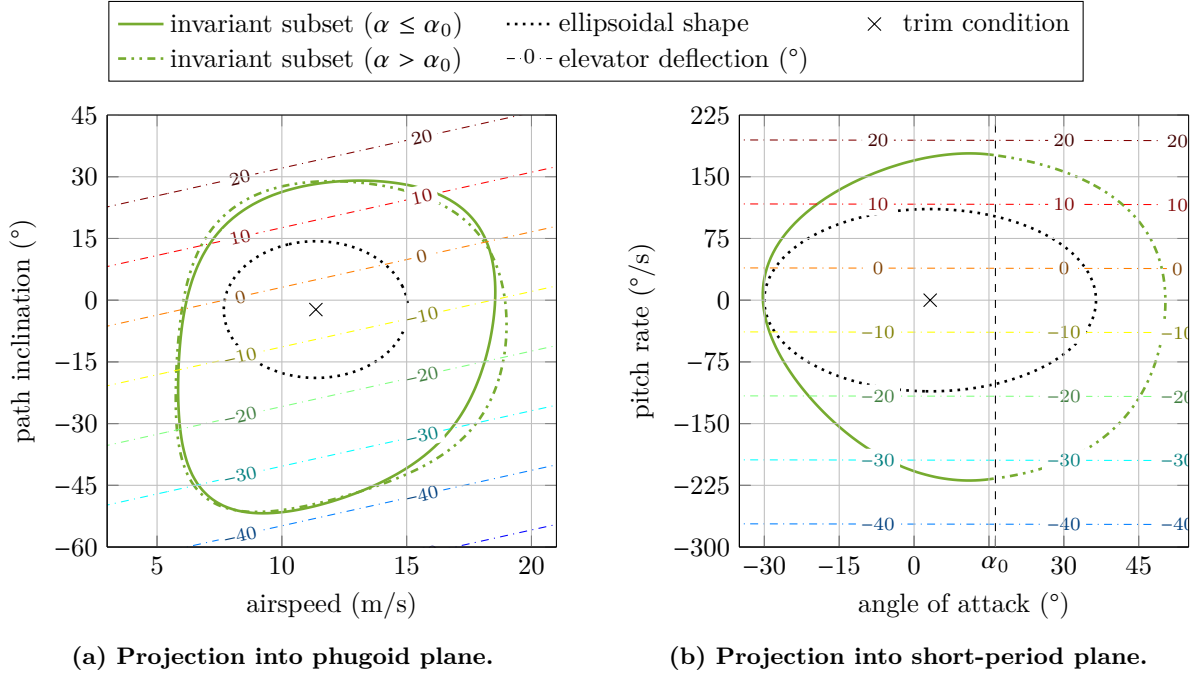


Fig. 4 Estimated control-invariant region of attraction of the linear quadratic regulator.

Even after descaling, the LQR gain on the angle of attack is significantly smaller, diminishing its contribution to the overall control feedback. With the additional gains on airspeed and path inclination, the estimated region of attraction contains the aircraft's high-angle of attack conditions in the lower-left corner of the phugoid plane. Consequently, the depicted LQR feedback is able to stably recover from deep-stall.

B. Synthesis of polynomial control laws

Until now, we have considered the control input to be determined by an *a priori* obtained state feedback law, which might have been designed by any means of control engineering. Synthesis of such a control law is subject to various objectives including desired closed-loop dynamics, disturbance rejection, and optimal reference tracking. From this section on, we treat the feedback law as decision variables of sum-of-squares analysis rather than as part of the initial problem formulation. Thus, the feedback control is synthesized with the aim of enlarging the region of attraction, again quantified by the size of the ellipsoidal shape \mathcal{E} . In the following, we subsequently derive a linear feedback similar to laws discussed in the previous section, a polynomial feedback, and piecewise feedback comparable to gain-scheduling control approaches. Further details can be found in Appendix B.

In order to reformulate the dynamics of section II into the companion form, we take the elevator deflection

η to be a state and introduce the rate of the actuator, $d\eta$, as new input:

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{\text{EOM}}(\mathbf{x}, \eta) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} d\eta, \quad (18)$$

where \mathbf{f}_{EOM} are the system dynamics of the previous sections. The thus extended state vector is denoted by \mathbf{x}_η and scaled by $\text{diag}(10 \text{ m/s}, 45^\circ, 150^\circ/\text{s}, 45^\circ, 20^\circ)^{-1}$; the new input $d\eta$ is scaled by $(100^\circ/\text{s})^{-1}$. The viable (unscaled) actuator rate inputs are constrained to $|d\eta| \leq 200^\circ/\text{s}$ in order to ensure realistic actuator dynamics.

The role of the ellipsoid \mathcal{E} , and in particular the polynomial P governing its shape, for control synthesis deserves a further discussion. In the last step of the extended \mathcal{V} - s -iteration that is employed for both analysis and control synthesis, the ellipsoid $\{\mathbf{x} \mid P(\mathbf{x}) \leq \rho^\circ\}$ serves as lower bound for the region of attraction estimate, both in size and shape. Recall further that each estimate $\{\mathbf{x} \mid \mathcal{V}(\mathbf{x}) \leq \lambda^\circ\}$ is itself invariant; if the feedback law is chosen prior analysis, as in the preceding section, all invariant sets of the aircraft are predetermined by the closed-loop system dynamics. That is, we “find” a certain invariant set by guessing a Lyapunov function and computing its largest stable level set. With the feedback law being a decision variable of the control synthesis now, the selection of a control feedback actively “shapes” the resulting invariant set. In consequence, we expect the estimated region of attraction to follow the chosen ellipsoidal shape more closely and therefore, we must carefully select its shape.

We will initially choose a polynomial P of second order that results in an ellipsoid which is rotated with respect to the normal vector of the phugoid plane, in order to enhance recovery from deep-stall trim conditions, where air speed is exceptionally low and the path inclination is oriented steeply downwards. As the elevator deflection constitutes a state of Eq. (18), the constraints $\eta \in [-60^\circ; 20^\circ]$ form asymmetric boundaries. Hence, to ensure recovery from deep-stall trim of conditions of largely negative elevator deflections is challenging. We will therefore employ an asymmetrically defined quasi-ellipsoidal shape governed by a fourth-order polynomial.

1. Linear feedback control

We start with synthesizing a linear feedback law $d\eta = \mathbf{K}_{\text{lin}} = \mathbf{G} \mathbf{x}$, where $\mathbf{G} \in \mathbb{R}^{5 \times 5}$ is a decision variable of the sum-of-squares program. Fig. 5 shows the estimated region of attraction for the synthesized linear control feedback. The ellipsoidal shape \mathcal{E} is governed by

$$\tilde{P} = 220\tilde{V}_A^2 - 360\tilde{V}_A\tilde{\gamma}_A + 100\tilde{a}^2 + 25\tilde{\eta}^2 + 220\tilde{\gamma}_A^2 + 100\tilde{q}^2$$

and rotated with respect to the phugoid plane. The synthesized linear feedback law is illustrated in Fig. 5 as contour plots with respect to states and elevator deflection.

