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Iterated foldings of discrete spaces and their limits: candidates for the role of Brownian map in higher dimensions

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Abstract

In this last decade, an important stochastic model emerged: the Brownian map. It is the limit of various models of random combinatorial maps after rescaling: it is a random metric space with Hausdorff dimension 4, almost surely homeomorphic to the 2-sphere, and possesses some deep connections with Liouville quantum gravity in 2D. In this paper, we present a sequence of random objects that we call D th-random feuilletages (denoted by $\mathbf{r}[D]$), indexed by a parameter $D \geq 0$ and which are candidate to play the role of the Brownian map in dimension D . The construction relies on some objects that we name iterated Brownian snakes, which are branching analogues of iterated Brownian motions, and which are moreover limits of iterated discrete snakes. In the planar $D = 2$ case, the family of discrete snakes considered coincides with some family of (random) labeled trees known to encode planar quadrangulations.

Iterating snakes provides a sequence of random trees $(\mathbf{t}^{(j)}, j \geq 1)$. The D th-random feuilletage $\mathbf{r}[D]$ is built using $(\mathbf{t}^{(1)}, \dots, \mathbf{t}^{(D)})$: $\mathbf{r}[0]$ is a deterministic circle, $\mathbf{r}[1]$ is Aldous' continuum random tree, $\mathbf{r}[2]$ is the Brownian map, and somehow, $\mathbf{r}[D]$ is obtained by quotienting $\mathbf{t}^{(D)}$ by $\mathbf{r}[D-1]$.

A discrete counterpart to $\mathbf{r}[D]$ is introduced and called the D th random discrete feuilletage with $n + D$ nodes $(\mathbf{R}_n[D])$. The proof of the convergence of $\mathbf{R}_n[D]$ to $\mathbf{r}[D]$ after appropriate rescaling in some functional space is provided (however, the convergence obtained is too weak to imply the Gromov-Hausdorff convergence). An upper bound on the diameter of $\mathbf{R}_n[D]$ is $n^{1/2^D}$. Some elements allowing to conjecture that the Hausdorff dimension of $\mathbf{r}[D]$ is 2^D are given.

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1 Introduction

1.1 Presentation of the main objects

The question at the origin of this paper is the following: are there any random continuous objects likely to play the role of the “Brownian map” in higher dimensions?

The question is probably ill posed since it may not be so clear what the dimension of the Brownian map is: on the one hand it is indeed a.s. homeomorphic to the 2-sphere (Le Gall & Paulin [48]), and on the other hand, by Le Gall [45, Thm. 6.1], it has Hausdorff dimension 4... and due to its huge fluctuations, it is likely not possible¹ to embed it isometrically in $(\mathbb{R}^D, \|\cdot\|_2)$ for any finite D .

Fixing the topological dimension as the base of our considerations, we are not aware of any family of combinatorial objects that would be the right candidates to play the role, in dimension $D > 2$, of the combinatorial maps which provide, in the $D = 2$ case, the Brownian map as a scaling limit. To our knowledge, the previous attempts, in a theoretical-physics context, either led to Aldous’ continuous random tree, to the Brownian map², or to a crumpled phase with “infinite Hausdorff dimension”³ [4, 66, 37]. We propose the construction, for every integer $D \geq 1$, of a continuous random object which we call the D th random feuilletage ($\mathbf{r}[D]$). By construction, $\mathbf{r}[D]$ will appear as the limit (for a topology discussed further) of a discrete analogue, which we call D th random discrete feuilletage $\mathbf{R}_n[D]$, when a size parameter n goes to $+\infty$. The latter can be viewed as obtained by $D - 2$ series of foldings of a random discrete surface. The sequence $(\mathbf{r}[D], D \geq 1)$ is encoded and built thanks to another new sequence of objects $(\mathbf{bs}[D], D \geq 1)$: the D th Brownian snake $\mathbf{bs}[D]$ is – *mutatis mutandis* – a branching analogue of the D th iterated Brownian motion. The first Brownian snake, $\mathbf{bs}[1]$, is the usual Brownian snake with lifetime process the normalized Brownian excursion \mathbf{e} . The D th Brownian snake appears as the limit of a discrete counterpart after some appropriate normalizations, which we call D th random discrete snake $\mathbf{BS}_n[D]$ (and $\mathbf{bs}_n[D]$ for the normalized version).

By construction, $\mathbf{r}[0]$ can be thought to be a deterministic cycle, $\mathbf{r}[1]$ and $\mathbf{r}[2]$ will respectively be seen to coincide with Aldous’ continuum random tree and with the Brownian map, and for $D \geq 3$, we think that $\mathbf{r}[D]$ is a candidate to play the role of the Brownian map “in dimension D ”.

Remark 1.1. *We do not claim that the topological dimension of the object $\mathbf{r}[D]$ we define in the paper (the D th random feuilletages) should be precisely D for any D . For $D = 3$ however, it could be argued that nodal surfaces whose number of nodes goes to infinity can be related to three dimensional objects. But indeed, we hope that the topological dimension of $\mathbf{r}[D]$ is a.s. well defined, and is an increasing function of D .*

What are these objects? The complete and rigorous construction will take pages, but let us try to provide some insights on this construction. First, some words about the french word “feuilletage”:

■ It is the french word for “foliation”, which is used in differential geometry to denote some equivalence relation on manifolds: depending on the context, the equivalence classes (the leaves) correspond to parts

¹On \mathbb{R}^D , the largest tuple of points (x_1, \dots, x_k) such that $i \neq j \Rightarrow d(x_i, x_j) = 1$ is bounded by $(D + 1)$ when it is not the case on maps.

²This is likely the case for some models of random 4-dimensional triangulations when using the distance on the dual graph [8, 50], and is conjectured for another model of random 3-dimensional triangulations using the distance in the triangulation.

³More precisely, this last case corresponds to families of random D -dimensional triangulations whose diameter is small or a.s. bounded when the number of simplices goes to infinity, for which a scaling limit cannot be defined.

of the initial manifold; they are themselves equivalent to some “regular” spaces.

■ In the “art français de la pâtisserie” (french art of pastry cooking), *le feuilletage* or *la pâte feuilletée* is the name for the puff pastry dough, which is the main ingredient of many sweet or salted pastries, as mille-feuille, galette des rois, bouchée à la reine, pâté lorrain, ... and even croissants (with some adjustments). The dough is obtained by placing some butter (6 mm thick say) on half a simple dough composed by flour, water and salt, shaped in a rectangular form (20 cm \times 40 cm, thick 5 mm say). Then, the dough is folded to cover the butter, and flattened into its initial shape R . It is then folded again, flattened, folded, flattened... Each time, the number of layers of butter is multiplied by 2. The pastry chef stops his/her work when the number of layers 2^n is large enough: 128 for example. After cooking, if this difficult recipe is well done, the layers are separated: we get the “feuilletage”.

The construction we will propose is similar to this feuilletage, of course, up to the ingredients and to the precise gestures of the cook... thus our choice of naming.

In the following paragraphs, we often omit some precisions, such as how the different objects involved are rooted for instance. The precise definitions of the various objects will be given in Sec. 2 for the continuous objects, and Sec. 3 and 6 for the discrete objects.

■ We start by discussing the content of Fig. 1, in which a well known and simple bijection is sketched: a planar tree having n edges and then $2n$ corners (say, rooted at a corner 0) can be encoded by a non-crossing partition⁴ on $\{0, \dots, 2n - 1\}$. The integers correspond to the corners of the tree when one turns around starting at 0, and then the partition is a way to present together the different corners of each node. As

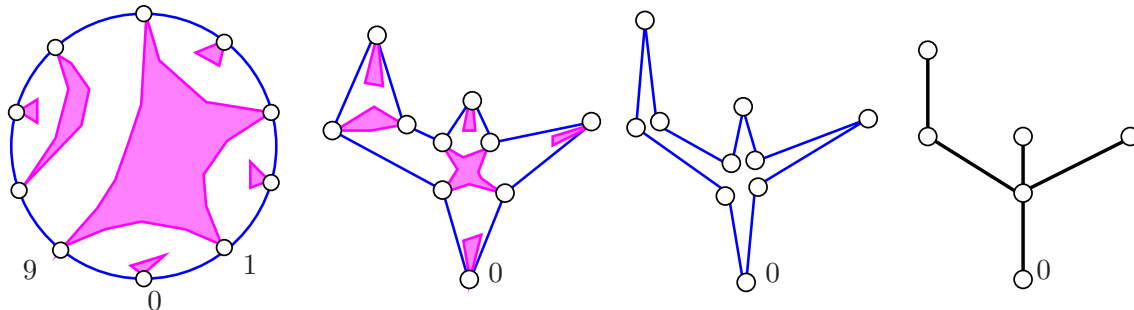


Figure 1: A planar tree with 6 nodes and then 10 corners, seen as a folded circle. The corresponding parts of the non-crossing partitions are $\{0\}$, $\{1, 3, 5, 9\}$, $\{2\}$, $\{6, 8\}$, $\{7\}$.

may be seen in Fig. 1, this bijection can be used to present a tree as a circle folded multiple times: these foldings are encoded by the non-crossing partition on a finite subset of the (continuous) circle (or on the discrete circle with $2n$ points). Through this bijection, trees and non-crossing partitions are essentially the same combinatorial objects. For the sake of studying asymptotics of trees, one usually prefers to use contour processes instead of non-crossing partitions (Fig. 2), since it allows gaining access to the toolbox of usual linear stochastic processes. It is now folklore that the contour process can be glued from below to recover the tree (Fig. 2), so that the two points of view of Fig. 1 and Fig. 2 are equivalent.

■ Let us now focus on Fig. 3, which is the analogue of Fig. 1, when the initial object is a tree instead of a circle. In this case, a tree is folded multiple times according to a non-crossing partition on its corners (for a corner numeration obtained by turning around it). Again, it is now classical that such a construction

⁴We use encodings of planar trees by non-crossing partitions whose Kreweras complements are matchings (disjoint sets

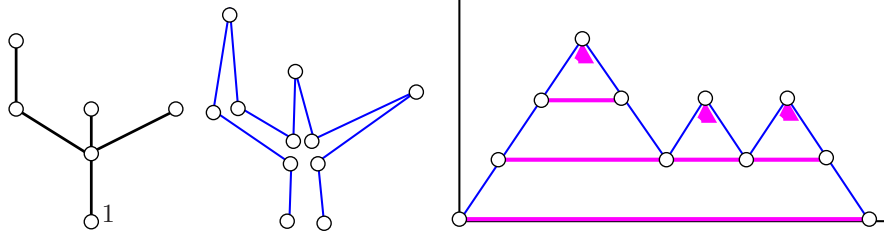


Figure 2: A planar tree and its contour process.

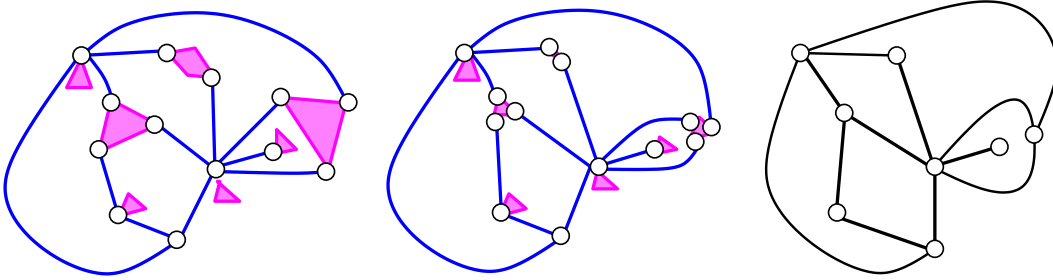


Figure 3: A planar tree equipped with a non-crossing partition on (some corners) of its vertices. The identification of the corners of the tree belonging to the same part allows constructing a map.

can be done on the plane by avoiding edge-crossings: the resulting object is a planar map (see Fig. 3). The converse is also true: it is possible to unplug the edges of any planar map in any order, until there are no more cycles, while preserving the connectedness. The resulting map is a tree, and keeping track of the ancient connections can be done thanks to a non-crossing partition on the corners of that tree. Hence, proper foldings of a tree, that is, two series of foldings of a circle, allow constructing a map. This powerful point of view is at the origin of the first definition of the Brownian map [54]: the non-crossing partition encodes a tree $T_n^{(1)}$ (right of Fig. 4), and together with the initial tree $T_n^{(2)}$, we have two objects that in turn can be encoded by linear processes. In Fig. 4, the black tree is $T_n^{(1)}$, the blue tree $T_n^{(2)}$: the nodes of $T_n^{(2)}$ are “glued” in the corners of $T_n^{(1)}$. Starting from a uniform planar quadrangulation with n

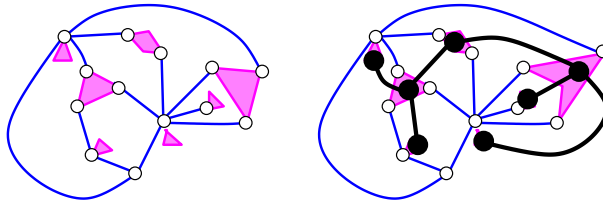


Figure 4: Identification of the non-crossing partition with a tree.

faces, it is possible to construct the random tree $\mathbf{T}_n^{(1)}$ (the non-crossing partition) as well as the random tree $\mathbf{T}_n^{(2)}$ (containing the edges of the map) using a bijection which allows controlling the distributions of $(\mathbf{T}_n^{(1)}, \mathbf{T}_n^{(2)})$: $\mathbf{T}_n^{(1)}$ is uniform in the set of rooted planar trees with n edges⁵, $\mathbf{T}_n^{(2)}$ has $2n$ edges; the standard diameter of $\mathbf{T}_n^{(1)}$ is \sqrt{n} and that of $\mathbf{T}_n^{(2)}$ is $n^{1/4}$ (see [54] for the representation of quadrangulations with $(\mathbf{T}_n^{(1)}, \mathbf{T}_n^{(2)})$, construction relying on the Cori-Vauquelin-Schaeffer [20, 65] correspondence).

encode all the vertices of the tree). For more details, see the end of Sec. 6.1.

⁵More precisely, this is the case when starting from a uniform rooted pointed quadrangulation with n faces.

■ We presented a tree and a map respectively as foldings of a circle and foldings of a tree: this will lead, up to some details (roots, sizes, degree of faces, etc), to the two first discrete random feuilletages $\mathbf{R}_n[1]$ and $\mathbf{R}_n[2]$. Moreover, $\mathbf{R}_n[0]$ can be considered to be the initial circle $\mathbb{Z}/2n\mathbb{Z}$. To build the sequence $(\mathbf{R}_n[D], D \geq 3)$, we will fold again and again: the D th object will be constructed by a series of foldings of the $(D - 1)$ th one, using an exterior source of randomness which will be a random non-crossing partition of the (nodes) corners of the $(D - 1)$ th object. In fact, we will rather use the $(D - 1)$ th feuilletage to fold a tree, even if at the level of this preliminary presentation, these two ways of doing appear similar.

In order to iterate the construction, a single possibility appears to resist all the requirements. Let us get a glimpse of a 3-discrete feuilletage $R_n[3]$ (a deterministic combinatorial object in the support of $\mathbf{R}_n[3]$): it is obtained by a series of foldings of a planar quadrangulation, using an additional “non-crossing” partition on its vertices. More precisely, the idea is to take three trees $(T_n^{(1)}, T_n^{(2)}, T_n^{(3)})$, with respectively n , $2n$ and $4n$ edges. The tree $T_n^{(3)}$ will contain all the edges of $R_n[3]$, $T_n^{(2)}$ will encode a non-crossing partition on $T_n^{(3)}$, and $T_n^{(1)}$ a non-crossing partition on $T_n^{(2)}$. Hence, folded by $T_n^{(2)}$, the pair $(T_n^{(2)}, T_n^{(3)})$ forms a planar quadrangulation $Q_n^{(2,3)}$, and the additional foldings of the nodes of $T_n^{(2)}$ by the tree $T_n^{(1)}$ (or the equivalent non-crossing partition) forms $R_n[3]$. Notice that the set of nodes of the quadrangulation $Q_n^{(2,3)}$ coincides with the nodes of $T_n^{(2)}$, so that identifying the nodes of $T_n^{(2)}$ (using $T_n^{(1)}$) provides an important number of additional identifications: the number of nodes in $R_n[3]$ roughly coincides with those of $T_n^{(1)}$, and then represents half the nodes of $Q_n^{(2,3)}$. This construction gives us access to the toolbox of stochastic processes, required for considering the asymptotics of these objects. Of course, in view of Fig. 5, the obtained structure is not planar, because the foldings of $Q_n^{(2,3)}$ using $T_n^{(1)}$

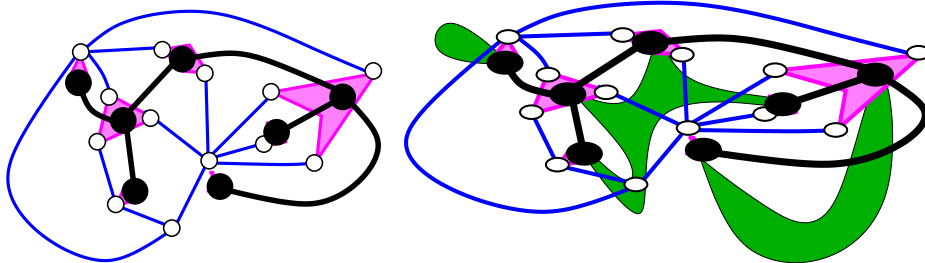


Figure 5: A planar map constructed from two trees $T_n^{(1)}$ and $T_n^{(2)}$. On the second picture, an additional non-crossing partition is added on the nodes (corners) of $T_n^{(1)}$. For our construction, the “green partition” will be encoded by the tree $T_n^{(1)}$, the pink one, corresponding to the black tree, will be the tree $T_n^{(2)}$, and the blue tree, the one whose edges remain finally, is the tree $T_n^{(3)}$.

create a linear number of additional node identifications.

To produce the subsequent $R_n[D]$ for $D \geq 3$, we will just take a sequence of trees $(T_n^{(1)}, T_n^{(2)}, \dots, T_n^{(D)})$, where $T_n^{(i)}$ will have $2^{i-1}n$ edges, and proceed to the identification of the nodes of $T_n^{(D)}$: for any $j < D$, the tree $T_n^{(j)}$ is used to identify the corners of $T_n^{(j+1)}$ (as do non-crossing partitions), producing D successive series of foldings of the vertices of $T_n^{(D)}$ starting from the circle. This allows identifying the nodes of the obtained object $R_n[D]$ as those of $T_n^{(1)}$ (roughly), and the edges of $R_n[D]$ as those of $T_n^{(D)}$.

■ An issue in the construction sketched above, is to define a distribution on the set of objects under investigation for which the main characteristics of interest are tractable. We propose a construction for which the natural scalings⁶ of the random trees $\mathbf{T}_n^{(1)}, \mathbf{T}_n^{(2)}, \dots, \mathbf{T}_n^{(D)}$ are respectively $n^{1/2}, n^{1/4}, \dots, n^{1/2^D}$.

⁶The asymptotic dependence of their diameters in the number of edges.

We think that this iterative construction of trees is interesting on its own: it is somewhat similar to the construction of the iterated Brownian motions.

Take a uniform rooted planar tree $\mathbf{T}_n^{(1)}$ with n edges, and use this tree as the underlying tree of a branching random walk with increments uniform in $\{0, 1, -1\}$. That is, conditionally on $\mathbf{T}_n^{(1)} = T$ some fixed planar tree, equip each node u of T (different from the root) with a random variable X_u uniform in $\{0, 1, -1\}$. A labeling of each node u is then obtained by summing the variables on the path from the root to u . On each branch, the labeling forms a random walk which starts at the label 0 of the root (see

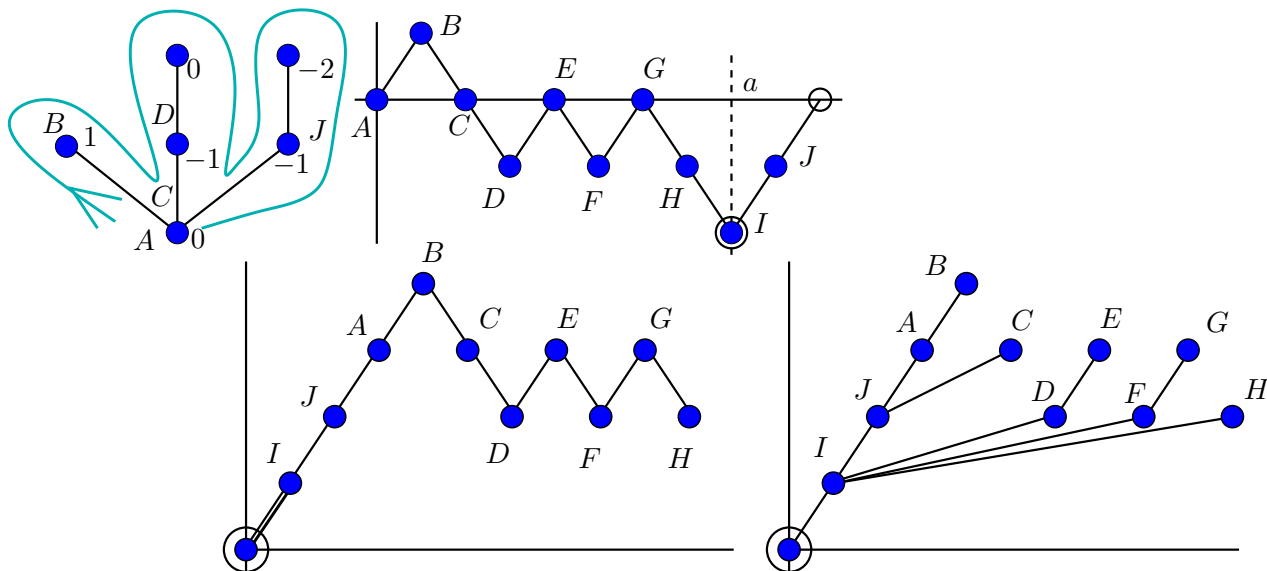


Figure 6: Definition of a new tree with the labels of a labeled tree. To a labeled tree (top left) is associated a label sequence (top right), and the conjugated labeled sequence (bottom left) is the height sequence of a new tree (bottom right).

Fig. 6). To construct a second tree $T_n^{(2)}$ from this labeling (see [20, 65, 54]), walk around the tree as done in Fig. 6, and record the successive labels encountered (one per corner); we get the so-called label process $\mathbf{L}_n^{(1)}$ (second picture in Fig. 6). Register $a = \min \arg \min \mathbf{L}_n^{(1)}$, the first time the label process reaches its minimum (dotted lines on the second picture). Now, on the third picture of Fig. 6, a new process starting at 0 is obtained by adding the point $(1, 1)$, and then, by appending the increments after a of $\mathbf{L}_n^{(1)}$, and then those before a (this is the so-called conjugation of paths). The process obtained this way is positive on $\llbracket 1, n \rrbracket$ and has increments $+1, -1, 0$: it is the height process of a random tree $\mathbf{T}_n^{(2)}$ (the height process of a tree t is the sequence the depth of the nodes in t , when the nodes of t are taken according to the lexicographical order; it encodes bijectively the tree, see last line of Fig. 6). Every realization $T_n^{(2)}$ of $\mathbf{T}_n^{(2)}$ is a planar tree in its own right: it can and will be used as the underlying tree of a *branching random walk* with the same increment distribution. The label process of this branching random walk $\mathbf{L}_n^{(2)}$, can in turn be used, after conjugation, as the height process of the third tree $\mathbf{T}_n^{(3)}$, which will be used as the underlying tree of a new branching random walk... These iterations allow building successively, $\mathbf{T}_n^{(1)}, \mathbf{L}_n^{(1)}, \mathbf{T}_n^{(2)}, \mathbf{L}_n^{(2)}, \mathbf{T}_n^{(3)}, \mathbf{L}_n^{(3)}, \dots$

For such a tree $T_n^{(2)}$ constructed “on” a tree $T_n^{(1)}$, since the process used to define the height process of $T_n^{(2)}$ is the sequence of labels of the corners of $T_n^{(1)}$, and since each node of $T_n^{(2)}$ (but its root) comes from one of the corners of $T_n^{(1)}$ (see Fig. 6), there is an obvious way to identify the nodes of $T_n^{(2)}$: identify

two nodes of $T_n^{(2)}$ if they come from different corners of the same node of $T_n^{(1)}$ (in Fig. 6, the nodes of each one of the following sets $\{A, C, G\}, \{B\}, \{D, F\}, \{E\}, \{H, J\}, \{I\}$ will be identified). Hence, when a sequence of trees $(T_n^{(1)}, \dots, T_n^{(D)})$ has been defined as above, it is possible to identify the nodes of $T_n^{(j)}$ using the corners of $T_n^{(j-1)}$ (which then defines a non-crossing partition of the corners of $T_n^{(j)}$), as wanted.

■ The joint distribution of the involved processes can be described, and the joint asymptotic distributions characterized. In fact, the construction is even simpler in the continuous setting, because many combinatorial details disappear in the limit. In a few words: start from $\mathbf{t}^{(1)}$, Aldous' continuum random tree. This tree is then used as the underlying tree of a spatial⁷ Brownian branching process, which amounts to equipping $\mathbf{t}^{(1)}$ with a compatible spatial Brownian motion indexed by the branches of $\mathbf{t}^{(1)}$. In the literature, the tree $\mathbf{t}^{(1)}$ equipped with this Brownian labeling is called Brownian snake with lifetime process the normalized Brownian excursion \mathbf{e} . The process \mathbf{e} is the contour process of the underlying tree $\mathbf{t}^{(1)}$, and the label process $(\ell_x^{(1)}, 0 \leq x \leq 1)$ in this setting is the process that gives the values of the spatial Brownian motion in accordance with the Brownian excursion. To iterate the construction, it suffices to conjugate the label process in order to get a non-negative process, $\mathbf{h}^{(2)}$, which can be used as the height process of a continuum random tree $\mathbf{t}^{(2)}$, which in turn, can be used as the underlying tree of a Brownian snake with label process $\ell^{(2)}$, and so on. Iteratively, we construct $\mathbf{t}^{(1)}, \ell^{(1)}, \mathbf{h}^{(2)}, \mathbf{t}^{(2)}, \ell^{(2)}, \mathbf{h}^{(3)}, \mathbf{t}^{(3)}, \dots$. We believe that these iterated Brownian snakes are interesting on their own. Once these objects are defined, it is possible to use them to define iterated continuous random feuilletages: for instance $\mathbf{r}[3]$ is defined as the random tree $\mathbf{t}^{(3)}$, whose corners are identified using $\mathbf{t}^{(2)}$, whose corners are in turn identified using $\mathbf{t}^{(1)}$, in the same way as we proceeded in the discrete setting (see Fig. 7).

1.2 Contents of the paper

Convention. *All the objects we will introduce in the paper will be rooted, unless specified otherwise (as for instance in Sec. 4).*

The last two sections of the introduction are dedicated to a discussion on the context in which our approach takes place, as well as the motivations from theoretical physics.

In Sec. 2, after providing some notions about real trees (Sec. 2.1) and snakes (Sec. 2.2), we describe directly in the continuum the iterative construction of the D th Brownian snake $\mathbf{bs}[D]$ and the D th random tree $\mathbf{t}^{(D)}$ (Sec. 2.3), and then of the D th random feuilletage $\mathbf{r}[D]$ (Sec. 2.4).

We start Sec. 3 by defining planar trees and their encodings using height and contour processes (Sec. 3.1), and then iterated discrete snakes $\mathbf{BS}_n[D]$ and the iterated random discrete trees $\mathbf{T}_n^{(D)}$ (Sec. 3.2), before introducing the iterated random discrete feuilletages $\mathbf{R}_n[D]$ (Sec. 3.3), as well as their normalized versions (Sec. 3.3.2).

While the objects introduced in the previous section are rooted, Sec. 4 is dedicated to the definition of *pointed* trees (Sec. 4.2), snakes (Sec. 4.2) and feuilletages (Sec. 4.3). In this last section, we show the convergence in law of the pointed iterated random discrete feuilletages to the pointed iterated continuous random feuilletage for a topology characterizing the convergence of the encoding trees (which does not imply the convergence for the Gromov-Hausdorff distance).

The proofs of the results in the previous sections are gathered in Section 5.

⁷“Spatial” is an adjective that is used to distinguish the different processes into play: prosaically, it is just a usual linear Brownian motion.

