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Turnpike in optimal shape design

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Abstract

We introduce and study the turnpike property for time-varying shapes, within the viewpoint of optimal control. We focus here on second-order linear parabolic equations where the shape acts as a source term and we seek the optimal time-varying shape that minimizes a quadratic criterion. We first establish existence of optimal solutions under some appropriate sufficient conditions. We then provide necessary conditions for optimality in terms of adjoint equations and, using the concept of strict dissipativity, we prove that state and adjoint satisfy the measure-turnpike property, meaning that the extremal time-varying solution remains essentially close to the optimal solution of an associated static problem. We show that the optimal shape enjoys the exponential turnpike property in term of Hausdorff distance for a Mayer quadratic cost. We illustrate the turnpike phenomenon in optimal shape design with several numerical simulations.

Keywords: optimal shape design, turnpike, strict dissipativity, direct methods, parabolic equation

1. Introduction

We start with an informal presentation of the turnpike phenomenon for general dynamical optimal shape problems. Let $T > 0$. We consider the problem of determining a time-varying shape $t \mapsto \omega(t)$ (viewed as a control, as in [2]) minimizing the cost functional

$$J_T(\omega) = \frac{1}{T} \int_0^T f^0(y(t), \omega(t)) dt + g(y(T), \omega(T)) \quad (1)$$

under the constraints

$$\dot{y}(t) = f(y(t), \omega(t)), \quad R(y(0), y(T)) = 0 \quad (2)$$

where (2) may be a partial differential equation with various terminal and boundary conditions.

We associate to the dynamical problem (1)-(2) a *static* problem, not depending on time,

$$\min_{\omega} f^0(y, \omega), \quad f(y, \omega) = 0 \quad (3)$$

i.e., the problem of minimizing the instantaneous cost under the constraint of being an equilibrium of the control dynamics.

According to the well known turnpike phenomenon, one expects that, for T large enough, optimal solutions of (1)-(2) remain most of the time “close” to an optimal (stationary) solution of the static problem (3).

The turnpike phenomenon was first observed and investigated by economists for discrete-time optimal control problems (see [10, 24]). There are several possible notions of turnpike properties, some of them being stronger than the others (see [37]). *Exponential turnpike* properties have been established in [17, 26, 27, 33, 34] for the optimal triple resulting of the application of Pontryagin’s maximum principle, ensuring that the extremal solution (state, adjoint and control) remains exponentially close to an optimal solution of the corresponding static controlled problem, except at the beginning and at the end of the time interval, as soon as T is large enough. This follows from hyperbolicity properties of the Hamiltonian flow. For discrete-time problems it has been shown in [8, 13, 15, 16, 32] that exponential turnpike is closely related to strict dissipativity. *Measure-turnpike* is a weaker notion of turnpike, meaning that any optimal solution, along the time frame, remains close to an optimal static solution except along a subset of times of small Lebesgue measure. It has been proved in [13, 32] that measure-turnpike follows from strict dissipativity or from strong duality properties.

Applications of the turnpike property in practice are numerous. Indeed, the knowledge of a static optimal solution is a way to reduce significantly the complexity of the dynamical optimal control problem. For instance it has been shown in [34] that the turnpike property gives a way to successfully initialize direct or indirect (shooting) methods in numerical optimal control, by initializing them with the optimal solution of the static problem. In shape design and despite of technological progress, it is easier to design pieces which do not evolve with time. Turnpike can legitimate such decisions for large-time evolving systems.

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2. Shape turnpike for the heat equation

Throughout the paper, we denote by:

- $|Q|$ the Lebesgue measure of a measurable subset $Q \subset \mathcal{R}^N$, $N \geq 1$;
- (p, q) the scalar product in $L^2(\Omega)$ of p, q in $L^2(\Omega)$;
- $\|y\|$ the L^2 -norm of $y \in L^2(\Omega)$;
- χ_ω the indicator (or characteristic) function of $\omega \subset \mathcal{R}^N$;
- d_ω the distance function to the set $\omega \subset \mathbf{R}^d$ and $b_\omega = d_\omega - d_{\omega^c}$ the *oriented distance function*.

Let $\Omega \subset \mathbf{R}^d$ ($d \geq 1$) be an open bounded Lipschitz domain. We consider a uniformly elliptic second-order operator

$$Au = - \sum_{i,j=1}^d \partial_{x_j} (a_{ij}(x) \partial_{x_i} u) + \sum_{i=1}^d b_i(x) \partial_{x_i} u + c(x)u$$

with $a_{ij}, b_i \in C^1(\Omega)$, $c \in L^\infty(\Omega)$ with $c \geq 0$, and its adjoint

$$A^*v = - \sum_{i,j=1}^d \partial_{x_i} (a_{ij}(x) \partial_{x_j} v) - \sum_{i=1}^d b_i(x) \partial_{x_i} v + \left(c - \sum_{i=1}^d \partial_{x_i} b_i \right) v$$

(which is also uniformly elliptic, see [11, Definition Chapter 6]), not depending on t and with a constant of ellipticity $\theta > 0$ (for A written in *nondivergence form*), i.e.:

$$\sum_{i,j=1}^d a_{ij}(x) \xi_i \xi_j \geq \theta |\xi|^2 \quad \forall x \in \Omega$$

Moreover, θ is such that

$$\theta > \theta_1 \quad (4)$$

where θ_1 is the largest root of the polynomial $P = \frac{\chi^2}{4 \min(1, C_p)} - \|c\|_{L^\infty(\Omega)} X - \frac{\sum_{i=1}^d \|b_i\|_{L^\infty(\Omega)}}{2}$ with C_p the Poincaré constant on Ω . This assumption is used to ensure that an energy inequality is satisfied with constants not depending on the final time T (see Appendix A for details).

Moreover, we assume throughout that A satisfies the classical maximum principle (see [11, sec. 6.4]) and that $c^* = c - \sum_{i=1}^d \partial_{x_i} b_i \in C^2(\Omega)$.

We define A_D as the differential operator A defined on the domain $D(A)$ encoding Dirichlet conditions $y|_{\partial\Omega} = 0$ (when Ω is C^2 or a convex polytop in \mathbf{R}^2 , we have $D(A) = H_0^1(\Omega) \cap H^2(\Omega)$). Let $(\lambda_j, \phi_j)_{j \in \mathbf{N}^*}$ be the eigenelements of A_D with $(\phi_j)_{j \in \mathbf{N}^*}$ an orthonormal eigenbasis of $L^2(\Omega)$:

- $\forall j \in \mathbf{N}^*$, $A\phi_j = \lambda_j \phi_j$, $\phi_j|_{\partial\Omega} = 0$
- $\forall j \in \mathbf{N}^*$, $j > 1$, $\lambda_1 < \lambda_j \leq \lambda_{j+1}$, $\lambda_j \rightarrow +\infty$

A typical example satisfying all assumptions above is the Dirichlet Laplacian, which we will consider in our numerical simulations.

