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Abstract: This paper presents a rigorous definition of the isolability of a defect in a flat system whose flat outputs are measured by sensors that are subject to faults. Particularly, we show that the isolation of higher-dimensional defects can be attained under a certain condition pertaining to the relation between the flat outputs. Accordingly, a detailed characterization of this relation is presented in a mathematical framework. Finally, the validity of the results is demonstrated using the three-tank system.

Keywords: Non-linear flat system, flat output, fault detection and isolation, three-tank system.

1. INTRODUCTION

The fault detection and isolation (FDI) problem has been introduced in automatic control as a paradigm for designing algorithms able to detect the outbreak of faults and isolate their causes. Various FDI techniques have been developed and can be found in survey papers (see *e.g.* Zhou et al. (2014), Thirumarimurugan et al. (2016)). The first proposed method is the hardware redundancy in which multiple sensors and actuators are used to measure and control a particular variable (Chen et al. (2015)). The drawbacks of this method are the extra equipment, maintenance cost and additional space required to accommodate the equipment. Later, this method has been replaced by the analytical redundancy, which is based on the notion of generating residual signals. These residues are defined as the difference between the measured variables and the estimated ones. In the case of no defect, the value of the residue is close to zero and it is different than zero otherwise. There exist several methods to generate residues, such as the observer-based approach (Tousi and Khorasani (2011)), the parity-space approach (Diversi et al. (2002)) or the Kalman-based approach (Izadian and Khayyer (2010)).

Recently, the flatness property has been introduced into the repertoire of FDI techniques ((Suryawan et al. (2010)), (Martínez-Torres et al. (2014))). Here, residues are calculated using the differential flatness property. Roughly speaking, a system is said to be flat if all the state and input variables can be expressed as function of a particular variable, called flat output, and a finite number of its successive derivatives. The method presented in (Suryawan et al. (2010)) is dedicated to linear flat systems and it uses B-splines parametrisation to estimate the time derivatives of the flat output, which may not be defined because of the presence of noises. The disadvantage of this method is that the derivative estimation could take time and might delay the reconfiguration process. This issue has been overcome later in (Martínez-Torres et al. (2014)), where a high-gain observer is used to evaluate the time derivative of noisy signals and a low-pass filter is synthesized to improve its performance. In addition, this method can be applied to both, linear and non-linear flat systems.

In the flatness-based FDI approach, residues betweeen the measured state and input variables and their expression using the measured flat output are computed online. Then, the fault detection algorithm is similar to that of other approaches: if a residue exceeds its threshold then a fault is detected. Therefore the problem of the isolability of a fault is directly related to the dependence of the generated residues with respect to the state variables (Kóscielny et al. (2016)). Moreover, if several flat outputs are required, these flat outputs must be such that, if a fault affects one flat output, the others are not totally affected (Martínez-Torres et al. (2014)).

In the present paper, a rigorous definition of the isolability of defects is presented as well as a characterization of the relations between the flat outputs used. The latter flat outputs may be computed by the method of unimodular completion of polynomial matrices (Franke and Röbenack (2013)) that, in some particular cases, directly provides the flat outputs (Fritzsche et al. (2016b)).

The main contributions of this paper are the above mentioned rigorous definition of the isolability of defects and

the characterization of the flat outputs to be used in the defect isolation.

This paper is organised as follows: section 2 introduces the basic concepts of FDI for non-linear differentially flat systems and their definitions. After a brief recall in section 3 of the direct flat output computation method, section 4 discusses the relations that exist between flat outputs. Section 5 presents the application of this FDI approach to the three-tank system. Finally, section 6 concludes the paper.

2. FLATNESS-BASED FDI

2.1 Differentially Flat System

Consider the following non-linear system

$$\dot{x} = f(x, u) \tag{1}$$

where x, the vector of states, evolves in a n-dimensional manifold X, $u \in \mathbb{R}^m$ is the vector of inputs, $m \leq n$ and $\text{rank}(\frac{\partial f}{\partial u}) = m$. Let $(x, \overline{u}) \triangleq (x, u, \dot{u}, \ddot{u}, \ldots)$ be a prolongation of the coordinates (x, u) to the manifold of jets of infinite order $\mathfrak{X} \triangleq X \times \mathbb{R}^m_\infty$ (Levine, 2009, Chapter 5).