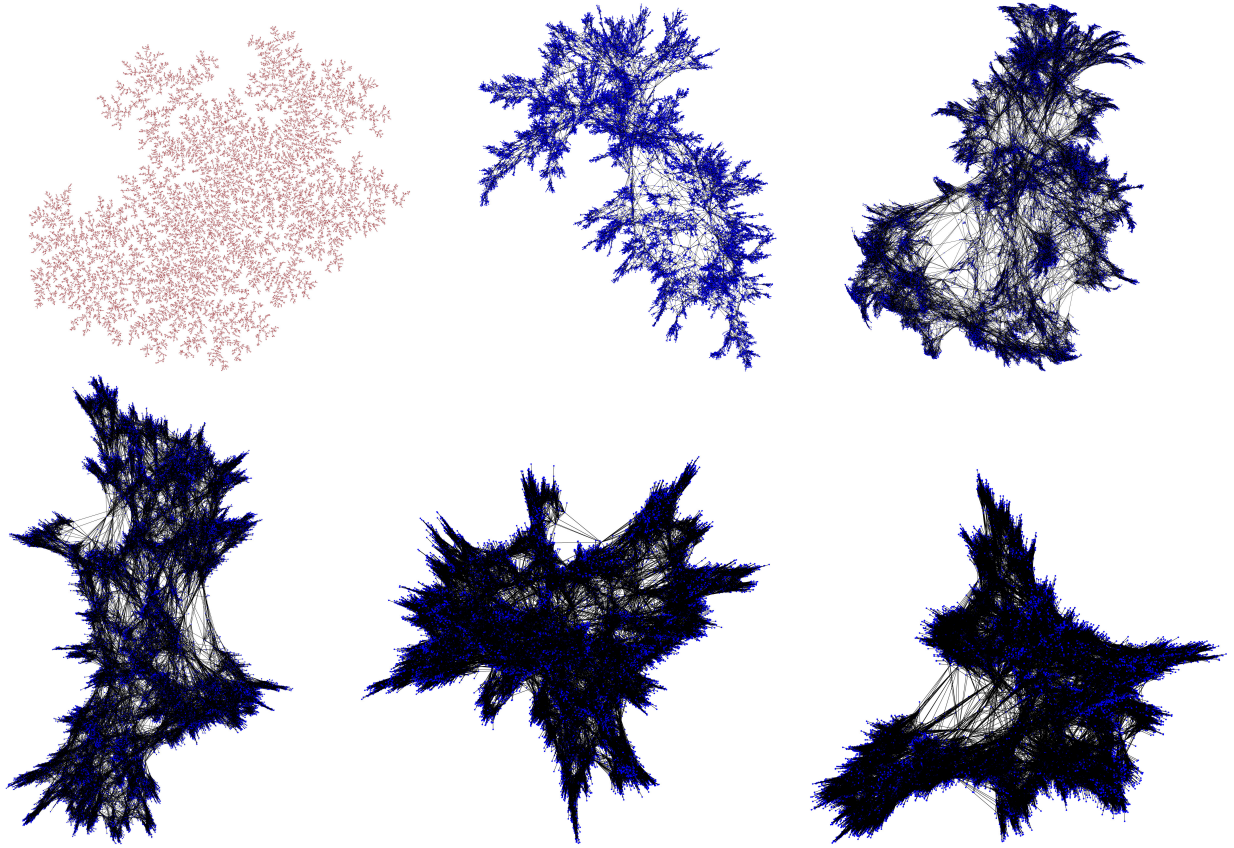


Figure 7: Some images of randomly generated $R_n[D]$ for D going from 1 to 6 (loops and multiple edges have been removed, keeping a unique single edge between adjacent vertices). The size of the vertex set is 50000.

While we rather use the encodings of trees by processes in the previous sections, in Sec. 6 we review in the combinatorial map picture the objects introduced previously. We define a class of combinatorial objects that generalize combinatorial maps using nested non-crossing partitions and that contains the iterated discrete feuilletages.

1.3 Some references and comments on the approach

The point of view we develop here could appear somewhat artificial, but it is actually very similar to the first works on the Brownian map [19, 54]. Planar quadrangulations and other similar simple families of maps were the only objects whose combinatorics was well understood at this time, particularly thanks to the existence of a bijection (CVS) between the set of planar quadrangulations with n faces and some sets of labelled trees (Cori & Vauquelin [20], Schaeffer [65]). Building on this, Chassaing & Schaeffer [19] made the first connection between the scaling limit of uniform planar quadrangulations and the Brownian snake with lifetime process the normalized excursion, from what they obtained the right scale $n^{1/4}$ for the diameter of uniform planar quadrangulations with n faces. The second author and Mokkadem introduced in [54] the Brownian map as the rescaled limit of random quadrangulations with n faces (under a distribution close to the uniform distribution). The question of the convergence to the Brownian map for a nice topology implying the metric convergence (as the Gromov-Hausdorff topology) was known by the authors, but out of reach at this time. Another topology was used, a topology absolutely faithful

to the CVS bijection: building on the latter, [54] proved that the set of quadrangulations with n faces is in bijection with a set of pairs of trees $(T_n^{(1)}, T_n^{(2)})$ as presented above. The pair of random trees $(\mathbf{T}_n^{(1)}, \mathbf{T}_n^{(2)})$ associated with uniform planar quadrangulations with n faces converges in distribution after normalization to a limiting pair of continuous random trees $(\mathbf{t}^{(1)}, \mathbf{t}^{(2)})$. The Brownian map is defined in [54] to be $\mathbf{t}^{(2)}$ quotiented by the non-crossing partition defined by $\mathbf{t}^{(1)}$ in a way analogous to what is done in the discrete case. This approach provides a direct construction of the Brownian map equipped with a natural distance.

In [26], Duplantier, Miller and Sheffield (DMS) investigate random surfaces obtained from the mating of two continuum random trees T_e and $T_{\tilde{e}}$ encoded by two independent normalized Brownian excursions e and \tilde{e} (see Sec. 3). They then identify in these trees the nodes with corner t , for all $t \in [0, 1]$. If (E_0, d_0) and (E_1, d_1) are two metric spaces, and \sim_R is an equivalence relation on $E_0 \cup E_1$, then the quotient space $E = (E_0 \cup E_1) / \sim_R$ is a topological space. One can try to define a distance d on E by: for $x, y \in E$,

$$d(x, y) = \inf_k \inf_{x_1, \dots, x_{2k}} \inf_{(\varepsilon_j, 1 \leq j \leq k) \in \{0, 1\}^k} \sum_{j=1}^k d_{\varepsilon_j}(x_{2j-1}, x_{2j}),$$

where $x_1 \sim_R x$, $x_{2k} \sim_R y$, and, for any j , $x_{2j} \sim_R x_{2j+1}$, and x_{2j-1} and x_{2j} are both elements on the same set E_{ε_j} . In other words, geodesic paths in E are “limits” of connected paths in E formed by alternating sections totally included in E_0 or in E_1 . It turns out that this way of defining a distance on E fails in general because it may happen that $d(u, v) = 0 \not\Rightarrow u = v$ (as detailed in footnote 12). In the case of the mating of trees, which is known to be homeomorphic to the 2-sphere, DMS do not consider the metric induced by this construction, but consider the topological properties of this space, together with a special space-filling path coming from the construction; they call this space the peanosphere. They study some stochastic processes defined on this rich structure, and make many connections, among others with the Brownian map, Gaussian free field, and Liouville quantum gravity in 2D.

Hence, the Brownian map has been defined before the proof of the convergence of its inner metric (Le Gall [46] and Miermont [56]), before the proof of its property to be homeomorphic to the 2-sphere (Le Gall and Paulin [48], Miermont [55]), and also before the proof of its connection with Liouville quantum gravity. To define the Brownian map before these considerations was probably a necessary step in this research field since knowing the limit even for a “bad topology” is always an advantage. In the same way the peanosphere is constructed by taking “a formal topological” limit of some discrete analogue constructed using binary trees with n leaves, somehow independently of standard considerations concerning invariance principles since no proof of convergence is given: only “the limit” is considered.

What we propose here is to proceed as in the first construction of the Brownian map: we present a combinatorial model and its limit (the D th random feuilletage). The first properties we are able to prove provide some first clues that this construction could indeed be analogous to the Brownian map. We hope that it could also lead to the construction of some peanospheres in higher dimensions.

1.4 Motivation from theoretical physics.

From a theoretical-physics perspective, the definition of an analogue of the Brownian map in higher dimensions is sought in the context of discrete approaches to quantum gravity, which aim at describing space-time, on which general relativity is formulated, at a microscopic “quantum” level. Such theories (see

e.g. [4, 22, 2, 35, 36]) attempt to describe quantum gravity as a continuum limit of a statistical system of random discrete space-times, namely some family of D -dimensional triangulations (or other cell pseudo-complexes), together with a probability of occurrence provided by Einstein’s general relativity, or more precisely by discretizing *à la Regge* the Einstein-Hilbert action, which is the field-theoretical formulation of general relativity. These “Boltzmann” weights depend exponentially on the discrete curvature⁸ (see e.g. [2, 50]).

Such theories are seen in analogy with the description of gas thermodynamics in statistical physics: the simplices are viewed as “particles of space-time” and the D -dimensional triangulations as accessible states for the statistical system, each with a given Boltzmann weight, obtained by discretizing the Einstein-Hilbert action. It is however the continuum/scaling limit that would be physically interpreted as a quantum theory of gravity, if it had consistent properties. From the scaling limit of these random discrete space-times, one then needs to recover general relativity as an effective theory by taking a suitable “coarse-graining” limit⁹.

In dimension $D = 2$, the link between combinatorial maps, matrix models, and later the Brownian map on one hand, and quantum gravity in dimension $D = 2$ on the other, has been investigated since the 80’s [24, 40, 21, 25, 27]. It was then proven in 2016 [57, 59, 58] that the Brownian map is indeed equivalent to Liouville quantum gravity [23], a theory of random continuum surfaces introduced by Polyakov in the context of string theory [62].

For $D > 2$, the only phases found asymptotically are the following¹⁰:

- A tree-phase (called branched-polymer phase in physics), in which a narrow subset of very highly curved D -spheres is selected, whose scaling limit is Aldous’ continuum random tree (CRT) [4, 66, 37].
- A phase in which very singular D -dimensional triangulations dominate asymptotically, whose diameter are bounded or small [4, 66]. This regime is called the crumpled phase in the physics literature, and it is expected that no scaling limit can be defined. This includes the uniform distribution for D -dimensional triangulations of the same size that are manifolds (colored, [15]), or spheres [4, 66].
- There are ways to restrict the set of colored D -dimensional triangulations that should lead to the Brownian map as a scaling limit (see the footnote 2).

In a well-defined continuum limit, one would like to recover something which resembles a random emergent continuum “ D -dimensional” space-time (as discussed previously, this is a very vague statement, however for many reasons, we should rule out the phases above in physics considerations in $D > 2$). The important question would then be how to find out whether it leads to general relativity in a certain

⁸A “canonical” discrete curvature [63] is defined on a D -dimensional triangulation by assuming that all edges have the same length. Then, the discrete curvature locally depends only on the number of D -simplices around each $(D - 2)$ -simplex.

⁹If these random discrete space-times converge towards a certain continuum limit (scaling limit), finding out if general relativity is recovered as an effective theory in a certain “non-quantum” limit could involve defining suitable observables on the scaling limit, that would converge to their classical (i.e. non-quantum) values throughout a coarse-graining process, or knowing how to describe this continuum limit in a field theoretic way, and then renormalizing this theory to translate it to our scales. *But there are no known spaces so far to serve as toy-models for addressing this question.*

¹⁰All results discussed here are in the Euclidean case, in which time is not considered. Introducing “time” can be done by requiring some additional causality condition on the D -dimensional triangulations [3, 51]. Numerical simulations seem to indicate that the continuum limit in dimension 4 has promising properties, however no exact result exists so far.

“coarse-grained” limit (see the footnote 9), however until now, there are no known examples of continuum random spaces that could serve this purpose. *At this level, regardless of the precise notion of dimension for scaling limits of random graphs, even the construction of scaling limits of random graphs that are neither trees, nor surfaces of any genus, would be an important step forward, by providing toy-models to address this question.* Having made this point, we would like to emphasize the following:

- It is not clear to us why choosing the Boltzmann weights at the discrete level should favor or not the obtention of general relativity after both the continuum limit and the coarse graining limit are taken. It seems also justified to study random cell-complexes with other probability distributions.
- Pushing this reasoning further, the *continuum limit* is what matters (the convergence to the Brownian map in dimension 2), so *any* sequence of random graphs converging to a suitable limit is a fortiori justified.
- The importance of the sphericity requirement for the scaling limit is not clear from the quantum gravity perspective. It could for instance be easier to produce random continuum spaces that are singular or do not have a well-defined topological dimension but still provide smooth manifolds after coarse-graining, and it seems like there is no reason to exclude random spaces obtained from limits of more general random graphs, at least to define a coarse-graining procedure and study its impact on the topology.

The facts listed above provide a strong motivation for trying to build interesting random continuum spaces as limits of more general sequences of random graphs, which is the aim of the present paper: Our approach is to construct scaling limits of random graphs, in a way which allows keeping track of the distances, to ensure a good control on the fractal (Hausdorff) dimension. The D th Brownian object is obtained from the $(D - 1)$ th by identifying many points, as explained in the introduction (see also Sec. 7.1). While this approach, based on the repeated use of the Cori-Vauquelin-Schaeffer bijection, renders the question of the topology of the scaling limit less intuitive than when considering scaling limits of discrete topological spaces, it is quite clear that the random feuilletages are neither random trees, nor random surfaces of any genus. Therefore, if the continuum random feuilletages are indeed non-trivial, they would provide the first examples of random continuum spaces that could be used to study what it means to “coarse-grain” a continuum Brownian space and what consequences it would impose for this space to require that the “non-quantum” limit of this coarse-graining process is general relativity (footnote 9).

It is worth adding here that the critical exponent we find for the D th random discrete feuilletages, associated with their asymptotic enumeration, is

$$\gamma_D = 3/2 - D. \tag{1.1}$$

It generalizes the well-known universality class exponents $\gamma_1 = 1/2$ for random trees, and $\gamma_2 = -1/2$ for random planar maps. It is important in physics, where it is sometimes called string susceptibility.

2 Iterated Brownian snakes and iterated random feuilletages

All the random variables are assumed to be defined on a common probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

Notations : We will denote by $\llbracket a, b \rrbracket$ the set $[a, b] \cap \mathbb{Z}$ equipped with its natural order. For an ordered finite set I , the notation $X(I)$ stands for the sequence $(X(i), i \in I)$ taken under the index order; hence, $X(\llbracket 0, 5 \rrbracket) = (X_0, X_1, \dots, X_5)$. Finally, we will denote by (x_n) the infinite sequence (x_1, x_2, \dots) . By convention $x + y \bmod p$ stands for $(x + y) \bmod p$.

For a function $g : \mathbb{R} \rightarrow \mathbb{R}$ (or defined on an interval $[a, b] \subset \mathbb{R}$ only), we denote by

$$\check{g}(x, y) = \min\{g(u), u \in [\min\{x, y\}, \max\{x, y\}]\}$$

the minimum of g on the interval with extremities x and y .

2.1 Continuous trees

Start by a digression concerning the so-called “iterated Brownian motion”: take a sequence of independent two-sided Brownian motions $(\mathbf{B}_i, i \geq 0)$, meaning that for any i , $(\mathbf{B}_i(s), s \geq 0)$ and $(\mathbf{B}_i(-s), s \geq 0)$ are two standard linear Brownian motions starting at 0. The D th Brownian motion (see e.g. Burdzy [14]) is the one dimensional process defined by

$$\mathbf{I}^{(D)}(t) = \mathbf{B}_D(\mathbf{B}_{D-1}(\dots(\mathbf{B}_1(t))\dots)), \quad t \in \mathbb{R}. \quad (2.1)$$

The construction we propose for the D th Brownian snake $\mathbf{bs}[D]$ can be viewed as a kind of tree-like counterpart to $\mathbf{I}^{(D)}$: as explained in Section 1.1, we will produce a sequence of labeled trees $(\mathbf{t}^{(i)}, \ell^{(i)})$, building $\mathbf{t}^{(i+1)}$ thanks to $\ell^{(i)}$, “a Brownian labeling of $\mathbf{t}^{(i)}$ ”. Up to some changes of roots, constructing $\mathbf{t}^{(D)}$ will require $D - 1$ iterated Brownian labelings. The construction is tuned in such a way that the pair $(\mathbf{t}^{(1)}, \mathbf{t}^{(2)})$ corresponds exactly to the random trees encoding the Brownian map.

Representation of (real) trees using continuous functions. We refer to Le Gall & Duquesne [30], Miermont [56], Le Gall [46, 47] or Miermont & Le Gall [43], for information on real and continuous trees. Compact \mathbb{R} -trees are compact metric spaces (T, d) such that for every a, b in T , there exists a unique injective function $f_{a,b} : [0, d(a, b)] \rightarrow T$, for which $f_{a,b}(0) = a$ and $f_{a,b}(d(a, b)) = b$. In the sequel, we present some continuous trees encoded by functions; these objects are rooted-ordered real trees (see Duquesne [28] and references therein for a complete discussion on the relation between compact real trees and trees encoded by real valued functions).

Consider $C[0, 1]$, the set of continuous functions $f : [0, 1] \rightarrow \mathbb{R}$. Let

$$C^+[0, 1] = \{f \in C[0, 1], f([0, 1]) \subset \mathbb{R}^+, f(0) = f(1) = 0\}. \quad (2.2)$$

Associate with each function g in $C^+[0, 1]$, an equivalence relation in $[0, 1]$ by

$$x \underset{g}{\sim} y \iff g(x) = g(y) = \check{g}(x, y).$$

The quotient space $T_g = [0, 1] / \underset{g}{\sim}$ is connected and possesses no cycle: it is a **tree**, and its elements are called **nodes** or **vertices** (see Fig. 8). The canonical surjection c_g from $[0, 1]$ into T_g is denoted

$$\begin{aligned} c_g : [0, 1] &\longrightarrow T_g \\ x &\longmapsto c_g(x) = \dot{x} := \{y \in [0, 1], x \underset{g}{\sim} y\} \end{aligned}$$

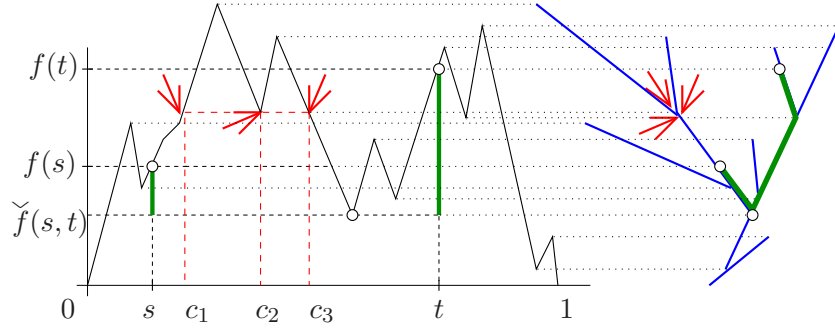


Figure 8: Tree associated with a function taken in $C^+[0, 1]$: the green path in the tree is visible in the functional encoding; the red arrows point toward the three corners c_1, c_2 and c_3 of a vertex in the tree.

It is the continuous analogue of the depth first traversal c_T for discrete trees defined below (3.3). Seeing $x \in [0, 1]$ as the **corner** of a node, $c_g(x)$ is precisely that node (and a node corresponds to a set of corners). The tree T_g can be turned into a totally ordered set, by setting

$$\dot{x} < \dot{y} \quad \text{iff} \quad \inf \dot{x} < \inf \dot{y},$$

that is, the order of two nodes is inherited from the order of their first corners. The class $\dot{0}$ is the root of T_g . Let x and y be elements of \dot{x} and \dot{y} . The node $\dot{z} \in T_g$ defined by $z \in [x, y]$ and $g(z) = \check{g}(x, y)$ does not depend on the chosen representatives x and y : \dot{z} is called the highest common ancestor of \dot{x} and \dot{y} . From there, we can define the notions of **ancestor**, **descendent**, **subtree**, **branch**, as in the discrete case (Sec. 3.1). A node \dot{x} is said to be an internal node if $\inf \dot{x} < \sup \dot{x}$, and it is said to be a leaf if $\inf \dot{x} = \sup \dot{x}$ (the leaf set includes positions of local maxima). The distance between \dot{x} and \dot{y} is defined as

$$d_{T_g}(\dot{x}, \dot{y}) := D_g(x, y), \quad (2.3)$$

where

$$D_g : [0, 1]^2 \longrightarrow \mathbb{R}^+ \\ (s, t) \longmapsto D_g(s, t) := g(s) + g(t) - 2\check{g}(s, t) . \quad (2.4)$$

It is an exercise to check that D_{T_g} is well defined and indeed a distance. The function g is called **the contour process** of T_g in accordance to the discrete case (see later Def. 3.4) since

$$d_{T_g}(\dot{0}, \dot{x}) = D_g(0, x) = g(x), \quad \text{for any } x \in [0, 1] \quad (2.5)$$

Trees as measured spaces. Consider $\mu \in \mathcal{M}[0, 1]$, the set of probability measures on $[0, 1]$, $g \in C^+[0, 1]$ and T_g the associated tree. The pair (T_g, μ) is called a measured tree. Since $c_g : [0, 1] \rightarrow T_g$ maps each corner x on the associated vertex $c_g(x) \in T_g$, the measure μ is a measure on the corner sets, and its push-forward measure by c_g is a measure on T_g .

There are two main reasons to enrich trees with measures, both being linked with discrete trees:

- In the discrete case, up to a time normalisation, for many models of random trees, the number of nodes visited in the contour process between time a and b is (approx.) a proportion $b - a$ of the nodes (and also a proportion $b - a$ of the corners). The measure component allows accounting for this and then expressing that, at the limit, the same property holds for the continuum random tree T_g (the “limiting measure”

being the Lebesgue measure).

■ The second reason is the need to distinguish between discrete and continuous trees! If $g \neq 0$, the set $[0, 1]/ \sim g$ has the cardinality of \mathbb{R} , regardless of whether T_g “is used to model” a discrete tree or not. When one embeds the set of discrete trees in the set of continuous trees using contour processes, their discrete nature is lost. Corner measures allow recovering corner positions, and then allow one to cover discrete and continuous objects by a single notion, which is compulsory to prove convergence results.

Let $K = \{T_g := (T_g, d_{T_g}, \mu_g) \text{ , } g \in C^+[0, 1], \mu \in \mathcal{M}[0, 1]\}$ be the set of such rooted trees, considered as metric spaces, and equipped with a corner measure (in the following, we use the same notation for a real tree and the corresponding measured metric space). The set of trees K is a metric space: we transport the metric and the topology from $(C[0, 1], \|\cdot\|_\infty) \times (\mathcal{M}[0, 1], d_P)$ (where d_P is the Prokhorov distance) onto K , by setting the following distance on K : for g and f in $C^+[0, 1]$,

$$d_K(T_g, T_f) = \|g - f\|_\infty + d_P(\mu_f, \mu_g).$$

This makes of the set of trees K a Polish space.

Note 1. *Formally, the measure μ_f is seen as a measure on the corners of the trees; if one prefers, one can put a measure on the push forward measure on the set of nodes (but this somehow brings some additional abstraction, which seems not useful in the present work).*

Definition 2.1. *We call Aldous’ CRT, the tree $T_e = (e, d_e, \lambda)$ where λ is the Lebesgue measure on $[0, 1]$ (and where e is a Brownian excursion).¹¹*

We end this introduction to continuous trees by defining the change of root:

Definition 2.2. *For $g \in C^+[0, 1]$ and $x \in [0, 1]$, the tree T_g rerooted at its corner x is the tree T_h for*

$$\begin{cases} h(s) & := D_g(x + s \bmod 1, x), \text{ for } s \in [0, 1], \\ \mu_h(\cdot) & = \mu_g(x + \cdot \bmod 1) \end{cases}.$$

This definition fits perfectly with (2.5), since the distance to the corner x in T_g is indeed the function “distance to the root” in T_h .

Remark 2.3 (Important). *All along the paper we will encounter many continuous trees $T_g = (g, d_g, \mu_g)$ for which the corner measure **will always be the Lebesgue measure on $[0, 1]$** . For a discrete normalized discrete trees with N edges, the corner measure **will always be the uniform corner measure** $\lambda_N = 1/(2N) \sum_{k=0}^{2N-1} \delta_{k/(2N)}$ on $[0, 1]$. When $N \rightarrow +\infty$,*

$$\lambda_N \rightarrow \lambda \tag{2.6}$$

¹¹The normalized Brownian excursion can be obtained by rescaling the excursion of the Brownian motion which straddles 1. From here, it can be seen that the trajectory of the Brownian excursion inherits from the Brownian motion of many features. For example, it has a countable number of local minima or maxima (Revuz-Yor [64, Chap. III, 3.26]). Besides, the Brownian motion has the strong Markov property, and the property that the set $Z = \{t : B_t = 0\}$ is a.s. a closed set without isolated point (see [64, Chap. III Prop 3.12]), allows seeing that a.s., 0 (or any other point $x \in [0, 1]$) is not a local maximum or minimum: for any $x \in [0, 1]$, one has a.s. $\inf \dot{x} = \sup \dot{x}$, even if a.s. x is not a local maximum. Hence, in T_e the set of leaves has a.s. Lebesgue measure 1, and T_e is a.s. a binary tree.

for the classical weak convergence in $\mathcal{M}[0, 1]$, so that this additional “measure component” does not modify the proof of convergence for trees, for snakes, and after that for feuilletages. Nevertheless, the presence of the measures allows defining properly the feuilletages, whose definition, in the discrete case, must take into account the actual nodes locations.

To avoid heavy notations, we will however often drop the measure component, but will recall its presence when it is crucial. In the proofs, the presence of measures will be simply completely dropped, since (2.6) alone allows taking care of the convergence of the measure components.

2.2 Brownian snake

The Brownian snake is a classical object (see e.g. [30], Le Gall [44] for continuous snakes, and Marckert & Mokkadem [53], Janson & Marckert [38] for discrete snakes and their convergence.

Consider the following set (of “bridges”)

$$C^0[0, 1] = \{g \in C[0, 1], g(0) = g(1) = 0\}. \quad (2.7)$$

Definition 2.4. Let $g \in C^+[0, 1]$ and T_g be the associated tree. A labeling of the rooted tree T_g is a map $\ell \in C^0[0, 1]$ which satisfies $s \sim_g t \Rightarrow \ell(s) = \ell(t)$. In other words, the labels of the corners of a node coincide. A pair (g, ℓ) , where ℓ is a labeling of T_g , is called tour of a continuous snake. We denote by

$$\overrightarrow{\text{Snakes}} = \{(g, \ell) \in C^+[0, 1] \times C^0[0, 1], \ell \text{ is a labeling of } T_g\} \quad (2.8)$$

the space of tours of continuous snakes, equipped with the uniform convergence topology.

Remark 2.5. As detailed previously, a corner measure is sometimes considered, so that the elements of $\overrightarrow{\text{Snakes}}$ will sometimes be viewed as 3-tuples (g, ℓ, μ) instead (here $\mu \in \mathcal{M}([0, 1])$, equipped with the distance

$$D((g, \ell, \mu), (g', \ell', \mu')) = \|g - g'\|_\infty + \|\ell - \ell'\|_\infty + d_P(\mu, \mu').$$

To avoid too much heaviness, we remove this third component as long as it is not explicitly needed.

Remark 2.6. In the literature, given (g, ℓ) in $\overrightarrow{\text{Snakes}}$, the snake with tour (g, ℓ) is the family of trajectories $(w_x, x \in [0, 1])$, where the lifetime of w_x is $g(x)$, and $w_x(h) = \ell(z)$ for $0 \leq h \leq g(x)$, where z is one of the corners of the ancestors of x at height h . The natural maps that associate tours of snakes and snakes are homeomorphic under natural topologies (see Marckert & Mokkadem [53]), making possible to transfer all convergence results obtained on tours of snakes to snakes. In the following, we will only deal with tours of snakes and we will often call them simply snakes, by abuse of language.

