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# Structural Invariants for Parametric Verification of Systems with Almost Linear Architectures

Marius Bozga, Radu Iosif and Joseph Sifakis

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We consider concurrent systems consisting of a finite but unknown number of components, that are replicated instances of a given set of finite state automata. The components communicate by executing interactions which are simultaneous atomic state changes of a set of components. We specify both the type of interactions (e.g. rendez-vous, broadcast) and the topology (i.e. architecture) of the system (e.g. pipeline, ring) via a decidable interaction logic, which is embedded in the classical weak sequential calculus of one successor (WS1S). Proving correctness of such system for safety properties, such as deadlock freedom or mutual exclusion, requires the inference of an inductive invariant that subsumes the set of reachable states and avoids the unsafe states. Our method synthesizes such invariants directly from the formula describing the interactions, without costly fixed point iterations. We applied our technique to the verification of several textbook examples, such as dining philosophers, mutual exclusion protocols and concurrent systems with preemption and priorities.

## 1 Introduction

The problem of parametric verification asks whether a system composed of  $n$  replicated processes is safe, for all  $n \geq 2$ . By safety we mean that every execution of the system stays clear of a set of global error configurations, such as deadlocks or mutual exclusion violations. Even if we assume each process to be finite-state and every interaction to be a synchronization of actions without data exchange, the problem remains challenging because we want a general proof of safety, that works for any number of processes.

In general, parametric verification is undecidable if unbounded data is exchanged [5], while various restrictions of communication (rendez-vous) and architecture<sup>1</sup> (ring, clique) define decidable subproblems [14,21,20,4]. Seminal works consider *rendez-vous* communication, allowing a fixed number of participants [14,21,20], placed in a ring [14,20] or a clique [21]. Recently, MSO-definable graphs (with bounded tree- and cliquewidth) and point-to-point rendez-vous communication were considered in [4].

Most approaches to decidability focus on computing a *cut-off* bound  $c$ , that reduces the verification problem from  $n \geq 2$  to at most  $c$  processes [14,20]. Other methods identify systems with well-structured transition relations, for which symbolic enumeration of reachable states is feasible [1] or reduce to known decidable problems, such as reachability in vector addition systems [21]. When theoretical decidability is not of concern, semi-algorithmic techniques such as *regular model checking* [25,2], SMT-based *bounded model checking* [3,17], *abstraction* [9,12] and *automata learning* [15] can be

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<sup>1</sup> We use the term architecture for the shape of the graph along which the interactions take place.

used to deal with more general classes of systems. An exhaustive chart of existing parametric verification techniques is drawn in [11].

The efficiency of a semi-algorithmic method crucially relies on its ability of synthesizing an *inductive safety invariant*, that is an infinite set of global configurations, which contains the initial configurations, is closed under the transition relation, and excludes the error configurations. In general, automatically synthesizing invariants requires computationally expensive fixpoint iterations [18]. In the particular case of parametric systems, invariants can be either *global*, relating the local states of all processes [19], or *modular*, relating the states of few processes, of unimportant identities [29,16].

We focus on parametric systems described using the Behavior-Interaction-Priorities (BIP) framework [8], in which processes are instances of finite-state *component types*, whose interfaces are sets of *ports*, labeling transitions between local states, and interactions are sets of strongly synchronizing ports, described by formulae of an *interaction logic*. An interaction formula captures the architecture of the interactions (pipeline, ring, clique, tree) and the communication scheme (rendez-vous, broadcast), which are not hardcoded, but rather specified by the system designer.

As a distinguishing feature, we synthesize invariants directly from the interaction formula of a system, without iterating its transition relation. Such invariants depend only on the structure (and not on the operational semantics) of the interaction network, described by a Petri Net of unbounded size, being thus *structural* invariants. Essentially, the invariants we infer use the *traps*<sup>2</sup> of the system, which are sets  $W$  of local states with the property that, if a process is in a state from  $W$  initially, then always some process will be in a state from  $W$ . We call these invariants *trap invariants* [10,13].

Inferring trap invariants from interaction formulae relies on two logical operations: (a) the *positivation* operation, producing a weaker formula with the same minimal models, and (b) the *dualization* operation, that essentially switches the conjunctions with disjunctions and the universal with existential quantifiers. Although dualization is just a linear time syntactic transformation of formulae, positivation is a more involved operation, depending on the semantics of the underlying logic. A definition of positivation for a simple interaction logic, relying on equalities and disequalities between process indices to describe clique architectures, is provided in [13].