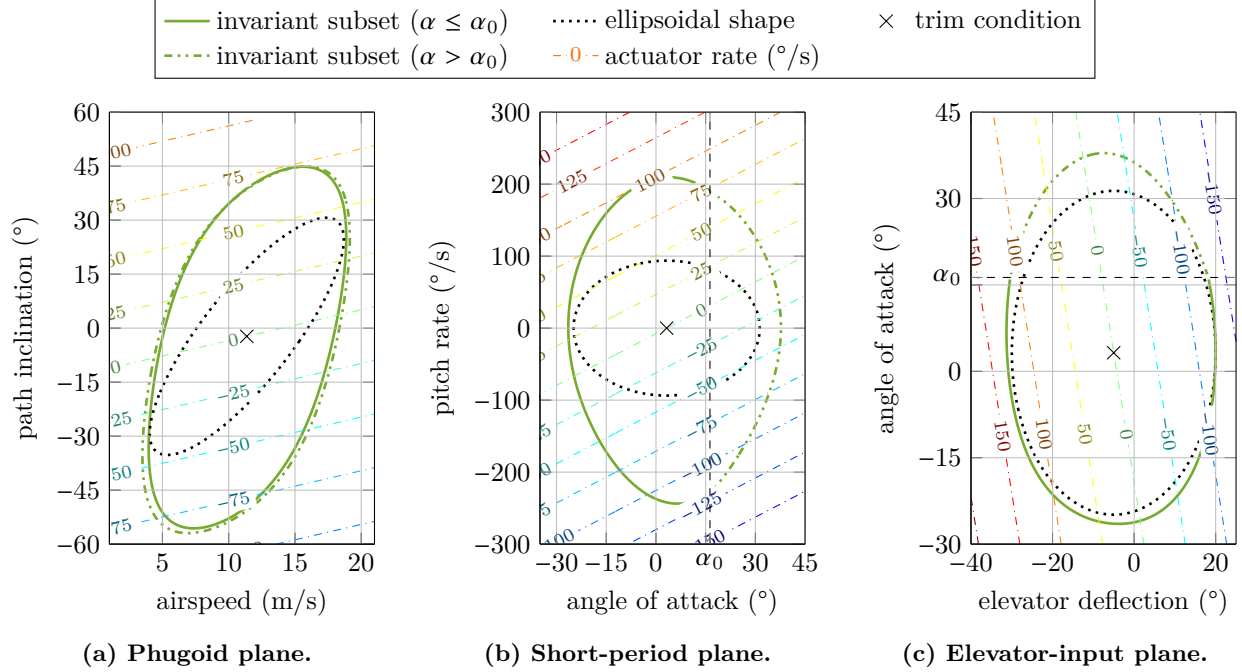


Fig. 5 Region of attraction of the synthesized linear control feedback matrix.

The synthesized linear feedback maintains a region of attracting that contains initial conditions of a wide range of airspeed, path inclination, and angle of attack. In phugoid and short-period plane, the invariant set only loosely follows the ellipsoidal shape, leading to an enlarged region of attraction. However, the ellipsoidal shape starkly affects the elevator deflection, as the dynamics of the actuator are decoupled, and the upper bound constraints both positive and negative deflections. Thus, negative elevator deflections observed in deep-stall flight are not contained by the estimated region of attraction of this linear feedback controller.

2. Polynomial feedback control

Indeed, sum-of-squares control synthesis benefits from its ability to synthesized polynomial feedback laws which are not represented by linear matrices. Whereas the Lyapunov function-candidate is conveniently represented by a polynomial without linear coefficients and of even degree to facilitate positivity of the Lyapunov function, it seems reasonable to have a polynomial feedback law without constant terms and of odd degree. Here, we choose a polynomial $d\eta = \mathbf{K}_{\text{poly}}$ with linear, quadratic, and cubic terms. Furthermore, to maintain a region of attraction including large negative elevator deflections, we select an asymmetric quasi-ellipsoidal shape \mathcal{E} that is governed by \tilde{P}_{poly} given in Appendix C. The estimated region of attraction for the synthesized third-order control feedback is shown in Fig. 6.

The differences between third-order (cubic) and first-order (linear) feedback laws can well be obtained from the isolines, that is, the contour lines of equal actuator rate inputs. Not only decreases the distance

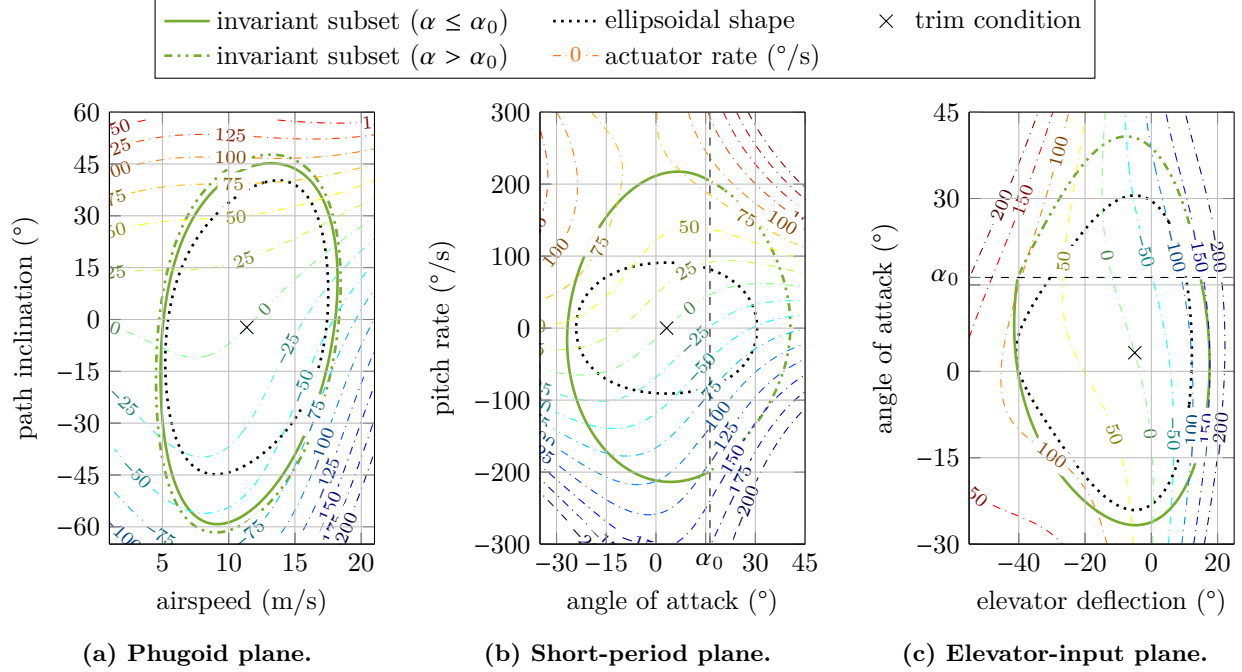


Fig. 6 Region of attraction of the synthesized polynomial control feedback law.

between two isolines with increasing distance from the trim condition, their course varies starkly between different sections of the state-space. Moreover, the invariant set seems to be “embedded” into the contour lines of equal feedback.

The asymmetric quasi-ellipsoid pushes the boundary of the invariant set towards larger negative elevator deflections, without violating the upper constraint. However, deep-stall trim conditions with their angles of attack of $\geq 30^\circ$ are not contained by the estimated region of attraction.