We recall that the Hausdorff distance between two compact subsets K_1, K_2 of \mathbf{R}^d is defined by

$$d_{\mathcal{H}}(K_1, K_2) = \sup \left(\sup_{x \in K_2} d_{K_1}(x), \sup_{x \in K_1} d_{K_2}(x) \right).$$

2.1. Setting

Let $L \in (0, 1)$. We define the set of admissible shapes

$$\mathcal{U}_L = \{ \omega \subset \Omega \text{ measurable} \mid |\omega| \leq L|\Omega| \}$$

Dynamical optimal shape design problem (OSD)_T. Let $y_0 \in L^2(\Omega)$ and let $\gamma_1 \geq 0, \gamma_2 \geq 0$ be arbitrary. We consider the parabolic equation controlled by a (measurable) time-varying map $t \mapsto \omega(t)$ of subdomains

$$\partial_t y + Ay = \chi_{\omega(\cdot)}, \quad y|_{\partial\Omega} = 0, \quad y(0) = y_0 \quad (5)$$

Given $T > 0$ and $y_d \in L^2(\Omega)$, we consider the dynamical optimal shape design problem **(OSD)_T** of determining a measurable path of shapes $t \mapsto \omega(t) \in \mathcal{U}_L$ that minimizes the cost functional

$$J_T(\omega(\cdot)) = \frac{\gamma_1}{2T} \int_0^T \|y(t) - y_d\|^2 dt + \frac{\gamma_2}{2} \|y(T) - y_d\|^2$$

where $y = y(t, x)$ is the solution of (5) corresponding to $\omega(\cdot)$.

Static optimal shape design problem (SSD). Besides, for the same target function $y_d \in L^2(\Omega)$, we consider the following associated static shape design problem **(SSD)**:

$$\min_{\omega \in \mathcal{U}_L} \gamma_1 \|y - y_d\|^2, \quad Ay = \chi_\omega, \quad y|_{\partial\Omega} = 0 \quad (6)$$

We are going to compare the solutions of **(OSD)_T** and of **(SSD)** when T is large.

2.2. Preliminaries

Convexification. Given any measurable subset $\omega \subset \Omega$, we identify ω with its characteristic function $\chi_\omega \in L^\infty(\Omega; \{0, 1\})$ and we identify \mathcal{U}_L with a subset of $L^\infty(\Omega)$ (as in [1, 28, 29]). Then, the convex closure of \mathcal{U}_L in L^∞ weak star topology is

$$\overline{\mathcal{U}_L} = \left\{ a \in L^\infty(\Omega; [0, 1]) \mid \int_\Omega a(x) dx \leq L|\Omega| \right\}$$

which is also weak star compact. We define the *convexified* (or *relaxed*) optimal control problem **(ocp)_T** of determining a control $t \mapsto a(t) \in \overline{\mathcal{U}_L}$ minimizing the cost

$$J_T(a) = \frac{\gamma_1}{2T} \int_0^T \|y(t) - y_d\|^2 dt + \frac{\gamma_2}{2} \|y(T) - y_d\|^2$$

under the constraints

$$\partial_t y + Ay = a, \quad y|_{\partial\Omega} = 0, \quad y(0) = y_0 \quad (7)$$

The corresponding convexified static optimization problem **(sop)** is

$$\min_{a \in \overline{\mathcal{U}_L}} \frac{\gamma_1}{2} \|y - y_d\|^2, \quad Ay = a, \quad y|_{\partial\Omega} = 0 \quad (8)$$

Note that the control a does not appear in the cost functionals of the above convexified control problems. Therefore the resulting optimal control problems are affine with respect to a . Once we have proved that optimal solutions a do exist, we expect that

any minimizer will be an extremal point of the compact convex set $\overline{\mathcal{U}}_L$, which is exactly \mathcal{U}_L : if this is true, then actually $a = \chi_\omega$ with $\omega \in \mathcal{U}_L$. Here, as it is usual in shape optimization, the interest of passing by the convexified problem is to allow us to derive optimality conditions, and thus to characterize the optimal solution. It is anyway not always the case that the minimizer a of the convexified problem is an extremal point of $\overline{\mathcal{U}}_L$ (i.e., a characteristic function): in this case, we speak of a *relaxation phenomenon*. Our analysis hereafter follows these guidelines.

Taking a minimizing sequence and by classical arguments of functional analysis (see, e.g., [23]), it is straightforward to prove existence of solutions a_T and \bar{a} respectively of **(ocp)_T** and of **(sop)** (see details in Section 3.1).

It can be noted that, when $a_T(\cdot) = \chi_{\omega_T}(\cdot)$ and $\bar{a} = \chi_{\bar{\omega}}$ with $\omega_T(t) \in \mathcal{U}_L$ (for a.e. $t \in [0, T]$) and $\bar{\omega} \in \mathcal{U}_L$, i.e., when $a_T(t)$ (for a.e. $t \in [0, T]$) and \bar{a} are characteristic functions of some subsets, then actually, $t \mapsto \omega_T(t)$ and $\bar{\omega}$ are optimal shapes, solutions respectively of **(ocp)_T** and of **(sop)**.

Our next task is to apply necessary optimality conditions to optimal solutions of the convexified problems, and infer from these necessary conditions that, under appropriate assumptions, the optimal controls are indeed characteristic functions.

Necessary optimality conditions for (ocp)_T. According to the Pontryagin maximum principle (see [23, Chapter 3, Theorem 2.1], see also [21]), for any optimal solution (y_T, a_T) of **(ocp)_T** there exists an adjoint state $p_T \in L^2(0, T; \Omega)$ such that

$$\partial_t y_T + A y_T = a_T, \quad y_{T|_{\partial\Omega}} = 0, \quad y_T(0) = y_0 \quad (9)$$

$$\partial_t p_T - A^* p_T = \gamma_1(y_T - y_d), \quad p_{T|_{\partial\Omega}} = 0, \quad p_T(T) = \gamma_2(y_T(T) - y_d) \quad (10)$$

$$\forall a \in \overline{\mathcal{U}}_L, \text{ for a.e. } t \in [0, T], \quad (p_T(t), a_T(t) - a) \geq 0 \quad (11)$$

Necessary optimality conditions for (sop). Similarly, applying [23, Chapter 2, Theorem 1.4], for any optimal solution (\bar{y}, \bar{a}) of **(sop)** there exists an adjoint state $\bar{p} \in L^2(\Omega)$ such that

$$\begin{aligned} A\bar{y} &= \bar{a}, & \bar{y}|_{\partial\Omega} &= 0 \\ -A^*\bar{p} &= \gamma_1(\bar{y} - y_d), & \bar{p}|_{\partial\Omega} &= 0 \end{aligned} \quad (12)$$

$$\forall a \in \overline{\mathcal{U}}_L \quad (\bar{p}, \bar{a} - a) \geq 0 \quad (13)$$

Using the bathtub principle (see, e.g., [22, Theorem 1.14]), (11) and (13) give

$$a_T(\cdot) = \chi_{\{p_T(\cdot) > s_T(\cdot)\}} + c_T(\cdot) \chi_{\{p_T(\cdot) = s_T(\cdot)\}} \quad (14)$$

$$\bar{a} = \chi_{\{\bar{p} > \bar{s}\}} + \bar{c} \chi_{\{\bar{p} = \bar{s}\}} \quad (15)$$

with

$$\begin{aligned} \text{a.e. } t \in [0, T], \quad c_T(t) &\in L^\infty(\Omega; [0, 1]) \text{ and } \bar{c} \in L^\infty(\Omega; [0, 1]) \\ s_T(\cdot) &= \inf \{ \sigma \in \mathbf{R} \mid |\{p_T(\cdot) > \sigma\}| \leq L|\Omega| \} \\ \bar{s} &= \inf \{ \sigma \in \mathbf{R} \mid |\{\bar{p} > \sigma\}| \leq L|\Omega| \} \end{aligned}$$

Note that, if $|\{\bar{p} = \bar{s}\}| = 0$, then it follows from (15) that the static optimal control \bar{a} is actually the characteristic function of a shape $\bar{\omega} \in \mathcal{U}_L$ and hence in that case we have existence of an optimal shape.