**Our Contribution** This paper describes a non-trivial generalization of the method from [13], that considers an interaction logic with equality and uninterpreted monadic predicate symbols, which is embedded into the combined theory of sets and Presburger cardinality constraints [27]. In addition, here we introduce a cyclic (modulo- $n$ , where  $n$  is the unbounded parameter of the system) successor function and embed our logic in the weak monadic logic of one successor (WS1S), for which validity of a formula boils down to proving language emptiness of a finite Rabin-Scott automaton built from that formula. This new logic naturally describes systems with ring and pipeline, as well as previously considered clique/multiset architectures. Moreover, we provide an example showing that the method can be easily generalized to handle tree-like architectures.

The trap invariants method is incomplete, meaning that there exists parametric systems that are safe for any number of components, but whose trap invariant does not suffice to prove safety. We deal with this problem by computing universal Ashcroft in-

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<sup>2</sup> Called in this way by analogy with the notion of traps for Petri Nets [30].

variants [6], able to add extra constraints inferred by restricting the interaction formula of the parametric system to a fixed set of symbolic components. This technique is orthogonal to the trap invariant computation and resembles the computation of *invisible invariants* [29], but tailored to the BIP framework we have chosen to work with.

**Running Example** Consider the dining philosophers system in Fig. 1, consisting of  $n \geq 2$  components of type Fork and Philosopher respectively, placed in a ring of size  $2n$ . The  $k$ -th philosopher has a left fork, of index  $k$  and a right fork, of index  $(k + 1) \bmod n$ . Each component is an instance of a finite state automaton with states  $f$ (ree) and  $b$ (usy) for Fork, respectively  $w$ (aiting) and  $e$ (ating) for Philosopher. A fork goes from state  $f$  to  $b$  via a  $t$ (ake) transition and from  $b$  to  $f$  via a  $\ell$ (eave) transition. A philosopher goes from  $w$  to  $e$  via a  $g$ (et) transition and from  $e$  to  $w$  via a  $p$ (ut) transition. In this example, we assume that the  $g$  action of the  $k$ -th philosopher is executed jointly with the  $t$  actions of the  $k$ -th and  $(k + 1) \bmod n$  forks, in other words, the philosopher takes both its left and right forks simultaneously. Similarly, the  $p$  action of the  $k$ -th philosopher is executed simultaneously with the  $\ell$  action of the  $k$ -th and  $[(k + 1) \bmod n]$ -th forks, i.e. each philosopher leaves both its left and right forks at the same time. We describe the interactions of the system by the following first order formula  $\Gamma_{philo} = \exists i . [g(i) \wedge t(i) \wedge t(succ(i))] \vee [p(i) \wedge \ell(i) \wedge \ell(succ(i))]$ , where transition labels (ports) are encoded as monadic predicate symbols and  $succ(\cdot)$  is the function symbol which denotes the successor of an index in the ring. Each interaction is defined by a model of this formula, for instance the structure interpreting  $g$  as the set  $\{k\}$  and  $t$  as the set  $\{k, (k + 1) \bmod n\}$  corresponds to the interaction of the  $k$ -th philosopher taking its forks, where  $0 \leq k < n$  is some index. The ring topology is implicit in the modulo- $n$  interpretation of the successor function  $s$  as each  $k$ -th component interacts with its  $k$ -th and  $succ(k)$ -th neighbours only.

Intuitively, the system is deadlock-free for any  $n \geq 2$  since there is no circular waiting scenario involving all the philosophers at once. A rigorous proof requires an invariant disjoint from the set of deadlock states, defined by the formula  $\Delta(\Gamma_{philo}) = \forall i . [\neg w(i) \vee \neg f(i) \vee \neg f(succ(i))] \wedge [\neg e(i) \vee \neg b(i) \vee \neg b(succ(i))]$ . Our method computes a trap invariant corresponding to the set of solutions of the following constraint

$\Theta(\Gamma_{philo}) = \forall i . w(i) \vee f(i) \vee f(succ(i)) \leftrightarrow e(i) \vee b(i) \vee b(succ(i))$ , derived from the interaction formula  $\Gamma$  and the local structure of the component types. Together with an automata-based decision procedure for the interaction logic, this invariant allows to prove deadlock freedom for the system in Fig. 1 in  $\sim 0.1$  seconds on an average machine.