3. Piecewise feedback control

It seems desirable to have alternative control laws for high and low angles of attack in order to adapt for changed dynamics beyond stall. With the aerodynamic model defined piecewise, it is convenient to synthesize a piecewise polynomial control law for the same regions, that is,

$$d\eta = \mathbf{K}_{pw}(\mathbf{x}) = \begin{cases} \mathbf{K}^{pre} & \text{if } \alpha \leq \alpha_0; \\ \mathbf{K}^{post} & \text{else;} \end{cases} \quad (19)$$

where $\mathbf{K}^{pre}, \mathbf{K}^{post}$ are third-order polynomials in \mathbf{x} . Note that we don’t require a boundary condition, as we command a change of deflection, but could enforce equality of \mathbf{K}^{pre} and \mathbf{K}^{post} along $\alpha \equiv \alpha_0$ similar to (12)–(13). When synthesizing polynomial feedbacks for a control-invariant region of attraction spanning both

low and high angles of attack, both must not violate state and input constraints within their respective domains. Fig. 7 shows the estimated region of attraction for the synthesized piecewise third-order control feedback. The ellipsoidal shape \mathcal{E} is extended towards the section of high angles of attack and large negative elevator deflections and therefore governed by \tilde{P}_{pw} detailed in Appendix C. For the sake of legibility of the phugoid-plane projection, we only show the contour lines of equal control feedback for the low angle of attack law.

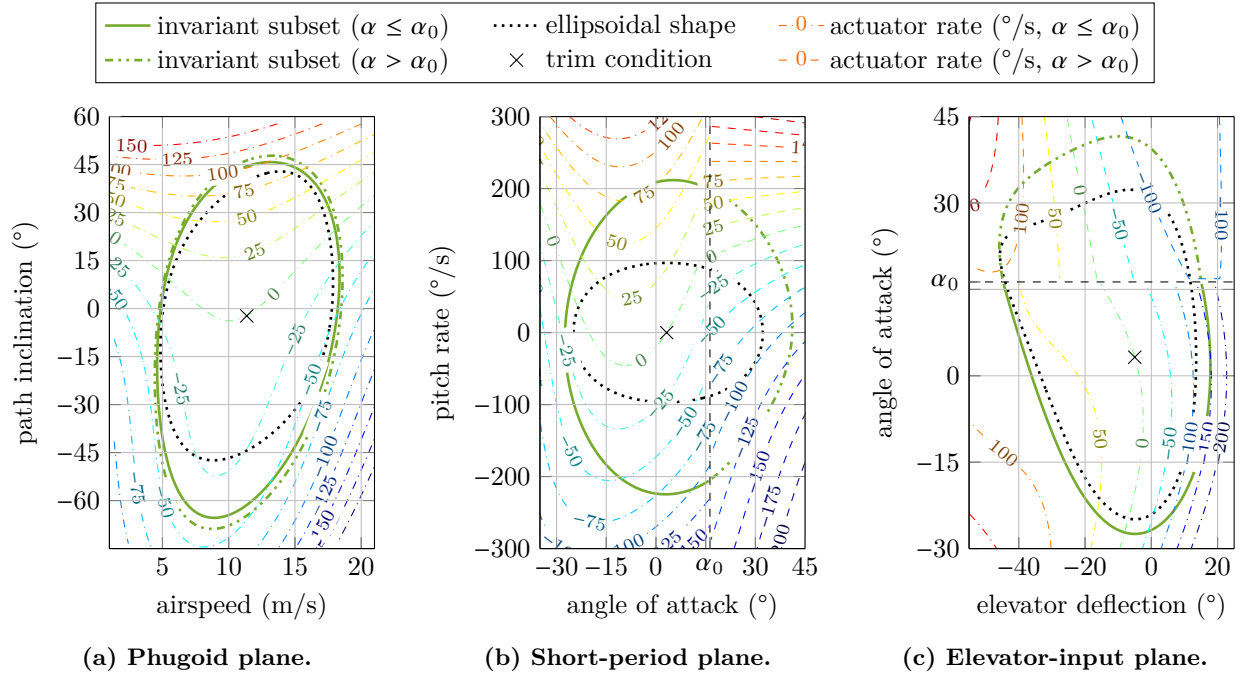


Fig. 7 Region of attraction of the synthesized piecewise polynomial control feedback law.

C. Discussion

We employed the sum-of-squares framework in order to synthesize linear, polynomial, and piecewise polynomial control feedback laws that, by design, grant an enlarged region of attraction subject to constraints on the deflection and rate of change of the elevator. Here, as discussed in the beginning, the choice of its shape P turns out to be crucial for the shape of the resulting provable invariant set and thus the synthesized feedback law. Alternative iteration approaches which remove the necessity of a shape function by a set-inclusion constraint of the estimates likewise lack the directional information to effectively synthesize a control feedback that enlarges the region of attraction towards the desired states. On the other hand, during the iteration the surface \mathcal{E}_p and the input constraints form lower and upper bounds, respectively, for the computation of a new feedback law \mathbf{K} . As the deflection constraint has been particularly asymmetric (-60° to 20°), we made use of increasingly complicated, asymmetrically shaped quasi-ellipsoidal surfaces. Thus we were able to

synthesize feedback laws for recovery from flight condition at increasingly steep descents at low airspeeds without violating the elevator deflection constraint. However, if the generated invariant sets do not contain the deep-stall flight conditions (which is the case), this is rather due to the choice of P as parameter of the \mathcal{V} - s -iteration than to the form of the control feedback. Despite a large number of iterations, when these bounds finally converge and the iteration terminates (see Table 2 in Appendix B), we have not succeeded in reaching the deep-stall flight conditions with the resulting set invariant under the synthesized control. In the next section, we will therefore propose an alternative algorithm that directly formulates the SOS control synthesis as reachability problem.

Sum-of-squares programming is notoriously limited by the size of the resulting matrices for the semidefinite problems, which in turn is a function of both the number of state variables and polynomial degree and scales badly [26]. A further partitioning of the state-space into piecewise defined polynomials could help reduce the necessary polynomial degree to accurately represent aircraft dynamics, thus also the underlying matrix size, but increases the number of decision variables in Eq. (10) and must be subject to a careful trade-off.

IV. Deep-stall Recovery

Reachability of deep-stall trim conditions via SOS control synthesis, as the previous section revealed, remains subject of a careful selection of the ellipsoidal shape \mathcal{E} . Defining a polynomial surface, in particular in higher dimensions and larger-than-quadratic order, is a nontrivial task (see also Appendix C for a sum-of-squares procedure for quasi-ellipsoids based on a selection of points). On the other hand, deep-stall recovery is often formulated as finite-horizon problem, namely, as part of a multi-mode paradigm where the flight controller switches to a local feedback after recovery. In this section, we modify the control synthesis into a simplified backwards reachability scheme, where the ellipsoidal shape is replaced by a target state whence recovery is to be ensured. An alternative approach for a finite-horizon backwards reachability analysis was presented in [27]; here, the ellipsoidal shape is replaced by a constraint enforcing that each prior estimate is nested in the next. However, this approach lacks any directional information for enlarging the closed-loop reachable set under the control law to be synthesized. We further provide a numerical comparison of deep-stall recovery by all presented feedback laws in the time domain.