2.3. Main results

Existence of optimal shapes. Proving existence of optimal shapes, solutions of **(OSD)_T** and of **(SSD)**, is not an easy task. We can find cases where there is no existence for a variant of **(SSD)** in [19, Sec. 4.2, Example 2]: this is the relaxation phenomenon. Therefore, some assumptions are required on the target function y_d to establish existence of optimal shapes.

We define:

- $y^{T,0}$ and $y^{T,1}$, the solutions of (7) corresponding respectively to $a(\cdot) = 0$ and to $a(\cdot) = 1$;
- $y^{s,0}$ and $y^{s,1}$, the solutions of (8) corresponding respectively to $a = 0$ and to $a = 1$;
- $y^0 = \min(y^{s,0}, \min_{t \in (0,T)} y^{T,0}(t))$ and $y^1 = \max(y^{s,1}, \max_{t \in (0,T)} y^{T,1}(t))$.

Theorem 1. *We distinguish between Lagrange and Mayer cases.*

1. $\gamma_1 = 0, \gamma_2 = 1$ (*Mayer case*): *If A is analytic hypoelliptic in Ω then there exists a unique optimal shape ω_T , solution of **(OSD)_T**.*
2. $\gamma_1 = 1, \gamma_2 = 0$ (*Lagrange case*): *Assuming that $y_0 \in D(A)$ and that $y_d \in H^2(\Omega)$:*
 - (i) *If $y_d < y^0$ or $y_d > y^1$ then there exist unique optimal shapes $\bar{\omega}$ and ω_T , respectively, of **(SSD)** and of **(OSD)_T**.*
 - (ii) *There exists a function β such that if $Ay_d \leq \beta$, then there exists a unique optimal shape $\bar{\omega}$, solution of **(SSD)**.*

We recall that A is said to be analytic hypoelliptic in the open set Ω if any solution of $Au = v$ with v analytic in Ω is also analytic in Ω . Analytic hypoellipticity is satisfied for the second-order elliptic operator A as soon as its coefficients are analytic in Ω (for instance it is the case for the Dirichlet Laplacian, without any further assumption).

Remark 2. *This result implies uniqueness of the optimal shapes. We deduce from (A.2) that we also have uniqueness of state and adjoint.*

In what follows, we denote by

- (y_T, p_T, ω_T) the optimal triple of **(OSD)_T** and

$$J_T = \frac{\gamma_1}{2T} \int_0^T \|y_T(t) - y_d\|^2 dt + \frac{\gamma_2}{2} \|y_T(T) - y_d\|^2;$$

- $(\bar{y}, \bar{p}, \bar{\omega})$ the optimal triple of **(SSD)** and $\bar{J} = \frac{\gamma_1}{2} \|\bar{y} - y_d\|^2$.

Integral turnpike in the Lagrange case.

Theorem 3. *For $\gamma_1 = 1, \gamma_2 = 0$ (Lagrange case), there exists $M > 0$ such that*

$$\int_0^T (\|y_T(t) - \bar{y}\|^2 + \|p_T(t) - \bar{p}\|^2) dt \leq M \quad \forall T > 0.$$

Measure-turnpike in the Lagrange case.

Definition 4. We say that (y_T, p_T) satisfies the state-adjoint measure-turnpike property if for every $\varepsilon > 0$ there exists $\Lambda(\varepsilon) > 0$, independent of T , such that

$$|P_{\varepsilon, T}| < \Lambda(\varepsilon) \quad \forall T > 0$$

where $P_{\varepsilon, T} = \{t \in [0, T] \mid \|y_T(t) - \bar{y}\| + \|p_T(t) - \bar{p}\| > \varepsilon\}$.

We refer to [5, 13, 32] (and references therein) for similar definitions. Here, $P_{\varepsilon, T}$ is the set of times along which the time optimal state-adjoint pair (y_T, p_T) remains outside of an ε -neighborhood of the static optimal state-adjoint pair (\bar{y}, \bar{p}) in L^2 topology.

We next recall the notion of dissipativity (see [36]).

Definition 5. We say that $(\text{OSD})_T$ is strictly dissipative at an optimal stationary point $(\bar{y}, \bar{\omega})$ of (6) with respect to the supply rate function

$$w(y, \omega) = \|y - y_d\|^2 - \|\bar{y} - y_d\|^2$$

if there exists a storage function $S : E \rightarrow \mathbf{R}$ locally bounded and bounded below and a \mathcal{K} -class function $\alpha(\cdot)$ such that, for any $T > 0$ and any $0 < \tau < T$, the strict dissipation inequality

$$S(y(\tau)) + \int_0^\tau \alpha(\|y(t) - \bar{y}\|) dt < S(y(0)) + \int_0^\tau w(y(t), \omega(t)) dt \quad (16)$$

is satisfied for any pair $(y(\cdot), \omega(\cdot))$ solution of (5).

Theorem 6. For $\gamma_1 = 1, \gamma_2 = 0$ (Lagrange case):

- (i) $(\text{OSD})_T$ is strictly dissipative in the sense of Definition 5.
- (ii) The state-adjoint pair (y_T, p_T) satisfies the measure-turnpike property.

Exponential turnpike. The exponential turnpike property is a stronger property and can be satisfied either by the state, by the adjoint or by the control or even by the three together.

Theorem 7. For $\gamma_1 = 0, \gamma_2 = 1$ (Mayer case): For Ω with C^2 boundary and $c = 0$ there exist $T_0 > 0, M > 0$ and $\mu > 0$ such that, for every $T \geq T_0$,

$$d_{\mathcal{H}}(\omega_T(t), \bar{\omega}) \leq M e^{-\mu(T-t)} \quad \forall t \in (0, T).$$

In the Lagrange case, based on the numerical simulations presented in Section 4 we conjecture the exponential turnpike property, i.e., given optimal triples $(y_T, p_T, \chi_{\omega_T})$ and $(\bar{y}, \bar{p}, \bar{\omega})$, there exist $C_1 > 0$ and $C_2 > 0$ independent of T such that

$$\|y_T(t) - \bar{y}\| + \|p_T(t) - \bar{p}\| + \|\chi_{\omega_T(t)} - \chi_{\bar{\omega}}\| \leq C_1 \left(e^{-C_2 t} + e^{-C_2(T-t)} \right)$$

for a.e. $t \in [0, T]$.