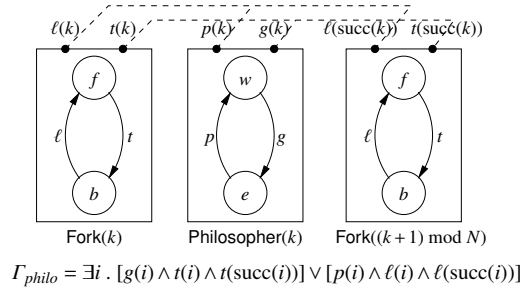


Fig. 1: Parametric Dining Philosophers

## 2 Parametric Component-based Systems

A *component type* is a tuple  $C = \langle P, S, s_0, \Delta \rangle$ , where  $P = \{p, q, r, \dots\}$  is a finite set of *ports*,  $S$  is a finite set of *states*,  $s_0 \in S$  is an initial state and  $\Delta \subseteq S \times P \times S$  is a set of *transitions*

$s \xrightarrow{p} s'$ . To simplify the upcoming technical details, we assume there are no two different transitions with the same port and write, for a port  $p$  that labels a transition  $s \xrightarrow{p} s' \in \mathcal{A}$ ,  $\bullet p$  and  $p\bullet$  for the source  $s$  and destination  $s'$  of that (unique) transition, respectively.

A *component-based system*  $\mathcal{S} = \langle C^1, \dots, C^K, \Gamma \rangle$  consists of a fixed number ( $K$ ) of component types  $C^k = \langle P^k, S^k, s_0^k, \mathcal{A}^k \rangle$  and an *interaction formula*  $\Gamma$ . We shall sometimes write  $P(C^k)$ ,  $S(C^k)$ ,  $s_0(C^k)$  and  $\mathcal{A}(C^k)$  for  $P^k$ ,  $S^k$ ,  $s_0^k$  and  $\mathcal{A}^k$ , respectively. Without loss of generality, we assume that  $P^i \cap P^j = \emptyset$ , for all  $1 \leq i < j \leq K$  and unambiguously denote by  $\text{type}(p) \stackrel{\text{def}}{=} C^k$  the component type of a port  $p \in P^k$ . For instance, in Fig. 1 we have  $\text{type}(g\ell) = \text{type}(gr) = \text{type}(p) = \text{Philosopher}$  and  $\text{type}(g) = \text{type}(\ell) = \text{Fork}$ .

The interaction logic intentionally uses the names of the ports and states, here viewed as monadic predicate symbols  $\text{Pred} = \bigcup_{k=1}^K (P^k \cup S^k)$ , where  $\text{pr} \in \text{Pred}$  is an arbitrary predicate symbol. In addition, we consider a countable set  $\text{Var}$  of first-order variables and a set of constant symbols  $\text{Const}$ . The only function symbol of arity greater than zero is  $\text{succ}(\cdot)$ , denoting the *successor* function. Interaction formulae are written in the syntax of *Interaction Logic with One Successor* (IL1S), given below:

$$\begin{aligned} t &:= x \in \text{Var} \mid c \in \text{Const} \mid \text{succ}(t) && \text{terms} \\ \phi &:= t_1 \leq t_2 \mid \text{pr}(t) \mid \phi_1 \wedge \phi_2 \mid \neg \phi_1 \mid \exists x. \phi_1 && \text{formulae} \end{aligned}$$

A *sentence* is a formula in which all variables are in the scope of a quantifier. A formula is *positive* if each predicate symbol occurs under an even number of negations and *ground* if it contains no variables. We write  $t_1 = t_2$  for  $t_1 \leq t_2 \wedge t_2 \leq t_1$ ,  $\phi_1 \vee \phi_2$  for  $\neg(\neg\phi_1 \wedge \neg\phi_2)$ ,  $\phi_1 \rightarrow \phi_2$  for  $\neg\phi_1 \vee \phi_2$ ,  $\phi_1 \leftrightarrow \phi_2$  for  $\phi_1 \rightarrow \phi_2 \wedge \phi_2 \rightarrow \phi_1$ ,  $\forall x. \phi$  for  $\neg\exists x. \neg\phi$ .

For a positive integer  $n > 0$ , we denote by  $[n]$  the set  $\{0, \dots, n-1\}$ . We interpret IL1S formulae over structures  $\mathcal{I} = ([n], \iota, \nu)$ , where  $[n]$  is the universe,  $\iota : \text{Const} \cup \text{Pred} \rightarrow [n] \cup 2^{[n]}$  maps constant symbols into elements and predicate symbols into subsets of  $[n]$ , respectively, and  $\nu : \text{Var} \rightarrow [n]$  maps variables into elements of  $[n]$ . The successor function symbol  $\text{succ}(\cdot)$  is always interpreted in  $\mathcal{I}$  by the function  $\text{succ}_{\text{IL1S}}^n(x) = (x+1) \bmod n$  and the inequality relation by the set  $\{(u, v) \in [n] \times [n] \mid u \leq v\}$ . With these definitions, the truth value of a formula  $\phi$  in  $\mathcal{I}$  is defined recursively on the structure of  $\phi$  and we write  $\mathcal{I} \models \phi$  when this value is true.