A. Backwards reachability control synthesis

Denote the target state by \mathbf{x}_1 and the distance vector from the nominal trim by $\bar{\mathbf{x}} = \mathbf{x}_1 - \mathbf{x}^*$. The target state can be *recovered* into the nominal trim condition by a control feedback \mathbf{K} (i.e., \mathbf{x}_1 is backwards reachable from \mathbf{x}^*), if there is a function $\mathcal{V}(\cdot)$ and level set λ such that Ω_λ is a control-invariant region of attraction estimate (cf. Eqs. (10)–(14)) and $\mathbf{x}_1 \in \Omega_\lambda$, that is, $\mathcal{V}(\bar{\mathbf{x}}) \leq \lambda$. Obtaining such \mathbf{K} , $\mathcal{V}(\cdot)$, and λ requires again

bisection: Given $\mathcal{V}(\cdot)$ and λ such that (10)–(14) hold, we define the *degree of reachability* ϑ° as the maximal distance ratio ϑ such that

$$\mathcal{V}(\vartheta \bar{\mathbf{x}}) \leq \lambda \tag{20}$$

and observe that \mathbf{x}_1 can be recovered if $\vartheta^\circ \geq 1$. For robustness, it might be desirable to have \mathbf{x}_1 well inside the interior of Ω_λ , that is, the degree of reachability is strictly larger than one or even satisfies a chosen margin. The \mathcal{V} - s -iteration for backwards reachability is then formulated as follows:

- 1) Find λ°, λ^* maximal such that (10)–(11) as well as (14) hold for \mathcal{V} and \mathbf{K} fixed;
- 2) Find ϑ° maximal such that (20) holds for \mathcal{V} and \mathbf{K} as well as $\lambda' = \min\{\lambda^\circ, \lambda^*\}$ fixed;
- 3a) Find $\mathbf{K} \in \mathbb{R}[\mathbf{x}]$ of fixed degree such that (10)–(11) and (16) hold for \mathcal{V} as well as ϑ° and λ' fixed;
- 3b) Find $\mathcal{V} \in \mathbb{R}[\mathbf{x}]$ of fixed degree such that (9)–(14) and (20) hold for \mathbf{K} as well as ϑ° and λ' fixed.

Instead of \mathcal{E}_ρ before, the degree of reachability ϑ° ensures that, in the last step, the region of attraction estimate grows towards the target state \mathbf{x}_1 , that is, the degree of reachability increases. The thus modified V - s -iteration is performed by Algorithm 2. Note that, unlike finite-horizon reachability the obtained feedback law \mathbf{K} stabilizes the target trim condition *beyond* recovery, too. Line 11 of Algorithm 2 cannot be solved as the sum-of-squares problem (20) is not linear in ϑ in general, but is efficiently obtained using a nonlinear solver such as MATLAB’s `fmincon`.

We choose now one of the stable deep-stall trim conditions as target state, namely, that at $\eta_1 = -40^\circ$ (see Appendix A). Again, states and inputs are scaled to $\tilde{\mathbf{x}}, \tilde{d}\eta$, etc. and subject to state and input constraints. For the sake of a demonstration here, we choose a single function-candidate $\mathcal{V}(\cdot)$ as well as a linear control feedback. Fig. 8 illustrates the estimated region of attraction, a five-dimensional surface, as slices projected into the phugoid plane; for this purpose, we assign the free parameters as $\alpha = t\bar{\alpha}$, $q = t\bar{q}$, and $\eta = t\bar{\eta}$ and use the ratio t for the out-of-plane drawing axis.

With a terminal degree of reachability of $\vartheta^\circ = 1.3027$, the synthesized feedback law robustly recovers the aircraft from the deep-stall target into nominal flight. Note that the invariant subset is shaped mainly around the trajectory from \mathbf{x}_1 to \mathbf{x}^* , as this has been the objective, rather than growing as large as possible. This is further illustrated in Fig. 9, which depicts the estimated regions of attraction in the course of the 39 iterations. With increasing iterations, the region of attraction estimate it stretched towards the target flight condition \mathbf{x}_1 until contained ($\vartheta^\circ \geq 1$).

Algorithm 2 Modified \mathcal{V} - s -iteration for backwards reachability of target $\bar{\mathbf{x}} = \mathbf{x}_1 - \mathbf{x}^*$.

1: **for** $N = 1$ to N_{\max}

2: **if** $N > 1$ **then**

3: find $\mathbf{K}_N \in \mathbb{R}[\mathbf{x}]$ s.t. for all $j \in [1, m], i \in \{1, 2\}$,

$$\begin{aligned} & -\mathbf{p}_j^T \mathbf{K}_N + (\mathcal{V}_N - \lambda^\dagger) s_{j,\lambda'} \in \Sigma[\mathbf{x}] \\ & -\nabla \mathcal{V}_N \mathbf{f}_{\mathbf{K}_N i} - l_b + (\mathcal{V}_N - \lambda^\dagger) s_{i,\lambda} + \varphi s_{i,\varphi} \in \Sigma[\mathbf{x}] \end{aligned}$$

4: find $\mathcal{V}_N \in \mathbb{R}[\mathbf{x}]$ s.t. for all $j \in [1, m], i \in \{1, 2\}$,

$$\begin{aligned} & \mathcal{V}_N(\mathbf{x}) - l_a \in \Sigma[\mathbf{x}] \\ & -\nabla \mathcal{V}_N \mathbf{f}_{\mathbf{K}_N i} - l_b + (\mathcal{V}_N - \lambda^\dagger) s_{i,\lambda} + \varphi s_{i,\varphi} \in \Sigma[\mathbf{x}] \\ & -\mathbf{p}_j^T \mathbf{K}_N + (\mathcal{V}_N - \lambda^\dagger) s_{j,\lambda'} \in \Sigma[\mathbf{x}] \\ & \mathcal{V}_N(\vartheta^\circ \bar{\mathbf{x}}) \leq \lambda^\dagger \end{aligned}$$

5: **end**

6: **for** $i \in \{1, 2\}$

7: find $\lambda_i^\circ := \max_{\lambda \geq 0} \lambda$ s.t. $s_{i,\lambda}, s_{i,\varphi} \in \Sigma[x]$ and

$$-\nabla \mathcal{V}_N \mathbf{f}_{\mathbf{K}_N i} - l_b + (\mathcal{V}_N - \lambda) s_{i,\lambda} + \varphi s_{i,\varphi} \in \Sigma[\mathbf{x}]$$

8: **end**

9: find $\lambda^* := \max_{\lambda' \geq 0} \lambda'$ s.t. $s_{j,\lambda'} \in \Sigma[\mathbf{x}]$ and for all $j \in [1, m]$,

$$-\mathbf{p}_j^T \mathbf{K}_N + (\mathcal{V}_N - \lambda') s_{j,\lambda'} \in \Sigma[\mathbf{x}]$$

10: $\lambda^\dagger := \min\{\lambda_1^\circ, \lambda_2^\circ, \lambda^*\}$

11: find $\vartheta^\circ := \max_{\vartheta \geq 0}$ s.t.

$$\mathcal{V}_N(\vartheta \bar{\mathbf{x}}) \leq \lambda^\dagger$$

12: **end**

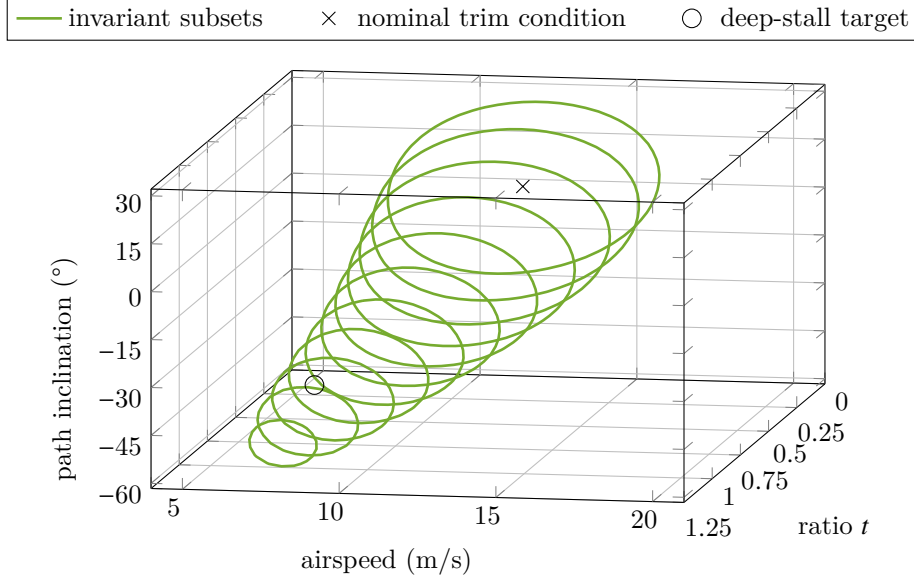


Fig. 8 Region of attraction of a linear control feedback law, synthesized by backwards reachability of the deep-stall target.