3. Proofs

3.1. Proof of Theorem 1

We first show existence of an optimal shape, solution for $(\text{ocp})_T$ and similarly for (sop) . We first see that the infimum exists. We take a minimizing sequence $(y_n, a_n) \in L^2(0, T; H_0^1(\Omega)) \times L^\infty(0, T; \overline{\mathcal{U}}_L)$ such that, for every $n \in \mathbf{N}$, the pair (y_n, a_n) satisfies (7) and $J_T(a_n) \rightarrow J_T$. The sequence (a_n) is bounded in $L^\infty(0, T; L^\infty(\Omega))$, so, using (A.2), the sequence (y_n) is bounded in $L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$. We show then, using (7), that the sequence $(\frac{\partial y_n}{\partial t})$ is bounded in $L^2(0, T; H^{-1}(\Omega))$. We subtract a sequence always denoted by (y_n, a_n) such that one can find a pair $(y, a) \in L^2(0, T; H_0^1(\Omega)) \times L^\infty(0, T; \overline{\mathcal{U}}_L)$ with

$$\begin{aligned} y_n &\rightharpoonup y && \text{weakly in } L^2(0, T; H_0^1(\Omega)) \\ \partial_t y_n &\rightharpoonup \partial_t y && \text{weakly in } L^2(0, T; H^{-1}(\Omega)) \\ a_n &\rightharpoonup a && \text{weakly * in } L^\infty(0, T; L^\infty(\Omega)) \end{aligned}$$

We deduce that

$$\begin{aligned} \partial_t y_n + A y_n - a_n &\rightarrow \partial_t y + A y - a && \text{in } \mathcal{D}'(\Omega) \\ y_n(0) &\rightarrow y(0) && \text{weakly in } L^2(\Omega) \end{aligned} \quad (17)$$

We get using (17) that (y, a) is a weak solution of (7). The pair (y, a) is then admissible. Since $H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega)$ and by using the Aubin-Lions compactness Lemma (see [3]), we obtain

$$y_n \rightarrow y \quad \text{strongly in } L^2(0, T; L^2(\Omega))$$

We get then by weak lower semi-continuity of J_T and of the volume constraint, and by the Fatou Lemma that

$$J_T(a) \leq \liminf J_T(a_n) \quad \text{and} \quad \int_\Omega a(t, x) dx \leq L|\Omega| \quad \forall t \in (0, T)$$

hence a an optimal control for $(\text{ocp})_T$, that we rather denote by a_T (and \bar{a} for (sop)).

We next proceed by proving existence of optimal shape designs.

1- We take $\gamma_1 = 0, \gamma_2 = 1$ (Mayer case). We consider an optimal triple (y_T, p_T, a_T) of $(\text{OSD})_T$. Then it satisfies (12) and (14). It follows from the properties of the parabolic equation and from the assumption of analytic hypoellipticity that p_T is analytic on $(0, T) \times \Omega$ and that all level sets $\{p_T(t) = \alpha\}$ have zero Lebesgue measure. We conclude that the optimal control a_T satisfying (12)-(14) is such that

$$\text{for a.e. } t \in [0, T] \quad \exists s(t) \in \mathbf{R} \mid a_T(t, \cdot) = \chi_{\{p_T(t) > s(t)\}} \quad (18)$$

i.e., $a_T(t)$ is a characteristic function. Hence, for a Mayer problem $(\text{OSD})_T$, existence of an optimal shape is proved.

2-(i) In the case $\gamma_1 = 1, \gamma_2 = 0$ (Lagrange case), we give the proof for the static problem (SSD) . We suppose $y_d < y^0$ (we proceed similarly for $y_d > y^1$). Having in mind (12) and (15), we have $A\bar{y} = \bar{c}$ on $\{\bar{p} = \bar{s}\}$. By contradiction, if $\bar{c} \leq 1$ on $\{\bar{p} = \bar{s}\}$, let us consider the solution y^* of (8) with the control a^* which is the same as \bar{a} verifying (15) except that $\bar{c} = 0$ ($\bar{c} = 1$ if $y_d > y^1$)

on $\{\bar{p} = \bar{s}\}$. We have then $A(\bar{y} - y^*) \leq 0$ (or $A(\bar{y} - y^*) \geq 0$ if $y_d > y^1$). Then, by the maximum principle (see [11, sec. 6.4]) and using the homogeneous Dirichlet condition, we get that the maximum (the minimum if $y_d > y^1$) of $\bar{y} - y^*$ is reached on the boundary and hence $y_d \geq y^* \geq \bar{y}$ (or $y_d \leq y^* \leq \bar{y}$ if $y_d > y^1$). We deduce $\|y^* - y_d\| \leq \|\bar{y} - y_d\|$. This means that a^* is an optimal control. We conclude by uniqueness.

We use a similar argument thanks to maximum principle for parabolic equations (see [11, sec. 7.1.4]) for existence of an optimal shape solution of **(OSD)_T**.

In view of proving the next part of the theorem, we first give a useful Lemma inspired from [20, Theorem 3.2] and from [12, Theorem 6.3].

Lemma 8. *Given any $p \in [1, +\infty)$ and any $u \in W^{1,p}(\Omega)$ such that $\{|u = 0|\} > 0$, we have $\nabla u = 0$ a.e. on $\{u = 0\}$.*

Proof of Lemma 8. A proof of a more general result can be found in [20, Theorem 3.2]. For completeness, we give here a short argument. Du denotes here the weak derivative of u . We need first to show that for $u \in W^{1,p}(\Omega)$ and for a function $S \in C^1(\mathbf{R})$ for which there exists $M > 0$ such that $\|S'\|_{L^\infty(\Omega)} < M$, we have $S(u) \in W^{1,p}(\Omega)$ and $DS(u) = S'(u)Du$. By the Meyer-Serrins theorem, we get a sequence $u_n \in C^\infty(\Omega) \cap W^{1,p}(\Omega)$ such that $u_n \rightarrow u$ in $W^{1,p}(\Omega)$ and pointwise too. We first get that $\int_\Omega |S(u)|^p dx \leq \|S'\|_{L^\infty(\Omega)}^p \|u\|_{L^p(\Omega)}^p$. Then $DS(u_n) = S'(u_n)Du_n$. We write

$$\begin{aligned} \int_\Omega |DS(u_n) - S'(u)Du|^p dx &= \int_\Omega |S'(u_n)Du_n - S'(u)Du|^p dx \\ &\leq \int_\Omega |S'(u_n)(Du_n - Du)|^p dx + \int_\Omega |S'(u_n) - S'(u)|^p |Du|^p dx \\ &\leq \|S'\|_{L^\infty(\Omega)}^p \|u_n - u\|_{W^{1,p}(\Omega)}^p + \int_\Omega |S'(u_n) - S'(u)|^p |Du|^p dx \end{aligned}$$

The first term tends to 0 since $u_n \rightarrow u$ in $W^{1,p}(\Omega)$. As regards the second term, we use that $|S'(u_n) - S'(u)|^p \rightarrow 0$ pointwise and $|S'(u_n) - S'(u)|^p \leq 2\|S'\|_{L^\infty(\Omega)}^p$. By the Lebesgue dominated convergence $\int_\Omega |S'(u_n) - S'(u)|^p |Du|^p dx \rightarrow 0$ and $DS(u) = S'(u)Du$. Then, we consider $u^+ = \max(u, 0)$ and $u^- = \min(u, 0) = -\max(-u, 0)$. We define

$$S_\varepsilon(s) = \begin{cases} (s^2 + \varepsilon^2)^{\frac{1}{2}} - \varepsilon & \text{if } s \geq 0 \\ 0 & \text{else} \end{cases}$$