Definition 2.7. Consider \mathbf{X} a random process taking its values in $C^+[0, 1]$; we call tour of the Brownian snake with lifetime process \mathbf{X} , the pair (\mathbf{X}, ℓ) where, conditionally on $\mathbf{X} = g \in C^+[0, 1]$, the process $(\ell(s), s \in [0, 1])$ is a centered Gaussian process with covariance function

$$\text{cov}(\ell(x), \ell(y)) = \check{g}(x, y). \quad (2.9)$$

■ Hence the pair (\mathbf{X}, ℓ) is a.s. the tour of a snake, since for $0 \leq x \leq x' \leq 1$,

$$g(x) = g(x') = \check{g}(x, x') \Rightarrow \text{cov}(\ell(x) - \ell(x'), \ell(x) - \ell(x')) = 0 \Rightarrow \ell(x) = \ell(x').$$

However, the a.s. continuity of ℓ is not granted: it depends on the regularity of \mathbf{X} (all considerations on the Hölder coefficients in the paper are developed for this reason),

■ The tour of the rooted standard Brownian snake with lifetime process g corresponds to the labeling of a continuum random tree with contour process g , by a Brownian motion starting at its root with the property that a node at height h is labeled by a Brownian motion at time h , and for \dot{u} and \dot{v} two nodes of the tree T_g , the Brownian trajectories $(\mathbf{B}_s, 0 \leq s \leq g_u)$ and $(\mathbf{B}'_s, 0 \leq s \leq g_v)$ coincide on $[0, \check{g}(u, v)]$, and

$$\left(\mathbf{B}_s - \mathbf{B}_{\check{g}(u,v)}, \check{g}(u, v) \leq s \leq g_u \right) \text{ and } \left(\mathbf{B}'_s - \mathbf{B}_{\check{g}(u,v)}, \check{g}(u, v) \leq s \leq g_v \right) \text{ are independent.}$$

■ The *Brownian snake with lifetime process the normalized Brownian excursion* is the process which corresponds to the case $\mathbf{X} = \mathbf{e}$, that is, when the underlying tree is Aldous' CRT $T_{\mathbf{e}}$ (see e.g. [30, 53, 38]).

2.3 Iteration of snakes

To iterate the construction, we will associate a tree to the label process ℓ of a snake with contour process (g, ℓ) . To this end assume for a moment that ℓ is continuous, and is an element of $C^0[0, 1]$.

If $g \not\equiv 0$ the tree T_g contains a non-trivial branch b , so that ℓ is a.s. not in $C^+[0, 1]$ (since its range contains the Brownian motion on b), and therefore ℓ is not the contour process of a tree. Pushed by combinatorial and technical considerations, we proceed as follows. For a function f in $C^0[0, 1]$, define

$$\begin{cases} m(f) & := \inf\{f(x), x \in [0, 1]\}, \\ a(f) & := \min \operatorname{argmin} f = \min\{x, f(x) = m(f)\}. \end{cases} \quad (2.10)$$

The value $a(f)$ is the first hitting time of the minimum $m(f)$ for the function f (left of Fig. 9).

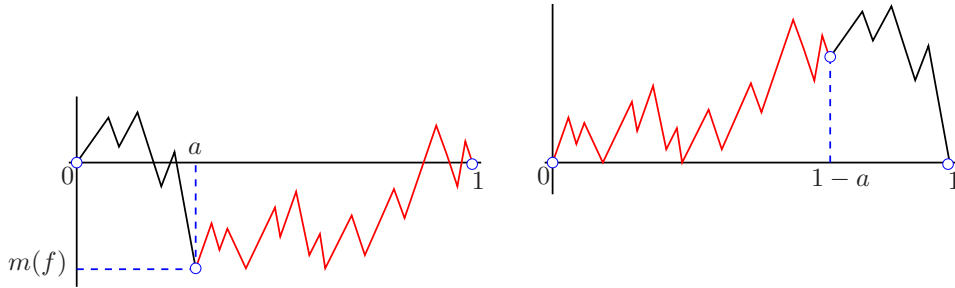


Figure 9: Illustration of the map $f \mapsto \operatorname{Conj}(f)$: it exchanges the part of the graph before and after $a(f)$.

Leading idea: The so-called conjugation of paths (see Fig. 9), is the map

$$\begin{aligned} \operatorname{Conj} : C^0[0, 1] &\longrightarrow C^+[0, 1] \\ f &\longmapsto x \mapsto f[(a(f) + x) \bmod 1] - m(f) \end{aligned} \quad (2.11)$$

A quite famous result is that if \mathbf{b} is a standard Brownian bridge, then $\operatorname{Conj}(\mathbf{b}) \stackrel{(d)}{=} \mathbf{e}$ (Vervaat [67]).

For $f \in C^0[0, 1]$, since $\operatorname{Conj}(f) \in C^+[0, 1]$, $\operatorname{Conj}(f)$ is naturally the contour process of a tree.

Remark 2.8. Let μ be the corner measure of a tree T_f and let $g = \operatorname{Conj}(f)$. The push-forward measure of μ by Conj is μ' satisfies $\mu'(\cdot) = \mu(a(f) + \cdot \bmod 1)$. The measure μ' is the “same corner measure” as μ in the sense that it puts the same weight to the corners that are in correspondence on T_g and T_f .

Starting from any function f in $C^0[0, 1]$, one can consider the tree with contour $\text{Conj}(f)$, and use it as the underlying tree of a branching Brownian motion.

Regarding the assumption that a label process $\ell \in C^0[0, 1]$, since $g(0) = g(1) = \check{g}(0, 1) = 0$, then a.s. $\ell(0) = \ell(1) = 0$. As for the continuity of ℓ , the a.s. existence (or not) of a continuous version for ℓ , depends on the regularity of g . This is one of the (relative) difficulties of this construction.

Definition 2.9. *The space of rooted continuous D -snakes is defined to be $\overrightarrow{\text{Snakes}}^D := (\overrightarrow{\text{Snakes}})^D$, equipped with the uniform topology. When we specify the corner measures, a rooted D -snake has the following form: $[(f^{(i)}, \ell^{(i)}, \mu^{(i)}), 1 \leq i \leq D]$.*

Iterated rooted Brownian snakes

Definition 2.10. *For any positive integer D , we call D th Brownian snake the process*

$$\mathbf{bs}[D] := \left(\left[\mathbf{h}^{(1)}, \ell^{(1)} \right], \dots, \left[\mathbf{h}^{(D)}, \ell^{(D)} \right] \right) \quad (2.12)$$

taking its values in the space $\overrightarrow{\text{Snakes}}^D$, where:

- (i) the first tree is the continuum random tree: $\mathbf{h}^{(1)} \stackrel{(d)}{=} \mathbf{e}$,
- (ii) for any j , conditionally on $\left[\mathbf{h}^{(1)}, \ell^{(1)}, \dots, \mathbf{h}^{(j-1)}, \ell^{(j-1)}, \mathbf{h}^{(j)} \right]$, the process $\ell^{(j)}$ is the label process of the rooted Brownian snake with lifetime process $\mathbf{h}^{(j)}$ (as defined in Def. 2.7),
- (iii) for $j \geq 2$, the contour of the j th random tree $\mathbf{t}^{(j)} := T_{\mathbf{h}^{(j)}}$ is

$$\mathbf{h}^{(j)} = \text{Conj}(\ell^{(j-1)}). \quad (2.13)$$

For any j , the corner measure on $\mathbf{t}^{(j)}$ is λ , the Lebesgue measure on $[0, 1]$.

Hence, the standard Brownian snake with lifetime process the normalized Brownian excursion coincides with $\mathbf{bs}[1]$ (see e.g. [44, 53, 38]). Definition 2.10 really defines an existing object only if the $\ell^{(j)}$'s are all a.s. continuous: this property is needed to define the continuous contour of the tree $\mathbf{t}^{(j+1)}$ using $\ell^{(j)}$. The following will be proven in Sec. 5.2:

Theorem 2.11. *For any $j \geq 1$, $\ell^{(j)}$ is a.s. continuous process and then for any $D \geq 1$, the process $\mathbf{bs}[D]$ is well defined.*

Remark 2.12. ■ *There is no natural process $\mathbf{bs}[0]$, since the Brownian bridge is not the label process of any tree. It is somehow the label process of the circle \mathbb{R}/\mathbb{Z} .*

- *The fact that $\ell^{(j)}$ reaches its minimum only once a.s. for $j \geq 2$ (on a leaf of $\mathbf{h}^{(j)}$) is unclear, but we conjecture that it is true; this property holds for $\ell^{(1)}$ (Le Gall & Weill [33, Prop. 2.5]).*
- *Constructing a version of $\mathbf{bs}[D]$ conditioned by the non-negativity of all the $\ell^{(i)}$, which is a singular conditioning, and which has been done by Le Gall & Weill [33] in the case of the Brownian snake $\mathbf{bs}[1]$, does not seem to be an easy task. This positive processes could be used to define more easily some feuilletages (as done below) without having to deal with what we call below “tree synchronisations”.*

2.4 The iterated random feuilletages $\mathbf{r}[D]$

2.4.1 $\mathbf{r}[2]$ is the Brownian map

Let $\mathbf{bs}[1] = (\mathbf{h}^{(1)}, \ell^{(1)})$ be the standard Brownian snake; let again $\mathbf{h}^{(2)} = \text{Conj}(\ell^{(1)})$. Consider the tree $\mathbf{t}^{(2)} = T_{\mathbf{h}^{(2)}}$, and set

$$\mathbf{a}^{(1)} = \min \text{argmin } \ell^{(1)}.$$

By definition (2.11) of the map Conj , since $\mathbf{h}^{(2)}(\cdot) = \ell^{(1)}(\mathbf{a}^{(1)} + \cdot \bmod 1) - \min \ell^{(1)}$, the corner $x - \mathbf{a}^{(1)} \bmod 1$ of $\mathbf{t}^{(2)}$ corresponds to the corner x in the tree $\mathbf{t}^{(1)}$.

Definition 2.13. Let $\mathbf{bs}[2] = \left(\left[\mathbf{h}^{(1)}, \ell^{(1)} \right], \left[\mathbf{h}^{(2)}, \ell^{(2)} \right] \right)$ be the 2nd Brownian snake. The Brownian map $\mathbf{r}[2]$ is the topological space defined as $[0, 1] / \sim_2$, where \sim_2 is the coarsest equivalence relation that extends the two following equivalence relations:

$$\begin{aligned} x &\sim_{\mathbf{h}^{(1)}} y, \\ (x - \mathbf{a}^{(1)}) \bmod 1 &\sim_{\mathbf{h}^{(2)}} (y - \mathbf{a}^{(1)}) \bmod 1. \end{aligned}$$

These two relations are equivalent to

$$\begin{aligned} D_{\mathbf{h}^{(1)}}(x, y) &= 0, \\ D_{\mathbf{h}^{(2)}}(x - \mathbf{a}^{(1)} \bmod 1, y - \mathbf{a}^{(1)} \bmod 1) &= 0. \end{aligned}$$

Let us discuss further the appearance of $\mathbf{a}^{(1)}$ in the considerations and the implications for iterations.

2.4.2 Trees synchronization and $\mathbf{r}[D]$

The worst hassle in the construction of the D th Brownian snake is the use of Conj , which brings some extra random shifts at each iteration: these shifts are inherited by the D random rooted feuilletages (in the $D = 2$ case, these considerations are present in relation with the rooted pointed Brownian map). Later on, the non-continuity of these synchronizations will lead to the definition of pointed counterparts to the rooted snakes and to the random feuilletages.

Take a D th Brownian snake $\mathbf{bs}[D]$ (with the same notation as in Def. 2.10). Set

$$\mathbf{a}^{(m)} = \min \text{argmin } \ell^{(m)}, \quad \text{for } m \in \llbracket 1, D \rrbracket, \quad (2.14)$$

so that again, $\mathbf{h}^{(m+1)}(\cdot) = \text{Conj}(\ell^{(m)}) = \ell^{(m)}((\mathbf{a}^{(m)} + \cdot) \bmod 1) - \min \ell^{(m)}$, and for any m ,

$$\mathbf{t}^{(m)} = T_{\mathbf{h}^{(m)}}.$$

In order to trace back all the shifts coming from the successive change of roots, we set

$$\mathbf{A}^{(m)} = \mathbf{a}^{(1)} + \dots + \mathbf{a}^{(m-1)}, \quad \text{for } m \geq 1. \quad (2.15)$$

For example, the corner $x - \mathbf{A}^{(3)} \bmod 1$ of the tree $\mathbf{t}^{(3)}$ corresponds to the corner $x - \mathbf{A}^{(2)} \bmod 1$ of $\mathbf{t}^{(2)}$, which in turn corresponds to the corner x of $\mathbf{t}^{(1)}$.

Definition 2.14. Let $\mathbf{bs}[D] = \left(\left[\mathbf{h}^{(1)}, \ell^{(1)} \right], \dots, \left[\mathbf{h}^{(D)}, \ell^{(D)} \right] \right)$ be the D th Brownian snake. We call D th random feuilletage $\mathbf{r}[D]$ the topological space

$$\mathbf{r}[D] := [0, 1] / \sim_D, \quad (2.16)$$

where \sim_D is the coarsest equivalence relation on $[0, 1]$ refining all the following equivalence relations $\sim_{[m]}$ for $1 \leq m \leq D$, defined for $x, y \in [0, 1]$ by

$$x \sim_{[m]} y \Leftrightarrow D_{\mathbf{h}^{(m)}} \left(x - \mathbf{A}^{(m)} \bmod 1, y - \mathbf{A}^{(m)} \bmod 1 \right) = 0. \quad (2.17)$$

Hence, $x \sim_D y$ if and only if there exists a finite sequence of identification points $((x_m, j_m), 1 \leq m \leq N) \in ([0, 1] \times [1, D])^N$ such that, for $x_0 := x, x_{N+1} := y$,

$$x_m \sim_{[j_m]} x_{m+1} \quad \text{for } 0 \leq m \leq N. \quad (2.18)$$

Remark 2.15. Formula (2.17) defines the feuilletage. In the discrete case we will transform (2.17) so that only corners corresponding to “discrete nodes” are identified. This will amount to restricting (2.17) to the support of the corner measure into play. This way of doing applies to continuous snakes too.

2.4.3 Some potential metrics on $\mathbf{r}[D]$

Here are two (potential) metrics on $\mathbf{r}[D]$ compatible with its topology:

$$d_{\mathbf{r}[D]}^{(1)}(x, y) = \inf_{r \geq 1} \inf_{1 \leq m_1, \dots, m_r \leq D} \inf_{0 \leq x_0, \dots, x_r \leq 1} \sum_{j=0}^r D_{\mathbf{h}^{(m_j)}} \left(x_j - \mathbf{A}^{(m_j)} \bmod 1, x_{j+1} - \mathbf{A}^{(m_j)} \bmod 1 \right) \quad (2.19)$$

where $x_0 \sim_D x, x_{m+1} \sim_D y$, and

$$d_{\mathbf{r}[D]}^{(2)}(x, y) = \inf_m \inf_{0 \leq x_0, \dots, x_{2m+1} \leq 1} \sum_{i=0}^m D_{\mathbf{h}^{(D)}} \left(x_{2i} - \mathbf{A}^{(D)} \bmod 1, x_{2i+1} - \mathbf{A}^{(D)} \bmod 1 \right) \quad (2.20)$$

where $x_{2i+1} \sim_D x_{2i+2}, x_0 \sim_D x, x_{2m+1} \sim_D y$. Identifications can be viewed as distance-free jumps in the space $\mathbf{r}[D]$: they combine identifications coming possibly from several different trees $\mathbf{t}^{(j_m)}$.

- The metric $d_{\mathbf{r}[D]}^{(1)}$ is more symmetric: a traveller who wants to go from x to y has to walk on one of the trees $\mathbf{t}^{(j)}$ for $1 \leq j \leq D$; when it does so, the distance is given by the metric on this tree. If he is at a given moment at $a \in [0, 1]$, he can jump at $b \in [0, 1]$ if $a \sim_D b$ without paying anything, or in other words, if a and b are two corners of the same node in one of the trees $\mathbf{t}^{(j)}$ for $1 \leq j \leq D$. He can change tree whenever he wants to go on his travel, and the final distance for a path is the minimum on all possible trips of the sums of all the distances made on each of these trees.

- For the metric $d_{\mathbf{r}[D]}^{(2)}$, the traveller can only walk on the tree $\mathbf{t}^{(D)}$, but whenever he wants, if he is at $a \in [0, 1]$, he can jump at $b \in [0, 1]$ without paying anything if $a \sim_D b$.

Open question 1. Are the distances $d_{\mathbf{r}[D]}^{(1)}$ and $d_{\mathbf{r}[D]}^{(2)}$ non-trivial for any $D > 2$? (that is, is the diameter of $\mathbf{r}[D]$ under $d_{\mathbf{r}[D]}^{(j)}$ a.s. positive)?

- The advantage of $d_{\mathbf{r}[D]}^{(1)}$ is that it is non-increasing in D , since when one passes from D to $D+1$, the set on which the minimum is taken is larger for the inclusion order. “Geometrically”, new identifications are provided by the $(D+1)$ th tree. Moreover, with each subset of indices $\{i_1, \dots, i_m\}$ included in $\llbracket 1, D \rrbracket$ one can associate the space;

$$\mathbf{r}[i_1, \dots, i_m] := ((\dots ([0, 1] / \sim_{[i_1]}) / \dots) / \sim_{[i_m]})$$

with the analogue of distance $d^{(1)}$ given by taking the infimum in (2.19) only on these indices.

- For $D = 1$, $\mathbf{r}[1]$ coincides topologically with Aldous’ continuum random tree, and the metrics $d_{\mathbf{r}[1]}^{(1)}$ and $d_{\mathbf{r}[1]}^{(2)}$ are equal and coincide with the standard metric on this space.
- For $D = 2$, $\mathbf{r}[2]$ coincides topologically with the Brownian map and $d_{\mathbf{r}[2]}^{(2)}$ corresponds to the standard metric on this space.

But we must say that we do not know the answers to the following questions:

Open question 2. *For $D > 2$, is it true that $d_{\mathbf{r}[D]}^{(j)}(x, y) = 0 \Rightarrow x \sim_D y$ for the distance $j = 1$ or 2 ? It is true for $D = 2$ as a consequence¹² of Miermont [56] and Le Gall [46].*

As a consequence of Theorem 5.7, which allows seeing that $\mathbf{h}^{(D)}$ is a.s. Hölderian with exponent $1/2^D - \varepsilon$, for any $\varepsilon > 0$, it may be shown that the tree $(\mathbf{t}^{(D)}, D_{\mathbf{h}^{(D)}})$ has Hausdorff dimension smaller than 2^D . For $D \in \{1, 2\}$, these upper bounds fit with the right values [45, 29]. These bounds are also the Hausdorff dimensions of $(\mathbf{r}[D], d_{\mathbf{r}[D]}^{(2)})$ for $D \in \{1, 2\}$.

Open question 3. *What are the Hausdorff dimensions of the random trees $(\mathbf{t}^{(D)}, D_{\mathbf{h}^{(D)}})$ and of the random feuilletages $(\mathbf{r}[D], d_{\mathbf{r}[D]}^{(2)})$ for $D > 2$? If 2^D is indeed the Hausdorff dimension of $(\mathbf{t}^{(D)}, D_{\mathbf{h}^{(D)}})$, then this value provides a lower bound for the dimension of $(\mathbf{r}[D], d_{\mathbf{r}[D]}^{(2)})$... We conjecture that both spaces indeed have Hausdorff dimension 2^D .*

Remark 2.16. *About the redundancy of the iterated Brownian snake: from f to $\text{Conj}(f)$, a change of origin has been done. If one has only $\text{Conj}(f)$ in hand, the “shift” $a(f)$ cannot be recovered. It turns out that for our applications to \mathbf{r} , the shift is needed to “synchronize” the identifications provided by the different trees. Working directly and only with f – which is possible since f determines $\text{Conj}(f)$ – is a bit annoying because it demands reintroducing $\text{Conj}(f)$ everywhere, since the iteration we propose relies on the tree encoded by $\text{Conj}(f)$.*

Remark 2.17. *When dealing with asymptotic discrete snakes, we will observe that the change of origin is not continuous, that is $\|f_n - f\|_\infty \rightarrow 0 \not\Rightarrow a(f_n) \rightarrow a(f)$ ¹³. As a consequence, even if the sequence (f_n) converges in $C[0, 1]$, the sequence of trees $(\text{Conj}(f_n))$ may not converge in the set of rooted trees, K . This issue explains the complications that will appear progressively in the sequel. The strategy we have adopted*

¹² When one quotients a topological space as we did when we introduced \sim_D , it may happen that the “projected distance” is not a distance: for example, let $E = [0, 1]$ equipped with the usual distance $|\cdot|$, and quotiented by the equivalence relation $x \sim y$ iff $x = y$ or $x, y \in \mathbb{Q}$ (identify rational numbers). Clearly, the quotient space E^* is not reduced to a single point, but $d^*(x, y) = 0$ for all $x, y \in E^*$ under “the inherited distance d^* ”. Hence, d^* is not a distance, since $d^*(x, y) = 0 \not\Rightarrow x = y$. If one further quotients E^* by $x \sim^* y$ when $d^*(x, y) = 0$, then the space E^* becomes trivial, reduced to a single point.

¹³To avoid this problem, it would suffice to prove that a.s., $\#\text{argmin}(\ell^{(j)}) = 1$, for the iterated process $\ell^{(j)}$.

to treat them is to use the redundancy provided by the presence of $h_{j+1} = \text{Conj}(f_j)$ together with f_j in the iterated Brownian snake. The discontinuity of the map $a(\cdot)$ will result in the loss of the identity of the root corner in the iterated trees while the root vertex will still be well known: pointing a tree amounts to considering as equivalent two trees rooted at different corners of the same root vertex. Pointing is compatible with the snake construction in which the root vertex is labeled 0, whatever the considered root corner. We will therefore progressively turn our intention to pointed snakes, pointed feuilletages, and finally, state our main theorems for these objects.

3 Iterated snakes and feuilletages: combinatorial objects

Notation : The i th increment of any sequence $(x_i, i \geq 0)$ is denoted $\Delta x_i = x_i - x_{i-1}$.

Convention : We make a great use of continuous processes X obtained by linear interpolation of some random sequence of the form $(X_k, k \in \llbracket 0, n \rrbracket)$ or of the form $(X_{k/n}, k \in \llbracket 0, n \rrbracket)$. We will keep the same notation X for the continuous and discrete version, but we will name “process” the interpolated version, and “sequence” the discrete one (without additional warning).

The main aim of this section is to present the discrete iterated snakes and discrete iterated feuilletages.

3.1 Planar trees and their encodings

Rooted planar trees. For $\mathbb{N}^* = \{1, 2, \dots\}$, consider $U = \{\emptyset\} \cup \bigcup_{k \geq 1} \mathbb{N}^{*k}$ the set of words on the alphabet \mathbb{N}^* . For any word $w = w_1 \dots w_k$ in U where $w_j \in \mathbb{N}^*$, $|w| = k$ is the length of w , also called the depth of w . For u and v in U , uv stands for the concatenation of u and v .

Definition 3.1. A rooted planar tree T is a finite subset of U , containing \emptyset , stable by prefix (if $uv \in T$ for $u, v \in U$, then $u \in T$), and such that if $ui \in U$ for $u \in U$ and $i \in \mathbb{N}^*$, then $uj \in T$ for $1 \leq j \leq i$.

An example is shown on the left of Fig. 10. The elements of T are called nodes or vertices. For $u \in T$ and $j \in \mathbb{N}^*$, if $uj \in T$, then uj is called a child of u , and u is the parent of uj . The number of children of u is $c_u(T) = \#\{j \in \mathbb{N}^*, uj \in T\}$. The prefixes of u are called the ancestors of u . The size of a tree T , denoted by $|T|$ is its cardinality (its number of nodes). We also set

$$\|T\| = |T| - 1,$$

the number of edges of T . Denote by \mathbb{T} the set of trees, and by \mathbb{T}_n the subset of those with n edges:

$$\mathbb{T}_n = \{T \in \mathbb{T}, \|T\| = n\}.$$

It may be proved by induction that the cardinality $\#\mathbb{T}_n = C_n = \binom{2n}{n}/(n+1)$ of \mathbb{T}_n is the n th Catalan number.

Height sequence. The lexicographical order on U induces an ordering on any tree, and allows to represent trees as sequences.

Definition 3.2. The height sequence of T is the sequence $H_T \llbracket 0, \|T\| \rrbracket$ of the successive heights of the nodes of T sorted in lex. order $u_0 = \emptyset, u_1, \dots, u_{\|T\|}$:

$$H_T(k) = |u_k|, \quad \text{for } 0 \leq k \leq \|T\|. \quad (3.1)$$

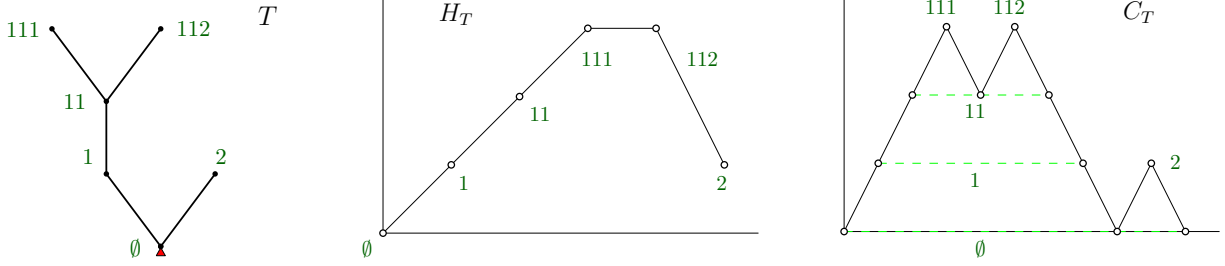


Figure 10: From left to right: a rooted planar tree T , and the corresponding height process and contour process.

Here is a classical result (see e.g. [30, 52]):

Lemma 3.3. *For any n , the map which to a tree associates its height sequence,*

$$\Phi_n^{T \rightarrow H} : \begin{array}{l} \mathbb{T}_n \longrightarrow \mathbb{H}_n \\ T \longmapsto H_T \end{array},$$

is a bijection, where $\mathbb{H}_n = \{H(\llbracket 0, n \rrbracket), H_0 = 0, \Delta H_i \leq 1 \text{ and } H_i > 0 \text{ for } i \geq 1\}$.

An example of a tree and corresponding height sequence is shown in Fig. 10. The reverse bijection

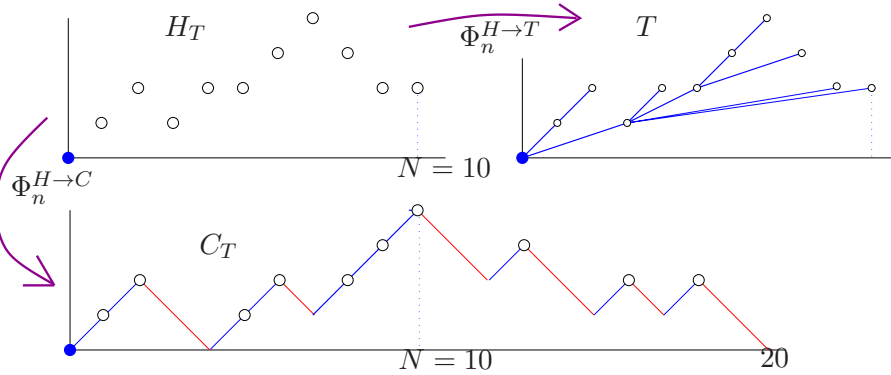


Figure 11: Illustration of the map $\Phi_n^{H \rightarrow T}$, and of the map $\Phi_n^{H \rightarrow C}$; the sequence of successive final heights of increasing steps in the contour process is exactly the height process.

$\Phi_n^{H \rightarrow T} : \mathbb{H}_n \rightarrow \mathbb{T}_n$ allows constructing a tree from its height sequence:

$$\Phi_n^{H \rightarrow T} = (\Phi_n^{T \rightarrow H})^{-1}. \quad (3.2)$$

It will play an important role (see Fig. 11): take any sequence $H(\llbracket 0, n \rrbracket)$ in \mathbb{H}_n , and draw the points $z_k = (k, H_k)$ for $k \in \llbracket 0, n \rrbracket$ in the plane: for any $1 \leq k \leq n$, draw the segments $[z_k, z_{\rho(k)}]$ with $\rho(k) = \max\{j < k, H_j = H_k - 1\}$. An example is given in Fig. 11 for the map $\Phi_n^{H \rightarrow T}$.