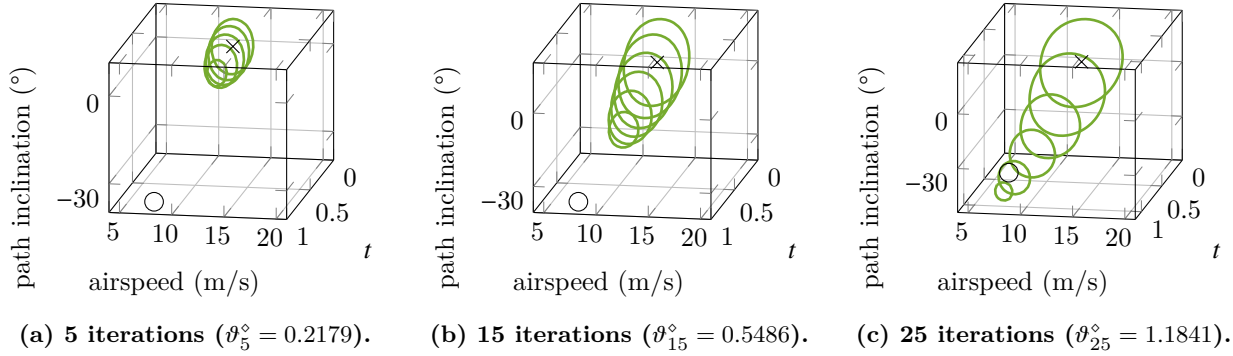


Fig. 9 Region of attraction estimates during iteration of the backwards-reachability control synthesis.

B. Numerical comparison

We consider recovery from the deep-stall trim condition \mathbf{x}_1 , given in Appendix A, and simulate the closed-loop behaviour for the feedback laws discussed in this study. Fig. 10 compares recovery under control synthesized for reachability to the previous synthesized linear, polynomial, and piecewise polynomial feedback laws as well as to the LQR feedback. In addition, the closed-loop response with a single-rate damping law, $\tilde{\eta} = \tilde{k}_{\eta q} \tilde{q}$, where $\tilde{k}_{\eta q} = 1$ is a positive, proportional gain on the (scaled) pitch rate. All trajectories have been computed against the non-polynomial longitudinal equations of motion given in Eqs. (1)–(4) rather than the polynomial approximations employed for the SOS iterations.

The LQR controller, as bespoke in Section III, recovers stably and fast; the pitch-damper (in dashed) too is eventually recovering though delayed and with large oscillations. As for the control laws of Section B,

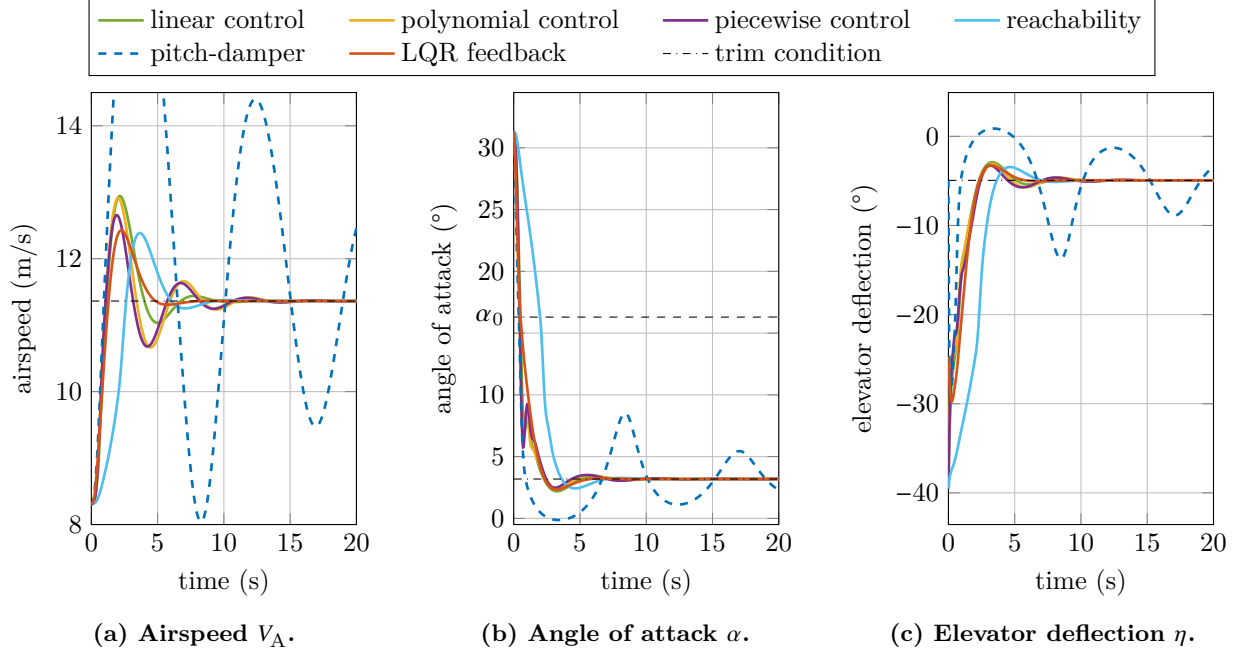
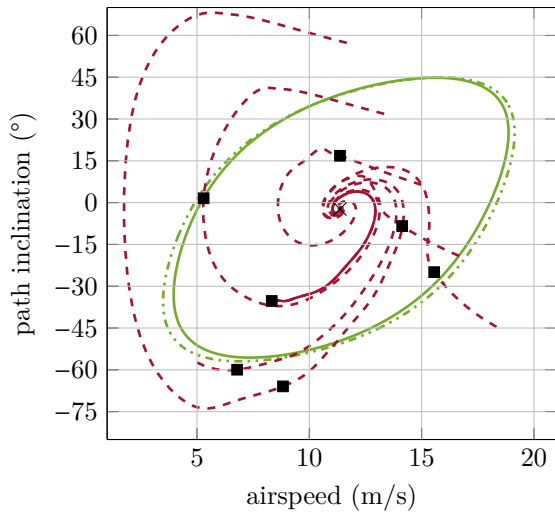


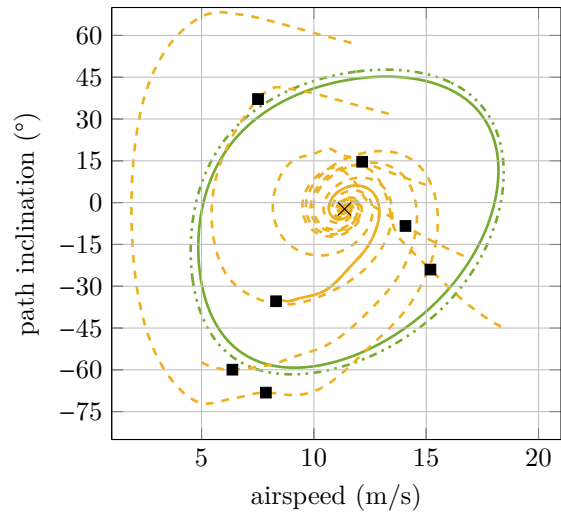
Fig. 10 Comparison of time responses starting in \mathbf{x}_1 .

the objective of synthesis has been an enlarged region of attraction rather than closed-loop performance; consequently, the recovery is performed rather slow and with considerable overshoot (in particular for the polynomial and piecewise polynomial feedback laws). The control law synthesized for reachability of the deep-stall condition also leads to a slow but straight recovery. An extended SOS control synthesis improving performance measures in addition to enlarging the region of attraction and/or ensuring backwards reachability could be achieved by further maximizing an exponential stability gain in Eqs. (10) and (10).