Note that $\|S'_\varepsilon\|_{L^\infty(\Omega)} < 1$. We deduce that $DS_\varepsilon(u) = S'_\varepsilon(u)Du$ for every $\varepsilon > 0$. For $\phi \in C_c^\infty(\Omega)$ we take the limit of $\int_\Omega S_\varepsilon(u)D\phi dx$ when $\varepsilon \rightarrow 0^+$ to get that

$$Du^+ = \begin{cases} Du & \text{on } \{u > 0\} \\ 0 & \text{on } \{u \leq 0\} \end{cases} \quad \text{and} \quad Du^- = \begin{cases} 0 & \text{on } \{u \geq 0\} \\ -Du & \text{on } \{u < 0\} \end{cases}$$

Since $u = u^+ - u^-$, we get $Du = 0$ on $\{u = 0\}$. We can find this Lemma in a weaker form in [12, Theorem 6.3]. \square

2-(ii) We assume that $Ay_d \leq \beta$ in Ω with $\beta = \bar{s}Ac^*$. Having in mind (12) and (15), we assume by contradiction that $\{|\bar{p} = \bar{s}|\} > 0$. By Lemma 8 and since A and A^* are differential operators, we have $A^*\bar{p} = c^*\bar{s}$ on $\{\bar{p} = \bar{s}\}$. We infer that $Ay_d - \bar{s}Ac^* = \bar{a} \in (0, 1)$

on $\{\bar{p} = \bar{s}\}$, which contradicts $Ay_d \leq \beta$. Hence $\{|\bar{p} = \bar{s}|\} = 0$ and thus $\bar{a} = \chi_{\bar{\omega}}$ for some $\bar{\omega} \in \mathcal{U}_L$. Existence of solution for **(SSD)** is proved.

Uniqueness of $\bar{a} = \chi_{\bar{\omega}}$ and of $a_T = \chi_{\omega_T}$ comes from the fact that the cost functionals of **(ocp)_T** and **(sop)** are strictly convex whatever $(\gamma_1, \gamma_2) \neq (0, 0)$ may be. Uniqueness of (\bar{y}, \bar{p}) follows by application of the Poincaré inequality and uniqueness of (y_T, p_T) follows from (A.3).

3.2. Proof of Theorem 3

For $\gamma_1 = 1, \gamma_2 = 0$ (Lagrange case), the cost is $J_T(\omega) = \frac{1}{2T} \int_0^T \|y(t) - y_d\|^2 dt$. We consider the triples $(y_T, p_T, \chi_{\omega_T})$ and $(\bar{y}, \bar{p}, \chi_{\bar{\omega}})$ satisfying the optimality conditions (9), (10) and (12). Since χ_{ω_T} is bounded at each time $t \in [0, T]$ and by application of (A.3) to y_T and p_T we can find a constant $C > 0$ depending only on A, y_0, y_d, Ω, L such that

$$\forall T > 0 \quad \|y_T(T)\|^2 \leq C \quad \text{and} \quad \|p_T(0)\|^2 \leq C$$

Setting $\tilde{y} = y_T - \bar{y}, \tilde{p} = p_T - \bar{p}, \tilde{a} = \chi_{\omega_T} - \chi_{\bar{\omega}}$, we have

$$\partial_t \tilde{y} + A\tilde{y} = \tilde{a}, \quad \tilde{y}|_{\partial\Omega} = 0, \quad \tilde{y}(0) = y_0 - \bar{y} \quad (19)$$

$$\partial_t \tilde{p} - A^*\tilde{p} = \tilde{y}, \quad \tilde{p}|_{\partial\Omega} = 0, \quad \tilde{p}(T) = -\bar{p} \quad (20)$$

First, using (9), (10) and (12) one has $(\tilde{p}(t), \tilde{a}(t)) \geq 0$ for almost every $t \in [0, T]$. Multiplying (19) by \tilde{p} , (20) by \tilde{y} and then adding them, one can use the fact that

$$(\bar{y} - y_0, \tilde{p}(0)) - (\bar{y}(T), \tilde{p}) = \int_0^T (\tilde{p}(t), \tilde{a}(t)) dt + \int_0^T \|\tilde{y}(t)\|^2 dt$$

By the Cauchy-Schwarz inequality we get a new constant $C > 0$ such that

$$\frac{1}{T} \int_0^T \|\tilde{y}(t)\|^2 dt + \frac{1}{T} \int_0^T (\tilde{p}(t), \tilde{a}(t)) dt \leq \frac{C}{T}$$

The two terms at the left-hand side are positive and using the inequality (A.2) with $\psi(t) = \tilde{p}(T - t)$, we finally obtain

$$\frac{1}{T} \int_0^T (\|y_T(t) - \bar{y}\|^2 + \|p_T(t) - \bar{p}\|^2) dt \leq \frac{M}{T}$$

3.3. Proof of Theorem 6

(i) Strict dissipativity is established thanks to the storage function $S(y) = (y, \bar{p})$ where \bar{p} is the optimal adjoint. Since $\|y\|_{L^\infty((0,T) \times \Omega)} < M$, the storage function S is locally bounded and bounded from below. Indeed, we consider an admissible pair $(y(\cdot), \chi_{\omega}(\cdot))$ satisfying (5), we multiply it by \bar{p} and we integrate over Ω . Then we integrate in time on $(0, T)$, we use the optimality conditions of static problem (12) and we get the strict dissipation inequality (16) with $\alpha(s) = s^2$:

$$(\bar{p}, y(\tau)) + \int_0^\tau \|y(t) - \bar{y}\|^2 dt < (\bar{p}, y(0)) + \int_0^\tau w(y(t), \omega(t)) dt \quad (21)$$

(ii) Now we prove that strict dissipativity implies measure-turnpike, by following an argument of [32]. Applying (21) to the optimal solution (y_T, ω_T) at $\tau = T$, we get

$$\frac{1}{T} \int_0^T \|y_T(t) - \bar{y}\|^2 dt \leq J_T - \bar{J} + \frac{(y(0) - y(T), \bar{p})}{T}$$

Considering then the solution y_s of (5) with $\omega(\cdot) = \bar{\omega}$ and $J_s = \frac{1}{T} \int_0^T \|y_s(t) - y_d\|^2$, we have $J_T - J_s < 0$ and we show that $J_s - \bar{J} \leq \frac{1-e^{-CT}}{CT}$, then

$$\frac{1}{T} \int_0^T \|y_T(t) - \bar{y}\|^2 dt \leq \frac{M}{T}$$

Applying (A.2) to $\psi(\cdot) = p_T(T - \cdot) - \bar{p}$, we get

$$\begin{aligned} \frac{1}{2C} \int_0^T \|p_T(t) - \bar{p}\|^2 dt &\leq C \int_0^T \|y_T(t) - \bar{y}\|^2 dt \\ &+ \frac{\|p_T(0) - \bar{p}\|^2 - \|p_T(T) - \bar{p}\|^2}{2} \end{aligned}$$

Using again the strict dissipativity (16) we get $\frac{\varepsilon^2 |P_{\varepsilon,T}|}{T} \leq \frac{M}{T}$. Hence we can find a constant $M > 0$ which does not depend on T such that $|P_{\varepsilon,T}| \leq \frac{M}{\varepsilon^2}$.