Contour sequence. The *depth first traversal* of T is a function

$$c_T : \llbracket 0, 2\|T\| \rrbracket \rightarrow T,$$

defined as follows: first $c_T(0) = \emptyset$. Assume that the image of $\llbracket 0, j \rrbracket$ has been defined for some $0 \leq j < 2\|T\|$, two cases arise:

- if $c_T(j)$ has some non-visited children, that is some children not in $c_T(\llbracket 0, j \rrbracket)$, then $c_T(j+1)$ is the smallest of these children for the lex. order,
- if all the children of $c_T(j)$ have been visited, then $c_T(j+1)$ is the parent of $c_T(j)$.

Definition 3.4. *The contour sequence $C_T(\llbracket 0, 2\|T\| \rrbracket)$ of T is the successive heights of the nodes of T when turning around clockwise (see Fig. 10 (iii)):*

$$C_T(k) = |c_T(k)|, \text{ for } k \in \llbracket 0, 2\|T\| \rrbracket. \quad (3.3)$$

For any $k \in \llbracket 0, 2\|T\| - 1 \rrbracket$, the pair $(c_T(k), c_T(k+1))$ is an edge of T . If T is drawn in the plane, it is suitable to consider that c_T is a walk around the tree, and that for any $0 \leq k \leq 2\|T\|$,

$$\left(c_T(k-1 \bmod 2\|T\|), c_T(k), c_T(k+1 \bmod 2\|T\|) \right) \quad (3.4)$$

is the k th *corner* of the tree. We will call this corner $c_T(k)$ for simplicity.

The following result is a classical result in combinatorics (see e.g. [52]).

Lemma 3.5. *For any $n \geq 1$, the map which to a tree associates its contour sequence,*

$$\begin{aligned} \Phi_n^{T \rightarrow C} : \mathbb{T}_n &\longrightarrow \text{Dyck}_{2n} \\ T &\longmapsto C_T \end{aligned}, \quad (3.5)$$

is a bijection, where Dyck_{2n} is the set of Dyck paths with $2n$ steps:

$$\text{Dyck}_{2n} = \{S(\llbracket 0, 2n \rrbracket), S_0 = S_{2n} = 0, \Delta S_i \in \{-1, 1\} \text{ and } S_i \geq 0, \forall i \in \llbracket 1, 2n \rrbracket\}. \quad (3.6)$$

The distance between two nodes $c_T(k)$ and $c_T(k')$ in the tree can be expressed in terms of C_T :

$$d_T(c_T(k), c_T(k')) = C_T(k) + C_T(k') - 2\check{C}_T(k, k'). \quad (3.7)$$

As represented in Fig. 11, there is a direct way to pass from the contour to its height process.

Lemma 3.6. *The map $\Phi_n^{H \rightarrow C} : \mathbb{H}_n \longrightarrow \text{Dyck}_{2n}$, which sends a height process $H\llbracket 0, n \rrbracket$ (of a tree T) to the corresponding contour process (the one of the tree T) is a bijection.*

Proof. The bijection is simple: both processes start at 0, and the contour process ends at 0. The successive values in $H\llbracket 1, n \rrbracket$ correspond to the first visit times of the nodes according to the lexicographical order. These heights are then the successive heights of the contour process $C\llbracket 0, 2n \rrbracket$ at the times t_i such that $C_{t_i} = C_{t_i-1} + 1$, since the contour process increases every time a new node is visited. \square

3.2 Discrete snakes and iteration of discrete snakes

Discrete snakes. Labelings and snakes are defined as in the continuous case.

Definition 3.7. *Let T be a planar tree, and $c_T = (c_T(k), 0 \leq k \leq 2\|T\|)$ be its contour sequence. A labeling of T is a sequence $\ell_T = (\ell_T(k), 0 \leq k \leq 2\|T\|)$ such that*

$$c_T(k) = c_T(k') \Rightarrow \ell_T(k) = \ell_T(k'), \text{ for all } k, k' \in \llbracket 0, 2\|T\| \rrbracket.$$

The tour of the corresponding discrete snake is defined as (C_T, L_T) , where

$$\begin{cases} C_T(k) &= |c_T(k)|, & 0 \leq k \leq 2\|T\|, \\ L_T(k) &= \ell(c_T(k)), & 0 \leq k \leq 2\|T\|, \end{cases}$$

meaning that $L_T(k) = \ell(c_T(k))$ is the label of the k th node visited by the depth first traversal.

Let μ be a probability distribution on \mathbb{R} , and let T be a given planar tree. The **standard branching random walk** with underlying tree T and μ -distributed spatial increments is defined as follows: consider $(\mathbf{Y}_u, u \in T \setminus \{\emptyset\})$ a family of i.i.d. random variables with common distribution μ and set $\mathbf{Y}_\emptyset = 0$. Now, consider the ‘‘spatial’’ labeling $\ell = (\ell(u), u \in T)$ of T defined by

$$\ell(u) = \sum_{v \preceq u} \mathbf{Y}_v, \quad \text{for any } u \in T, \quad (3.8)$$

where the sum is taken on the set of ancestors v of u . Hence, $\ell(\emptyset) = 0$, and along each branch of T the labels evolve as a random walk with increment distribution μ . This definition extends to random trees, by sampling first the underlying tree T at random, and by constructing the branching random walk using spatial increments independent of T .

In the sequel, we will consider only branching random walks with increment distribution

$$\nu := \frac{1}{3} (\delta_1 + \delta_0 + \delta_{-1}), \quad (3.9)$$

so that the child of a vertex with label l has label $l - 1, l$, or $l + 1$ with equal probability. For each branching random walk with underlying tree T and spatial increments ν -distributed, consider the spatial labeling $\ell = (\ell(u), u \in T)$. If T has n edges, the label process \mathbf{L}_T of a labeled tree (T, ℓ) is an element of \mathbb{L}_{2n} where, $\mathbb{L}_N = \{L(\llbracket 0, N \rrbracket)\}$, $L_0 = L_N = 0$ and $\Delta L_j \in \{-1, 0, 1\}$ for any j , for any $N \geq 0$.

The (tour) of the random discrete snake with n edges, is $(C_{\mathbf{T}}, \mathbf{L}_{\mathbf{T}})$ for \mathbf{T} taken uniformly in \mathbb{T}_n .

Discrete conjugation map. We here define a map $\Phi_N^{L \rightarrow H}$ similar to the conjugation map (2.11) in the discrete setting: it sends \mathbb{L}_N onto \mathbb{H}_N . An example is shown in Fig. 12.

Definition 3.8. For each $N \geq 0$, the discrete conjugation map $\Phi_N^{L \rightarrow H}$ is the map

$$\begin{aligned} \Phi_N^{L \rightarrow H} : \quad \mathbb{L}_N &\longrightarrow \mathbb{H}_N \\ L(\llbracket 0, N \rrbracket) &\longmapsto H(\llbracket 0, N \rrbracket) := \Phi_N^{L \rightarrow H}(L(\llbracket 0, N \rrbracket)) \end{aligned} \quad (3.10)$$

where H is defined by $H(0) = 0$ and

$$H(j) = 1 + L(A + j - 1 \bmod N) - L(A), \quad \text{for } j \in \llbracket 1, N \rrbracket, \quad (3.11)$$

where $A = \min \arg \min L(\llbracket 0, N \rrbracket)$ (since $L(0) = L(N)$, necessarily $0 \leq A < N$).

- In the discrete case, height and contour processes are different objects, and cannot be obtained by a simple conjugation of Łukasiewicz walks (which are random walks with increments in $\mathbb{N} \cup \{-1\}$ conditioned to end at -1), since the height process ends at a positive position, and contour processes of discrete trees have only steps ± 1 (see e.g. [52], in which Łukasiewicz walks are called depth first queue processes).
- When $A + j - 1$ passes from $N - 1$ to N , $A + j - 1 \bmod N$ passes from $N - 1$ to 0 . Since $L(N) = L(0) = 0$, there is no bad border effect on the increments $(\Delta H_j, 1 \leq j \leq N)$, which all belong to $\{+1, -1, 0\}$.
- Height processes are not exactly elements of the set $C^0[0, N]$ (of continuous functions on $[0, N]$ starting and ending at 0), so that the nature of $\Phi_N^{L \rightarrow H}$ is a bit different from that of **Conj**. The composition $\Phi_N^{H \rightarrow C} \circ \Phi_N^{L \rightarrow H}$ will be used and is closer in nature to the conjugation map (see e.g. in Prop.5.5).

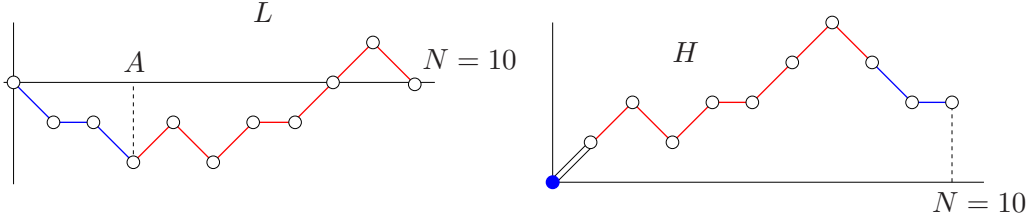


Figure 12: The map $\Phi_{10}^{L \rightarrow H}$.

Iterated discrete snakes. We define *iterated discrete snakes* as we did in the continuous case.

Definition 3.9. For any positive integer D , we call D th random discrete snake of size n , the process

$$\mathbf{BS}_n[D] := \left(\left[\mathbf{C}_n^{(1)}, \mathbf{L}_n^{(1)} \right], \dots, \left[\mathbf{C}_n^{(D)}, \mathbf{L}_n^{(D)} \right] \right), \quad (3.12)$$

such that

- $\mathbf{C}_n^{(1)}$ is uniform in Dyck_{2n} (the contour process of a uniform planar tree with n edges),
- if $\mathbf{C}_n^{(j)}$ is defined for some $j \geq 1$, $\mathbf{L}_n^{(j)}$ is the label process of a branching random walk with increment distribution ν as given in (3.9) with underlying tree $\mathbf{T}_n^{(j)}$, the tree with contour process $\mathbf{C}_n^{(j)}$,
- if for some $j \geq 2$, $\mathbf{L}_n^{(j-1)} \in \mathbb{L}_{2^{j-1}n}$ is known, then the j th tree $\mathbf{T}_n^{(j)}$ is defined by its height process, obtained by conjugation of $\mathbf{L}_n^{(j-1)}$,

$$\mathbf{H}_n^{(j)} := \Phi_{2^{j-1}n}^{L \rightarrow H}(\mathbf{L}_n^{(j-1)}) \in \mathbb{H}_{2^{j-1}n}, \quad (3.13)$$

and its contour process is $\mathbf{C}_n^{(j)} = \Phi_{2^{j-1}n}^{H \rightarrow C}(\mathbf{H}_n^{(j)}) \in \text{Dyck}_{2^j n}$.

Some simulations for the processes of $\mathbf{BS}_{5000}[3]$ are shown in Fig. 13.

- The discrete conjugation map $\Phi_N^{L \rightarrow H}$ is length preserving; its image is the set of height processes of trees with N edges (they are indexed by $\llbracket 0, N \rrbracket$).
- If one observes $L \llbracket 0, N \rrbracket$, the two extremal values L_0 and L_N correspond to the labels of the same corner of the root. This redundancy is suppressed during the construction of $H = \Phi_N^{L \rightarrow H}(L \llbracket 0, N \rrbracket)$: the -1 and $+1$ shifts in (3.11) correspond to the creation of a new vertex (which is drawn in blue in Fig. 12). This size difference is not “a problem”: the standard encoding of quadrangulations with n faces by pairs of trees shares this characteristic [54]. Hence, the construction we propose fits in the case $D = 2$ with the case of rooted pointed quadrangulations. This is the main reason why we use this convention.
- The map $\Phi_N^{H \rightarrow C}$ sends \mathbb{H}_N onto Dyck_{2N} . Dyck paths indexed by $\llbracket 0, 2N \rrbracket$ are used to construct branching random walks whose label processes are indexed by $\llbracket 0, 2N \rrbracket$. Each iteration in the construction of the iterated snake multiplies the number of edges by 2. The number of edges of the tree $\mathbf{T}_n^{(j)}$ is thus

$$N_n^{(j)} = 2^{j-1}n.$$

Normalized versions. Let $\mathbf{BS}_n[D]$ be the D th random discrete snake of size n . We need to fix some normalizing sequences to state the convergence of $\mathbf{C}_n^{(j)}$ and $\mathbf{L}_n^{(j)}$: the normalization is fixed so that $\mathbf{C}_n^{(1)}$, after normalization, converges to the Brownian excursion e . When one iterates, since $\mathbf{C}_n^{(j+1)}$ (at first

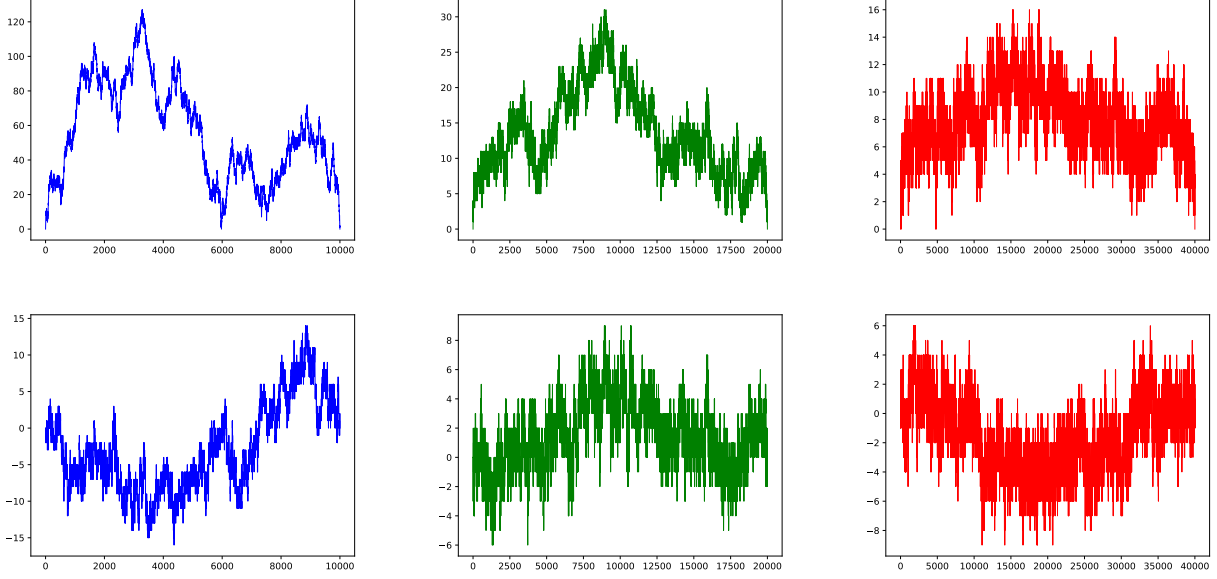


Figure 13: Simulation of $\mathbf{BS}_{5000}[3]$. On the k th column are the processes $\mathbf{C}_{5000}^{(k)}$ and $\mathbf{L}_{5000}^{(k)}$. The range decreases when k increases. The “irregularities” of the processes increase with k .

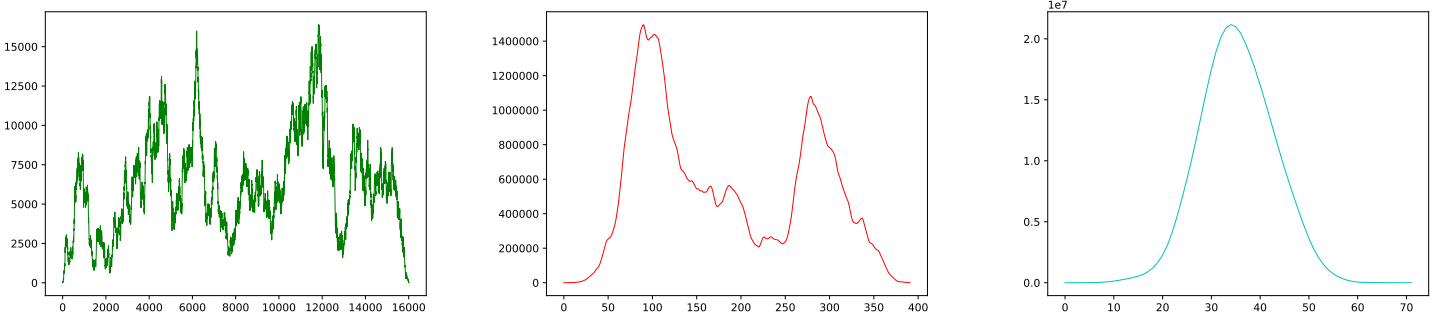


Figure 14: Profiles of the iterated trees 0, 1, 2 starting from a tree with 50×10^6 nodes (the profile $(N_i, i \geq 0)$ is the sequence giving the number of nodes N_i at distance i to the root). Observe how the profiles become smoother and smoother: the first profile converges to the local time of the Brownian excursion (Biane & Yor [6]), which is itself the Brownian excursion up to a change of time. The second one converges to the density of ISE, the integrated super Brownian excursion (see [53, 38] and Chassaing & Louchard [19] for the same result for random rooted quadrangulations with n faces): it is known to be differentiable (Bousquet-Mélou & Janson [11]), but is expected not to possess a second derivative. It appears on this simulation that the following profile is even smoother but it is still random, as can be observed on different simulations; this randomness can be proved for any κ , by just considering that the range of the κ th profile (of the tree \mathbf{T}_n^κ) corresponds to the range of the process $\mathbf{L}_n^{(\kappa-1)}$ which is random.

order) is obtained by conjugation of $\mathbf{L}_n^{(j)}$, the normalization of $\mathbf{C}_n^{(j+1)}$ will be the same as that of $\mathbf{L}_n^{(j)}$. Hence, if $\alpha_n^{(j)}$ and $\beta_n^{(j)}$ are the natural normalizations of $\mathbf{C}_n^{(j)}$ and $\mathbf{L}_n^{(j)}$, we have $\alpha_n^{(1)} = \sqrt{2n}$, and for $j \geq 1$,

$$\beta_n^{(j)} = \sqrt{(2/3)\alpha_n^{(j)}}, \quad \alpha_n^{(j+1)} = \beta_n^{(j)},$$

which gives

$$\alpha_n^{(j)} = (2n)^{1/2^j} (2/3)^{1-1/2^{j-1}}, \quad \beta_n^{(j)} = (2n)^{1/2^{j+1}} (2/3)^{1-1/2^j}.$$

We then fix

$$\begin{cases} \mathbf{c}_n^{(j)}(t) &= \mathbf{C}_n^{(j)}(2^j nt) / \alpha_n^{(j)}, \\ \boldsymbol{\ell}_n^{(j)}(t) &= \mathbf{L}_n^{(j)}(2^j nt) / \beta_n^{(j)}. \end{cases} \quad (3.14)$$

The main important feature, is the order $n^{1/2^j}$ of (the normalization of) $\mathbf{C}_n^{(j)}$ and $n^{1/2^{j+1}}$ of $\mathbf{L}_n^{(j)}$.

We call *normalized D th random discrete snake* the process

$$\mathbf{bs}_n[D] := \left(\left[\mathbf{c}_n^{(1)}, \boldsymbol{\ell}_n^{(1)} \right], \dots, \left[\mathbf{c}_n^{(D)}, \boldsymbol{\ell}_n^{(D)} \right] \right). \quad (3.15)$$

Each of these processes $\mathbf{C}_n^{(j)}$ and $\mathbf{L}_n^{(j)}$ is considered as being interpolated between discrete points, and then $\mathbf{bs}_n[D]$ belongs to $\overrightarrow{\text{Snakes}}^D$. The corner set of $\mathbf{c}_n^{(j)}$ is $\text{AS}_{n,j} = \left\{ k / (2N_n^{(j)}), 0 \leq k \leq 2N_n^{(j)} \right\}$. The natural corner measure $\lambda_n^{(j)}$ on the normalized tree (denoted by $\mathbf{t}_n^{(j)}$) encoded by $\mathbf{c}_n^{(j)}$ is

$$\lambda_n^{(j)} = \frac{1}{2N_n^{(j)}} \sum_{x \in \text{AS}_{n,j}} \delta_x, \quad (3.16)$$

which is the uniform measure on the (normalized) corner set of the j th tree. The measured version of the normalized D th random discrete snake is $\mathbf{bs}_n[D] := \left(\left[\mathbf{c}_n^{(i)}, \boldsymbol{\ell}_n^{(i)}, \lambda_n^{(j)} \right], 1 \leq j \leq n \right)$.

3.3 Feuilletages as measured spaces

We now give the definition of a feuilletage associated to a D -snake. This generalizes Def. 2.14 for measured (non-necessarily random) objects.

Definition 3.10. *The D -feuilletage associated to a D -snake $E_D = ([f^{(j)}, w^{(j)}, \mu^{(j)}], 1 \leq j \leq D)$ is the space denoted by $\text{Feuilletage}(E_D) = [0, 1] / \sim_D$, where \sim_D is the coarsest equivalence relation on $[0, 1]$ refining all the following equivalence relations $\sim_{[m]}$ for $1 \leq m \leq D$, defined for $x, y \in [0, 1]$ by*

$$x \sim_{[m]} y \Leftrightarrow \left[D_{f^{(m)}} \left(x - A^{(m)} \bmod 1, y - A^{(m)} \bmod 1 \right) = 0 \text{ for } x, y \in \text{Support}(\mu^{(m)}) \right], \quad (3.17)$$

where $A^{(m)} = \min \text{argmin } w^{(1)} + \dots + \min \text{argmin } w^{(m-1)}$, in other words, only the points of $[0, 1]$ encoding some corners for $f^{(m)}$ will be identified under condition (3.17).

Remark 3.11. *The D th random feuilletage as defined in Def. 2.14) satisfies the identity*

$$\mathbf{r}[D] \stackrel{(d)}{=} \text{Feuilletage} \left([\mathbf{h}^{(i)}, \boldsymbol{\ell}^{(i)}, \lambda], 1 \leq i \leq D \right). \quad (3.18)$$

3.3.1 Discrete iterated feuilletages (before normalization)

The construction of the D th discrete feuilletage uses the D th discrete snake. It can be viewed as a procedure “for gluing” the $N_n^{(j)} + 1$ nodes of the j th tree in the $N_n^{(j)} = 2N_n^{(j-1)}$ corners of the $(j - 1)$ th tree, one node per corner, except for the root which is not glued. The nodes of the j th tree glued in different corners of the same node of the $(j - 1)$ th tree are identified. This point of view is detailed in Sec. 6. In the sequel, for $\mathbf{BS}_n[D]$ the D th random discrete snake, we will set

$$\mathbf{a}_n^{(j)} := \min \text{argmin } \mathbf{L}_n^{(j)}, \quad \text{for any } 1 \leq j \leq D. \quad (3.19)$$

The discrete conjugation map $\Phi_{N_n^{(j+1)}}^{L \rightarrow H}$ sends the label of the corner c in $\mathbf{L}_n^{(j)}$ (the index of the corner of the tree $\mathbf{T}_n^{(j)}$ on which $\mathbf{L}_n^{(j)}$ has been defined) to the index $\mathbf{a}_n^{(j)} + c - 1$ of $\mathbf{H}_n^{(j+1)}$, in other words, the $(\mathbf{a}_n^{(j)} + c - 1 \bmod N_n^{(j+1)})$ th node of the tree $\mathbf{T}_n^{(j+1)}$ encoded by $\mathbf{H}_n^{(j+1)}$. Hence, in the tree $\mathbf{T}_n^{(j+1)}$, the c th node, for $c \geq 1$ corresponds to the corner $\mathbf{a}_n^{(j)} + c - 1 \bmod N_n^{(j+1)}$ of $\mathbf{L}_n^{(j)}$. Again, when we deal with iterated identifications as we will do, it is easier to use as reference, the index set of the initial tree $\mathbf{T}_n^{(1)}$.

Definition 3.12. Let $D \geq 1$ and $n \geq 1$ be two fixed parameters, and $\mathbf{BS}_n[D] = ([\mathbf{C}_n^{(j)}, \mathbf{L}_n^{(j)}], 1 \leq j \leq D)$ a D th random discrete snake. Let $\mathbf{T}_n^{(j)}$ be the random planar tree with contour process $\mathbf{C}_n^{(j)}$ (is number of edges of $\mathbf{T}_n^{(j)}$ is $N_n^{(j)} = 2^{j-1}n$). We call D th random discrete feuilletage $\mathbf{R}_n[D]$ of size n the graph obtained by identification of the nodes of the $\mathbf{T}_n^{(j)}$'s, for $j = 1, \dots, D$ as follows. For all $j \geq 2$, for all pairs $(c, c') \in \llbracket 1, N_n^{(j)} \rrbracket^2$ such that the corners c and c' of $\mathbf{T}_n^{(j-1)}$ are both corners of the same node, identify in $\mathbf{T}_n^{(j)}$ the two nodes $\mathbf{a}_n^{(j-1)} + c - 1 \bmod N_n^{(j)}$ and $\mathbf{a}_n^{(j-1)} + c' - 1 \bmod N_n^{(j)}$.

- The edges of $\mathbf{R}_n[D]$ coincide with those of $\mathbf{T}_n^{(D)}$, so that the number of edges of $\mathbf{R}_n[D]$ is $N_n^{(D)} = 2^{D-1}n$.
- The vertex set of $\mathbf{R}_n[D]$ coincides with that of $\mathbf{T}_n^{(1)}$ union the set $\{r_n^{(j)}, 2 \leq j \leq D\}$ where $r_n^{(j)}$ is the root of $\mathbf{T}_n^{(j)}$, since as discussed above the definition, the root of $\mathbf{T}_n^{(j)}$ is new and won't be identified with any node of the $\cup_{m \leq j-1} \mathbf{T}_n^{(m)}$. The number of vertices of $\mathbf{R}_n[D]$ is thus $n + D$.

It remains to normalize this object and to represent it using the map Feuilletage.

3.3.2 Normalized discrete feuilletage as foldings of normalized discrete snakes

Recall the definition of the normalizations (3.14).

Definition 3.13. We call D th normalized random discrete feuilletage the space

$$\mathbf{r}_n[D] = \text{Feuilletage} \left(\left[\mathbf{c}_n^{(j)}, \ell_n^{(j)}, \lambda_n^{(j)} \right], \text{ for } j \in \llbracket 1, D \rrbracket \right). \quad (3.20)$$

Two important remarks have to be done:

Remark 3.14. The tree $\mathbf{T}_n^{(j)}$ encoded by the contour $\mathbf{C}_n^{(j)}$ defined in Definition 3.12 must be submitted to a rescaling of order $1/n^{1/2^j}$ to converge, so that the scaling depends strongly on j . A priori, this fact makes unnatural the distance $d^{(1)}$ for which walking on $\mathbf{t}^{(j)}$ has a cost “independent from j ”, when these trees appear as limits of the $\mathbf{T}_n^{(j)}$ after much different normalizations. The distance $d^{(2)}$ does not have this flaw, since all trees $\mathbf{T}_n^{(j)}$ for $j < D$ are used to make identifications, to somehow create shortcuts: their inner distances are not really used.

Remark 3.15. It is possible to equip the space of feuilletages with the induced distance between the measured snakes encoding them, or with the (Prohorov)-Gromov-Hausdorff distance between isometry classes of compact metric spaces, and try to prove that, for one of these metrics, $\mathbf{r}_n[D] \xrightarrow[n]{(d)} \mathbf{r}[D]$, which is reasonable guess for both topologies. However we are not able to prove it for the moment. We will prove the convergence for a pointed version instead, for a metric defined between corresponding pointed snakes.

4 Pointed variants

The previous constructions, discrete or continuous have the disadvantage of relying on Conj and its discrete analogue $\Phi^{H \rightarrow C} \circ \Phi^{L \rightarrow H}$, which is not continuous, as said previously. The discontinuity of Conj

comes from the following situation: consider $f \in C^0[0, 1]$ such that

$$0 < a = \min \operatorname{argmin} f < b = \max \operatorname{argmin} f < 1.$$

It is easy to construct a sequence of functions (f_n) in $C^0[0, 1]$ such that $\|f_n - f\|_\infty \rightarrow 0$, but $\min \operatorname{argmin} f_n \rightarrow b$, or or even, that does not have a nor b as an accumulation point if $\# \operatorname{argmin} f > 2$. This implies that $[\|f_n - f\|_\infty \rightarrow 0] \not\Rightarrow [(f_n, \operatorname{Conj}(f_n)) \rightarrow (f, \operatorname{Conj}(f))]$. Even if it is known that $(f_n, \operatorname{Conj}(f_n))$ converges uniformly to (f, g) , we cannot deduce that $g = \operatorname{Conj}(f)$. The minimal property which would remove this problem would be the (unavailable) proof of the a.s. uniqueness of $\operatorname{argmin} \ell^{(j)}$ for all j 's.