The system (1) is flat at a point $(x_0, \overline{u}_0) \in \mathfrak{X}$ if and only if there exist a vector $y = (y_1, \ldots, y_m) \in \mathbb{R}^m$, two integers r and s and mappings ψ defined on a neighbourhood $\mathcal{V}$ of $(x_0, \overline{u}_0)$ in $\mathfrak{X}$ and ϕ defined on a neighbourhood $\mathcal{W} \subset \psi(\mathcal{V})$ of $\overline{y}_0 \triangleq (y_0, \dot{y}_0, \ddot{y}_0, \ldots) \triangleq \psi(x_0, \overline{u}_0)$ in $\mathbb{R}^m_\infty$ such that:

(1) $y = \psi(x, u, \dot{u}, \ldots, u^{(s)}) \in \mathcal{W}$
(2) $y_1, \ldots, y_m$ and their successive derivatives are linearly independent in $\mathcal{W}$
(3) $(x, u) = \varphi(\overline{y})$
$= (\varphi_0(y, \dot{y}, \ldots, y^{(r)}, \varphi_1(y, \dot{y}, \ldots, y^{(r+1)})) \in \text{pr}_{X \times \mathbb{R}^m}(\mathcal{V})$
where $\text{pr}_{X \times \mathbb{R}^m}(\mathcal{V})$ is the projection of $\mathcal{V}$ in $X \times \mathbb{R}^m$
(4) The differential equation $\dot{\varphi}_0(\overline{y}) = f(\varphi_0(\overline{y}), \varphi_1(\overline{y}))$ is identically satisfied in $\mathcal{W}$.

The vector y is called flat output of the system. The mappings ψ and φ are called isomorphisms of Lie-Bäcklund and are inverse of one another.

After elimination of the input u in equation (1), the implicit system associated with the system (1) is given by:

$$F(x, \dot{x}) = 0 \tag{2}$$

where F is supposed to be meromorphic (Levine (2009)) and $\text{rank}(\frac{\partial F}{\partial \dot{x}}) = n - m$. Integral curves of the systems (1) and (2) coincide on the set $\mathfrak{X}_0$ defined by

$$\mathfrak{X}_0 = \left\{ \overline{x} \in \mathfrak{X} \mid \frac{d^k}{dt^k} F(x, \dot{x}) = 0, \forall k \in \mathbb{N} \right\} \setminus \{ \overline{x} \in \mathfrak{X} \mid \nexists\, u \in \mathbb{R}^m \text{ s.t. } \dot{x} - f(x, u) = 0 \}.$$

Remark 1. The property of flatness is not defined globally. The Lie-Bäcklund isomorphisms ψ and ϕ are only locally defined. Thus, there might exist points in $\mathfrak{X}_0$ where no such isomorphisms exist or, otherwise stated, where the system is not flat. It has been proven in (Kaminski et al. (2018)) that the set of intrinsic singularities contains the set of equilibrium points of the system that are not first order controllable.

2.2 Fault Detection and Isolation

We recall the residue design method of (Martínez-Torres et al. (2014)), using the expressions of the state and input variables in function of the flat output.

Suppose that the system (1) is flat with
$y = \psi(x, u, \dot{u}, \ldots, u^{(s)})$ a flat output, then the full state and input read:

$$x = \varphi_0(y, \dot{y}, \ldots, y^{(r)}) \tag{2a}$$
$$u = \varphi_1(y, \dot{y}, \ldots, y^{(r+1)}). \tag{2b}$$

Definition 1. (Martínez-Torres et al. (2014). The k^{th}-state (resp. l^{th}-input) residue r_{x_k} (resp. r_{u_l}) is defined by the difference between the state (resp. input) measurement ξ_k (resp. ν_l) and the one calculated by (2a) (resp. (2b)) using the measured flat output y:

$$\begin{aligned} r_{x_k} &= \xi_k - x_k, , \quad k = 1, \ldots, n \\ r_{u_l} &= \nu_l - u_l , \quad l = 1, \ldots, m, \end{aligned} \tag{3}$$

with $x_k = \varphi_{0,k}(\overline{y})$ and $u_l = \varphi_{0,l}(\overline{y})$.