3.4. Proof of Theorem 7

We take $\gamma_1 = 0, \gamma_2 = 1$ (Mayer case). We want to characterize optimal shapes as being the level set of some functions as in [7]. Let $(y_T, p_T, \chi_{\omega_T})$ be an optimal triple, coming from Theorem 1-(i). Then $\psi(t, x) = p_T(T - t, x)$ satisfies

$$\partial_t \psi + A^* \psi = 0, \quad \psi|_{\partial\Omega} = 0, \quad \psi(0) = y_1 - y_T(T) \quad (22)$$

We write $y_1 - y(T)$ in the basis $(\phi_j)_{j \in \mathbf{N}^*}$. There exists $(a_j) \in \mathbf{R}^{\mathbf{N}^*}$ such that $y_1 - y(T) = \sum_j a_j \phi_j$. We can solve (22) and get $p(t, x) = \sum_{j \geq 1} a_j \phi_j(x) e^{-\lambda_j(T-t)}$. By the maximum principle for parabolic equations, there exists $M > 0$ such that, for every $T > 0$, for every $t \in (0, T)$, the solution of (5) satisfies $\|y(t)\|^2 \leq M$. Hence $|a_j|^2 \leq M$. Let us consider the index $j_0 = \inf\{j \in \mathbf{N}, a_j \neq 0\}$. Take $\lambda = \lambda_{j_0}$ and $\mu = \lambda_k$ where k is the first index for which $\lambda_k > \lambda$. We define $\Phi_0 = \sum_{\lambda_j = \lambda_{j_0}} \phi_j$ which is a finite sum of

the eigenfunctions associated to the eigenvalue λ_{j_0} . Using the bathtub principle ([22, Theorem 1.16]), we define the stationary shape ω_0 and the constant s_0 such that

$$\omega_0 = \{\Phi_0 < s_0\}, \quad \chi_{\omega_0} \text{ solves: } \max_{u \in \mathcal{U}_L} \int \Phi_0(x) u(x) dx \quad (23)$$

Since $(\phi_j)_{j \in \mathbf{N}^*}$ is an orthonormal basis of $L^2(\Omega)$ we get

$$\|p_T(t) - e^{-\lambda(T-t)} \Phi_0\|_{L^2(\Omega)}^2 \leq C e^{-\mu(T-t)} \quad \forall t \in [0, T]$$

Let us now write, for every $x \in \Omega$ and every $t \in [0, T]$,

$$\begin{aligned} |p(t, x) - e^{-\lambda(T-t)} \Phi_0(x)| &= \left| \sum_{j \geq k} a_j \phi_j(x) e^{-\lambda_j(T-t)} \right| \\ &\leq \sum_{j \geq k} |a_j \phi_j(x) e^{-\lambda_j(T-t)}| \end{aligned}$$

By the Weyl Law and sup-norm estimates for the eigenfunctions of A (see [30, Chapter 3]), there exists $\alpha \in (0, 1)$ such that $\alpha\mu > \lambda$ and thus

$$|p(t, x) - e^{-\lambda(T-t)} \Phi_0(x)| \leq e^{-\alpha\mu(T-t)} \sum_{j \geq k} M j^{\frac{N-1}{2N}} e^{-Cj^{\frac{1}{N}}(T-t)}$$

where M, C are positive constants not depending on x, t, T . Let $\varepsilon > 0$ be arbitrary. We claim that there exists $C_\varepsilon > 0$ independent of x, t, T such that, for every $x \in \Omega$,

$$|p(t, x) - e^{-\lambda(T-t)} \Phi_0(x)| \leq C_\varepsilon e^{-\alpha\mu(T-t)} \quad \forall t \in (0, T - \varepsilon)$$

$$|p(t, x) - e^{-\lambda(T-t)} \Phi_0(x)| \leq C_\varepsilon \quad \forall t \in (T - \varepsilon, T)$$

To conclude we take an arbitrary value for ε and we write μ instead of $\alpha\mu$ but always with $\mu > \lambda$ to get

$$\|p_T(t) - e^{-\lambda(T-t)} \Phi_0\|_{L^\infty(\Omega)} \leq C e^{-\mu(T-t)} \quad \forall t \in [0, T] \quad (24)$$

with C not depending on the final time T . This is an exponential turnpike property on the adjoint-state. Moreover we get from (24) that

$$|s(t) - e^{-\lambda(T-t)} s_0| \leq C e^{-\mu(T-t)} \quad \forall t \in [0, T] \quad (25)$$

We write $\Phi = \Phi_0 - s_0$ and $\psi_0(t) = e^{-\lambda(T-t)} \Phi$ and using (24) with (25), we get

$$\|\psi_T(t, x) - \psi_0(t, x)\|_{L^\infty(\Omega)} \leq C e^{-\mu(T-t)}, \quad \forall t \in [0, T] \quad (26)$$

We now follow arguments of [7] to establish the exponential turnpike property for the control and then for the state by using some information on the control χ_{ω_T} . We first remark that for all $t_1, t_2 \in [0, T]$, $\{\psi_0(t_1, \cdot) < 0\} = \{\psi_0(t_2, \cdot) < 0\} = \{\Phi < 0\}$. Then we take $t \in [0, T]$ and we compare the sets $\{\psi_0(t, \cdot) < 0\}$, $\{\psi_T(t, \cdot) < 0\}$ and $\{\psi_0(t, \cdot) + C e^{-\mu(T-t)} < 0\}$. Thanks to (25) and (26) we get, for every $t \in [0, T]$,

$$\{\Phi \leq -C e^{-(\mu-\lambda)(T-t)}\} \subset \{\psi_T(t, \cdot) \leq 0\} \subset \{\Phi \leq C e^{-(\mu-\lambda)(T-t)}\} \quad (27)$$

$$\{\Phi \leq -C e^{-(\mu-\lambda)(T-t)}\} \subset \{\psi_0(t, \cdot) \leq 0\} \subset \{\Phi \leq C e^{-(\mu-\lambda)(T-t)}\} \quad (28)$$

We infer from [7, Lemma 2.3] that, for every $t \in [0, T]$,

$$\begin{aligned} d_{\mathcal{H}}(\{\psi_T(t, \cdot) \leq 0\}, \{\Phi \leq 0\}) \\ \leq d_{\mathcal{H}}(\{\Phi \leq -C e^{-(\mu-\lambda)(T-t)}\}, \{\Phi \leq C e^{-(\mu-\lambda)(T-t)}\}) \end{aligned} \quad (29)$$

Since $d_{\mathcal{H}}$ is a distance, we only have to estimate $d_{\mathcal{H}}(\{\Phi \leq 0\}, \{\Phi \leq \pm C e^{-(\mu-\lambda)(T-t)}\})$.