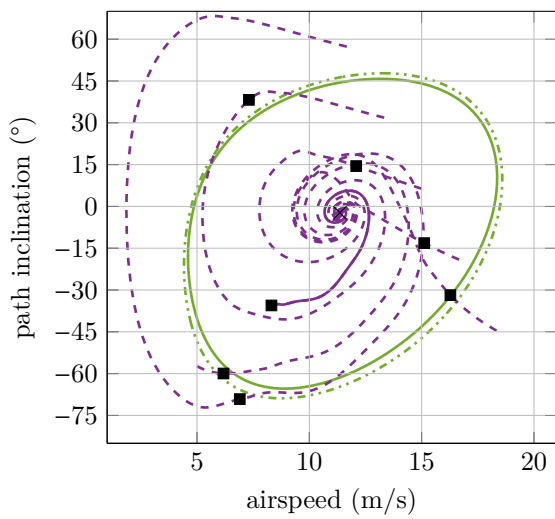
In Fig. 11, we detail trajectories starting in various deep-stall flight conditions in the phugoid plane and additionally show projections of the respective region of attraction estimates. The response for the trim condition \mathbf{x}_1 is drawn in solid; the other responses (in dashed) have been computed for non-trim conditions given in Appendix A. In addition, for each trajectory the point it enters the region of attraction estimate is marked by a black square (due to the projection into the phugoid plane, these points might not necessarily appear inside the ellipsoid). For the feedback law synthesized by backwards reachability, further out-of-plane slices of the estimated region of attraction are projected into the phugoid plane as well. Except for one trajectory of the LQR feedback (drawn dotted), all controllers are able to recover from each deep-stall condition into the gliding trim condition. However, for the high-inclination conditions \mathbf{x}_6 and \mathbf{x}_7 this has partially led to saturations of the elevator deflection and its rate of change. Therefore, even though the saturation has not prevented recovery, these conditions could not have been included in any region of attraction estimate subject to the imposed constraints in the methodology presented in this paper. In



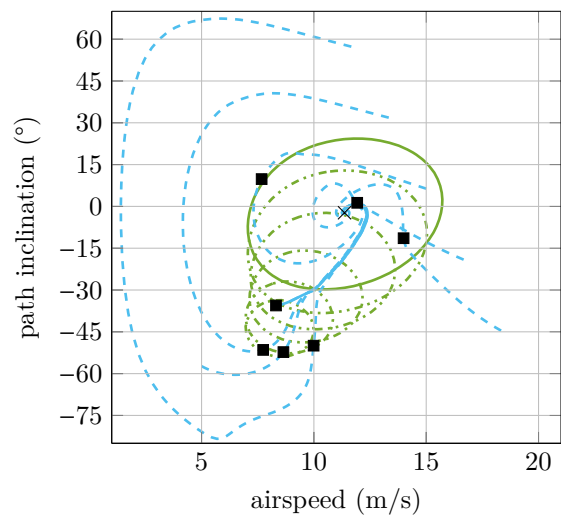
(a) Linear control law.



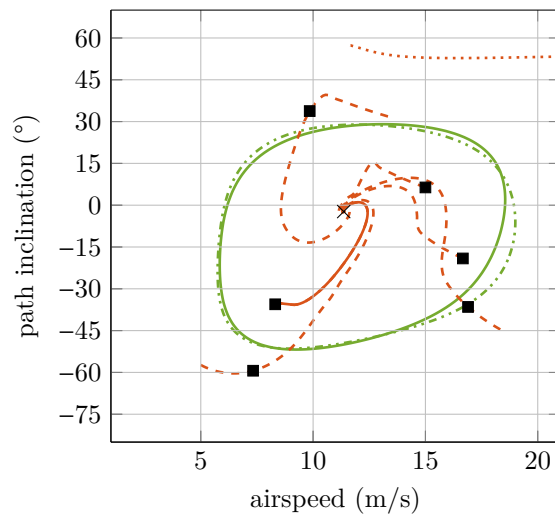
(b) Polynomial control law.



(c) Piecewise control law.



(d) Reachability-synthesis law.



(e) LQR feedback law.

Fig. 11 Time responses and region of attraction estimates in the phugoid plane.

the other cases, the fact that stable recovery trajectories lie outside the estimates highlights the inevitable conservatism of any sum-of-squares analysis approach.

V. Conclusion

Sum-of-squares techniques provide exact certificates for stability and the region of attraction, but the application to accurate, full-envelope aircraft models such as multivariate splines is computationally demanding. Simple piecewise polynomial models, on the other hand, accommodate the aerodynamic coefficients well in both domains of low and high angles of attack, while only moderately increasing the costs of sum-of-squares analysis. In this note, we have applied sum-of-squares techniques to a fixed-wing aircraft model in order to verify stable recovery from deep-stall. As an intermediate result, we verified that a classical LQR feedback law onto the elevator deflection is sufficient for recovery without violation of input constraints. We then further extended the sum-of-squares analysis for control synthesis, using the actuator rate instead of deflection. Here, the choice of an ellipsoidal shape function for the sum-of-squares program turned out to be crucial, yet a selection was challenged by asymmetric constraints on the elevator deflection. Despite all three of the synthesized linear, polynomial, and piecewise polynomial feedback laws could not be verified to recover the aircraft from its deep-stall trim conditions, this does not imply no such control feedback exist. Concluding our study, we proposed a reformulated sum-of-squares control synthesis based on backwards-reachability of a target trim condition without the necessity of an ellipsoidal shape. Indeed, a simple linear control feedback could thus be verified to recover the aircraft. We therefore held sum-of-squares programming combined with piecewise polynomial models a powerful tool for aircraft analysis and verification.

Appendix

We provide details of the polynomial aircraft model, the sum-of-squares computations, and the computation of quasi-ellipsoidal surfaces using SOS programming.