The components $(y, \dot{y}, \ddot{y}, \ldots)$ of the flat output take their values from the sensors and actuators. However, due to the presence of noises on sensors and actuators, the derivatives of the flat outputs may not be defined. For this purpose, a high-gain observer is used to estimate these derivatives and a low-pass filter is synthesized to improve its performance. Conditions of robustness of this method are detailed in (Martínez-Torres et al. (2014)).

Also, due to the existence of noises, a threshold is fixed for each residue. In the case of no defects, the calculated states x_k (resp. inputs u_l) and their measurements ξ_k (resp. ν_l) have about the same values and their residues do not exceed their thresholds. In contrast, if, at least, one of the calculated residues exceeds its threshold then a fault is detected on the corresponding sensor or actuator. However, for an arbitrary flat output, that may depend on all the system variables, several thresholds may be exceeded simultaneously and the possibility of isolating a defect thus highly depends on the choice of flat output. The definition of the isolability for the structured residual approach in the framework of general polynomial systems has been introduced in (Staroswiecki and Comtet-Varga (2001)). We pose the following rigorous fault isolability definition in the flatness context:

Definition 2. (Isolability) A fault on the state x_i is isolable if it verifies the following conditions:

(1) $\dfrac{\partial r_{x_i}}{\partial x_i} \neq 0$;
(2) $\dfrac{\partial r_{x_j}}{\partial x_i} = 0 \;\forall\, j \neq i$ and $\dfrac{\partial r_{u_l}}{\partial x_i} = 0 \;\forall\, l \in \{1, \ldots, m\}$.

An isolable fault on the input u_l is defined in the same way. This definition of isolability reflects the fact that a fault on a state (resp. input) is isolable if this state is not a component of the flat output y.

Hypothesis: From now on, we assume that there is only one fault at a time affecting the sensors or actuators.

2.3 Application on the Three-Tank System

The three-tank system represents the dynamics of three cylindrical tanks of cross-sectional area S, connected to each other by means of cylindrical pipes of section S_n, and two pumps P_1 and P_2 that supply tanks T_1 and T_3. These three tanks are also connected to a central reservoir through pipes (see Fig. 1).

The explicit system of equations of the three-tank model is given by:

$$S\dot{x}_1 = -Q_{10}(x_1) - Q_{13}(x_1, x_3) + u_1 \quad (4)$$
$$S\dot{x}_2 = -Q_{20}(x_2) + Q_{32}(x_2, x_3) + u_2 \quad (5)$$
$$S\dot{x}_3 = Q_{13}(x_1, x_3) - Q_{32}(x_2, x_3) - Q_{30}(x_3) \quad (6)$$

where the state variables x_i, $i = 1, 2, 3$ represent the water level of each tank, Q_{i0}, $i = 1, 2, 3$ the outflow between each tank and the central reservoir, Q_{13} is the outflow between tank 1 and tank 3 and Q_{32} the outflow between tank 3 and tank 2, u_1 and u_2 are the input variables, namely the incoming flow of each pump.

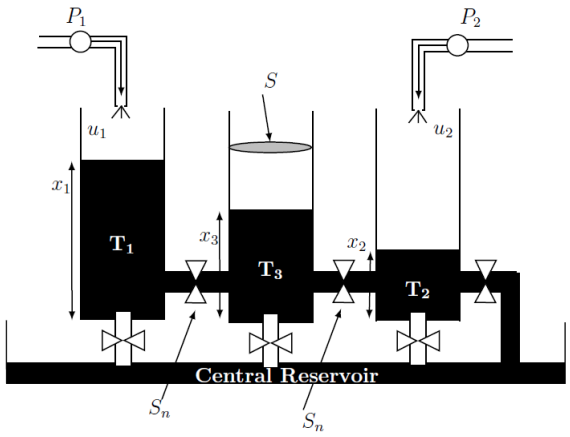

Fig. 1. *Three Tank System, Source: Noura et al. (2009)*

Hypothesis: The following configuration is considered to avoid singularities [1] :