Lemma 9. *Let $f : \Omega \rightarrow \mathbf{R}$ be a continuously differentiable function and set $\Gamma = \{f = 0\}$. Under the assumption (S): there exists $C > 0$ such that*

$$\|\nabla f(x)\| \geq C \quad \forall x \in \Gamma,$$

there exist $\varepsilon_0 > 0$ and $C_f > 0$ only depending on f such that for any $\varepsilon \leq \varepsilon_0$

$$d_{\mathcal{H}}(\{f \leq 0\}, \{f \leq \pm \varepsilon\}) \leq C_f \varepsilon.$$

Proof of Lemma 9. We consider f satisfying (S) with $\Gamma = \{\Phi = 0\}$. We assume by contradiction that for every $\varepsilon > 0$, there exists $x \in \{|f| \leq \varepsilon\}$ such that $\|\nabla f(x)\| = 0$. We take $\varepsilon = \frac{1}{n}$ and we subtract a subsequence $(x_n) \rightarrow x \in \{|f| \leq 1\}$ (which is compact). By continuity of f and of $\|\nabla f\|$, we have $x \in \Gamma$ and

$\|f(x)\| = 0$, which raises contradiction with (S). Hence we find $\varepsilon_0 > 0$ such that $\|\nabla f(x)\| \geq \frac{c}{2}$ for every $x \in \{|f| \leq \varepsilon\}$. We apply [4, Corollary 4] (see also [4, Theorem 2]) to get

$$d_{\mathcal{H}}(\{f \leq 0\}, \{f \leq \pm\varepsilon\}) \leq \frac{2}{C}\varepsilon$$

A more general statement can be found in [4, 7]. \square

We infer that Φ satisfies (S) on $\|\nabla_x \psi_0(t, x)\| = e^{-\lambda(T-t)}\|\nabla_x \Phi(x)\|$ for $x \in \Omega$. We first remark that Φ_0 satisfies $A\Phi_0 = \lambda_j_0 \Phi_0, \Phi_{0\Gamma} = s_0$ and that the set $\Gamma = \{\Phi_0 = 0\}$ is compact. Since Ω has a C^2 boundary and $c = 0$ the Hopf lemma (see [11, sec. 6.4]) gives

$$x_0 \in \Gamma_0 \implies \|\nabla_x \Phi(x_0)\| = \|\nabla_x \Phi_0(x_0)\| > 0$$

Hence there exists $C_0 > 0$ not depending on t, T such that for every $x \in \Gamma_0, \|\nabla_x \Phi(x_0)\| \geq C_0 > 0$. We take $\nu > 0, e^{-\mu\nu} \leq \varepsilon_0$. We remark that $e^{-\mu(T-t)} \leq \varepsilon_0, \forall t \in (0, T - \nu)$ and we use Lemma 9 combined with (29) to get that, for every $t \in (0, T - \nu)$,

$$d_{\mathcal{H}}(\{\psi_T(t, \cdot) \leq 0\}, \{\Phi \leq 0\}) \leq C_0 e^{-(\mu-\lambda)(T-t)}$$

We adapt the constant C_0 such that on the compact interval $t \in (T - \nu, T)$ the sets are the same whatever $T \geq T_0 > 0$ may be, to get that, for every $t \in (0, T)$,

$$d_{\mathcal{H}}(\{\psi_T(t, \cdot) \leq 0\}, \{\Phi \leq 0\}) \leq C_0 e^{-(\mu-\lambda)(T-t)}.$$

We obtain therefore an exponential turnpike property for the control in the sense of the Hausdorff distance:

$$d_{\mathcal{H}}(\omega(t), \omega_0) \leq C_0 e^{-(\mu-\lambda)(T-t)} \quad \forall t \in [0, T] \quad (30)$$

To establish the further turnpike property on state and adjoint we could use a similar argument as in [14, Theorem 1-(i)]: $\|\chi_{\omega(t)} - \chi_{\omega_0}\| \leq C d_{\mathcal{H}}(\omega(t), \omega_0)$. We follow [9, Theorem 4.1-(ii)] and [9, Theorem 5.1-(iii)(iv)] and we use the inequality $\|\chi_{A_1} - \chi_{A_2}\| \leq \|d_{A_1} - d_{A_2}\|_{W^{1,2}(\Omega)} \leq \|b_{A_1} - b_{A_2}\|_{W^{1,2}(\Omega)} = \|b_{A_1} - b_{A_2}\| + \|\nabla b_{A_1} - \nabla b_{A_2}\|$, where $\|\chi_{\omega(t)} - \chi_{\omega_0}\|$ is the measure of the symmetric difference of the sets K_1, K_2 . Therefore, applying the energy inequality (A.3), we get

$$\|y(t) - \bar{y}\|_{L^2(\Omega)} \leq C_0 e^{-\frac{(\mu+\lambda)}{2}(T-t)} \quad \forall t \in (0, T) \quad (31)$$

with \bar{y} solution of $Ay = \chi_{\omega_0}, y|_{\partial\Omega} = 0$. Taking $\kappa = \frac{\mu+\lambda}{2} > 0$ and by application of (A.2) for the adjoint, we finally get the exponential turnpike property for the state, adjoint and control.

4. Numerical simulations: optimal shape design for the 2D heat equation

We take $\Omega = [-1, 1]^2, L = \frac{1}{8}, T \in \{1, \dots, 5\}, y_d = \text{Cst} = 0.1$ and $y_0 = 0$. We focus on the heat equation and consider the minimization problem

$$\min_{\omega(\cdot)} \int_0^5 \int_{[-1,1]^2} |y(t, x) - 0.1|^2 dx dt \quad (32)$$

under the constraints

$$\partial_t y - \Delta y = \chi_{\omega}, \quad y(0, \cdot) = 0, \quad y|_{\partial\Omega} = 0 \quad (33)$$

We compute numerically a solution by solving the equivalent convexified problem $(\text{ocp})_T$ thanks to a direct method in optimal control (see [31]). We discretize here with an implicit Euler method in time and with a decomposition on a finite element mesh of Ω using FREEFEM++ (see [18]). We express the problem as a quadratic programming problem in finite dimension. We use then the routine IpOpt (see [35]) on a standard desktop machine.

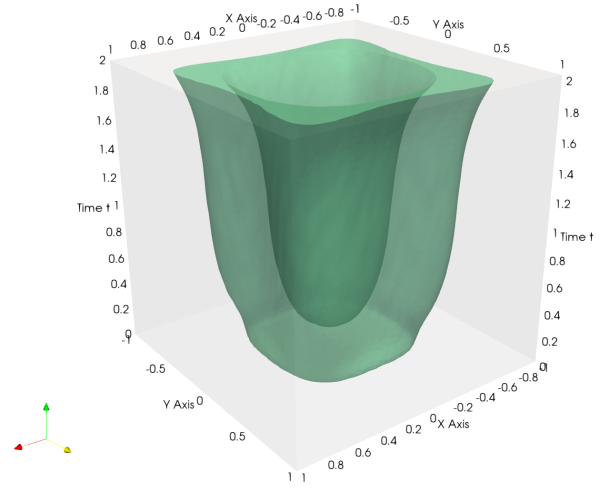


Figure 1: Time optimal shape's evolution cylinder - $T = 2$

We plot in Figure 1 the evolution in time of the optimal shape $t \rightarrow \omega(t)$ which appears like a cylinder whose section at time t represents the shape $\omega(t)$. At the beginning ($t = 0$) we notice that the shape concentrates at the middle of Ω in order to warm as soon as possible near to y_d . Once it is acceptable the shape is almost stationary during a long time. Finally, close to the final time, the shape moves to the boundary of Ω in order to flatten the state y_T because y_d is taken here as a constant.

We plot in Figure 2 the comparison between the optimal shape at several times (in red) and the optimal static shape (in yellow). We see the same behavior when $t = \frac{T}{2}$.

Now in order to highlight the turnpike phenomenon, we plot the evolution in time of the distance between the optimal dynamic triple and the optimal static one $t \mapsto \|y_T(t) - \bar{y}\| + \|p_T(t) - \bar{p}\| + \|\chi_{\omega_T(t)} - \chi_{\omega}\|$. In Figure 3 we observe that this function is exponentially close to 0. This behavior leads us to conjecture that the exponential turnpike property should be satisfied.