**Remark** We adopted a circular modulo- $n$  interpretation of the successor function, to naturally accomodate ring-like architectures, common in distributed system design practice. This is not a restriction, because clique architectures, where every component can interact with any other, can be described using only equality and disequality atoms. Moreover, acyclic pipeline architectures can be described using the order relation, as follows: we identify a least and a greatest element in the domain, namely  $\text{inf}(x) \stackrel{\text{def}}{=} \forall y. x \leq y$  and  $\text{sup}(x) \stackrel{\text{def}}{=} \forall y. y \leq x$ , and describe interactions only for indices that are between those elements. For the set of indices  $x$  such that  $\exists \zeta \exists \xi. \text{inf}(\zeta) \wedge \text{sup}(\xi) \wedge \zeta \leq x \wedge x < \xi$  holds, the graph of the successor function is acyclic. Further, in §3.1 we show that IL1S can be extended with equalities modulo constants, such as the *even* and *odd* predicates, without changing the invariant synthesis method upon which our verification technique is based. Finally, in §3.2, we argue that tree architectures can be fitted in this framework, with minimal changes. This claim is sustained by an example in §5. ■

One of the consequences of the modulo- $n$  interpretation of the successor function symbol is the existence of a IL1S formula that states the exact cardinality of the model:  $\exists x. \text{succ}^k(x) = x \wedge \bigwedge_{i=1}^{k-1} \neg \text{succ}^i(x) = x$ . This formula is true if and only if the cardinality

of the universe equals the constant  $k$ . Since the purpose of IL1S is to specify interactions in a system whose number of components is arbitrary, we shall restrict interaction formulae to finite disjunctions of formulae of the form below:

$$\exists x_1 \dots \exists x_\ell . \varphi \wedge \bigwedge_{j=1}^{\ell} p_j(x_j) \wedge \bigwedge_{j=\ell+1}^{\ell+m} \forall x_j . \psi_j \rightarrow p_j(x_j) \quad (1)$$

where  $\varphi, \psi_{\ell+1}, \dots, \psi_{\ell+m}$  are conjunctions of inequalities involving index variables, such that no comparison between terms with the same variable is allowed, i.e.  $\varphi$  and  $\psi_j$  do not contain atomic propositions of the form  $\text{succ}^i(x) \leq \text{succ}^j(x)$  for  $i, j > 0$ . Moreover, we assume that  $\text{type}(p_i) = \text{type}(p_j) \Rightarrow p_i = p_j$ , for all  $1 \leq i < j \leq \ell + m$ , i.e. the formula does not specify interactions between different ports of the same component type<sup>3</sup>.

Informally, the formula (1) states that at most  $\ell$  components can simultaneously engage in a multiparty rendez-vous, together with a broadcast to the ports  $p_{\ell+1}, \dots, p_{\ell+m}$  of the components whose indices satisfy the constraints  $\psi_{\ell+1}, \dots, \psi_{\ell+m}$ , respectively. An example of peer-to-peer rendez-vous with no broadcast is the dining philosophers system in Fig. 1, whereas examples of broadcast are found among the test cases in §5.

## 2.1 Execution Semantics of Component-based Systems

The semantics of a component-based system is defined by a 1-safe Petri Net, whose (reachable) markings and actions characterize the (reachable) global states and transitions of the system. For reasons of self-completeness, we recall below several basic definitions relative to Petri Nets.

Formally, a *Petri Net* (PN) is a tuple  $\mathbf{N} = \langle S, T, E \rangle$ , where  $S$  is a set of *places*,  $T$  is a set of *transitions*,  $S \cap T = \emptyset$ , and  $E \subseteq S \times T \cup T \times S$  is a set of *edges*. The elements of  $S \cup T$  are called *nodes*. Given nodes  $x, y \in S \cup T$ , we write  $E(x, y) \stackrel{\text{def}}{=} 1$  if  $(x, y) \in E$  and  $E(x, y) \stackrel{\text{def}}{=} 0$ , otherwise. For a node  $x$ , let  $\bullet x \stackrel{\text{def}}{=} \{y \in S \cup T \mid E(y, x) = 1\}$ ,  $x^\bullet \stackrel{\text{def}}{=} \{y \in S \cup T \mid E(x, y) = 1\}$  and lift these definitions to sets of nodes, as usual.