$$x_1 > x_3 > x_2.$$

We consider that the valves connecting tanks 1 and 3 with the central reservoir are closed, *i.e.* $Q_{10} \equiv 0$ and $Q_{30} \equiv 0$. The expressions of Q_{13}, Q_{32} and Q_{20} are given by:

$$Q_{13}(x_1, x_3) = a_{z1} S_n \sqrt{2g(x_1 - x_3)} \quad (7)$$
$$Q_{20}(x_2) = a_{z2} S_n \sqrt{2g(x_2)} \quad (8)$$
$$Q_{32}(x_2, x_3) = a_{z3} S_n \sqrt{2g(x_3 - x_2)} \quad (9)$$

where a_{zr}, $r = 1, 2, 3$, is the flow coefficient and g the gravitational force.

This system is flat with $y = (x_1, x_3)^T = (y_1, y_2)^T$ as flat output. The state and input variables of the system are then constructed using (2a) and (2b) as follows:

$$x_1 = y_1$$
$$x_2 = y_2 - \frac{1}{2g}\Big(\frac{a_{z1} S_n \sqrt{2g(y_1 - y_2)} - S\dot{y}_2}{a_{z3} S_n}\Big)^2$$
$$x_3 = y_2$$
$$u_1 = S\dot{y}_1 + a_{z1} S_n \sqrt{2g(y_1 - y_2)}$$
$$u_2 = S\dot{x}_2 - a_{z3} S_n \sqrt{2g(y_2 - x_2)} + a_{z2} S_n \sqrt{2g x_2}.$$

Since the flat outputs take their values from the sensors then the variables x_1 and x_3 being measured as flat output, they are equal to the measurements ξ_1 and ξ_3. Hence $r_{x_1} \equiv 0$ and $r_{x_3} \equiv 0$, so that they can be eliminated from the vector of residues, that thus reads:

$$\begin{pmatrix} r_{x_2} \\ r_{u_1} \\ r_{u_2} \end{pmatrix} = \begin{pmatrix} \xi_2 \\ \nu_1 \\ \nu_2 \end{pmatrix} - \begin{pmatrix} x_2 \\ u_1 \\ u_2 \end{pmatrix}. \quad (10)$$

In this system, the state x_2 is not a component of the flat output y, then, according to the *Definition 2*, only the residue r_{x_2} is triggered by a fault on the sensor x_2. As a consequence, a fault on the sensor x_2 can be detected and isolated. But it can be easily verified that if a fault affects one of the components of the flat output y then all the residues will be affected and the fault cannot be isolated at this stage.

In (Nagy et al. (2009)), it has been shown that the system of three-tank is observable with only the state variable x_1. This means that the other state variables x_2 and x_3 can be estimated using x_1 while given the measurements of u_1 and u_2. In this case, two additional residues $r'_{x_2} = \xi_2 - \widehat{x}_2$ and $r'_{x_3} = \xi_3 - \widehat{x}_3$ are added to (10), where $\widehat{x}_2$ and $\widehat{x}_3$ are the estimated values of x_2 and x_3, respectively. If a fault affects the component of the flat output x_3, then all the residues, except r'_{x_2} which is independent of x_3, will be triggered and, consequently, this fault is detected and isolated. Finally, if all the above residues exceed their thresholds then we conclude that a fault appears on the sensor x_1. Table 1 summarizes the residues triggered by each fault.

Fault	r_{x_2}	r_{u_1}	r_{u_2}	r'_{x_2}	r'_{x_3}
F_{x_1}	1	1	1	1	1
F_{x_2}	1	0	0	1	0
F_{x_3}	1	1	1	0	1
F_{u_1}	0	1	0	1	1
F_{u_2}	0	0	1	1	1

Table 1. Faults signatures

Due to the difficulty of estimating x_2 and x_3 as functions of x_1, and the need to know the measurements of u_1 and u_2, this method can be replaced by calculating another flat output of the considered flat system. In this case, the measurements of the inputs are not need to be known. This approach is available only if these flat outputs verify the condition of isolability represented above, so that they are algebraically independent in the sense that if a fault affects one of them the others will not be totally affected.

In the following, the concept of algebraically independent flat outputs is defined. Moreover, we show that this feature is valid for the class of direct flat systems that are introduced in the next section.