To complete this work, we need to clarify the existence of optimal shapes for $(\text{OSD})_T$ when y_d is convex. We see numerically in Figure 2 the time optimal shape's existence for y_d convex on Ω . Otherwise we can sometimes observe a relaxation phenomenon due to the presence of \bar{c} and $c_T(\cdot)$ in the optimality conditions (9), (10), (12).

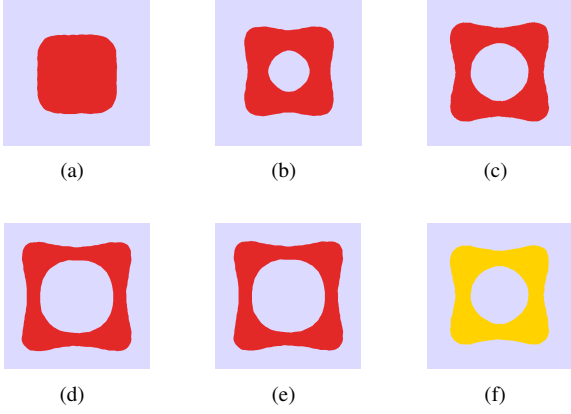


Figure 2: Time optimal shape $T = 5$ - Static shape: (a) $t = 0$; (b) $t = 0.5$; (c) $t \in [1, 4]$; (d) $t = 4.5$; (e) $t = T$; (f) static shape

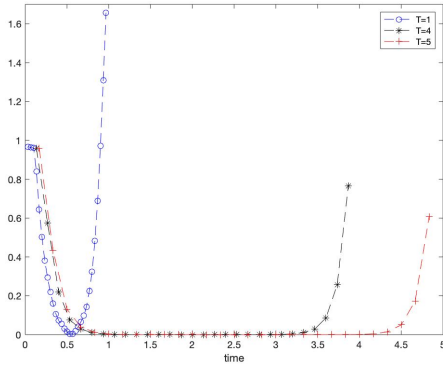


Figure 3: Error between time optimal triple and static one

We consider the same problem $(\mathbf{ocp})_T$ in 2D with $\Omega = [-1, 1]^2$, $L = \frac{1}{8}$, $T = 5$ and the static one associated (\mathbf{sop}) . We take $y_d(x, y) = -\frac{1}{20}(x^2 + y^2 - 2)$.

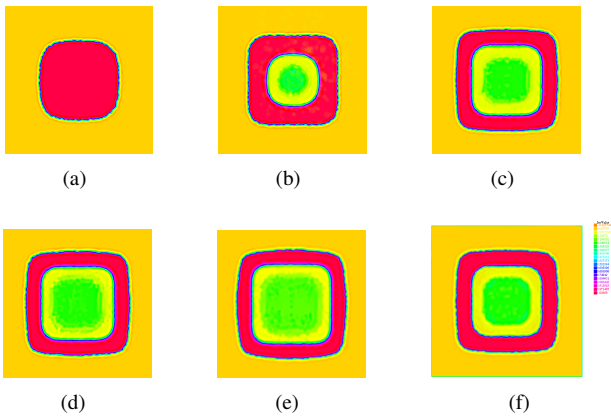


Figure 4: Relaxation phenomenon : (a) $t = 0$; (b) $t = 0.5$; (c) $t \in [1, 4]$; (d) $t = 4.5$; (e) $t = T$; (f) static shape

In Figure 4 we see that optimal control (a_T, \bar{a}) of $(\mathbf{ocp})_T$ and (\mathbf{sop}) take values in $(0, 1)$ in the middle of Ω . This illustrates that relaxation occurs for some y_d . Here, y_d was chosen such that $-\Delta y_d \in (0, 1)$. We have tuned the parameter L to observe

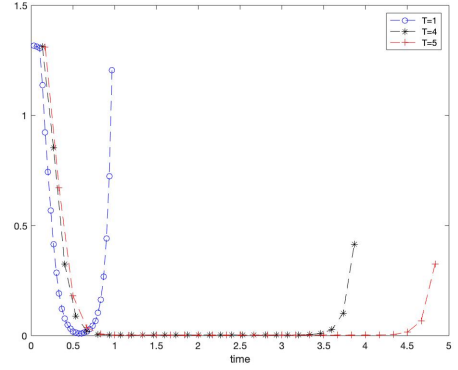


Figure 5: Error between time optimal triple and static one (Relaxation case)

the relaxation phenomenon, but for same y_d and smaller L , optimal solutions are shapes. Despite the relaxation we see in Figure 5 that turnpike still occurs.

5. Further comments

Numerical simulations when $\Delta y_d > 0$ lead us to conjecture existence of an optimal shape for $(\mathbf{OSD})_T$, because we have not observed any relaxation phenomenon in that case. Existence might be proved thanks to arguments like maximal regularity properties and Hölder estimates for solutions of parabolic equations.

Moreover, still based on our simulations and particularly on Figure 3, we conjecture the exponential turnpike property.

The work that we presented here is focused on second-order parabolic equations and particularly on the heat equation. Concerning the Mayer case, we have used in our arguments the Weyl law, sup-norm estimates of eigenelements (see [30]) and analyticity of solutions (analytic hypoellipticity). Nevertheless, concerning the Lagrange case and having in mind [27, 32] it seems reasonable to extend our results to general local parabolic operators which satisfy an energy inequality (A.2) and the maximum principle to ensure existence of solutions. However, some results like Theorem 1.2-(ii) should be adapted. Moreover we consider a linear partial differential equation which gives uniqueness of the solution thanks to the strict convexity of the criterion. At the contrary, if we do not have uniqueness, as in [32], the notion of measure-turnpike seems to be a good and soft way to obtain turnpike results.

To go further with the numerical simulations, our objective will be to find optimal shapes evolving in time, solving dynamical shape design problems for more difficult real-life partial differential equations which play a role in fluid mechanics for example. We can find in the recent literature some articles on the optimization of a wavemaker (see [6, 25]). It is natural to wonder what can happen when considering a wavemaker whose shape can evolve in time.

Appendix A. Energy inequality

We recall some useful inequalities to study existence and turnpike. Since θ satisfies (4), we can find $\beta > 0, \gamma \geq 0$ such that $\beta \geq \gamma$ and

$$(Au, u) \geq \beta \|u\|_{H_0^1(\Omega)} - \gamma \|u\|_{L^2(\Omega)} \quad (\text{A.1})$$

From this follows the energy inequality (see [11, Chapter 7, Theorem 2]): there exists $C > 0$ such that, for any solution y of (7), for almost every $t \in [0, T]$,

$$\|y(t)\|^2 + \int_0^t \|y(s)\|_{H_0^1(\Omega)}^2 ds \leq C \left(\|y_0\|^2 + \int_0^t \|a(s)\|^2 ds \right) \quad (\text{A.2})$$

We can improve this inequality by using the Poincaré inequality and the Gronwall Lemma to obtain $C_1, C_2 > 0$ such that, for almost every $t \in [0, T]$,

$$\|y(t)\|^2 \leq C_1 \left(\|y_0\|^2 e^{-\frac{t}{C_2}} + \int_0^t e^{-\frac{t-s}{C_2}} \|a(s)\|^2 ds \right) \quad (\text{A.3})$$

The constants C, C_1, C_2 depend only on the domain Ω (Poincaré inequality) and on the operator A and not on final time T since (A.1) is satisfied with $\beta \geq \gamma$.

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