The strategy we adopt instead is to use the fact that $\operatorname{Conj}(f_n)$ and f_n seen as 1-periodic functions are equal up to a change of origin: instead of $(C[0, 1], \|\cdot\|_\infty)$, we will use a topology which allows us to identify functions that are equal up to the change of origin. The chance is that this topology coincides with the right topology on pointed trees, that is, which makes equivalent two rooted trees, rooted at corners of the same root vertex. This permits to still get an intuitive understanding of the phenomenon into play.

4.1 Pointed real trees

By definition, a tree $T_g = [0, 1] / \sim_g$ is rooted at a corner. The different corners of the root are $g^{-1}(0) \setminus \{1\}$. We call *pointed tree* an equivalence class of trees rooted at the different corners of the same root vertex. For any $a \in [0, 1]$, formally define the a -shift Ψ_a as the following map defined on $C^0[0, 1]$:

$$\begin{aligned} \Psi_a : C^0[0, 1] &\longrightarrow C^0[0, 1] \\ g &\longmapsto x \mapsto \Psi_a(g)(x) = g(x + a \bmod 1) - g(a), \end{aligned} \tag{4.1}$$

which is a conjugation map, meaning the exchange of “two sections” of the graph of g , but starting from a given corner a instead of $\min \operatorname{argmin} g$. Of course, for any $f \in C^0[0, 1]$,

$$\operatorname{Conj}(f) = \Psi_{\min \operatorname{argmin}(f)}(f). \tag{4.2}$$

Introduce the following equivalence relation on $C^0[0, 1]$: we say that $f \sim_\Psi g$ if there exists $a \in [0, 1]$ such that $g = \Psi_a(f)$. Denote the quotient space by

$$QC^0[0, 1] = C^0[0, 1] / \sim_\Psi, \tag{4.3}$$

and denote by \bar{f} the class of a function f .

Remark 4.1. *A measure component can be added: Ψ_a then acts on $C^0[0, 1] \times \mathcal{M}([0, 1])$ with values in $C^0[0, 1] \times \mathcal{M}([0, 1])$, with (a slight abuse of language): $\Psi_a(f, \mu) = (f', \mu')$ for $f' = \Psi_a(f)$ as defined above and $\mu'(\cdot) = \mu(\cdot + a \bmod 1)$.*

The following proposition is proven in Sec. 5.3.

Proposition 4.2. *The map $D_\Psi : QC^0[0, 1]^2 \rightarrow \mathbb{R}^+$ defined by*

$$D_\Psi(\bar{g}_1, \bar{g}_2) = \inf_a \|g_1 - \Psi_a(g_2)\|_\infty \tag{4.4}$$

is a distance on $QC^0[0, 1]$ and equipped with this distance, $QC^0[0, 1]$ is a Polish space (for the measured extension, add the Prohorov distance between the measure components).

For any $f \in C^0[0, 1]$ denote by

$$\text{PointedTreeClass}(f) = \{\Psi_a(f), a \in \text{argmin}(f)\}. \quad (4.5)$$

The following result is immediate:

Lemma 4.3. *Two functions f and g in $C^+[0, 1]$ are in the same tree class (that is, $\text{PointedTreeClass}(f) = \text{PointedTreeClass}(g)$) iff there exists $a \in [0, 1]$ such that*

$$f(\cdot) = g(\cdot + a \bmod 1), \quad (4.6)$$

in which case

$$a \in \text{argmin}(g) = g^{-1}(0). \quad (4.7)$$

Hence, two rooted trees are in the same pointed tree class if each tree can be obtained from the other by a rerooting at a corner of the root vertex (for the measured version, the push-forward measure of the root corner of μ_f is required to coincide with that of μ_g , as explained in Rem. 2.8).

The next lemma is a direct consequence of Proposition 4.2 and its proof.

Lemma 4.4. *The map D_Ψ is a distance on the set of pointed tree classes (with a simple adaptation for measured pointed tree classes).*

Definition 4.5. *A sequence of pointed trees with classes of contour processes \bar{c}_n is said to converge to \bar{c} if $D_\Psi(\bar{c}_n, \bar{c}) \rightarrow 0$.*

The following proposition will be proven in Sec. 5.4.

Proposition 4.6. *Let $\mathbf{y}, \mathbf{y}_0, \mathbf{y}_1, \dots$ be a sequence of processes taking their values in $C^0[0, 1]$ such that $\mathbf{y}_n \xrightarrow[n]{(d)} \mathbf{y}$ in $(C[0, 1], \|\cdot\|_\infty)$.*

(i) *The sequence $(\text{Conj}(\mathbf{y}_n), n \geq 0)$ is tight in $C[0, 1]$. The limits of the converging subsequences (in $C[0, 1]$) are all in the tree class of $\text{Conj}(\mathbf{y})$.*

(ii) *If a.s. $\#\text{argmin } \mathbf{y} = 1$, then $\text{Conj}(\mathbf{y}_n) \xrightarrow[n]{(d)} \text{Conj}(\mathbf{y})$ in $C[0, 1]$.*

(iii) *$\bar{\mathbf{y}}_n \xrightarrow[n]{(d)} \bar{\mathbf{y}}$ in $QC^0[0, 1]$.*

(iv) *$\overline{\text{Conj}}(\mathbf{y}_n) \xrightarrow[n]{(d)} \overline{\text{Conj}}(\mathbf{y})$ in $QC^0[0, 1]$.*

4.2 Pointed snakes

We extend the notion of pointed trees to pointed snakes:

Definition 4.7. *Two snakes (f_1, w_1) and (f_2, w_2) taken in $\overrightarrow{\text{Snakes}}$ are said to be the same pointed snake if there exists $a \in [0, 1]$ such that,*

$$(f_1(x), w_1(x)) = (f_2(x + a \bmod 1), w_2(x + a \bmod 1)), \quad \text{for all } x \in [0, 1]. \quad (4.8)$$

Again, as for the case of pointed trees (Def. 4.3), if (4.8) holds, then $a \in \operatorname{argmin}(f_2) = f_2^{-1}(0)$ (notice that $w_2(a) = 0$ as a is a corner of the root, so that $\Psi_a(w_2)(x) = w_2(x + a \bmod 1)$). “To be the same pointed snake” is an equivalence relation \sim_\bullet . The state space of pointed snakes is

$$\text{Snakes}^\bullet = \overrightarrow{\text{Snakes}} / \sim_\bullet. \quad (4.9)$$

Denote by π_\bullet the canonical projection from $\overrightarrow{\text{Snakes}}$ to Snakes^\bullet . A distance on Snakes^\bullet is given by the following extension of D_Ψ (we keep the same notation):

$$D_\Psi((f_1, w_1), (f_2, w_2)) = \inf_a (\|\Psi_a(f_1) - f_2\|_\infty + \|\Psi_a(w_1) - w_2\|_\infty). \quad (4.10)$$

We call D -pointed snake an element of $\text{Snakes}^{\bullet, D} := (\text{Snakes}^\bullet)^D$. We equip this set with the distance

$$D_{\Psi, D} \left[([f_j, w_j], 1 \leq j \leq D), ([f'_j, w'_j], 1 \leq j \leq D) \right] = \sum_{j=1}^D D_\Psi((f_j, w_j), (f'_j, w'_j)). \quad (4.11)$$

Definition 4.8. An element $[(f_i, w_i), 1 \leq i \leq D]$ of $\overrightarrow{\text{Snakes}}^D$ is said to be consistent if $f_{i+1} \sim_\Psi w_i$ for every $1 \leq i \leq D - 1$.

Again this extends to measured pointed snakes: two measured (rooted) snakes (f, w, μ) and (f', w', μ') are said to be in the same measured pointed snake class, if for some $a \in \operatorname{argmin} f$,

$$(f(a + x \bmod 1), w(a + x \bmod 1), \mu(a + \cdot \bmod 1)) = (f'(x), w'(x), \mu'), \quad \text{for all } x \in [0, 1]. \quad (4.12)$$

We extend the projection π_\bullet to measured (rooted) snakes. The set of measured pointed snakes obtained in such a way can be equipped with a metric which extends $D_{\Psi, D}$ by adding the Prokhorov distance between the corner measures of the trees.

Branching random walks and pointed snakes Consider a branching random walk with underlying tree a rooted tree T (recall Section 3.2), defined using some spatial increments placed on the vertices of T different from the root. A change of root corner preserving the spatial increments amounts to shifting the encoding processes of this labeled tree as follows. Let (C_T, L_T) be the tour of a discrete snake with underlying rooted tree T , where $C_T = (|c_t(k)|, 0 \leq k \leq 2\|T\|)$ is the contour process of T , and let

$$R = (\operatorname{argmin} C_T) \setminus \{2\|T\|\}$$

be the set of corners of the root vertex of T . For each element $a \in R$, call $T(a)$ the tree T rerooted at its corner a . Since $C_{T(a)} = L_{T(a)} = 0$, the tour of the discrete snake now indexed by $T(a)$ is simply,

$$\begin{cases} C_{T(a)}(k) &= C_T(a + k \bmod 2\|T\|), \\ L_{T(a)}(k) &= L_T(a + k \bmod 2\|T\|) \text{ for } 0 \leq k \leq 2\|T\|. \end{cases} \quad (4.13)$$

The set of discrete snakes $((C_{T(a)}, L_{T(a)}), a \in R)$ forms a discrete (non-normalized) pointed snake.

Remark 4.9. The normalized processes $\mathbf{c}_n^{(j)}, \mathbf{\ell}_n^{(j)}$ were defined in (3.14). $\mathbf{h}_n^{(j+1)}$ is obtained by conjugating $\mathbf{\ell}_n^{(j)}$. We do not impose the consistence condition (Def. 4.8) in the initial definition of D th pointed snakes, because in the discrete case, the processes $\mathbf{c}_n^{(j+1)}$ and $\mathbf{\ell}_n^{(j)}$ are not in general in the same class modulo \sim_Ψ (while the continuous D th Brownian snake is consistent).

Definition 4.10. We call D th pointed Brownian snake a process taking its values in $\text{Snakes}^{\bullet, D}$,

$$\mathbf{bs}^{\bullet}[D] := \left(\pi_{\bullet} \left[\mathbf{h}^{(j)}, \ell^{(j)} \right], 1 \leq j \leq D \right), \quad (4.14)$$

where $\mathbf{bs}[D] = \left(\left[\mathbf{h}^{(j)}, \ell^{(j)} \right], 1 \leq j \leq D \right)$ is a D th Brownian snake. For the measured version, the measure of each tree is λ .

The D th (normalized) random pointed discrete snake is

$$\mathbf{bs}_n^{\bullet}[D] := \left(\pi_{\bullet} \left[\mathbf{c}_n^{(j)}, \ell_n^{(j)} \right], 1 \leq j \leq D \right). \quad (4.15)$$

For the measured version, the (root-invariant) measure of the j th tree is $\lambda_n^{(j)}$ as defined in (3.16).

Also, we only define pointed random discrete objects for the normalized versions, so that we will drop the “normalized” in the name of the objects to avoid lengthy names.

D th random pointed discrete snake and convergence

Theorem 4.11. The sequence of (normalized) D th random pointed discrete snake converges in distribution towards the D th pointed Brownian snake:

$$\mathbf{bs}_n^{\bullet}[D] \xrightarrow[n]{(d)} \mathbf{bs}^{\bullet}[D] \text{ in } (\text{Snakes}^{\bullet, D}, D_{\Psi, D}).$$

The result extends to the measured version.

This will be proven in Sec. 5.5.

Remark 4.12. Theorem 4.11 may seem somewhat weaker than it is: recall Section 2.4.2 about the synchronization of trees. Since the contour $\mathbf{h}^{(1)}$ of the rooted tree $\mathbf{t}^{(1)}$ is a Brownian excursion, and thus a.s. reaches 0 only at 0 and 1, the pointed tree class of $\mathbf{t}^{(1)}$ possesses a unique rooted tree. Again, using Section 2.4.2, a.s. there is a single well-defined way to synchronize the trees $\mathbf{t}^{(1)}, \dots, \mathbf{t}^{(D)}$ since none of the $\mathbf{h}^{(j)}$ are periodic (see the argument of the third point of Theorem 5.7). However, the synchronization does not apply to the discrete snakes $\mathbf{bs}_n^{\bullet}[D]$, at least not directly, because $\mathbf{c}_n^{(j)}$ is not obtained from $\ell_n^{(j-1)}$ by a simple change of origin (as explained in Rem. 4.9).

Remark 4.13. Theorem 4.11 implies that the right scale for the tree encoded by $\mathbf{h}_n^{(D)}$ has to be $n^{1/2^D}$, since normalized by this quantity, it converges (as a pointed tree) to a non-trivial limit.

We are not able to write an analogue of Theorem 4.11 for *rooted* iterated snakes as said several times, because of the non-continuity of Conj: pointed iterated snakes are simpler and it is the case even for $\mathbf{bs}^{\bullet}[2]$, which appears in relation with the rooted-pointed quadrangulations.

4.3 Convergence of random pointed iterated discrete feuilletages

We introduce the notions of iterated random feuilletages, rooted and pointed, discrete and continuous. All these objects are encoded by iterated snakes (rooted and pointed), in which some corner measures encode the positions of the nodes. In this subsection, we define a topology for which the convergence of discrete pointed snakes extends to the convergence of pointed discrete random feuilletages.

Definition 4.14. Consider a measured pointed D -snake $E_D^\bullet = (\pi_\bullet[f^{(i)}, w^{(i)}, \mu^{(i)}], 1 \leq i \leq D)$ in $\text{Snakes}^{\bullet, D}$. We call pointed feuilletage associated with E_D^\bullet , denoted by $\text{Feuilletage}(E_D^\bullet)$, the set of (rooted) feuilletages $\{\text{Feuilletage}(E_D) \text{ for } E_D \in E_D^\bullet\}$ associated with the (rooted) snakes in the class of E_D^\bullet .

Recall the Definitions 2.14 and 3.13 of $\mathbf{r}[D]$ and of $\mathbf{r}_n[D]$ defined using $\mathbf{bs}[D]$ and $\mathbf{bs}_n[D]$.

Definition 4.15. We call:

■ (measured) D th pointed random feuilletage, the space

$$\mathbf{r}^\bullet[D] = \text{Feuilletage}\left(\pi_\bullet\left[\mathbf{h}^{(i)}, \ell^{(i)}, \lambda\right], 1 \leq i \leq D\right),$$

■ (measured normalized) D th pointed random discrete feuilletage, the space

$$\mathbf{r}_n^\bullet[D] = \text{Feuilletage}\left(\pi_\bullet\left[\mathbf{c}_n^{(i)}, \ell_n^{(i)}, \lambda_n^{(i)}\right], 1 \leq i \leq D\right).$$

For this topology, the convergence of discrete pointed snakes (Theorem 4.11) extends immediately to the convergence of pointed discrete random feuilletages:

Theorem 4.16. For any $D \geq 1$,

$$\mathbf{r}_n^\bullet[D] \xrightarrow[n]{(d)} \mathbf{r}^\bullet[D] \quad (4.16)$$

for the topology induced by D_Ψ .

However, this convergence does not imply the convergence for the Gromov-Hausdorff distance. In terms of feuilletages it means in fact that their codings converge; we are not able at this point to deduce from this fact even the non triviality of $\mathbf{r}^\bullet[D]$ for $D \geq 3$.

Remark 4.17. Theorem 4.16 implies that the sequence of diameters of $\mathbf{r}_n^\bullet[D]$ (seen as graphs with edge-lengths) is tight, so that, without edge normalization, an upper bound on the scale of the D th random discrete feuilletage is $n^{1/2^D}$ (we expect $n^{1/2^D}$ to be the right normalization).

4.4 A variant for the feuilletage

We present here a variant which has the property to provide a feuilletage which scales clearly in $n^{1/2^D}$. However, we will just sketch this alternative construction, as it seems to us that it breaks the invariance by change of pointed vertex; moreover the representations of the trees $T_n^{(D)}$ using linear processes and the study of their asymptotics seems less tractable.

In the construction detailed in this paper, the tree $T_n^{(3)}$ is first constructed using the label process $L_n^{(2)}$ of the tree $T_n^{(2)}$. Two nodes of $T_n^{(3)}$ are subject to two identifications: gluing the nodes of $T_n^{(3)}$ in the corners of $T_n^{(2)}$ forms a quadrangulation $Q_n^{(2,3)}$, and the labels $\mathbf{L}_n^{(3)}$ can be used to recover the distances in this quadrangulation: hence the right scale of $Q_n^{(2,3)}$ is $n^{1/8}$. However, the corners of $T_n^{(2)}$ that are identified using $T_n^{(1)}$ may have different labels (for $L_n^{(2)}$): the identification by $T_n^{(1)}$ produces some shortcuts in $Q_n^{(2,3)}$; it is not easy to prove the scale $n^{1/8}$ for $R_n[3]$.

We propose the following variant: we can adapt the labeling of $T_n^{(2)}$, so that when identifying the vertices of the variant tree $T_n^{(3),*}$ according to the vertices of $T_n^{(1)}$, only vertices that have the same label are identified. To do this, we can for instance keep in memory for each vertex v of $T_n^{(1)}$, the list $c_1^v, \dots, c_{\deg(v)}^v$

of its corners (in $T_n^{(1)}$). Every vertex of $T_n^{(2)}$ but one corresponds to a corner of $T_n^{(1)}$. Then, to construct the variant labeling of $T_n^{(2)}$, proceed as follows: following the contour sequence, turn around the tree $T_n^{(2)}$; when visiting the k th node u_k of $T_n^{(2)}$, two cases arise:

- If the node u_k is identified by $T_n^{(1)}$ with a node u_j of $T_n^{(2)}$ with smaller index ($j < k$), then label u_k with the label of u_j .
- If the node u_k is not identified by $T_n^{(1)}$ with a node of smaller index, then label this node (in $T_n^{(2)}$) with the label of its father (in $T_n^{(2)}$) in which is added a random variable ν -distributed. By this construction, the labeling attributed to the k first vertices of $T_n^{(2)}$ forms an interval of \mathbb{Z} , from what we can see that still a tree $T_n^{(3),\star}$ can be constructed by adding (in $T_n^{(2)}$) an edge between u_k and the last corner with label that of u_k minus 1).

It allows building a planar map $Q_n^{(2,3),\star}$ when identifying the vertices of $T_n^{(3),\star}$ according to those of $T_n^{(2)}$, but now when identifying the vertices of $Q_n^{(2,3),\star}$ according to those of $T_n^{(1)}$, we identify vertices which are at the same distance to the pointed vertex in $Q_n^{(2,3),\star}$, so that the distances are conserved. We now sample $n+2$ random variables ν distributed to label $\mathbf{T}_n^{(2)}$ instead of $2n+1$ times in the usual construction, however the standard diameter of $\mathbf{T}_n^{(3),\star}$ and of $\mathbf{Q}_n^{(2,3),\star}$, and now of the resulting feuilletage has scale $n^{1/8}$.

A similar construction can be proposed for $R_n[D]$ and $D > 3$: the labeling of $T_n^{(D-1),\star}$ is produced so that the identifications by the trees $(T_n^{(j),\star}, j \leq D-2)$ only identify nodes in $T_n^{(D-1),\star}$ with same label.

5 Proofs

5.1 Some cornerstones of the proofs

We factorize here some ingredients of the proofs of the theorems.

5.1.1 Main arguments to prove the continuity of processes, and of limits of processes

The q -Hölder coefficient of a function $f : [0, 1] \rightarrow \mathbb{R}$ is defined as $\text{Hol}_q(f) := \sup_{0 \leq x < y \leq 1} |f(x) - f(y)|/|x - y|^q$. The function f is said to be q -Hölder for some $q \in (0, 1)$ (shorthand for Hölder with exponent q) if $\text{Hol}_q(f) < +\infty$. Needless to say that a q -Hölder function is continuous.

A process \mathbf{X} is said to be Hölder continuous with exponent q , if a.s. $\text{Hol}_q(\mathbf{X})$ is finite. We recall Kolmogorov's continuity criterion (see e.g. Kallenberg [39, Theo.2.23] or Revuz-Yor [64, Theo.2.1 p.26]):

Theorem 5.1. *Let \mathbf{X} be a process indexed by $[0, 1]$ taking its values in \mathbb{R} ; if for some $a, b > 0$,*

$$\mathbb{E}(|\mathbf{X}_s - \mathbf{X}_t|^a) \leq C|s - t|^{1+b}, \text{ for all } s, t \in [0, 1],$$

then \mathbf{X} has a continuous version, and for any $c \in (0, b/a)$, it is a.s. Hölder continuous with exponent c .

Definition 5.2. *A sequence of processes (\mathbf{X}_n) defined on $[0, 1]$ is q -Hölder tight, if $(\mathbf{X}_n(0))$ is tight, and if, for any $\varepsilon > 0$, there exists $M > 0$ and $N \geq 1$, such that for any $n \geq N$,*

$$\mathbb{P}(\text{Hol}_q(\mathbf{X}_n) \leq M) \geq 1 - \varepsilon. \tag{5.1}$$

Under this condition, if (\mathbf{X}_n) is q -Hölder tight for some $q \in (0, 1]$, then (\mathbf{X}_n) is tight on $C[0, 1]$ (since $\{f : f \in C[0, 1], |f(0)| \leq C, \text{Hol}_q(f) \leq M\}$ is a compact in $C[0, 1]$). Another criterion for tightness is the

famous ‘‘moment condition’’ which is a bit more restrictive: additionally to the tightness of $(\mathbf{X}_n(0))$, the existence of positive constants α, β, γ such that

$$\sup_{n \geq N} \mathbb{E}(|\mathbf{X}_n(x) - \mathbf{X}_n(y)|^\alpha) \leq \gamma|x - y|^\beta, \quad \text{for any } x, y \in [0, 1] \quad (5.2)$$

ensures that for any $c \in (0, \beta/\alpha)$, any limit of any converging subsequence of the sequence (\mathbf{X}_n) is a.s. Hölder continuous with exponent c (Kalleberg [39, Cor.14.9]).

5.1.2 Height and contour processes are asymptotically indistinguishable

The next theorem is folklore (see [52] for the case of critical Galton-Watson trees). For the sake of completeness we provide a proof in the degree of generality needed here.

Theorem 5.3. *Let $(r(n))$ be a sequence in $(0, +\infty)$, such that $r(n) \rightarrow +\infty$ and $r(n) = o(n)$. For any n , let μ_n be a distribution on \mathbb{T}_n , the set of trees with n edges. Denote by \mathbf{h}_n and \mathbf{c}_n the normalized height and contour processes of \mathbb{T}_n picked according to μ_n , and defined by*

$$\begin{cases} \mathbf{h}_n(x) & := H_{\mathbb{T}_n}(nx)/r(n), \quad \text{for } x \in [0, 1], \\ \mathbf{c}_n(x) & := C_{\mathbb{T}_n}(2nx)/r(n), \quad \text{for } x \in [0, 1]. \end{cases} \quad (5.3)$$

If $\mathbf{h}_n \xrightarrow[n]{(d)} \mathbf{h}$ in $(C[0, 1], \|\cdot\|_\infty)$, then $(\mathbf{h}_n, \mathbf{c}_n) \xrightarrow[n]{(d)} (\mathbf{h}, \mathbf{h})$ in $C[0, 1]^2$ (in other words limiting contour and height processes are asymptotically indistinguishable, $\mathbf{c} = \mathbf{h}$).

Proof. Let $u_0, \dots, u_{\|T\|}$ be the nodes of T sorted according to the lexicographical order. Denote by $m_T(k) = \inf\{j, c_T(j) = u_k\}$ the first visit time of u_k by the depth first traversal. A simple induction shows that

$$m_T(k) = 2k - H_T(k), \quad \text{for any } k \in \llbracket 0, \|T\| \rrbracket \quad (5.4)$$

(see [52, Lemma 2]), and for any p , and any $k \in \llbracket m_T(p), m_T(p+1) - 1 \rrbracket$,

$$H_T(p+1) - 1 \leq C_T(k) \leq H_T(p), \quad (5.5)$$

by [52, Lemma 3] (this can be proved by a simple induction again). Hence, for any k ,

$$\begin{cases} 2k - \max H_T \leq m_T(k) \leq 2k, \quad \forall k \in \llbracket 0, \|T\| \rrbracket, \\ m_T(k) \leq 2k \leq m_T(k) + \max H_T \leq m_T(\min\{k + \max H_T, \|T\|\}), \end{cases} \quad (5.6)$$

since $m_T(k+1) \geq m_T(k) + 1$.

Now, defining $\Delta_T := \max_p |H_T(p) - \mathcal{H}_T(p+1)| + 1$, from (5.6) and (5.5),

$$\sup_{p \in \llbracket 0, \|T\| \rrbracket} \{|C_T(m_T(p)) - C_T(2p)|\} \leq \Delta_T + w_{\max H_T}(H_T), \quad (5.7)$$

where $w_\delta(f) = \max\{|f(x) - f(y)|, |x - y| \leq \delta\}$ is the modulus of continuity of f . Indeed, $C_T(m_T(p)) = H_T(p)$ and $C_T(2p)$ coincide, up to an additive term bounded by Δ_T , with $H_T(j)$ for j such that $m_T(j) \leq 2p \leq m_T(j+1) - 1$ (by (5.5)). Therefore, by (5.6), $2j - \max H_T \leq 2p \leq 2(j+1) - 1$, so that $j \in \llbracket (2p-1)/2, p + \max H_T/2 \rrbracket = \llbracket p, p + \max H_T/2 \rrbracket$.

Let us now prove that under the hypotheses of the theorem, for a random tree \mathbf{T}_n taken under μ_n ,

$$\sup_p |C_{\mathbf{T}_n}(2p) - H_{\mathbf{T}_n}(p)| / r(n) \xrightarrow{(d)} 0. \quad (5.8)$$

Taking into account the convergence $\mathbf{h}_n \xrightarrow{(d)} \mathbf{h}$, this implies that $\|\mathbf{c}_n - \mathbf{h}_n\|_\infty \xrightarrow{(d)} 0$, and the continuity of \mathbf{c}_n allows concluding. By (5.7), it suffices to prove that $\Delta_{\mathbf{T}_n}/r(n) \xrightarrow{(d)} 0$, and $w_{\max H_{\mathbf{T}_n}}(H_{\mathbf{T}_n})/r(n) \xrightarrow{(d)} 0$.

■ By (5.5), since $\mathbf{h}_n \xrightarrow{(d)} \mathbf{h}$, for any $\varepsilon', \varepsilon > 0$, there exists $\delta > 0$ such that

$$\lim_n \mathbb{P}(w_\delta(\mathbf{h}_n) > \varepsilon) \leq \varepsilon'. \quad (5.9)$$

In particular, for any $\varepsilon > 0$, $\lim_n \mathbb{P}(\Delta_{\mathbf{T}_n} \geq \varepsilon r(n)) \rightarrow 0$.