[1] According to the *Remark 1*, the point $\overline{x} \in \mathfrak{X}_0$ s.t. $x_1 = x_2 = x_3$ is an equilibrium point which is not first order controllable, then it is a point of intrinsic singularity.

3. DIRECT FLAT SYSTEM

3.1 Unimodular Completion Algorithm

The variation of the implicit system (2) gives the following tangent system

$$0 = dF(x, \dot{x}) = P\Big(\frac{d}{dt}\Big)dx, \tag{11}$$

with

$$P\Big(\frac{d}{dt}\Big) = \begin{pmatrix} \frac{\partial F_1}{\partial_{x_1}} + \frac{\partial F_1}{\partial \dot{x_1}}\frac{d}{dt} & \cdots & \frac{\partial F_1}{\partial_{x_n}} + \frac{\partial F_1}{\partial \dot{x_n}}\frac{d}{dt} \\ \vdots & & \vdots \\ \frac{\partial F_{n-m}}{\partial_{x_1}} + \frac{\partial F_{n-m}}{\partial \dot{x_1}}\frac{d}{dt} & \cdots & \frac{\partial F_{n-m}}{\partial_{x_n}} + \frac{\partial F_{n-m}}{\partial \dot{x_n}}\frac{d}{dt} \end{pmatrix} \tag{12}$$

and $dx = (dx_1, \ldots, dx_n)$.

The matrix $P\Big(\frac{d}{dt}\Big) \in \mathcal{M}_{(n-m)\times n}\Big[\frac{d}{dt}\Big]$, the ring of polynomial matrices in the operator $\frac{d}{dt}$ with meromorphic coefficients (Levine (2009)). An invertible matrix U in $\mathcal{M}_{p\times p}\Big[\frac{d}{dt}\Big]$ whose inverse is also in $\mathcal{M}_{p\times p}\Big[\frac{d}{dt}\Big]$ is called unimodular. The set of unimodular matrices is denoted by $\mathcal{U}_p\Big[\frac{d}{dt}\Big]$. The degree in $\frac{d}{dt}$ of a matrix $K \in \mathcal{M}_{p\times q}\Big[\frac{d}{dt}\Big]$, denoted by $\deg(K)$, is defined by:

$$\deg(K) = \max\{\deg_{row}(K_i), i = 1, \ldots, p\},$$

where

$$\deg_{row}(K_i) = \max\{\deg(K_{i,j}), j = 1, \ldots, q\}.$$

Definition 3. (Lévine and Nguyen (2003)). The matrix $P\Big(\frac{d}{dt}\Big)$ is said to be hyper-regular if and only if there exists a unimodular matrix $U \in \mathcal{U}_n\Big[\frac{d}{dt}\Big]$ such that

$$P\Big(\frac{d}{dt}\Big)U = \begin{pmatrix} I_{(n-m)} & 0_{(n-m)\times m} \end{pmatrix}. \tag{13}$$

The main property of the flatness is given by the following proposition:

Proposition 1. (Kaminski et al. (2018)). If the system (2) is flat at a point $\overline{x}_0 \in \mathfrak{X}_0$, then there exists a neighbourhood $\mathcal{V}$ of $\overline{x}_0$ where $P\Big(\frac{d}{dt}\Big)$ is hyper-regular.

Let $\widehat{U} = U\begin{pmatrix} 0_{(n-m)\times m} \\ I_m \end{pmatrix}$ and $\omega = (\omega_1, \ldots, \omega_m)$ a vector of m independent 1-forms defined by

$$\omega = \widehat{U}^{\dagger}dx, \tag{14}$$

with

$$\widehat{U}^{\dagger} \triangleq (0_{m\times(n-m)}\ I_m)U^{-1}, \tag{15}$$

the pseudo-inverse of $\widehat{U}$. The vector ω is a flat output of the variational system (11) or simply a tangent flat output.

Remark 2. The pseudo-inverse $\widehat{U}^{\dagger}$ is not unique, then the tangent flat output ω, associated to $\widehat{U}^{\dagger}$, is not unique too.