A. Details on the polynomial aircraft model

The low-angle of attack trim condition \mathbf{x}^* for gliding descent in Section III is given by

$$V_A^* = 11.3631 \text{ m/s}, \quad \gamma_A^* = -2.2834^\circ, \quad q^* = 0^\circ/\text{s}, \quad \alpha^* = 3.2240^\circ, \quad \eta^* = -5^\circ,$$

and the deep-stall target trim condition \mathbf{x}_1 in Section IV, by

$$V_{A1} = 8.3131 \text{ m/s}, \quad \gamma_{A1} = -35.5372^\circ, \quad q_1 = 0^\circ/\text{s}, \quad \alpha_1 = 31.6781^\circ, \quad \eta_1 = -40^\circ.$$

The additional deep-stall conditions are given by

$$\begin{aligned}
V_{A2} &= 5.0000 \text{ m/s}, & \gamma_{A2} &= -57.2958^\circ, & q_2 &= 0^\circ/\text{s}, & \alpha_2 &= 29.8081^\circ, & \eta_2 &= -38^\circ; \\
V_{A3} &= 18.3333 \text{ m/s}, & \gamma_{A3} &= -44.5634^\circ, & q_3 &= 0^\circ/\text{s}, & \alpha_3 &= 32.5836^\circ, & \eta_3 &= -41^\circ; \\
V_{A4} &= 16.6667 \text{ m/s}, & \gamma_{A4} &= -19.0986^\circ, & q_4 &= 0^\circ/\text{s}, & \alpha_4 &= 35.1836^\circ, & \eta_4 &= -44^\circ; \\
V_{A5} &= 15 \text{ m/s}, & \gamma_{A5} &= 6.3662^\circ, & q_5 &= 0^\circ/\text{s}, & \alpha_5 &= 37.6092^\circ, & \eta_5 &= -47^\circ; \\
V_{A6} &= 13.3333 \text{ m/s}, & \gamma_{A6} &= 31.8310^\circ, & q_6 &= 0^\circ/\text{s}, & \alpha_6 &= 39.8574^\circ, & \eta_6 &= -50^\circ; \\
V_{A7} &= 11.6667 \text{ m/s}, & \gamma_{A7} &= 57.2958^\circ, & q_7 &= 0^\circ/\text{s}, & \alpha_7 &= 41.9214^\circ, & \eta_7 &= -53^\circ.
\end{aligned}$$

The piecewise model of Eqs. (1)–(4) is not immediately suitable for sum-of-squares analysis. The non-polynomial sine and cosine functions in γ_A and α are therefore approximated by 5th-order and 4th-order Taylor series expansions,

$$\sin a \approx a - \frac{a^3}{3!} + \frac{a^5}{5!}; \quad \cos a \approx 1 - \frac{a^2}{2} + \frac{a^4}{4!};$$

respectively, which have an error of less than ± 0.02 for $\gamma_A, \alpha \in]-90^\circ; +90^\circ[$. The inversion of V_A in Eq. (2) is likewise replaced by

$$a^{-1} \approx a_0^{-1} - a_0^{-2}(a - a_0) + 2a_0^{-3}(a - a_0)^2 - 3!a_0^{-4}(a - a_0)^3,$$

where a_0 denotes the airspeed in trim. This approximation is less accurate, however, the resulting piecewise polynomial seems to be conservative, that is, it diverges rather than the nonlinear model.

The uncontrolled piecewise polynomial model with the polynomial approximations above is of order 13. In order to ease the complexity of the SOS computation, we have removed any polynomial term of 6th order or higher or with coefficients absolutely smaller than 10^{-6} .

B. Details on the SOS computations

We provide details about the sum-of-squares computations, polynomial degrees of the problems, and the decision variables involved, respectively. Table 2 details the results of the different SOS computations in Sections III and IV in terms of the pseudo-radii ρ° of the ellipsoidal shapes \mathcal{E}_ρ and degree of reachability ϑ° , respectively, as well as the level sets for invariance and invariance under control, λ° and λ^* . The number of involved states is further given as well as the number of repetitions of the (extended) \mathcal{V} -s-iteration and

Table 2 Results of SOS computations.

(a) **Stability analysis** (Intel Core i7, 3 GHz, 16 GB).

Section	n	ρ°	λ°	λ^*	Iterations	Time
Sec. III.A	4	54.5349	1.0513	1.0513	135	3.8295 h

(b) **Control synthesis** (Intel Core E5, 3 GHz, 16 GB).

Section	n	ρ°	λ°	λ^*	Iterations	Time
Sec. III.B	1	39.0320	38.7939	38.5986	109	36.4526 h
	2	251.9531	35.3516	38.6230	70	18.4957 h
	3	295.6543	17.7979	19.1650	73	21.1351 h

(c) **Deep-stall recovery** (Intel Core i7, 3 GHz, 16 GB).

Section	n	θ°	λ°	λ^*	Iterations	Time
Sec. IV	5	1.3027	57.5684	99.9023	29	4.5832 h

Table 3 Polynomial degrees of SOS problems.

Section	\mathbf{f}	\mathbf{K}	\mathbf{f}_K	$\mathcal{V}_{1,2}$	$l_{a,b}$	P	p_η
Sec. III.A	5	1	5	4	2	2	2
Sec. III.B	1	1					
	2	5	3	5	4	2	2
	3		3				
Sec. IV	5	1	5	4	2	-	2

Table 4 Polynomial degrees of SOS multipliers.

Section	$s_{i,\lambda}$	$s_{i,\varphi}$	$s_{\varphi,i}$	$s_{-\varphi,i}$	$s_{i,P}$	$s_{\eta,\lambda'}$
Sec. III.A	2	2	2	2	1	1
Sec. III.B	1					
	2	2	2	2	1	1
	3					
Sec. IV	2	2	2	2	-	1

the total computation time. Table 3 reports the polynomial degrees of control system, Lyapunov functions, positive terms $l_{a,b}$, ellipsoidal shape functions, and constraints. In addition, Table 4 gives the chosen degrees for the sum-of-squares multipliers in Eqs. (9)–(16).

C. Computing quasi-ellipsoids with SOS

If the cartesian dimensions, that is, number of variables, or the desired polynomial order of a quasi-ellipsoid grow, so does the number of independent coefficients. Thus, selecting an ellipsoidal manually becomes increasingly difficult. In order to obtain the asymmetric shapes in Sections III.B-2 and III.B-3, we have solved the following sum-of-squares problem given a sequence of points $\mathbf{x}_1^\circ, \dots, \mathbf{x}_k^\circ \in \mathbb{R}^n$, $k \in \mathbb{N}$ finite: find $P \in \mathbb{R}[\mathbf{x}]$ such that P is positive-definite and

$$P(\mathbf{x}_i^\circ) = 1 \quad (21)$$

for all $1 \leq i \leq k$. As the degree of P is predefined, the equality constraints in (21) are linear in the decision variables (the coefficients of P).

The 4th-order, asymmetric shape functions \tilde{P}_{poly} and \tilde{P}_{pw} are thus derived to (coefficients are subject to rounding; coefficients $< 10^{-3}$ are omitted):