A *marking* of  $\mathbf{N}$  is a function  $m : S \rightarrow \mathbb{N}$ . A transition  $t$  is *enabled* in  $m$  if and only if  $m(s) > 0$  for each place  $s \in \bullet t$ . The *transition relation* of  $\mathbf{N}$  is defined as follows. For all markings  $m, m'$  and all transitions  $t$ , we write  $m \xrightarrow{t} m'$  whenever  $t$  is enabled in  $m$  and  $m'(s) = m(s) - E(s, t) + E(t, s)$ , for all  $s \in S$ . Given two markings  $m$  and  $m'$ , a finite sequence of transitions  $\sigma = t_1, \dots, t_n$  is a *firing sequence*, written  $m \xrightarrow{\sigma} m'$  if and only if either (i)  $n = 0$  and  $m = m'$ , or (ii)  $n \geq 1$  and there exist markings  $m_1, \dots, m_{n-1}$  such that  $m \xrightarrow{t_1} m_1 \dots m_{n-1} \xrightarrow{t_n} m'$ .

A *marked Petri net* is a pair  $\mathcal{N} = (\mathbf{N}, m_0)$ , where  $m_0$  is the *initial marking* of  $\mathbf{N}$ . A marking  $m$  is *reachable* in  $\mathcal{N}$  if and only if there exists a firing sequence  $\sigma$  such that  $m_0 \xrightarrow{\sigma} m$ . We denote by  $\mathcal{R}(\mathcal{N})$  the set of reachable markings of  $\mathcal{N}$ . A set of markings  $\mathcal{M}$  is an *invariant* of  $\mathcal{N} = (\mathbf{N}, m_0)$  if and only if  $m_0 \in \mathcal{M}$  and for each  $m \xrightarrow{t} m'$  such that  $m \in \mathcal{M}$ , we have  $m' \in \mathcal{M}$ . A marked PN  $\mathcal{N}$  is *1-safe* if  $m(s) \leq 1$ , for each  $s \in S$  and  $m \in \mathcal{R}(\mathcal{N})$ . All PNs considered in the following will be 1-safe and we shall silently blur the distinction between a marking  $m : S \rightarrow \{0, 1\}$  and the valuation  $v_m : S \rightarrow \{\perp, \top\}$  defined as  $v_m(s) = \top \iff m(s) = 1$ .

Turning back to the definition of the semantics of component-based parametric systems, let  $\mathcal{S} = \langle C^1, \dots, C^K, \Gamma \rangle$  be a system with component types  $C^k = \langle P^k, S^k, s_0^k, \Delta^k \rangle$ , for

<sup>3</sup> This restriction simplifies the technical presentation of the results and can be removed w.l.o.g.

all  $k = 1, \dots, K$ . For each parameter  $n \geq 1$ , we define a marked PN  $\mathcal{N}_S^n$ , of size  $O(n)$ , that characterizes the set of executions of the instance of  $\mathcal{S}$  having  $n$  replicas of each component type. Formally, given a positive integer  $n \geq 1$ , we have  $\mathcal{N}_S^n = (\mathbf{N}, \mathbf{m}_0)$ , where  $\mathbf{N} \stackrel{\text{def}}{=} \langle \bigcup_{k=1}^K \mathbf{S}^k \times [n], T, E \rangle$  and whose sets of transitions  $T$  and edges  $E$  are defined from the interaction formula  $\Gamma$ , as follows.