The next definition is borrowed from (Fritzsche et al. (2016a)) and (Fritzsche et al. (2016b)):

Definition 4. Given a hyper-regular matrix $M \in \mathcal{M}_{p,q}\Big[\frac{d}{dt}\Big]$ with $p \leq q$, we say that $N \in \mathcal{M}_{(q-p)\times q}\Big[\frac{d}{dt}\Big]$ is a unimodular completion of M if and only if

$$\begin{pmatrix} M \\ N \end{pmatrix} \in \mathcal{U}_q\Big[\frac{d}{dt}\Big]. \tag{16}$$

Proposition 2. The matrix $\widehat{U}^{\dagger}$ is a unimodular completion of $P\Big(\frac{d}{dt}\Big)$.

Proof. Let $U \in \mathcal{U}_n[\frac{d}{dt}]$ such that

$$P\Big(\frac{d}{dt}\Big)U = \begin{pmatrix} I_{(n-m)} & 0_{(n-m)\times m} \end{pmatrix}, \tag{17}$$

then $P\Big(\frac{d}{dt}\Big) = \begin{pmatrix} I_{n-m} & 0_{(n-m)\times m} \end{pmatrix}U^{-1}$ constitutes the first $n-m$ rows of the matrix U^{-1}, i.e.

$$U^{-1} = \begin{pmatrix} P\Big(\frac{d}{dt}\Big) \\ W \end{pmatrix} \tag{18}$$

with $W \in \mathcal{M}_{m\times n}[\frac{d}{dt}]$. Moreover, according to (12), the matrix $\widehat{U}^{\dagger}$ constitutes the last m rows of U^{-1}, hence

$$U^{-1} = \begin{pmatrix} P\Big(\frac{d}{dt}\Big) \\ \widehat{U}^{\dagger} \end{pmatrix}, \tag{19}$$

which proves that $\widehat{U}^{\dagger}$ is a unimodular completion of $P\Big(\frac{d}{dt}\Big)$. □

An algorithm of computation of a unimodular completion of the matrix $P\Big(\frac{d}{dt}\Big)$ is developed in (Fritzsche et al. (2016a)). The flat output $y = (y_1, \ldots, y_m)$ of the system (2) is given by

$$dy = \omega \tag{20}$$

provided that ω is integrable, i.e. $d\omega = 0$. The conditions of integrability are detailed in (Levine (2009), Lévine (2011)) and they require the resolution of PDEs whose complexity depends on the system itself.

3.2 Direct Flat Representation

Definition 5. (Pomet (1997)). Let (1) be a flat system, it is called $(-1)-$flat or $x-$flat if and only if there is a flat output y such that y depends only on x, i.e.

$$y = \psi(x). \tag{21}$$

Consider a subclass of $(-1)-$flat systems called direct flat systems defined as follows:

Definition 6. (Fritzsche et al. (2016b)). We say that a $(-1)-$flat system is a direct flat system if there exists a permutation $\sigma : \{1, \ldots, n\} \longmapsto \{\sigma(1), \ldots, \sigma(n)\}$ such that there exists a flat output given by $y = (x_{\sigma(1)}, \ldots, x_{\sigma(m)})$. Such flat output is called direct flat output.

Proposition 3. (Fritzsche et al. (2016b)). Let $P\Big(\frac{d}{dt}\Big)$ defined by (12) be hyper-regular. Assume that there exists Π, a column permutation matrix such that

$$\widetilde{P}\Big(\frac{d}{dt}\Big) \triangleq P\Big(\frac{d}{dt}\Big)\Pi = (A\ B) \tag{22}$$

with $A \in \mathcal{U}_{(n-m)}\Big[\dfrac{d}{dt}\Big]$ and $B \in \mathcal{M}_{(n-m)\times m}\Big[\dfrac{d}{dt}\Big]$. Then denoting by $\widetilde{H} = \big(0_{m\times(n-m)}\ I_m\big)$ and $H = \widetilde{H}\Pi^T$, which are constant matrices, $\widetilde{H}$ (resp. H) is a unimodular completion of $\widetilde{P}\Big(\dfrac{d}{dt}\Big)$ (resp. $P\Big(\dfrac{d}{dt}\Big)$). A tangent flat output ω is given by $\omega = Hdx$ and always satisfies the integrability condition $d\omega = 0$. Hence, a (direct) flat output y of the non linear system is given by

$$y = Hx. \tag{23}$$

Definition 7. $\widetilde{P}\Big(\dfrac{d}{dt}\Big)$, defined in *Proposition 3*, is called a direct flat representation, for which $\widetilde{y} \triangleq \widetilde{H}x$ is a direct flat output.