$$\begin{aligned} \tilde{P}_{\text{poly}} = & 5.37 \times 10^2 \tilde{V}_A^4 - 2.90 \times 10^1 \tilde{V}_A^3 \tilde{\gamma}_A - 1.48 \times 10^{-3} \tilde{V}_A^3 \tilde{q} + 5.42 \times 10^2 \tilde{V}_A^2 \tilde{\alpha}^2 + 2.41 \times 10^2 \tilde{V}_A^2 \tilde{\eta}^2 \\ & + 4.11 \times 10^2 \tilde{V}_A^2 \tilde{\gamma}_A^2 + 5.42 \times 10^2 \tilde{V}_A^2 \tilde{q}^2 - 2.51 \times 10^{-3} \tilde{V}_A \tilde{\alpha}^2 \tilde{\eta} - 2.84 \times 10^{-3} \tilde{V}_A \tilde{\alpha} \tilde{q}^2 - 2.61 \times 10^2 \tilde{V}_A \tilde{\gamma}_A^3 \\ & + 1.09 \times 10^{-3} \tilde{V}_A \tilde{\gamma}_A \tilde{q}^2 + 5.00 \times 10^2 \tilde{\alpha}^4 - 1.33 \times 10^{-3} \tilde{\alpha}^3 \tilde{\gamma}_A - 2.65 \times 10^{-3} \tilde{\alpha}^3 \tilde{q} + 2.41 \times 10^2 \tilde{\alpha}^2 \tilde{\eta}^2 \\ & - 1.89 \times 10^{-3} \tilde{\alpha}^2 \tilde{\eta} \tilde{\gamma}_A + 5.42 \times 10^2 \tilde{\alpha}^2 \tilde{\gamma}_A^2 - 1.93 \times 10^{-3} \tilde{\alpha}^2 \tilde{\gamma}_A \tilde{q} + 5.42 \times 10^2 \tilde{\alpha}^2 \tilde{q}^2 + 2.04 \times 10^1 \tilde{\eta}^4 \\ & + 2.41 \times 10^2 \tilde{\eta}^2 \tilde{\gamma}_A^2 + 2.41 \times 10^2 \tilde{\eta}^2 \tilde{q}^2 + 1.35 \times 10^{-3} \tilde{\eta} \tilde{\gamma}_A \tilde{q}^2 + 1.38 \times 10^{-3} \tilde{\eta} \tilde{q}^3 + 1.50 \times 10^2 \tilde{\gamma}_A^4 \\ & + 5.42 \times 10^2 \tilde{\gamma}_A^2 \tilde{q}^2 - 2.46 \times 10^{-3} \tilde{\gamma}_A \tilde{q}^3 + 5.00 \times 10^2 \tilde{q}^4 - 4.93 \times 10^{-3} \tilde{\alpha}^2 \tilde{\eta} - 2.34 \times 10^{-3} \tilde{\alpha} \tilde{q}^2 \\ & + 1.17 \times 10^2 \tilde{\eta}^3 + 5.04 \times 10^2 \tilde{V}_A^2 - 2.24 \times 10^2 \tilde{V}_A \tilde{\gamma}_A + 5.00 \times 10^2 \tilde{\alpha}^2 + 6.63 \times 10^{-3} \tilde{\alpha} \tilde{q} \\ & + 2.23 \times 10^2 \tilde{\eta}^2 + 2.06 \times 10^2 \tilde{\gamma}_A^2 + 5.00 \times 10^2 \tilde{q}^2; \end{aligned}$$

$$\begin{aligned} \tilde{P}_{\text{pw}} = & 5.41 \times 10^2 \tilde{V}_A^4 - 2.93 \times 10^1 \tilde{V}_A^3 \tilde{\gamma}_A - 1.21 \times 10^{-3} \tilde{V}_A^3 \tilde{q} + 5.46 \times 10^2 \tilde{V}_A^2 \tilde{\alpha}^2 + 2.07 \times 10^{-3} \tilde{V}_A^2 \tilde{\alpha} \tilde{q} \\ & + 2.43 \times 10^2 \tilde{V}_A^2 \tilde{\eta}^2 - 1.44 \times 10^{-3} \tilde{V}_A^2 \tilde{\eta} \tilde{q} + 4.15 \times 10^2 \tilde{V}_A^2 \tilde{\gamma}_A^2 + 1.69 \times 10^{-3} \tilde{V}_A^2 \tilde{\gamma}_A \tilde{q} + 5.46 \times 10^2 \tilde{V}_A^2 \tilde{q}^2 \\ & - 1.12 \times 10^{-3} \tilde{V}_A \tilde{\alpha}^2 \tilde{q} - 2.02 \times 10^{-3} \tilde{V}_A \tilde{\alpha} \tilde{\eta} \tilde{q} + 1.32 \times 10^{-3} \tilde{V}_A \tilde{\alpha} \tilde{\gamma}_A^2 - 2.63 \times 10^2 \tilde{V}_A \tilde{\gamma}_A^3 - 2.86 \times 10^{-3} \tilde{V}_A \tilde{\gamma}_A \tilde{q}^2 \\ & - 1.33 \times 10^{-3} \tilde{V}_A \tilde{q}^3 + 5.24 \times 10^2 \tilde{\alpha}^4 + 1.51 \times 10^1 \tilde{\alpha}^3 \tilde{\eta} + 1.10 \times 10^{-3} \tilde{\alpha}^3 \tilde{\gamma}_A + 1.11 \times 10^{-3} \tilde{\alpha}^3 \tilde{q} \end{aligned}$$

$$\begin{aligned}
& + 1.91 \times 10^2 \tilde{\alpha}^2 \tilde{\eta}^2 + 1.05 \times 10^{-3} \tilde{\alpha}^2 \tilde{\eta} \tilde{\gamma}_A + 5.46 \times 10^2 \tilde{\alpha}^2 \tilde{\gamma}_A^2 + 4.00 \times 10^{-3} \tilde{\alpha}^2 \tilde{\gamma}_A \tilde{q} + 5.46 \times 10^2 \tilde{\alpha}^2 \tilde{q}^2 \\
& + 7.84 \times 10^1 \tilde{\alpha} \tilde{\eta}^3 - 1.58 \times 10^{-3} \tilde{\alpha} \tilde{\eta} \tilde{q}^2 - 1.32 \times 10^{-3} \tilde{\alpha} \tilde{\gamma}_A^2 \tilde{q} + 1.05 \times 10^{-3} \tilde{\alpha} \tilde{\gamma}_A \tilde{q}^2 + 2.89 \times 10^1 \tilde{\eta}^4 \\
& + 2.43 \times 10^2 \tilde{\eta}^2 \tilde{\gamma}_A^2 + 2.43 \times 10^2 \tilde{\eta}^2 \tilde{q}^2 - 2.17 \times 10^{-3} \tilde{\eta} \tilde{q}^3 + 1.51 \times 10^2 \tilde{\gamma}_A^4 + 5.46 \times 10^2 \tilde{\gamma}_A^2 \tilde{q}^2 \\
& + 3.83 \times 10^{-3} \tilde{\gamma}_A \tilde{q}^3 + 5.00 \times 10^2 \tilde{q}^4 + 1.69 \times 10^{-3} \tilde{V}_A \tilde{\alpha} \tilde{\eta} - 1.86 \times 10^{-3} \tilde{V}_A \tilde{\gamma}_A \tilde{q} - 1.15 \times 10^{-3} \tilde{V}_A \tilde{q}^2 \\
& - 4.61 \times 10^1 \tilde{\alpha}^3 + 3.65 \times 10^1 \tilde{\alpha}^2 \tilde{\eta} + 1.58 \times 10^{-3} \tilde{\alpha}^2 \tilde{\gamma}_A + 2.24 \times 10^{-3} \tilde{\alpha}^2 \tilde{q} - 8.84 \times 10^1 \tilde{\alpha} \tilde{\eta}^2 \\
& + 1.57 \times 10^{-3} \tilde{\alpha} \tilde{q}^2 + 1.06 \times 10^2 \tilde{\eta}^3 + 1.07 \times 10^{-3} \tilde{\gamma}_A \tilde{q}^2 + 5.08 \times 10^2 \tilde{V}_A^2 - 2.27 \times 10^2 \tilde{V}_A \tilde{\gamma}_A \\
& + 1.19 \times 10^{-3} \tilde{V}_A \tilde{q} + 5.22 \times 10^2 \tilde{\alpha}^2 + 2.25 \times 10^1 \tilde{\alpha} \tilde{\eta} + 2.21 \times 10^2 \tilde{\eta}^2 + 2.05 \times 10^2 \tilde{\gamma}_A^2 \\
& + 1.06 \times 10^{-3} \tilde{\gamma}_A \tilde{q} + 5.00 \times 10^2 \tilde{q}^2;
\end{aligned}$$

shape functions are defined in scaled variables.

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