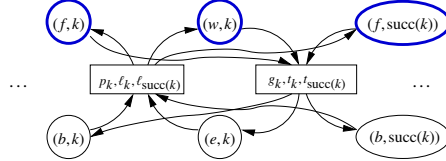


Fig. 2: Unbounded Marked Petri Net for the Dining Philosophers Example

First, we define the set of minimal models of  $\Gamma$ , where minimality is with respect to the pointwise inclusion of the sets that interpret the predicate symbols. Formally, given structures  $\mathcal{S}_1 = ([n], \nu_1, \iota_1)$  and  $\mathcal{S}_2 = ([n], \nu_2, \iota_2)$  sharing the same universe  $[n]$ , we have  $\mathcal{S}_1 \sqsubseteq \mathcal{S}_2$  if and only if  $\iota_1(\text{pr}) \subseteq \iota_2(\text{pr})$ , for all  $\text{pr} \in \text{Pred}$ . Given a formula  $\phi$ , a structure  $\mathcal{S}$  is a *minimal model* of  $\phi$  if  $\mathcal{S} \models \phi$  and, for all structures  $\mathcal{S}'$  such that  $\mathcal{S}' \sqsubseteq \mathcal{S}$  and  $\mathcal{S}' \neq \mathcal{S}$ , we have  $\mathcal{S}' \not\models \phi$ . We denote by  $\llbracket \phi \rrbracket^{\min}$  the set of minimal models of  $\phi$ . Two formulae  $\phi_1$  and  $\phi_2$  are *minimally equivalent*, written as  $\phi_1 \equiv^{\min} \phi_2$ , if and only if  $\llbracket \phi_1 \rrbracket^{\min} = \llbracket \phi_2 \rrbracket^{\min}$ .

Back to the definition of  $\mathcal{N}_S^n$ , for each minimal model  $\mathcal{I} = ([n], \nu, \iota) \in \llbracket \Gamma \rrbracket^{\min}$ , we have a transition  $t_I \in T$  and edges  $((s, i), t_I), (t_I, (s', i)) \in E$ , for all  $s \xrightarrow{p} s' \in \bigcup_{k=1}^K \Delta^k$ , such that  $i \in \iota(p)$ , and nothing else is in  $T$  or  $E$ . The initial marking of  $\mathcal{N}_S^n$  corresponds to the initial state of each components, formally for each  $1 \leq k \leq K$ , each  $s \in \mathbf{S}^k$  and each  $1 \leq i \leq n$ ,  $m_0((s, i)) = 1$  if  $s = s_0^k$  and  $m_0((s, i)) = 0$ , otherwise. For instance, Fig. 2 shows the PN for the system in Fig. 1, with the initial marking highlighted.

Below we give a property of the marked PNs that define the semantics of parametric component-based systems.

**Definition 1.** Given a component-based system  $\mathcal{S}$ , a marked PN  $\mathcal{N} = (\mathbf{N}, \mathbf{m}_0)$ , with  $\mathbf{N} = (\mathbf{S}, T, E)$ , is  $\mathcal{S}$ -decomposable if and only if there exists an integer  $n > 0$  such that  $\mathbf{S} = \bigcup_{k=1}^K \mathbf{S}^k \times [n]$  and in every reachable marking  $\mathbf{m} \in \mathcal{R}(\mathbf{N})$ , for each  $1 \leq i \leq n$  and each  $1 \leq k \leq K$  there exists exactly one state  $s \in \mathbf{S}^k$  such that  $\mathbf{m}((s, i)) = 1$ .

**Lemma 1.** The marked PN  $\mathcal{N}_S^n$  is  $\mathcal{S}$ -decomposable, for each component-based system  $\mathcal{S}$  and each integer  $n > 0$ .

*Proof:* Let  $\mathcal{S} = \langle C^1, \dots, C^K, \Gamma \rangle$  be a system with component types  $C^k = \langle \mathbf{P}^k, \mathbf{S}^k, s_0^k, \Delta^k \rangle$ , for all  $k = 1, \dots, K$ , and let  $n > 0$  be a parameter. Let  $\mathcal{N}_S^n = (\mathbf{N}, \mathbf{m}_0)$  and  $\mathbf{m} \in \mathcal{R}(\mathbf{N}_S^n)$  be a reachable marking. Then  $\mathbf{N} = (\bigcup_{k=1}^K \mathbf{S}^k \times [n], T, E)$ . We prove the property by induction on the length  $\ell$  of the shortest path from  $\mathbf{m}_0$  to  $\mathbf{m}$ . If  $\ell = 0$  the property holds because each component type  $1 \leq k \leq K$  has exactly one initial state  $s_0^k$  and only the states  $(s_0^k, i)$  are initially marked, for all  $1 \leq i \leq n$ . For the induction step  $\ell > 0$ , assume that  $\mathbf{m}' \xrightarrow{t} \mathbf{m}$  and the property of Definition 6 holds for  $\mathbf{m}'$ . Then there exists  $\mathcal{I} = ([n], \nu, \iota, \mu) \in \llbracket \Gamma \rrbracket^{\min}$ , such that  $t = t_I$  and, for each  $i \in [n]$  and each  $p \in \mathbf{P}^k$  such that  $s' \xrightarrow{p} s \in \Delta^k$  and  $i \in \iota(p)$ ,

there are edges  $((s', i), t_I), (t_I, (s, i)) \in E$ . Suppose, for a contradiction, that there exists  $1 \leq i_0 \leq n$  and  $1 \leq k_0 \leq K$  such that  $m((s, i_0)) = m((s'', i_0)) = 1$ , for two distinct states  $s, s'' \in \mathbf{S}^k$ . Then  $(t_I, (s, i_0)), (t_I, (s'', i_0)) \in E$  and  $i_0 \in \iota(p) \cap \iota(q)$ , for two transition rules  $s' \xrightarrow{p} s, s' \xrightarrow{q} s'' \in \mathcal{A}^k$ . However, this comes in contradiction with the assumption that a transition does not involve two different ports from the same component type (1).  $\square$