■ For some $M > 0$, consider the event $E_{n,M} = \{\max H_{\mathbf{T}_n} \leq Mr(n)\}$. Since the normalized height process converges in $(C[0,1], \|\cdot\|_\infty)$, then $r(n)^{-1} \max H_{\mathbf{T}_n}$ is tight, so that for any $\varepsilon > 0$, there exists M such that $\mathbb{P}(E_{n,M}) \geq 1 - \varepsilon$ for n large enough. Hence, for some $a > 0$, write

$$\begin{aligned} \mathbb{P}(w_{\max H_{\mathbf{T}_n}}(H_{\mathbf{T}_n})/r(n) \geq a) &\leq \varepsilon + \mathbb{P}(w_{\max H_{\mathbf{T}_n}}(H_{\mathbf{T}_n})/r(n) \geq a, E_{n,M}) \\ &\leq \varepsilon + \mathbb{P}(w_{Mr(n)}(H_{\mathbf{T}_n})/r(n) \geq a). \end{aligned}$$

Since $w_{Mr(n)}(H_{\mathbf{T}_n})/r(n) = w_{Mr(n)/n}(\mathbf{h}_n)$ and $r(n) = o(n)$ by hypothesis (we have $Mr(n)/n \rightarrow 0$), and by (5.9), $\mathbb{P}(w_{\max H_{\mathbf{T}_n}}(H_{\mathbf{T}_n})/r(n) \geq a) \rightarrow 0$ for any $a > 0$. \square

5.2 Proof of Theorem 2.11

By Kolmogorov's continuity criterion the Brownian motion, is q -Hölder for any $q \in (0, 1/2)$, because $\mathbb{E}(|B_t - B_s|^\gamma) \leq c_\gamma |t - s|^{\gamma/2}$, where $c_\gamma = \mathbb{E}(|B_1|^\gamma) < +\infty$. The Brownian excursion \mathbf{e} is also q -Hölder for any $q \in (0, 1/2)$, since $(\mathbf{e}_t, 0 \leq t \leq 1) \stackrel{(d)}{=} (|B_{d+(g-d)t}| / \sqrt{d-g}, 0 \leq t \leq 1)$, where $g = \sup\{t < 1, B_t = 0\}$ and $d = \inf\{t > 1, B_t = 0\}$. Since $\sqrt{d-g}$ is a.s. finite, the Brownian excursion is q -Hölder.

Now, assume by induction that we have proven that $\mathbf{h}^{(j)}$ is q -Hölder for any $q \in [0, 1/2^j)$ and more precisely that we proved that for any $q \in [0, 1/2^j)$, for any $\varepsilon > 0$, there exists M such that

$$\mathbb{P}(\text{Hol}_q(\mathbf{h}^{(j)}) \geq M) \leq \varepsilon. \quad (5.10)$$

To prove that this property is also true for $j+1$, consider the event $E_{j,q,M} = \{\text{Hol}_q(\mathbf{h}^{(j)}) \leq M\}$.

Since $\ell^{(j)}(x) - \ell^{(j)}(y) \stackrel{(d)}{=} \mathcal{N}(0, D_{\mathbf{h}^j}(x, y))$, for any $a \geq 1$, any $x, y \in [0, 1]$,

$$\mathbb{E} \left(\left| \ell^{(j)}(x) - \ell^{(j)}(y) \right|^a \mid E_{j,q,M} \right) = \mathbb{E}(|N_1|^a) \mathbb{E} \left(D_{\mathbf{h}^j}(x, y)^{a/2} \mid E_{j,q,M} \right) \quad (5.11)$$

$$\leq \mathbb{E}(|N_1|^a) 2^{a/2} M^{a/2} (x-y)^{qa/2} \quad (5.12)$$

$$\leq C(x-y)^{qa/2} \quad (5.13)$$

because $D_{\mathbf{h}^j}(x, y) \leq 2M|x-y|^q$ on $E_{j,q,M}$, as for any u in $[s, t]$, $|\mathbf{h}^j(s) - \mathbf{h}^j(u)| + |\mathbf{h}^j(t) - \mathbf{h}^j(u)| \leq 2M(t-s)^q$. Hence, conditionally on $E_{j,q,M}$, $\ell^{(j)}$ satisfies Kolmogorov's criterion, and it is then $q/2$ -Hölder. It follows readily that for M' large enough, $\mathbb{P}(\text{Hol}_{q/2}(\ell^{(j)}) \geq M') \leq \varepsilon$. Now, if a function f is q -Hölder, so does $\text{Conj}(f)$, from what we conclude. \square

5.3 Proof of Proposition 4.2

First, D_Ψ is a distance on $QC^0[0, 1]$: since D_Ψ is symmetric, in fact the main point is that in (4.4), the inf can be replaced by a min. For this, take a sequence a_n such that $\|g_1 - \Psi_{a_n}(g_2)\|_\infty \rightarrow \inf_a \|g_1 - \Psi_a(g_2)\|_\infty$. Extract from this sequence a converging subsequence (this is possible since (a_n) lives in the compact set $[0, 1]$); let b be the limit of this subsequence ($a_{n_k}, k \geq 0$). Since equicontinuity and pointwise convergence imply uniform convergence (on a compact set), $\|\Psi_{a_{n_k}}(g_2) - \Psi_b(g_2)\|_\infty \rightarrow 0$ (since the functions in $\{\Psi_a(g_2), a \in [0, 1]\}$ are equicontinuous). Therefore, $D_\Psi(\overline{g_1}, \overline{g_2}) = \lim_k \|g_1 - \Psi_{a_{n_k}}(g_2)\|_\infty = \|g_1 - \Psi_b(g_2)\|_\infty$. This shows that the infimum is reached and can be replaced by a min. From here, we see that if $D_\Psi(\overline{f}, \overline{g}) = 0$, then there exists $b \in [0, 1]$ such that $f = \Psi_b(g)$, so that $\overline{f} = \overline{g}$ in $QC^0[0, 1]$. Using this property, the triangular inequality for D_Ψ follows: take b_1, b_2 such that

$$\begin{aligned} D_\Psi(\overline{f_1}, \overline{f_2}) + D_\Psi(\overline{f_2}, \overline{f_3}) &= \|f_1 - \Psi_{b_1}(f_2)\|_\infty + \|f_2 - \Psi_{b_2}(f_3)\|_\infty \\ &= \|f_1 - \Psi_{b_1}(f_2)\|_\infty + \|\Psi_{b_1}(f_2) - \Psi_{b_2+b_1}(f_3)\|_\infty \\ &\geq \|f_1 - \Psi_{b_2+b_1}(f_3)\|_\infty \geq \min_c \|f_1 - \Psi_c(f_3)\|_\infty = D_\Psi(\overline{f_1}, \overline{f_3}), \end{aligned}$$

from what we conclude.

Now, $QC^0[0, 1]$ is Polish: just use the fact that the canonical projection from $C[0, 1]$ to $QC^0[0, 1]$ is 1-Lipschitz: $D_\Psi(\overline{f}, \overline{g}) \leq \|f - g\|_\infty$, so that the image of a countable dense subset of $C[0, 1]$ is countable and dense in $QC^0[0, 1]$. Now take a Cauchy sequence $(\overline{f_n})$ in $QC^0[0, 1]$. Take any sequence (b_n) in $[0, 1]$, and an element f_n in $\overline{f_n}$ for each n . We claim that the sequence $(\Psi_{b_n}(f_n))$ contains a converging subsequence in $C[0, 1]$. By the Arzela-Ascoli theorem it suffices to check that this sequence is bounded and uniformly continuous. First, the sequence $(\|\Psi_{b_n}(f_n)\|_\infty)$ is bounded (because $\text{Range}(f_n) := \max f_n - \min f_n$ is a class invariant: if along a subsequence, $\text{Range}(f_n) \rightarrow +\infty$, then $(\overline{f_n})$ cannot be Cauchy, since $D_\Psi(f, g) \geq |\text{Range}(f) - \text{Range}(g)|$).

In the same way, the following ‘‘circular’’ class invariant continuity modulus of f , defined by $\overline{w}_\delta(f) := \max\{|f(x) - f(y)|, d_{\mathbb{R}/\mathbb{Z}}(x, y) \leq \delta\}$, considering in this formula f as 1-periodic over \mathbb{R} , can be compared to the standard modulus of continuity:

$$w_\delta(f) \leq \overline{w}_\delta(f) \leq 2w_\delta(f). \quad (5.14)$$

Assume that $(\overline{f_n})$ is Cauchy in $QC^0[0, 1]$: there exists an array $(b_{n,m}, n, m \geq 0)$ such that $\|f_n - \Psi_{b_{n,m}}(f_m)\|_\infty \rightarrow 0$, for $n, m \geq N$ and $N \rightarrow +\infty$. Let $\varepsilon > 0$ be fixed, and N be large enough (and fixed) such that

$$\sup_{n, m \geq N} \|f_n - \Psi_{b_{n,m}}(f_m)\|_\infty \leq \varepsilon \quad (5.15)$$

and let also $\delta > 0$ be small enough such that $w_\delta(f_N) \leq \varepsilon$. By (5.15), $w_\delta(\Psi_{b_{n,m}}(f_m)) < 2\varepsilon$, and then $w_\delta(f_m) \leq 4\varepsilon$ for all $m \geq N$. Hence the sequence (f_m) is bounded and equicontinuous, so that it is relatively compact (by the Arzela-Ascoli theorem). Hence, there exists a converging subsequence (f_{n_k}) in $C[0, 1]$. Let f be the limit of this sequence. From here, it is easy to conclude that $\overline{f_n} \rightarrow \overline{f}$ in $QC[0, 1]$: by the triangular inequality, $D_\Psi(\overline{f_n}, \overline{f}) \leq D_\Psi(\overline{f_n}, \overline{f_{n_k}}) + D_\Psi(\overline{f_{n_k}}, \overline{f})$, and then taken the converging subsequence for (f_{n_k}) , we get that $\overline{f_{n_k}}$ converges to \overline{f} in $QC^0[0, 1]$ and then $(\overline{f_n})$ converges to \overline{f} too.

5.4 Proof of Proposition 4.6

(i) If $\mathbf{y}_n \xrightarrow[n]{(d)} \mathbf{y}$, by Skorohod's theorem, there exists a probability space $\tilde{\Omega}$ where some copies $\tilde{\mathbf{y}}_n \stackrel{(d)}{=} \mathbf{y}_n$, $\tilde{\mathbf{y}} \stackrel{(d)}{=} \mathbf{y}$, are defined, and so that $\tilde{\mathbf{y}}_n \xrightarrow[n]{(as.)} \tilde{\mathbf{y}}$. By Arzela-Ascoli, a subset \mathcal{K} of $C[0, 1]$ is relatively compact iff $\sup_{g \in \mathcal{K}} \|g\|_\infty < +\infty$, and if for any $\delta > 0$, $\sup_{g \in \mathcal{K}} w_\delta(g) < +\infty$. Since for any $g \in C[0, 1]^+$, $g(0) = g(1) = 0$, then for any $a \in [0, 1]$, $w_\delta(\Psi_a(g)) \leq 2w_\delta(g)$. It follows that if \mathcal{K} is a subset of $C[0, 1]^+$ relatively compact in $C[0, 1]$, then $\mathcal{K}' := \bigcup_{a \in [0, 1]} \Psi_a(\mathcal{K})$ is also relatively compact. From the tightness of $(\mathbf{y}_n, n \geq 0)$, we can therefore deduce the tightness of $(\text{Conj}(\mathbf{y}_n), n \geq 0)$.

Consider the sets $A = \text{argmin } \tilde{\mathbf{y}}$ and $a_n = \min \text{argmin } \tilde{\mathbf{y}}_n$. First, $\|\tilde{\mathbf{y}}_n - \tilde{\mathbf{y}}\|_\infty \rightarrow 0 \Rightarrow \min \tilde{\mathbf{y}}_n \rightarrow \min \tilde{\mathbf{y}}$, and this entails that $d(a_n, A) \rightarrow 0$. Indeed, if for a subsequence $d(a_{n_m}, A) \not\rightarrow 0$, then, by compactness a subsequence a'_{n_m} (of this subsequence) would converge in $[0, 1]$ to a point $x \notin A$ and at this point $\tilde{\mathbf{y}}_x \leq \min \tilde{\mathbf{y}}$, a contradiction. Hence, the accumulation points of (a_n) belong to A . If a converging subsequence a_{m_n} tends to some $a \in A$, $\text{Conj}(\ell_{m_n}) \rightarrow \Psi_a(\ell)$ in $C[0, 1]$.

(ii) Use the same Skorohod embedding as in (i). Now, in $C[0, 1]$, the map $g \mapsto \min \text{argmin } g$ is not continuous, but if $g \in C[0, 1]$ reaches its minimum only once, then $\|g_n - g\|_\infty \rightarrow 0 \Rightarrow \min \text{argmin } g_n \rightarrow \min \text{argmin } g$. Hence, on the space $\tilde{\Omega}$, $\min \text{argmin } \tilde{\mathbf{y}}_n \xrightarrow[n]{(as.)} \min \text{argmin } \tilde{\mathbf{y}}$, and since $\|\tilde{\mathbf{y}}_n - \tilde{\mathbf{y}}\|_\infty \xrightarrow[n]{(as.)} 0$, from (4.2), we conclude.

(iii) This is a consequence of (i): since the sequence $(\text{Conj}(\mathbf{y}_n), n \geq 1)$ is tight in $C[0, 1]$, from each subsequence $(\text{Conj}(\mathbf{y}_{n_m}), m \geq 1)$ of this sequence, one can extract a weakly converging subsequence $(\text{Conj}(\mathbf{y}_{n_{m_k}}), k \geq 1)$, and still by (i) the accumulation point in $QC^0[0, 1]$ is $\bar{\mathbf{y}}$. Using the Skorohod embedding, we can find a probability space on which the copies of these random variables converge a.s. (written with an extra \star):

$$D_\Psi(\text{Conj}(\mathbf{y}_{n_{m_k}}^\star), \text{Conj}(\mathbf{y}^\star)) \rightarrow 0.$$

Since $D_\Psi(\text{Conj}(\mathbf{y}_{n_{m_k}}^\star), \text{Conj}(\mathbf{y}^\star)) = D_\Psi(\mathbf{y}_{n_{m_k}}^\star, \mathbf{y}^\star)$, along this subsequence, $\overline{\mathbf{y}_{n_{m_k}}^\star} \xrightarrow[n]{(as.)} \overline{\mathbf{y}^\star}$ in $QC^0[0, 1]$, from what we deduce that $\overline{\mathbf{y}_{n_{m_k}}^\star} \xrightarrow[n]{(d)} \overline{\mathbf{y}^\star}$ in $QC^0[0, 1]$. Since the limit coincides with $\bar{\mathbf{y}}$ in distribution in $QC^0[0, 1]$, its distributions does not depend on the extracted subsequence, (iii) holds.

(iv) This is a consequence of (iii), since the class of $\text{Conj}(\mathbf{y}_n)$ (resp. $\text{Conj}(\mathbf{y})$) in $QC^0[0, 1]$ is the same as that of \mathbf{y}_n (resp. \mathbf{y}).

5.5 Proof of Theorem 4.11

The proof is done by induction. Before writing the proof, we need to state several propositions. The convergence for the case $D = 1$, is a consequence of a result already known:

Proposition 5.4. [Marckert & Mokkadem [53], Janson & Marckert [38]] *Let ν be a centered distribution having moments of order $4 + \varepsilon$ for some $\varepsilon > 0$, and variance $\sigma^2 > 0$. Consider $(\mathbf{T}_n, \mathbf{L}_{\mathbf{T}_n})$ a branching random walk, constructed on a random tree \mathbf{T}_n picked uniformly in \mathbb{T}_n . The following convergence in distribution holds in $C([0, 1], \mathbb{R}^2)$ equipped with the topology of uniform convergence:*

$$\left(\frac{C_{\mathbf{T}_n}(2nx)}{\sqrt{2n}}, \frac{\mathbf{L}_{\mathbf{T}_n}(2nx)}{n^{1/4}2^{1/4}\sigma} \right)_{x \in [0, 1]} \rightarrow (\mathbf{e}, r), \quad (5.16)$$

where \mathbf{e} is the normalized Brownian excursion and, conditionally on \mathbf{e} , r is distributed as a centered Gaussian process with covariance function $\text{cov}(r_x, r_y) = \check{\mathbf{e}}(x, y)$.

The convergence of the first marginal in Proposition 5.4 is equivalent to the convergence of uniform planar trees to Aldous' continuum random tree. This theorem can be found in Aldous [1, 52].

For the second marginal convergence, the main ingredient is the central limit theorem: for $2nx \in \mathbb{N}$, conditionally on C_{2nx} , R_{2nx} is a sum of C_{2nx} centered i.i.d. r.v. with variance σ^2 ; hence

$$\frac{R_{2nx}}{\sigma\sqrt{C_{2nx}}} = \frac{R_{2nx}}{\sigma(2n)^{1/4}} \frac{(2n)^{1/4}}{\sqrt{C_{2nx}}}$$

is close to a normal random variable $\mathcal{N}(0, 1)$. One perceives, according to the convergence $\frac{(2n)^{1/4}}{\sqrt{C_{2nx}}} \xrightarrow[n]{(d)}$ $1/\sqrt{\bar{e}_x}$, the one-dimensional convergence of the second marginal, as stated in (5.16). (The finite dimensional convergence can be proved using this argument, but the tightness needs additional work).

Now, in order to complete the induction we need the three following propositions.

Proposition 5.5. *Let $(r(n))$ be a sequence such that $r(n) \rightarrow +\infty$ and $r(n) = o(n)$. Assume that $(\mathbf{L}_n, n \geq 0)$ is a sequence of processes such that for every n , $\mathbf{L}_n = \mathbf{L}_n([0, n])$ takes its values in \mathbb{L}_n , and such that the normalized and interpolated process $\ell_n := \left(\frac{\mathbf{L}_n(nt)}{r(n)}, 0 \leq t \leq 1 \right)$ satisfies*

$$\ell_n \xrightarrow[n]{(d)} \ell \text{ in } C[0, 1],$$

where a.s., ℓ has no period (meaning $\ell(a + x \bmod 1) = \ell(x)$ for all $x \in [0, 1]$, implies $a \in \mathbb{Z}$). For

$$\begin{cases} \mathbf{C}_n & := \Phi_n^{H \rightarrow C}(\Phi_n^{L \rightarrow H}(\mathbf{L}_n)) \\ \mathbf{c}_n(t) & = \mathbf{C}_n(2nt)/r(n), t \in [0, 1] \\ \mathbf{c}'_n & = \text{Conj}(\ell_n) \end{cases}$$

we have $\|\mathbf{c}_n - \mathbf{c}'_n\|_\infty \rightarrow 0$ in probability, as $n \rightarrow +\infty$.

This proposition does not imply the convergence of \mathbf{c}_n or \mathbf{c}'_n ; again, the convergence of (ℓ_n) does not imply that of $(\text{Conj}(\ell_n))$.

Proof. Write $\mathbf{H}_n = \Phi_n^{L \rightarrow H}(\mathbf{L}_n)$, $\mathbf{h}_n(\cdot) = \mathbf{H}_n(n\cdot)/r(n)$ and $\mathbf{C}_n = \Phi_n^{H \rightarrow C}(\mathbf{H}_n)$. From Proposition 4.6, the sequence $(\text{Conj}(\ell_n))$ is tight in $C[0, 1]$. Consider a converging subsequence $(\text{Conj}(\ell_{n_k}))$, and let us observe that $\text{Conj}(\ell_{n_k})$ and (\mathbf{h}_{n_k}) are asymptotically indistinguishable, in the sense that

$$\|\mathbf{c}'_{n_k} - \mathbf{h}_{n_k}\|_\infty = \|\text{Conj}(\ell_{n_k}) - \mathbf{h}_{n_k}\|_\infty \rightarrow 0 \text{ in probability when } k \rightarrow +\infty.$$

The reason is that $\|\text{Conj}(\ell_n) - \mathbf{h}_n\|_\infty \leq 2/r(n)$ since there is at most one abscissa discrepancy of one (normalized) step between the two constructions, and since in the definition of $\Phi_n^{L \rightarrow H}$ (Defi. 3.8) there is an additional +1 (which after normalization becomes $1/r(n)$).

Now, Theorem 5.3 allows us to write $\|\mathbf{h}_{n_k} - \mathbf{c}_{n_k}\| \rightarrow 0$ (in proba.) and then to conclude. \square

Proposition 5.6. *Under the hypotheses of Proposition 5.5, $(\ell_n, \bar{\mathbf{c}}_n)$ converges in $C[0, 1] \times QC[0, 1]$ to some limiting process $(\ell, \bar{\mathbf{c}})$, and a.s., for any $\mathbf{c} \in \bar{\mathbf{c}}$, there exists a unique $a \in [0, 1)$ such that*

$$\mathbf{c}(\cdot - a \bmod 1) = \ell,$$

in other words, there is almost surely a unique shift sending ℓ onto \mathbf{c} .

Proof. The convergence of $\ell_n \xrightarrow{(d)} \ell$ in $C^0[0, 1]$ implies the convergence of $\bar{\ell}_n \xrightarrow{(d)} \bar{\ell}$ in $QC^0[0, 1]$. Since for any n , $D_\Psi(\bar{\mathbf{c}}'_n, \bar{\ell}_n) = 0$, we can deduce that $D_\Psi(\bar{\mathbf{c}}, \bar{\ell}) = 0$. Since ℓ has a.s. no period, for \mathbf{c} fixed in $\bar{\mathbf{c}}$, there exists a unique $a \in [0, 1]$ such that $\mathbf{c} = \Psi_a(\ell)$. \square

Theorem 5.7. *Let $(r(n))$ be a sequence such that $r(n) \rightarrow +\infty$. For any n , let μ_n be a distribution on \mathbb{T}_n . Consider a branching random walk with underlying tree \mathbf{T}_n , a random tree with law μ_n , and spatial increment ν (see (3.9)). Denote by \mathbf{L}_n the associated corner label process. Let (\mathbf{c}_n, ℓ_n) be the normalized versions of the contour and label processes defined by*

$$\begin{cases} \mathbf{c}_n(x) &= C_{\mathbf{T}_n}(2nx) / r(n), \text{ for } x \in [0, 1] \\ \ell_n(x) &= \mathbf{L}_n(2nx) / \sqrt{2r(n)/3}, \text{ for } x \in [0, 1]. \end{cases} \quad (5.17)$$

If $\mathbf{c}_n \xrightarrow{(d)} \mathbf{c}$ in $(C[0, 1], \|\cdot\|_\infty)$ and if the sequence (\mathbf{c}_n) is q -Hölder tight for some $q \in (0, 1)$, then:

■ the pair (\mathbf{c}_n, ℓ_n) converges in distribution to (\mathbf{c}, ℓ) in $(C[0, 1]^2, \|\cdot\|_\infty)$, where conditionally on \mathbf{c} , ℓ is a centered Gaussian process with covariance matrix

$$\text{cov}(\ell(x), \ell(y)) = \check{\mathbf{c}}(x, y); \quad (5.18)$$

- the sequence (ℓ_n) is q' -Hölder tight, for any $q' \in (0, q/2)$.
- If $\mathbf{c} \neq 0$ a.s., then a.s. ℓ has no period (in the sense of Proposition 5.5).

Proof. ■ Since $\mathbf{c}_n \xrightarrow{(d)} \mathbf{c}$, by Skorohod representation theorem, there is a probability space on which some copies $\tilde{\mathbf{c}}_n \stackrel{(d)}{=} \mathbf{c}_n$, $\tilde{\mathbf{c}} \stackrel{(d)}{=} \mathbf{c}$ are defined, such that $\tilde{\mathbf{c}}_n \xrightarrow{(as.)} \tilde{\mathbf{c}}$. Let us work on this space. Now, by the standard central limit theorem, one proves easily the convergence of the finite dimensional distributions of the sequence of processes ℓ_n to those of the Gaussian process with covariance matrix specified in (5.18).

■ To prove the property about the tightness, consider the following set $F_{M,q} = \{f \in C^+[0, 1], \text{Hol}_q(f) \leq M\}$ for some $M > 0$, and $q \in (0, 1)$. Then, as already explained below (5.13), for $f \in F_{M,q}$, the distance D_f defined in (2.4) satisfies

$$D_f(s, t) \leq 2M|t - s|^q, \quad \text{for any } (s, t) \in [0, 1]^2. \quad (5.19)$$

Let us assume that (\mathbf{c}_n) is q -Hölder tight, i.e. for any $\varepsilon > 0$, there exists $M > 0$, $N \geq 1$, such that, for any $n \geq N$, $\mathbb{P}(\mathbf{c}_n \in F_{M,q}) \geq 1 - \varepsilon$. Conditionally on $\mathbf{c}_n \in F_{M,q}$, for x and y such that $x \leq y$ and $2nx$ and $2ny$ are integers, $\ell_n(x) - \ell_n(y)$ is a sum of $D_{C_{\mathbf{T}_n}}(2nx, 2ny)$ i.i.d. r.v. $(\mathbf{X}_j, j \geq 1)$ with distribution ν , divided by $\sqrt{2r(n)/3}$. Since ν is centered and has all its moments, the Marcinkiewicz–Zygmund inequality gives

$$\mathbb{E}(|\mathbf{X}_1 + \cdots + \mathbf{X}_p|^s) \leq m(s)p^{s/2}, \quad (5.20)$$

where $m(s)$ is a constant (independent from p). Therefore, conditionally on $C_{\mathbf{T}_n}$, for x, y such that $2nx$ and $2ny$ are integers¹⁴

$$\mathbb{E} \left(\left| \frac{\mathbf{L}_n(2nx) - \mathbf{L}_n(2ny)}{\sqrt{r(n)}} \right|^s \middle| C_{\mathbf{T}_n} \right) \leq m(s) \left(\frac{D_{C_{\mathbf{T}_n}}(2nx, 2ny)}{r(n)} \right)^{s/2} = m(s) (D_{\mathbf{c}_n}(x, y))^{s/2},$$

¹⁴Between integer points, \mathbf{L}_n is linear, and it is folklore and easy to check, that the tightness can be proved by proving the moment condition only at these discretization points.

by the normalization (5.17). Using the fact that (\mathbf{c}_n) is q -Hölder tight and (5.19), this gives

$$\mathbb{E} \left(\left| \sqrt{2/3} (\ell_n(x) - \ell_n(y)) \right|^s \middle| C_{\mathbf{T}_n, F_{M,q}} \right) \leq m'(s, M) |x - y|^{sq/2},$$

with $m'(s, M) = m(s)(2M)^{s/2}$. The standard moment criterium (5.2) allows us to conclude.

■ Use again the probability space of the first point, on which some copies $\tilde{\mathbf{c}}_n \stackrel{(d)}{=} \mathbf{c}_n$, $\tilde{\mathbf{c}} \stackrel{(d)}{=} \mathbf{c}$ are defined, such that $\tilde{\mathbf{c}}_n \xrightarrow[n]{(as.)} \tilde{\mathbf{c}}$. Take any element $c \in C^+[0, 1]$ different from the zero function. Conditionally on $\tilde{\mathbf{c}} = c$, the tree T_c possesses a non-trivial branch, so that a.s. ℓ is not identically zero (its range contains that of the Brownian motion living on any non-trivial branch of T_c). Now, under this condition, if ℓ is periodic, its period must be rational: otherwise, since $\ell_0 = 0$, ℓ would be identically 0 which is excluded.

Since \mathbb{Q} is countable, it suffices then to show that for a rational number $q \in (0, 1)$, with probability 1, q is not a period of ℓ . We start to show that there exists $x \in [0, 1]$ so that $D_c(x, x + q) \neq 0$. Indeed, $D_c(x, x + q) = 0$ implies that x and $x + q$ are corners of the same node in T_c ; since nodes are non-crossing, $D_c(x, x + q) = 0$ for every x , implies that $[0, 1]$ is the set of corners of a unique node, so that c is identically 0, which is excluded. Now, take x such that $D_c(x, x + q) \neq 0$: for this x , since ℓ is the label process of c , $\ell(x) - \ell(x + q) \sim \mathcal{N}(0, D_c(x, x + q))$, and then, a.s. $\ell(x) \neq \ell(x + q)$ and then q is not a period of ℓ . \square

We go on with a kind of converse of Proposition 4.6:

Lemma 5.8. *Let $\bar{c}, \bar{c}_1, \bar{c}_2, \dots$ be a deterministic sequence in $QC^0[0, 1]$ such that $\bar{c}_n \rightarrow \bar{c}$ in $(QC^0[0, 1], D_\Psi)$. Consider for each n an element c_n taken in \bar{c}_n , and such that $c_n \in C^+[0, 1]$. The sequence (c_n) is relatively compact in $C[0, 1]$: every accumulation point of this sequence belongs to $\text{PointedTreeClass}(\bar{c})$.*

Proof. As in the proof of Proposition 4.2 (section 5.3), we start by observing that the sequence (c_n) is relatively compact: the reason is that $\bar{c}_n \rightarrow \bar{c}$ in $QC^0[0, 1]$ is equivalent to the existence of a sequence (a_n) such that $\|c_n(a_n + \cdot \bmod 1) - c\|_\infty \rightarrow 0$; from there follows the fact that $(\|c_n\|_\infty)$ is bounded and (c_n) equicontinuous. The sequence (c_n) is therefore relatively compact by Arzela-Ascoli, and owns some accumulation points (that are clearly elements of $C^+[0, 1]$). Consider an accumulation point $c^* \in C^+[0, 1]$. Since along a subsequence $\|c_{n_k} - c^*\|_\infty \rightarrow 0$, clearly $D_\Psi(\bar{c}_{n_k}, \bar{c}^*) \rightarrow 0$ and since by hypothesis $D_\Psi(\bar{c}_n, \bar{c}) \rightarrow 0$, we deduce that $D_\Psi(\bar{c}^*, \bar{c}) = 0$ so that \bar{c}^* and $\bar{c} = QC^0[0, 1]$. \square

Lemma 5.9. *Let \mathbf{C}_n be a uniform Dyck path taken in Dyck_{2n} , and $\mathbf{c}_n(\cdot) = \frac{\mathbf{C}_n(2n\cdot)}{\sqrt{2n}}$. The sequence (\mathbf{c}_n) is q -Hölder tight for any $q < 1/2$.*

As stated in (5.16) (first marginal convergence), $\mathbf{c}_n \xrightarrow[n]{(d)} \mathbf{e}$.