Remark 3. A direct flat representation is indeed not unique.

4. ALGEBRAICALLY INDEPENDENT FLAT OUTPUTS

As mentioned in (Torres et al. (2013)), in the aim to provide a total isolation of defects on a systems sensors or actuators, we need to increase the number of residues by calculating several flat outputs. These flat outputs must be algebraically independent in the sense that a fault that affect one of them, the others will not be totally affected. In this section, we present a way to characterize the relation between different flat outputs.

Proposition 4. Let $\Omega_{\overline{x}}$ be the set of all tangent flat outputs at $\overline{x}$ of a flat system. Then, for all ω_1 and $\omega_2 \in \Omega_{\overline{x}}$, there exists a unimodular matrix $\mathcal{K} \in \mathcal{U}_m[\frac{d}{dt}]$ such that

$$\omega_1 = \mathcal{K}\,\omega_2. \tag{24}$$

Proof. Since the matrix $P(\frac{d}{dt})$ is supposed to be hyper-regular, there exists a hyper-regular matrix $\widehat{U}$ such that

$$\omega = \widehat{U}^{\dagger}dx \tag{25}$$

and, indeed, according to (14) $dx = \widehat{U}\omega$.

Let $\omega_1 = \widehat{U}_1^{\dagger}dx$ and $\omega_2 = \widehat{U}_2^{\dagger}dx$ be two different tangent flat outputs at $\overline{x}$, then

$$\omega_1 = \widehat{U}_1^{\dagger}\widehat{U}_2\,\omega_2 \triangleq \mathcal{K}\,\omega_2 \tag{26}$$

with

$$\mathcal{K} = \widehat{U}_1^{\dagger}\widehat{U}_2. \tag{27}$$

Let us prove that $\mathcal{K}$ is unimodular. $\widehat{U}_1^{\dagger}$ and $\widehat{U}_2$ are hyper-regular by construction, then if ξ is a vector such that $\widehat{U}_1^{\dagger}\widehat{U}_2\xi = 0$ with $\xi \neq 0$ then $\zeta = \widehat{U}_2\xi$ is also $\neq 0$ and $\widehat{U}_1^{\dagger}\zeta = 0$ which contradicts the hyper-regularity of $\widehat{U}_1^{\dagger}$. Hence, $\mathcal{K}$ is hyper-regular and square, which proves the unimodularity. □

From here two types of relations between the tangent flat outputs are introduced:

(1) if $\deg(\mathcal{K}) = 0$ then the tangent flat outputs ω_1 and ω_2 are said to be algebraically dependent;
(2) if $\deg(\mathcal{K}) \geq 1$ then the tangent flat outputs ω_1 and ω_2 are said to be algebraically independent.

It is easy to see that, according to the implicit form (2) of the system, at least one state variable is an integral of order larger than or equal to 1 of the other state variables. As a consequence, we have the following corollary:

Corollary 1. Two different direct tangent flat outputs of a direct flat system are algebraically independent.

Remark 4. A direct flat output is given by $dy = \omega$, then consequently, two direct flat outputs are algebraically independent.

5. APPLICATION ON THE THREE TANK SYSTEM

For the reasons explained in section 2.3, and in order to detect and isolate faults on sensors and actuators of the three-tank system, we need to calculate another flat output that will be algebraically independent of $y = (x_1, x_3)^T$. In this section, we show that this system is a directly flat with two direct flat outputs. After that we will show how these two flat outputs are useful for the flatness-based FDI.

The implicit equation of this system is calculated as follows: equation (6) of the explicit system is free of inputs, so after elimination of equations (4) and (5) and replacing (7) and (9) in (6), we obtain:

$$F(x,\dot{x}) = \dot{x}_3 - a_{z1}\,\mathcal{S}\sqrt{2g(x_1 - x_3)} + a_{z3}\,\mathcal{S}\sqrt{2g(x_3 - x_2)} = 0. \tag{28}$$

with $\mathcal{S} = S_n/S$.