### 3 Computing Trap Invariants

We leverage from a standard notion in the theory of Petri Nets to define a class of invariants, that are useful for proving certain safety properties. Given a Petri Net  $\mathbf{N} = (S, T, E)$ , a set of places  $W \subseteq S$  is called a *trap* if and only if  $W^\bullet \subseteq {}^\bullet W$ . A trap  $W$  of  $\mathbf{N}$  is an *initially marked trap* (IMT) of the marked PN  $\mathcal{N} = (\mathbf{N}, m_0)$  if and only if  $m_0(s) = \top$  for some  $s \in W$ . An IMT of  $\mathcal{N}$  is minimal if none of its nonempty strict subsets is an IMT of  $\mathcal{N}$ . We denote by  $\text{Imt}(\mathcal{N}) \subseteq 2^S$  the set of IMTs of  $\mathcal{N}$ .

*Example 1.* Consider an instance of the marked PN in Fig. 2 for  $n = 2$ . For simplicity, we denote places  $(f, k)$ ,  $(w, k)$ ,  $(b, k)$  and  $(e, k)$  as  $f_k, w_k, b_k$  and  $e_k$  for  $k = 0, 1$ , respectively. The local states of each component form a minimal trap, i.e.  $\{f_k, b_k\}$  and  $\{w_k, e_k\}$  are traps, for  $k = 0, 1$ . In addition,  $\{w_0, e_0, w_1\}, \{w_0, b_1, w_1\}, \{f_0, b_0, e_1\}$  and  $\{f_1, b_0, e_1\}$  are also minimal traps.  $\blacksquare$

An IMT defines an invariant of the PN, because some place in the trap will always be marked, no matter which transition is fired. The *trap invariant* of  $\mathcal{N}$  is the set of markings that mark each IMT of  $\mathcal{N}$ . The trap invariant of  $\mathcal{N}$  subsumes the set of reachable markings of  $\mathcal{N}$ , because the latter is the least invariant of  $\mathcal{N}$ <sup>4</sup>. To prove that a certain set of markings is unreachable, it is sufficient to prove that the this set has empty intersection with the trap invariant. For self-completeness, we briefly discuss the computation of trap invariants for a given marked PN of fixed size, before explaining how this can be done for marked PNs defining the executions of parametric systems, which are of unknown sizes.

**Definition 2.** The trap constraint of a PN  $\mathbf{N} = (S, T, E)$  is the formula:

$$\Theta(\mathbf{N}) \stackrel{\text{def}}{=} \bigwedge_{t \in T} (\bigvee_{x \in {}^\bullet t} x) \rightarrow (\bigvee_{y \in t^\bullet} y)$$

where each place  $x, y \in S$  is viewed as a propositional variable.

It is not hard to show<sup>5</sup> that any boolean valuation  $\beta : S \rightarrow \{\perp, \top\}$  that satisfies the trap constraint  $\Theta(\mathbf{N})$  defines a trap  $W_\beta$  of  $\mathbf{N}$  in the obvious sense  $W_\beta = \{s \in S \mid \beta(s) = \top\}$ . Further, if  $m_0 : S \rightarrow \{0, 1\}$  is the initial marking of a 1-safe PN  $\mathbf{N}$  and  $\mu_0 \stackrel{\text{def}}{=} \bigvee_{m_0(s)=1} s$  is a propositional formula, then each minimal satisfying valuation of  $\mu_0 \wedge \Theta(\mathbf{N})$  defines a minimal IMT of  $(\mathbf{N}, m_0)$ , where minimality of boolean valuations is considered with respect to the usual partial order  $\beta_1 \leq \beta_2 \iff \forall s \in S. \beta_1(s) \rightarrow \beta_2(s)$ .

Usually, computing invariants requires building a sequence of underapproximants whose limit is the least fixed point of an abstraction of the transition relation of the

<sup>4</sup> Since invariants are closed under intersection, the least invariant is unique.