Proof. This property is folklore: it can be transferred from the classical simple random walk (with increment $+1$ or -1) which owns the same property (with the same rescaling), fact that follows a simple application of the moment condition. To transfer this property from the random walk to the Dyck paths there is a two steps argument ([38, Proof of Lemma 1]).

– There is a 1 to $2n + 1$ map which sends Dyck paths (at which an additional step -1 is appended) to bridges of size $2n + 1$, which are paths with steps $+1$ and -1 with length $2n + 1$ ending at -1 : the $2n + 1$ bridges associated to a (single) Dyck path are obtained by conjugations at one of the $2n + 1$ abscissa from 0 to $2n$. These maps multiply q -Hölder exponents by at most 2, so that, it suffices to transfer the tightness property from random walks to bridges.

– Now, bridges are invariant by translation, so that it suffices to prove that their first half are q -Hölder tight for any $q < 1/2$. Consider then a simple random walk (S_0, \dots, S_{2n+1}) , and A be an event (S_0, \dots, S_{n+1}) measurable (which depends on the first “half” of the trajectory).

Claim: Up to a multiplicative constant, the probability of A under the bridge condition is bounded by the probability of A for the random walk:

$$\mathbb{P}(A \mid S_{2n+1} = -1) \leq c \mathbb{P}(A) \quad (5.21)$$

for a universal finite constant c (independent from n and from A).

Taking for a moment this claim as granted, the fact that the rescaled random walk is q -Hölder tight on $[0, 1/2]$ (for any $q \in (0, 1/2)$) implies that it is also the case for the rescaled bridge (taking some events of the type $A = \{|S_{ns} - S_{nt}|/\sqrt{n} \leq C|s - t|^q, s, t \leq n + 1\}$) to use the claim). To prove the claim, write

$$\frac{\mathbb{P}(A, S_{2n+1} = -1)}{\mathbb{P}(S_{2n+1} = -1)} = \sum_{k=-n-1}^{n+1} \frac{\mathbb{P}(A, S_{2n+1} = -1, S_{n+1} = k)}{\mathbb{P}(S_{2n+1} = -1)} \quad (5.22)$$

$$= \sum_{k=-n-1}^{n+1} \frac{\mathbb{P}(S_{2n+1} = -1 \mid S_{n+1} = k, A)}{\mathbb{P}(S_{2n+1} = -1)} \mathbb{P}(A, S_{n+1} = k) \quad (5.23)$$

(sum which must be restricted to the k for which $\mathbb{P}(A, S_{n+1} = k) > 0$), and by the Markov property of random walk,

$$\sup_n \sup_k \frac{\mathbb{P}(S_{2n+1} = -1 \mid S_{n+1} = k, A)}{\mathbb{P}(S_{2n+1} = -1)} \leq c := \sup_n \sup_k \frac{\mathbb{P}(S_{2n+1} = -1 \mid S_{n+1} = k)}{\mathbb{P}(S_{2n+1} = -1)} = \sup_n \sup_k \frac{\mathbb{P}(S_n = -k - 1)}{\mathbb{P}(S_{2n+1} = -1)},$$

since $\sup_k \mathbb{P}(S_n = -k - 1) = \mathbb{P}(S_n \in \{0, 1\})$ (depending on the parity of n), using Stirling formula, one sees that $c < +\infty$. Using this bound in (5.23) allows us to get (5.21). \square

End of proof of Theorem 4.11. In the case $D = 1$, the convergence in $C[0, 1]^2$ stated in Proposition 5.4 can be restated as $(\mathbf{c}_n^{(1)}, \ell_n^{(1)}) \xrightarrow[n]{(d)} (\mathbf{c}^{(1)}, \ell^{(1)})$ in $C[0, 1]^2$: it implies the convergence in $\text{Snakes}^{\bullet, 1}$ of $\pi_{\bullet}(\mathbf{c}_n^{(1)}, \ell_n^{(1)}) \xrightarrow[n]{(d)} \pi_{\bullet}(\mathbf{c}^{(1)}, \ell^{(1)})$. Lemma 5.9, allows us to see that $(\mathbf{c}_n^{(1)})$ is q -Hölder tight for $q \in (0, 1/2)$; Theorem 5.7 allows us to deduce that $(\ell_n^{(1)})$ is q -Hölder tight for $q \in (0, 1/4)$, and since $\mathbf{c}^{(1)} \stackrel{(d)}{=} \sqrt{2}\mathbf{e}$ is a.s. different from the zero process, that $\ell^{(1)}$ has a.s. no period. Propositions 4.6 and 5.5 allow deducing from the convergence $\ell_n^{(1)} \rightarrow \ell^{(1)}$, the fact that $\mathbf{c}_n^{(2)} \rightarrow \mathbf{c}^{(2)} := \text{Conj}(\ell^{(1)})$ in $QC^0[0, 1]$.

Now, the proof is basically a proof by induction: by the incremental nature of the construction of the iterated discrete snakes, we have for any $j \geq 2$:

$$\text{Law} \left(\left(\mathbf{c}_n^{(j)}, \ell_n^{(j)} \right) \mid \left(\left(\mathbf{c}_n^{(m)}, \ell_n^{(m)} \right), m \leq j - 1 \right) \right) = \text{Law} \left(\left(\mathbf{c}_n^{(j)}, \ell_n^{(j)} \right) \mid \left(\mathbf{c}_n^{(j-1)}, \ell_n^{(j-1)} \right) \right).$$

In order to prove the convergence of $\pi_{\bullet}(\mathbf{c}_n^{(j)}, \ell_n^{(j)})$ knowing the convergence of $(\pi_{\bullet}(\mathbf{c}_n^{(m)}, \ell_n^{(m)}), m \leq j - 1)$ the simplest method consists in working on a space on which some copies of these processes are defined, that converge almost surely (whose existence is guaranteed by Skorohod representation theorem). On this space, since $\overline{\ell_n^{(j-1)}} \rightarrow \overline{\ell^{(j-1)}}$ a.s. (in $QC^0[0, 1]$), by the same argument as above (mainly Propositions 4.6 and 5.5), $\overline{\mathbf{c}_n^{(j)}}$ converges to some process, $\overline{\mathbf{c}^{(j)}}$ in $QC^0[0, 1]$. We suppose also by induction that $(\mathbf{c}_n^{(m)})$ is q -Hölder tight for $q \in (0, 1/2^m)$ and $(\ell_n^{(m)})$ is q -Hölder tight for $q \in (0, 1/2^{m+1})$ for $m \leq j - 1$.

First, the fact that $(\mathbf{c}_n^{(j)})$ is q -Hölder tight for any $q \in (0, 1/2^j)$ is a consequence of the preservation of this property by conjugation, and is then inherited from this property of $(\ell_n^{(j-1)})$. To discuss the convergence of the label process of the normalized branching random walk having $\overline{\mathbf{c}_n^{(j)}}$ as an underlying tree, let us come back to the world of rooted trees: by Lemma 5.8, choose an element $\mathbf{c}_n \in C^+[0, 1] \cap \overline{\mathbf{c}_n^{(j)}}$ (for example take $\mathbf{c}_n = \mathbf{c}_n^{(j)}$) for each n , build a rooted tree with this contour, and on this tree a branching random walk as explained in (3.2). After that, build the corresponding normalized tour of snake (\mathbf{c}_n, ℓ_n) associated to this rooted tree with the normalization specified in Theorem 4.11 (recall Section 4.2 for the effect of choosing \mathbf{c}_n or another tree in the tree class of $\overline{\mathbf{c}_n^{(j)}}$). From Lemma 5.8, (\mathbf{c}_n) has all its accumulation points in $\text{PointedTreeClass}(\overline{\mathbf{c}^{(j)}})$. Take a converging subsequence (\mathbf{c}_{n_k}) in $(C[0, 1], \|\cdot\|_\infty)$, and let $\mathbf{c}^* \in C^+[0, 1]$ be its limit. By Theorem 5.7, we get $(\mathbf{c}_{n_k}, \ell_{n_k}) \xrightarrow[n]{(d)} (\mathbf{c}^*, \ell)$, where ℓ is a Gaussian process with covariance matrix $\text{cov}(\ell(x), \ell(y)) = \check{\mathbf{c}}^*(x, y)$. We get also by this same theorem, that (ℓ_n) is q -Hölder tight for any $q \in (0, 1/2^{j+1})$ (for a bound on the q -Hölder exponent independent from the subsequence, since the q -Hölder exponents of the different elements $\mathbf{c}_n \in \overline{\mathbf{c}_n^{(j)}}$ have their ratio bounded by 2). Hence, taking into account that $\pi^\bullet(\mathbf{c}^*, \ell)$ is a random variable in Snakes^\bullet having a distribution which does not depend on the element $\mathbf{c}^* \in \overline{\mathbf{c}_n^{(j)}}$, we conclude.

6 Combinatorial aspects of maps and iterated discrete feuilletages

The aim of this section is to emphasize certain combinatorial aspects of the construction: we review our construction in the combinatorial map picture, putting the accent on the combinatorial encodings of the objects involved, before giving the asymptotic enumeration of discrete feuilletages in Sec. 6.5, and a combinatorial encoding of the discrete feuilletages in Sec. 6.6.

6.1 Generalities and different encodings of combinatorial maps

Some references on maps include [60, 42, 34, 43]. There are several equivalent definitions of combinatorial maps, suitable for different purposes and considerations: combinatorial, topological, or to produce asymptotic results and limit objects. Let us first provide two of them without entering into much details:

- A map is a connected graph drawn on a surface, such that the edges do not cross and the connected components of the complement of the graph on the surface are homeomorphic to disks.
- It can be seen as the result of gluing together a collection of polygons by pairing the edges on their boundaries, to form an orientable surface.

The map is then seen up to continuous deformation of the drawing on the surface (Fig. 15). Maps as

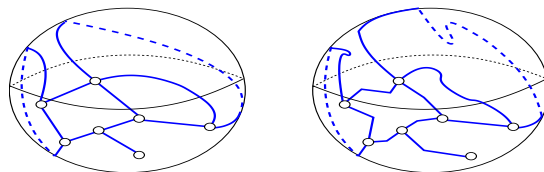


Figure 15: Two representations of the same planar map.

graph drawings up to homeomorphisms are combinatorial objects: there is a finite number of maps with n edges. Indeed, consider a map with n edges, and label each half-edge by a number – an index – from 0 to $2n - 1$. The set of edges is then naturally encoded by a set α of pairs of half-edges, and each vertex by the cyclic ordering of half-edges around it, that defines a permutation σ on the set of half-edges through its cycles. These permutations fully characterize the map, and thereby provide a third definition:

Definition 6.1. *A connected indexed map¹⁵ with n edges, is a pair $\mathcal{M} = (\sigma, \alpha)$, where σ is a permutation on $\llbracket 0, 2n - 1 \rrbracket$, α is a fixed-point free involution on $\llbracket 0, 2n - 1 \rrbracket$, and the group $\langle \sigma, \alpha \rangle$ generated by σ and α acts transitively on $\llbracket 0, 2n - 1 \rrbracket$ (the map is connected).*

The numbers V , n and F of vertices, edges, and faces of a map (σ, α) satisfy Euler’s formula:

$$V - n + F = 2 - 2g,$$

where g is a non-negative integer, called the genus of a map (the genus of the surface on which the graph is drawn). Planar maps are maps with genus 0.

The *distance* in a map is the length of the shortest path between two vertices. A map is said to be *pointed* if a particular vertex is distinguished. Two consecutive half-edges $j = \sigma(i)$ and i define a *corner* of the map. A map is said to be *rooted* if a particular corner is distinguished. It is equivalent to provide an orientation to a given edge, say from i to $\alpha(i)$ (i.e. to specify the first element of one of the disjoint pairs in α). We call root vertex (or simply root) the vertex incident to the root corner.

Combinatorial maps are usually considered up to reordering of the half-edges. In this case, an unindexed map is an equivalence class $\{(\rho \circ \sigma \circ \rho^{-1}, \rho \circ \alpha \circ \rho^{-1}) \mid \rho \in \mathcal{S}_{2n}\}$. In this paper, we generally consider rooted unindexed maps.¹⁶

Unicellular maps and trees. A rooted unicellular map is a rooted (connected) map with one face $\sigma \circ \alpha$. A tree is a unicellular map of genus 0. The face of a unicellular map with n edges is a cycle of length $2n$ which organizes all the corners: using the root as a starting point (indexed 0), the corners can be re-indexed from 0 to $2n - 1$: the “counterclockwise corner sequence” of the tree.¹⁷

Labelings of trees have been defined in Def. 3.7. A labeling provides an integer to every vertex (its label), such that the label of the root vertex is 0, and the labels of the extremities of any edge differ by at most 1. The set of labeled rooted trees (T, ℓ) with n edges will be denoted by $\mathbb{L}\mathbb{T}_n$, and we define

$$m(T, \ell) = \min_{v \in \mathcal{V}(T)} \ell(v). \tag{6.1}$$

Non-crossing partitions and trees. Another encoding of a rooted planar tree with n edges, which we will use later in this section, is as a pair (C, σ) , where C is an ordered set $C = \{1, \dots, 2n\}$ (the corner sequence of the tree), and σ is a non-crossing permutation on this set, that is, a permutation for which the disjoint cycles respect the cyclic ordering of C and have supports which do not cross for this ordering.

¹⁵The common denomination would be *labeled map*, but we chose *indexed* to avoid confusion with the labeling of the trees and the label processes in the rest of the text.

¹⁶For trees, we rather consider rooted trees together with a canonical indexing of the half-edges, as detailed below.

¹⁷This is not the convention used for processes in Sec. 3, in which the corners of the trees are labeled from 0 to $2n - 1$ going around the tree *clockwise*. This should be kept in mind when comparing the various examples. Apart from that, one then recovers the contour and height sequences of the tree, as defined in Sec. 3.

Definition 6.2. We say that a permutation σ on a totally ordered set C respects the ordering of C if for each one of its cycles c , there are only two consecutive elements $a < b$ in c for which $\sigma(a) > \sigma(b)$.

Definition 6.3. A partition $C = \sqcup_i V_i$ of C is said to be non-crossing if there are no elements $p_1 < q_1 < p_2 < q_2$ such that $p_1, p_2 \in V_i$ and $q_1, q_2 \in V_j$ with $i \neq j$. We say that a permutation σ on C is non-crossing if the partition it induces is non-crossing, and if in addition it respects the ordering of C .

In addition, we require that the *Kreweras complement* [41, 61] of σ on C is a (non-crossing) *matching*, where a matching is a partition in pairs, and:

Definition 6.4. Consider the totally ordered set $C = \{1 < 2 < \dots < N\}$ and make a copy $\bar{C} = \{\bar{1} < \bar{2} < \dots < \bar{N}\}$ of this set, so that $C \sqcup \bar{C}$ is ordered as $\{1 < \bar{1} < 2 < \bar{2} \dots < N < \bar{N}\}$. If π is a non-crossing partition on C , we define its *Kreweras complement* $\bar{\pi}$ as the maximal non-crossing partition on \bar{C} for the inclusion order such that $\pi \cup \bar{\pi}$ is a non-crossing partition on $C \sqcup \bar{C}$, where the inclusion order is defined as $C = \sqcup_i V_i \leq C' = \sqcup_i V'_i$ if each V'_i is the union of one or several V_j . See Fig. 16.

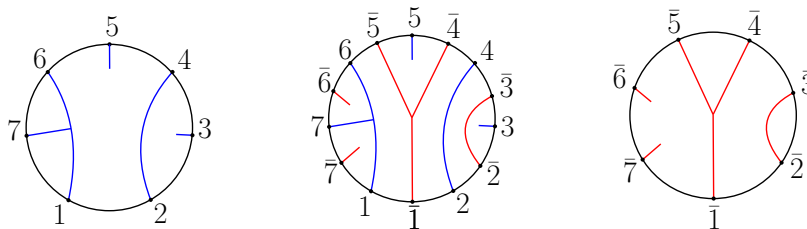


Figure 16: A partition π on an ordered set C (left), and its Kreweras complement $\bar{\pi}$ on \bar{C} (right), and the non-crossing partition $\pi \cup \bar{\pi}$ on $C \sqcup \bar{C}$ (middle).

The rooted planar tree is then obtained from (C, σ) by drawing the ordered set C on a circle and the cycles of σ as shaded regions, adding vertices in the shaded regions, and edges between two vertices whenever the corresponding shaded regions face each other, as shown in Fig. 17¹⁸.

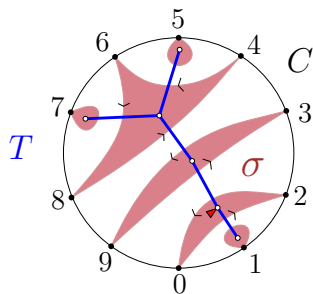
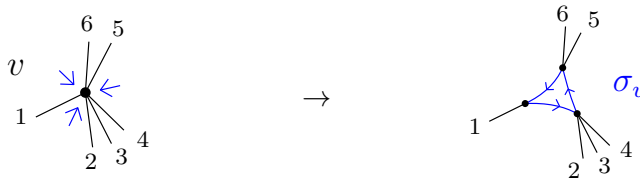


Figure 17: Encoding of a rooted planar tree by its counterclockwise corner sequence, together with a non-crossing permutation, whose Kreweras complement is a matching.

Note that more generally, if the permutation σ is not required to be non-crossing, then (C, σ) encodes a *unicellular map* (a map with only one face) of *positive genus*.

¹⁸Planar trees with n edges are also mapped bijectively to non-crossing partitions on n ordered elements. To obtain the description in the text from this last one, the non-crossing partition on n elements is completed by its Kreweras complement $C \rightarrow C \sqcup \bar{C}$. In the context of this paper, the permutations we obtain naturally have matchings as Kreweras complements.

Graph explosions. Maps can be decomposed yet in another way, “exploding” some of their vertices [20, 65, 54, 17, 13, 5, 16]. More precisely, for each cycle of the permutation encoding the vertices in Def. 6.1, we may choose to split it in two or more disjoint consecutive cycles e.g. $(123456) \rightarrow (1)(234)(56)$.



When doing so, a corner is distinguished on each copy of the vertex (the corners (1) , $(4, 2)$, and $(6, 5)$ in this example). Exploding some of the vertices of a map M to obtain a map M' and gathering the disjoint cycles σ_v for each exploded vertex v , the explosion is thus encoded as a permutation $\sigma = \prod_v \sigma_v$ defined on the corners of M' , which allows reversing the explosion and reconstructing the vertices of M .

Now suppose that the map M' obtained after exploding some of the vertices of M is a unicellular map, encoded by its corner sequence (an ordered set C') and a permutation σ' on C' . Then the support of the permutation $\sigma = \prod_v \sigma_v$ just described is a subset of C' , and the map M can be encoded by the triplet (C', σ', σ) . In addition, there is at most one element of the support of σ per disjoint cycle of σ' .

Any map M can always be encoded as such a triplet (C', σ', σ) by exploding some vertices to obtain a unicellular map. *The original map M is planar if and only if both σ and σ' are non-crossing permutations on C' , i.e. if both (C', σ) and (C', σ') are trees.*

To our opinion however, for an encoding of a map M as a tree $T = (C', \sigma)$ together with a permutation σ' encoding the explosion of vertices, it is less obvious to read the dimension and the topology of M directly on the encoding (T, σ') : the 2-dimensional faces of the map for instance are created from portions of the tree (that can be viewed as a one dimensional object) that are assembled to form the faces. This explains to our opinion why it is not obvious at first glance to identify elementary cells of a certain dimension in our iterative construction of random graphs, which is based on iterating such an encoding, see Sec. 6.6.

6.2 Planar maps and labeled trees: the Cori-Vauquelin-Schaeffer bijection

A quadrangulation is a map whose faces all have length four. We denote by $\mathbb{Q}_n^{\bullet \rightarrow}$ the set of rooted and pointed quadrangulations with n faces. The minimum $m(T, \ell)$ of ℓ was defined in (6.1).

Theorem 6.5 (Cori-Vauquelin 1981 [20], Schaeffer 1998 [65]). *There is a 1-to-2 mapping between rooted and pointed quadrangulations with n faces in $\mathbb{Q}_n^{\bullet \rightarrow}$, and labeled trees with n edges in \mathbb{LT}_n .*

Furthermore, if the quadrangulation Q is the image of (T, ℓ) , then the set of nodes of T is sent bijectively onto the set of nodes of Q deprived from the pointed vertex ν ; if the node $u \in T$ is sent onto v , then, up to a global translation, $\ell(u)$ coincides with the distance of v to the pointed vertex ν in Q :

$$d_Q(\nu, w) = \ell(w) - m(T, \ell) + 1.$$

For our purpose, we only detail the mapping from $\mathbb{LT}_n \times \{0, 1\}$ to $\mathbb{Q}_n^{\bullet \rightarrow}$: consider a labeled rooted tree (T, ℓ) , and a parameter η in $\{0, 1\}$. As usual, the depth traversal function c_T of T associates with each integer k in $\llbracket 0, 2n - 1 \rrbracket$, the node $c_T(k)$ (which can be seen as a corner, recall (3.4)). Note that here however, we use the *counterclockwise* contour sequence of T to label the corners from 0 to $2n - 1$.

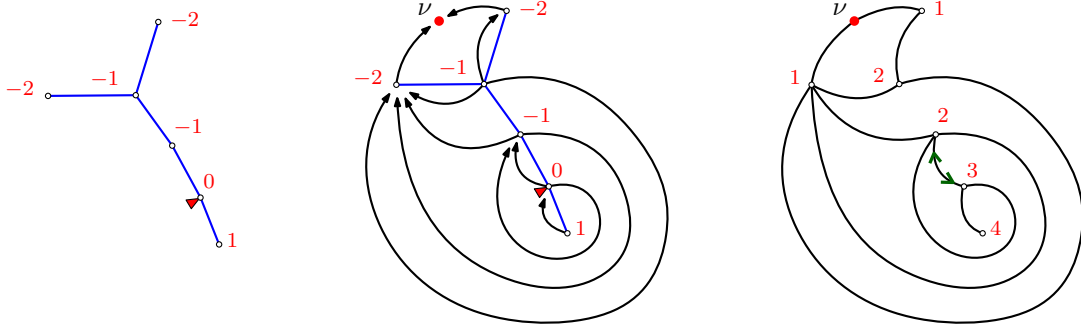


Figure 18: Building two rooted pointed planar quadrangulations from a rooted labeled tree. They differ due to the orientation of the root-edge, that depends on the value of the parameter η .

Consider also $L(k) = \ell(c_T(k))$, the label of the k th corner of T , for k in $\llbracket 0, 2n - 1 \rrbracket$. First draw in **red** a special vertex ν in the plane, and draw in **blue** the tree T in the plane. Then, treat successively all the corners $(c_T(k), 0 \leq k < 2n - 1)$ (starting from the root corner $c_T(0)$) as follows: to each corner $c_T(k)$,

- if $L(k) = m(T, \ell)$ (see (6.1)), add a **black** edge between $c_T(k)$ and the special vertex ν ,
- if not, add a **black** edge between $c_T(k)$ and the *preceding* corner with label $L(k) - 1$.

At each step, this is done so that the **black** map remains planar. There is always a unique corner on either ν or the preceding vertex carrying the label $L(k) - 1$ such that it is possible. We now remove the tree T (the **blue edges**), keeping only the vertices and the **black** edges. It can be proved that the resulting map is a connected planar quadrangulation. The map is pointed at ν , and rooted by choosing an orientation for the edge that had been added at the very first step, as follows: if $\eta = 0$, this edge is oriented *from* the corner indexed 0, and if $\eta = 1$, this edge is oriented *towards* the corner indexed 0.

6.3 The CVS bijection as a graph explosion

Let us consider a quadrangulation $Q \in \mathbb{Q}_n^{\bullet, \rightarrow}$ with n faces, and denote by $(T_n^{(1)}, L_n^{(1)})$ the corresponding labeled rooted tree in $\mathbb{L}\mathbb{T}_n$. We can consider Q together with the tree $T_n^{(1)}$, as on the middle of Fig. 18. The corresponding map, which contains both $T_n^{(1)}$ and Q as submaps, is denoted by \tilde{Q} .

The trees $T_n^{(2)}$ and $\tau_n^{(2)}$. Following [54], another tree $\tau_n^{(2)}$ with $2n$ edges is extracted from \tilde{Q} , by following the contour of $T_n^{(1)}$, and “ungluing” the vertices of Q along the corners of $T_n^{(1)}$ as shown in Fig. 19. In this section, we choose to root $\tau_n^{(2)}$ on the corner that was glued to the root corner of $T_n^{(1)}$, as it simplifies the combinatorial encoding of feuilletages in Sec. 6.6. To obtain the tree $T_n^{(2)}$ defined in Sec. 3, whose choice of root was needed to show the convergence, simply reroot the tree $\tau_n^{(2)}$ on the corner of the pointed vertex that faces the min argmin of the labeling $L_n^{(1)}$ of $T_n^{(1)}$ ¹⁹.

Exploding Q . The vertices of $T_n^{(1)}$ encode the gluing of the vertices of $\tau_n^{(2)}$ to form the vertices of Q (middle of Fig. 19), and this gluing can also be encoded by a permutation $\sigma^{(1)}$ defined on a subset of corners of $\tau_n^{(2)}$ (right of Fig. 19), with one corner per non-pointed vertex of $\tau_n^{(2)}$ (or non-rooted for $T_n^{(2)}$), see

¹⁹The min argmin is a corner of $T_n^{(1)}$, and if e is the edge added from this corner towards the pointed vertex ν , $\tau_n^{(2)}$ has to be rerooted on the corner of ν which precedes e on ν counterclockwise in order to obtain $T_n^{(2)}$ (the orange arrow in Fig. 19).

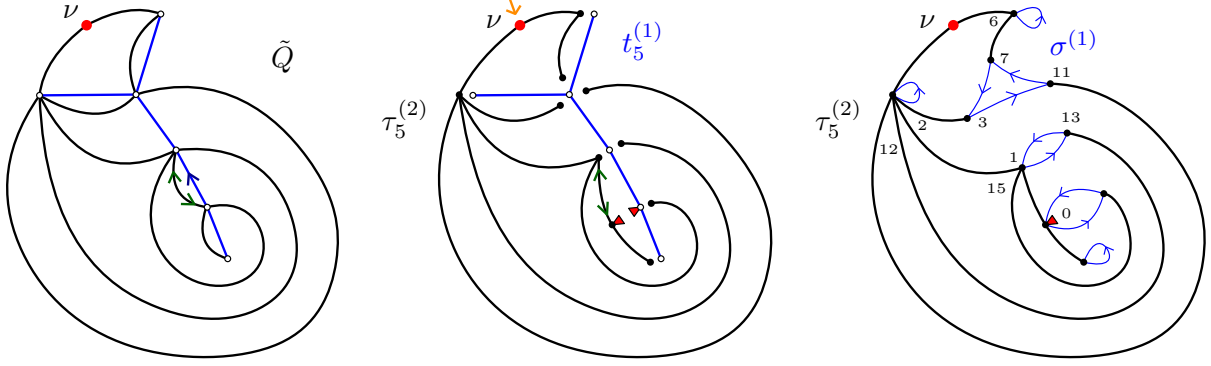


Figure 19: Left: the map \tilde{Q} , with the edges of both Q and $T_n^{(1)}$. Middle: \tilde{Q} can be understood as a discrete “mating” of two trees $T_n^{(1)}$ and $\tau_n^{(2)}$. The red triangles indicate the root-corners of the two trees, while the green arrows indicate the two possible orientations for the root edge in the quadrangulation Q , and the orange arrow indicates the corner on which to reroot to obtain the tree $T_n^{(2)}$ of Sec. 3, Fig. 11. Right: the tree $T_n^{(1)}$ can be seen as a permutation $\sigma^{(1)}$ gluing some corners of $\tau_n^{(2)}$ to form the vertices of Q (see Fig. 17 to recover $T_n^{(1)}$ from $\sigma^{(1)}$).

the end of Sec. 6.1). The planarity of Q is encoded in the fact that $T_n^{(1)}$ is a tree, or equivalently that $\sigma^{(1)}$ is a non-crossing permutation on the *clockwise* corner sequence $\overline{C}_2 = \{0 < 4n - 1 < 4n - 2 < \dots < 1\}$ ²⁰. The fact that the Kreweras complement of $\sigma^{(1)}$ is a matching is equivalent to Q being a quadrangulation.