According to (12), the matrix $P\Big(\dfrac{d}{dt}\Big)$, associated to (28), is given by:

$$P\Big(\frac{d}{dt}\Big) = \Big(-\frac{a_{z1}\mathcal{S}g}{\sqrt{2g(x_1 - x_3)}} \quad -\frac{a_{z3}\mathcal{S}g}{\sqrt{2g(x_3 - x_2)}} \quad \frac{a_{z1}\mathcal{S}g}{\sqrt{2g(x_1 - x_3)}} + \frac{a_{z3}\mathcal{S}g}{\sqrt{2g(x_3 - x_2)}} + \frac{d}{dt}\Big). \tag{29}$$

The matrix $P\big(\frac{d}{dt}\big)$ is of the form $P\big(\frac{d}{dt}\big) = \big(S(\frac{d}{dt})\ \ T(\frac{d}{dt})\big)$ with $S(\frac{d}{dt}) \in \mathcal{U}_1\Big[\dfrac{d}{dt}\Big]$, then according to the *Proposition 3*, the following matrix

$$H\Big(\frac{d}{dt}\Big) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{30}$$

is a unimodular completion of $P\big(\frac{d}{dt}\big)$. Hence, a tangent flat output is given by

$$\omega = H\Big(\frac{d}{dt}\Big)dx = \begin{pmatrix} dx_2 \\ dx_3 \end{pmatrix} \tag{31}$$

and it is integrable, i.e. $d\omega = 0$. Then the system (28) is a direct flat system with

$$y = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \tag{32}$$

a direct flat output. One can also find the flat output $y = (x_1, x_3)^T$ using the direct flat representation.

In the following, we denote by $y_1 = (x_1, x_3)^T$ and $\omega_1 = (dx_1, dx_3)^T$ the corresponding tangent flat output, $y_2 = (x_2, x_3)^T$ and $\omega_2 = (dx_2, dx_3)^T$. In fact, the direct flat outputs y_1 and y_2 are algebraically independent: let

$$\mathcal{K} = \begin{pmatrix} \gamma & \eta + \kappa\dfrac{d}{dt} \\ 0 & 1 \end{pmatrix} \tag{33}$$

be a unimodular matrix with

$$\begin{aligned}\gamma &= -\Big(\frac{a_{z1}}{a_{z3}}\Big)^2 - \frac{a_{z1}}{a_{z3}^2}\Delta \\ \eta &= 1 + \Big(\frac{a_{z1}}{a_{z3}}\Big)^2 + \frac{a_{z1}}{a_{z3}^2}\Delta \\ \kappa &= \frac{\dot{x}_3}{a_{z1}^2\mathcal{S}^2 g} + \frac{a_{z3}}{a_{z1}^2\mathcal{S}}\sqrt{2g(x_3 - x_2)} \\ \Delta &= \frac{g}{\mathcal{S}}\frac{\dot{x}_3}{\sqrt{2g(x_3 - x_2)}}\end{aligned} \tag{34}$$

and $\deg(\mathcal{K}) = 1$. With respect to the implicit form (28), one can easily verify that

$$\omega_1 = \mathcal{K}\,\omega_2, \tag{35}$$

which proves the algebraic independence.

According to the *Definition 2* of the isolability, the state x_1 is not a component of the flat output $y_2 = (x_2, x_3)^T$, then a fault on the sensor x_1 can be detected and isolated. Finally, by associating the two flat outputs y_1 and y_2, if a fault is detected but can not be isolated on x_1 and x_2, then x_3 is inevitably faulty, which allows the complete isolation of defects.

6. CONCLUSION

The current paper introduced a novel and rigorous definition of the isolability of faults affecting a system's sensors and actuators, using the flatness-based FDI approach. The described condition of isolability provided an efficient way to select a handful of flat outputs useful for fault isolation from an infinite number of possible flat outputs. Our results were tested and validated using the three-tank system. Future work needs to focus on providing new definitions of residue signals in order to overcome the reliance on a high-gain observer when estimating the time derivatives of flat outputs.

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