Considering the encoding of the tree $\tau_n^{(2)}$ as a pair $(C_2, \sigma^{(2)})$, where C_2 is the ordered set $\{0 < \dots < 4n - 1\}$ and $\sigma^{(2)}$ is a non-crossing permutation on C_2 , then Q is encoded as the triplet

$$(C_2, \sigma^{(2)}, \sigma^{(1)}),$$

where $\sigma^{(1)}$ is also a non-crossing permutation on a subset \overline{C}_1 of \overline{C}_2 . From $(C_2, \sigma^{(2)}, \sigma^{(1)})$, the tree $T_n^{(1)}$ is recovered as $(\overline{C}_1, \sigma^{(1)})$. In Fig. 19 for instance, we have $\tau_5^{(2)} = (C_2, \sigma^{(2)})$ and $T_5^{(1)} = (\overline{C}_1, \sigma^{(1)})$, with

$$C_2 = \{0 < 1 < \dots < 19\}, \quad \sigma^{(2)} = (0, 18)(1, 15, 17)(2, 4, 10, 12, 14)(3)(5, 9)(6, 8)(7)(11)(13)(16),$$

$$\overline{C}_1 = \{0 < 19 < 16 < 13 < 11 < 8 < 7 < 4 < 3 < 1\}, \quad \sigma^{(1)} = (0, 16)(1, 13)(3, 11, 7)(4)(8)(19),$$

where we recall that C_2 (resp. \overline{C}_1) is the *counterclockwise* corner sequence of $\tau_5^{(2)}$ (resp. $T_5^{(1)}$).

With this encoding, the corners of $T_5^{(1)}$ have the same indices in C_2 as a subset of the corners in $\tau_5^{(2)}$: in the following we will say that the corners that have the same index are *dual* one to another.

6.4 Discrete iterated feuilletages in the map picture

Consider a first rooted planar tree $T_n^{(1)}$, label it, and build a planar quadrangulation $Q_n^{(1,2)} \in \mathbb{Q}_n^{\bullet, \rightarrow}$ (Q in Sec. 6.3) using the CVS bijection (Sec. 6.2). Then delete $T_n^{(1)}$ and extract a tree $\tau_n^{(2)}$ with $2n$ edges.

Now label in turn $\tau_n^{(2)}$, and build a planar quadrangulation $Q_n^{(2,3)} \in \mathbb{Q}_{2n}^{\bullet, \rightarrow}$ from the CVS bijection (Fig. 20). Identifying the vertices of $Q_n^{(2,3)}$ according to the vertices of $T_n^{(1)}$ provides a 3-discrete feuilletage $R_n[3]$ (right of Fig. 21). Just as for $\tau_n^{(2)}$ at the previous step, we can extract a rooted planar tree $\tau_n^{(3)}$ with $4n$ edges from the map $\tilde{Q}_n^{(2,3)}$, which is $Q_n^{(2,3)}$, together with the edges of $\tau_n^{(2)}$ (right of Fig. 20). $R_n[3]$ is encoded by the tree $\tau_n^{(3)}$, identified by both the vertices of $\tau_n^{(2)}$ and of $T_n^{(1)}$. The discretized 2-sphere

²⁰The cycles of $\sigma^{(1)}$ induce a non-crossing partition on $C_2 = \{0 < \dots < 4n - 1\}$, but it is $(\sigma^{(1)})^{-1}$ that respects the ordering of C_2 , since when going around $T_n^{(1)}$ counterclockwise, we go around $\tau_n^{(2)}$ clockwise

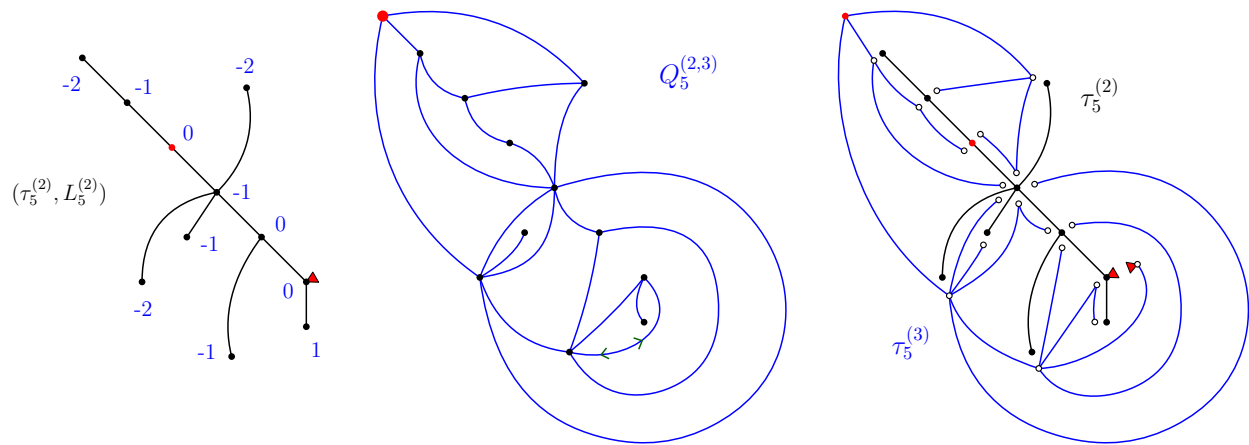


Figure 20: The tree $\tau_n^{(2)}$ is in turn provided with a labeling $L_5^{(2)}$ (left), from which a second rooted pointed planar quadrangulation $Q_5^{(2,3)}$ is built (middle), as well as a third tree $\tau_5^{(3)}$ (right).

$Q_n^{(2,3)}$, which has $2n + 2$ vertices, is “folded” repeatedly into an object that has only $n + 3$ vertices (those of $T_n^{(1)}$ plus the pointed vertices of $\tau_n^{(2)}$ and $\tau_n^{(3)}$), thus the name *discrete feuilletage*.

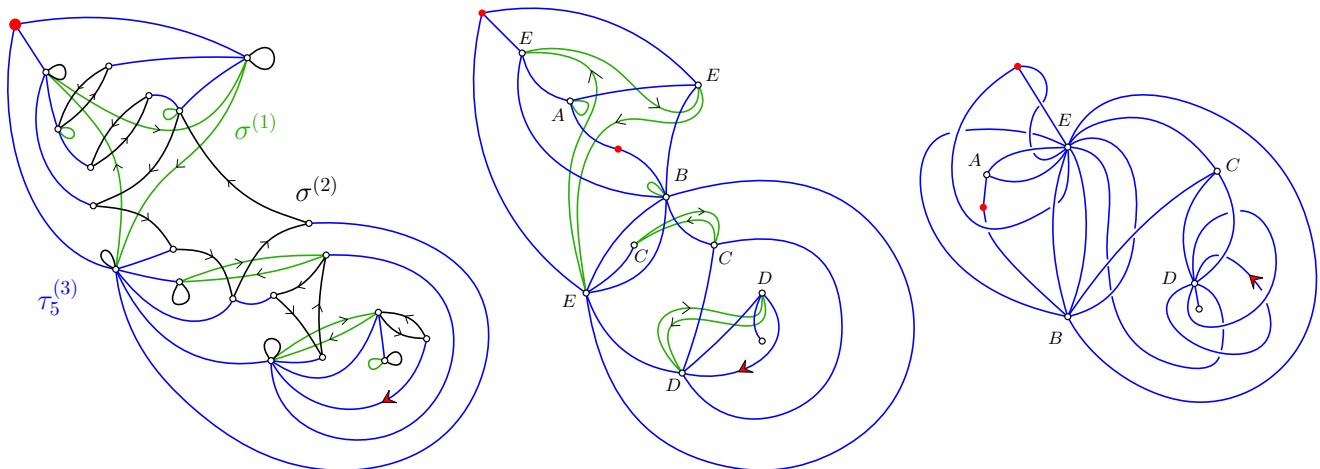


Figure 21: The 3-discrete feuilletage $R_5[3]$, represented as $\tau_5^{(3)}$ glued by $\sigma^{(2)}$ and $\sigma^{(1)}$ (left), as $Q_5^{(2,3)}$ (up to a rerooting) glued by $\sigma^{(1)}$ (middle), and as the completely folded object (right).

With the convention of this section, *the discrete feuilletage $R_n[3]$ has one oriented edge and is pointed twice*, from the pointed vertices of $Q_n^{(2,3)}$ and $Q_n^{(1,2)}$. Its oriented edge induces a rooting on $Q_n^{(2,3)}$, $Q_n^{(1,2)}$, $\tau_n^{(3)}$, $\tau_n^{(2)}$, and $T_n^{(1)}$: the oriented edge in $R_n[3]$ is also an oriented edge of $\tau_n^{(3)}$ (a root corner) of $\tau_n^{(3)}$. The root corner of $\tau_n^{(2)}$ is the one facing the root corner of $\tau_n^{(3)}$ (its dual corner), and the root corner of $T_n^{(1)}$ is that dual to the root corner of $\tau_n^{(2)}$. In $\tau_n^{(3)}$, there is only one edge incident to the root vertex, and whose other endpoint is closer to the pointed vertex. To root of $Q_n^{(2,3)}$, orient this edge in one or the other way, according to the parameter η of the CVS bijection. The root of $Q_n^{(1,2)}$ is obtained the same way in $\tau_n^{(2)}$.

This process can be iterated: dressing $\tau_n^{(D)}$ with a labeling, we build a rooted planar tree $\tau_n^{(D+1)}$ with $2^D n$ edges and $n + D$ vertices, such that $\tau_n^{(D+1)}$ identified according to the vertices of $\tau_n^{(D)}$ (up to a

rerooting) forms a pointed rooted planar map $Q_n^{(D,D+1)} \in \mathbb{Q}_{2^{D+1}}^{\bullet, \rightarrow}$, and a discrete feuilletage $R_n[D+1]$ is obtained by identifying the vertices of every $\tau_n^{(j)}$ for $3 \leq j \leq D+1$ according to the vertices of $\tau_n^{(j-1)}$ (and $T_n^{(1)}$ for $j=2$). A D -discrete feuilletage has one oriented edge, which induces a rooting of all the iterated trees $\tau_n^{(j)}$ and all the quadrangulations $Q_n^{(j,j+1)}$, and is pointed $D-1$ times, once per quadrangulation $Q_n^{(j,j+1)}$. Some examples of planar maps $Q_n^{(j,j+1)}$ are shown in Fig. 22 for $j=1, 2, 3, 4$.

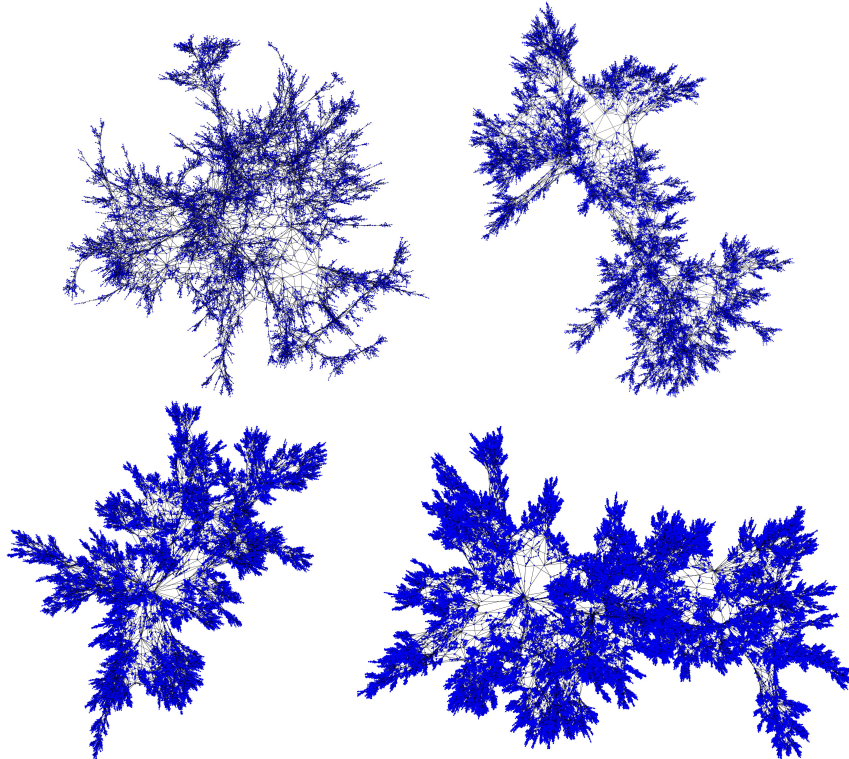


Figure 22: Planar maps $Q_n^{(j,j+1)}$ obtained thanks to some labeled trees $(C_{25000}^{(j)}, L_{25000}^{(j)})$ for j from 1 to 4. Loops and multiples edges have been discarded for the drawing (first map has 25000 vertices, and for the next ones 50k, 100k, 200k). The diameter of $Q_n^{(j,j+1)}$ scales asymptotically as $n^{1/2^{j+1}}$.

6.5 Asymptotic enumeration of discrete iterated feuilletages - pointed or not

From the CVS bijection, the numbers $m_n^{\bullet(2)}$ and $m_n^{(2)}$ of rooted planar quadrangulations respectively pointed or not are obtained as

$$m_n^{\bullet(2)} = 2 \times 3^n C_n \quad \text{and} \quad (n+2)m_n^{(2)} = 2 \times 3^n C_n, \quad (6.2)$$

where the factor 2 is due to the parameter η in the CVS bijection, $n+2$ comes from the choice of a pointed vertex in the quadrangulation $Q_n^{(1,2)}$ which has $n+2$ vertices, and 3^n from the choice of labeling.

Using the fact that asymptotically, $C_n \sim \pi^{-1/2} 4^n n^{-3/2}$, we obtain the asymptotics for $m_n^{\bullet(2)}$ and $m_n^{(2)}$:

$$m_n^{\bullet(2)} \sim 2\pi^{-1/2} \cdot 12^n \cdot n^{\gamma_2-1} \quad \text{and} \quad m_n^{(2)} \sim 2\pi^{-1/2} \cdot 12^n \cdot n^{\gamma_2-2}, \quad (6.3)$$

with $\gamma_2 = -1/2$. As this critical exponent is important in theoretical physics discussions, we compute it in the case of iterated discrete feuilletages, pointed or not.

We do not provide exact formulas for the numbers $m_n^{\bullet(D)}$ and $m_n^{(D)}$ of rooted D -discrete feuilletages, pointed or not. This depends for instance on how exactly the $D - 1$ pointed vertices are chosen in $R_n[D]$: among the $n + D$ vertices of $R_n[D]$ (laying a factor $(n + D)^{D-1}$) or on the various quadrangulations (laying a factor $\prod_{j=0}^{D-2} (2^j n + 2)$). In any case, the asymptotics for these numbers behave as

$$m_n^{\bullet(D)} \sim c_D \cdot \lambda_D^n \cdot n^{\gamma_D + D - 3} \quad \text{and} \quad m_n^{(D)} \sim c_D \cdot \lambda_D^n \cdot n^{\gamma_D - 2}, \quad (6.4)$$

where λ_D has a contribution of the form $3^{\sum_{j=0}^{D-2} 2^j}$ coming from the enumeration of the iterative labelings of the trees, and the critical exponent is now

$$\gamma_D = \frac{3}{2} - D. \quad (6.5)$$

which generalizes the well-known exponents for rooted planar trees and maps: $\gamma_1 = 1/2$ and $\gamma_2 = -1/2$.

6.6 New proposition for the notion of D -combinatorial map

6.6.1 A simple encoding using nested permutations.

In the previous sections, a combinatorial structure has emerged in the iterated encoding of the discrete feuilletages. We propose to define a notion of “ D -general feuilletages” which encompasses our notion of D -discrete feuilletages. We omit some constraints on the trees $\tau_n^{(j)}$ or the permutations $\sigma^{(j)}$ (their Kreweras complements are matchings, for instance), but we keep the fact that the permutations involved are non-crossing, as it provides an important restriction, which for $D = 2$ is equivalent to the planarity of the map. Denoting the non-zero elements of the support of $\sigma^{(j)}$ by $i_1, \dots, i_{p_j} \in \mathbb{N}^*$, where $p_j \geq 1$, we consider the ordered set $C_j = \{0 < i_1 < \dots < i_{p_j}\}$ and introduce the notation C_j^* which refers to the ordered set C_j if j and D have the same parity, and to the ordered set $\overline{C}_j = \{0 < i_{p_j} < i_{p_j-1} \dots < i_1\}$ otherwise, so that $\sigma^{(j)}$ is a non-crossing permutation on C_j^* iff it induces a non-crossing partition on C_j (and thus on C_D), and it respects the ordering of C_D if $j \equiv 0 \pmod{D}$ and of \overline{C}_D if $j \equiv 1 \pmod{D}$.

Definition 6.6. *We call D -general feuilletage a $(D + 1)$ -uplet*

$$\mathcal{M}_D = (C_D, \sigma^{(D)}, \dots, \sigma^{(1)}),$$

where C_D is the totally ordered set $C_D = \{0 < 1, \dots < N - 1\}$, and for $1 \leq j \leq D$, $\sigma^{(j)}$ is a permutation on C_D whose support C_j is included in that of $\sigma^{(j+1)}$ (the permutations are nested), $\sigma^{(j)}$ is a non-crossing permutation on C_j^* , and it has one element per disjoint cycle of $\sigma^{(j+1)}$ at most.

For any j , $(C_j^*, \sigma^{(j)})$ is a planar tree (the tree $\tau_n^{(j)}$ for a discrete feuilletage). Gluing the vertices of this planar tree according to the cycles of $\sigma^{(j-1)}$ provides a planar map, encoded by the triplet $(C_j^*, \sigma^{(j)}, \sigma^{(j-1)})$ (for a discrete feuilletage, it is a planar quadrangulation $Q_n^{(j-1, j)}$)²¹.

Given a D -general feuilletage \mathcal{M}_D and for $k > 1$, $\mathcal{M}_{D-k+1} = (C_D, \sigma^{(D)}, \dots, \sigma^{(k)})$ is of the same form as a $(D - k + 1)$ -general feuilletage. For $k = D$, it is a planar tree, for $k = D - 1$ it is a planar map, and \mathcal{M}_{D-k+1} is obtained from \mathcal{M}_{D-k} by folding it as many times as there are disjoint cycles in $\sigma^{(k)}$.

²¹The reason why we don't have to shift the first element of the set C_j to obtain the various iterated trees is precisely because the root of $\tau_n^{(j+1)}$ is chosen as dual to the root of $\tau_n^{(j)}$.

6.6.2 Comparison with other notions of D -dimensional combinatorial maps

Colored triangulations D -dimensional triangulation whose vertices are colored from 0 to D are called colored triangulations (in this section, by “triangulation” we always mean D -dimensional triangulation). Such triangulations appear in the topology (see [31] for a survey) and combinatorial (see e.g. [7, 18]) literature, and in the context of random tensor models [35, 36]. Orientable colored triangulations are fully encoded by the 1-skeleton of their dual graphs, which are $(D + 1)$ -regular bipartite graphs, whose edges carry a color $c \in \{0, \dots, D\}$, so that the vertices have one incident edge of each color, and one only. Labeling for instance the edges of color 0 in the graph, we see that a D -dimensional orientable colored triangulation is fully encoded by a D -uplet of permutations (ρ_1, \dots, ρ_D) acting on the n white vertices, so that $\rho_c(k) = p$ iff an edge of color c goes between the white vertex k and the black vertex p .²²

D -G-maps and D -maps In the constructions by Lienhardt [49], two generalizations of maps are defined on a set of darts \mathcal{D} . We give the encodings without further comments, and refer to the references for more details. A **D -G-map** without boundary is defined by a $(D + 2)$ -uplet $(\mathcal{D}, \alpha_0, \dots, \alpha_D)$, where the α_i are involutions on \mathcal{D} without fixed points. In addition, for $0 \leq i \leq D - 2$ and $i + 2 \leq j \leq D$, $\alpha_i \alpha_j$ is also an involution. The involution α_0 gathers the darts into edges, α_1 groups edges into vertices of valency 2, thus forming faces, α_2 groups pairs of edges together thus gluing faces, and so on. When two darts are identified by α_2 , the other two-darts of the edges they belong to, must also be paired by α_2 , so that $\alpha_0 \alpha_2$ is an involution, and so on. A **D -map** is defined similarly as a $(D + 1)$ -uple $(\mathcal{D}, \alpha_0, \dots, \alpha_{D-1})$, where for $1 \leq i \leq D - 2$, α_i is an involution on \mathcal{D} without fixed point, α_{D-1} is a permutation on \mathcal{D} , and for $0 \leq i < i + 2 < j \leq D - 1$, $\alpha_i \alpha_j$ is an involution. D -maps are suited to describe oriented spaces.

6.6.3 Concluding remarks regarding the physics motivations

The notion of D -generalized maps we introduced in Defi. 6.6 has in common with the other notions reviewed above, their encodings by D or $(D + 1)$ -uplets of permutations, with different types of constraints. As mentioned in Sec. 1.4 of the introduction, the issue in trying to obtain interesting asymptotic objects as limits of random generalized maps in the cases just reviewed, is to find a suitable criterion of selection for generalized maps such that uniform generalized maps in the corresponding subset have interesting asymptotic properties. Indeed, as mentioned in the introduction, the approaches based on the curvature or on the topology so far lead to Aldous’ CRT, to the Brownian map, or to degenerated spaces for which likely a continuum limit cannot be obtained [15, 4, 66] (and references in the introduction).

So far, there are no known ways to obtain continuum scaling limits other than trees or surfaces from the generalized maps just reviewed, a necessary and important step towards building discrete statistical theories with interest for quantum gravity.

On the other hand, although we do not have a representation of the iterated random discrete feuilletages as gluings of elementary D -dimensional volumes, because of our construction based on iterated trees which renders difficult the identification of elementary volumes (see the end of Sec. 6.1), we do have the convergence to the iterated random continuous feuilletages, which are not expected to be continuum random trees or surfaces. This has not been obtained in any physics approach to the problem, and in this

²²Note that there is a bijection [10, 9, 50, 32] between orientable colored triangulations labeled this way and so-called $(D + 1)$ -constellations [12] (but without any assumption on the genus of the constellation).

respect, we believe that our construction provides an important step in the context of discrete approaches to quantum gravity. We recall here that as mentioned in the introduction, it is not clear from a physics point of view whether the continuum limiting space should have a spherical topology, only that general relativity should be recovered in some “coarse-grained” limit (see the footnote 9). Clearly, the definition of (toy-)models leading to limiting spaces other than trees or surfaces is a necessary step towards defining such a coarse-graining procedure and understanding the properties (topology, fractal properties...) that the limiting continuum space should have to recover general relativity.

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7 Appendix

7.1 Folding a planar map to get a 3D object?

This informal section is devoted to explaining that a construction similar to that of $R_N[3]$ allow constructing “3D discrete objects”, for a certain notion of 3D explained below.

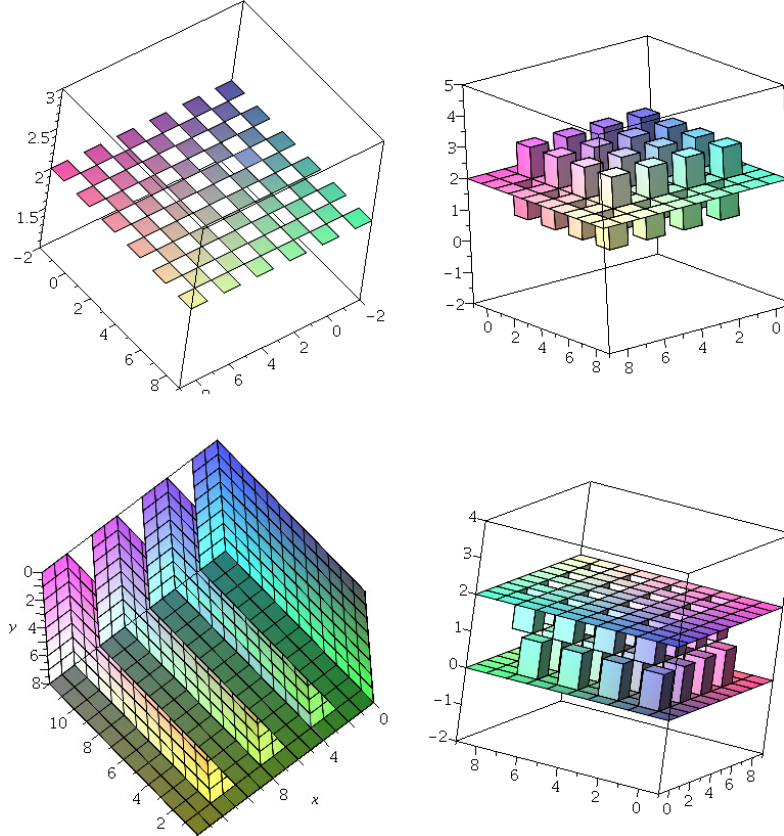


Figure 23: Folding a planar map to get a 3D object.

First, consider the graph G_n having as vertex set the points of the discrete cube $\{1, \dots, n\}^3$, and as edges, the pairs of points at Euclidean distance 1 from each other. In the sense above, G_n is a “3D-spherical-like” graph, as it is the 1-skeleton of a gluing of cubes which discretizes a 3-ball. And more generally, it may be argued that any reasonable definition of spherical graph dimension should give dimension 3 or higher to a graph having G_n as subgraph. Now, we will present G_n as a refolded map, very similar to some elements of $R_N(3)$ for some N .

The idea is the following: consider the chessboard type figure represented in Picture 1, Fig. 23, obtained by taking a section of \mathbb{R}^2 (or \mathbb{Z}^2), in which a unit square out of two has been removed. Each of the lacking square is then used in Picture 2 as the basis of a cube with the same lacking face: half of these cubes are placed above the plane, half below in such a way that the west-south corners of the cubes above (resp. below) forms, up to some translation, a section of $(2\mathbb{Z})^2$. The obtained object is a quadrangulation (with a big square boundary) since all its inner faces have degree 4. Now, change a bit of point of view in Picture 2, and view what is represented as a kind of fabric with a texture: the fabric being the plane, the texture being made of cubes above and below the fabric. Now, imagine a large piece of fabric which is folded and sewn as on Picture 3: it is a quadrangulation with 4 “layers”, each of them made by two large strips of fabric “at distance 2”. Each layer is also at distance 2 from the next layer. Here, “distance 2” has to be understood for the usual metric in \mathbb{R}^3 since clearly this picture can be embedded isometrically in \mathbb{R}^3 . Now add the texture to the fabric! Picture 4 figures what happens inside one layer (between two strips that are face to face) or between two layers of the fabric (outside the big quadrangulations,

between the two strips that are face to face). Up to some details concerning the parity of the strips, the cubes that have size 1 and which are placed on “planes” at distance 2, intersect at their vertices only. Observe again Picture 4, and imagine the \mathbb{Z}^3 lattice in between the two planes: notice that every edge of the lattice belongs to exactly one of the cubes. What is shown on Picture 4 represents what happens inside each of the 4 layers, and between 3 inter-layers of Picture 3, so, it concerns a total section of width 13 of \mathbb{Z}^3 .

Hence, it is possible to fold and identify vertices in quadrangulations to construct some objects whose underlying graphs contain G_n .

As a matter of fact, it may be argued that this example is not in the set $\mathbf{R}_N[3]$ for any N large (one of the reasons being that the extracted trees (t_2, t_3) of the textured fabric with the Cori-Vauquelin-Schaeffer bijection provides a tree t_2 whose height process has not the required increments $+1, -1$ or 0). But we hope that it illustrates the fact that such “3D-like objects” can arise in the construction we propose.