<sup>5</sup> See e.g. [7] for a proof.



system [18]. This is however not the case with trap invariants, that can be directly computed by looking at the structure of the system, captured by the trap constraint, and to the initial marking. To this end, we introduce two operations on propositional formulae. First, given a propositional formula  $\phi$ , we denote by  $(\phi)^+$  the result of deleting (i.e. replacing with  $\top$ ) the negative literals from the DNF of  $\phi$ . It is not hard to show that  $\phi \equiv^{\min} (\phi)^+$ , i.e. this transformation preserves the minimal satisfying valuations of  $\phi$ . We call this operation *positivation*.

Second, let  $\phi^\sim$  denote the result of replacing, in the negation normal form of  $\phi$ , all conjunctions by disjunctions and viceversa. Formally, assuming that  $\phi$  is in NNF, let:

$$(\phi_1 \wedge \phi_2)^\sim \stackrel{\text{def}}{=} \phi_1^\sim \vee \phi_2^\sim \quad (\phi_1 \vee \phi_2)^\sim \stackrel{\text{def}}{=} \phi_1^\sim \wedge \phi_2^\sim \quad (\neg s)^\sim \stackrel{\text{def}}{=} \neg s \quad s^\sim \stackrel{\text{def}}{=} s$$

For any boolean valuation  $\beta$ , we have  $\beta \models \phi \iff \bar{\beta} \models \neg(\phi^\sim)$ , where  $\bar{\beta}(s) \stackrel{\text{def}}{=} \neg\beta(s)$  for each propositional variable  $s$ . This operation is usually referred to as *dualization*.

The following lemma gives a straightforward method to compute trap invariants, logically defined by a CNF formula with positive literals only, whose clauses correspond to the (enumeration of the elements of the) traps. It is further showed that such a formula defines an invariant of the finite marked PN:

**Lemma 2.** *Given a marked PN  $\mathcal{N} = (\mathbf{N}, m_0)$ , we have  $\text{TrapInv}(\mathcal{N}) \equiv (((\mu_0 \wedge \Theta(\mathbf{N})))^+)^\sim$ , where  $\text{TrapInv}(\mathcal{N}) \stackrel{\text{def}}{=} \bigwedge_{W \in \text{Int}(\mathcal{N})} \bigvee_{s \in W} s$  and  $\mu_0 \stackrel{\text{def}}{=} \bigvee_{m_0(s)=1} s$ . Moreover,  $[\![\text{TrapInv}(\mathcal{N})]\!]$  is the trap invariant of  $\mathcal{N}$ .*

*Proof:* Let  $\mathbf{N} = (S, T, E)$  and  $W \subseteq S$  be a trap of  $\mathbf{N}$ . We have the following equivalences:

$$\begin{aligned} W^\bullet &\subseteq \bullet W && \iff \\ \bigwedge_{p \in S} [p \in W \rightarrow \{t \in T \mid (p, t) \in E\} \subseteq \{t \in T \mid \bigvee_{q \in S} (t, q) \in E\}] &&& \iff \\ \bigwedge_{p \in S} [p \in W \rightarrow (\bigwedge_{t \in T} p \in \bullet t \rightarrow \bigvee_{q \in S} q \in W \wedge q \in t^\bullet)] &&& \iff \\ \bigwedge_{p \in S} \bigwedge_{t \in T} (p \in W \wedge p \in \bullet t \rightarrow \bigvee_{q \in S} q \in W \wedge q \in t^\bullet) &&& \iff \\ \bigwedge_{t \in T} (\bigvee_{p \in \bullet t} p \in W \rightarrow \bigvee_{q \in t^\bullet} q \in W) \end{aligned}$$

If we use propositional variables  $p$  and  $q$  to denote  $p \in W$  and  $q \in W$ , respectively, we obtain the trap constraint  $\Theta(\mathbf{N})$  from the last formula. Hence, any boolean valuation  $\beta \in [\![\mu_0 \wedge \Theta(\mathbf{N})]\!]$  corresponds to an initially marked trap  $W_\beta \stackrel{\text{def}}{=} \{p \in S \mid \beta(p) = \top\}$ . Further, since  $\mu_0 \wedge \Theta(\mathbf{N})$  is a propositional formula, each satisfying valuation corresponds to a conjunctive clause of its DNF. Hence the set of propositional variables in each conjunctive clause in the DNF of  $(\mu_0 \wedge \Theta(\mathbf{N}))^+$  corresponds to an IMT and, moreover, every IMT has a corresponding conjunctive clause. Thus  $\text{TrapInv}(\mathcal{N}) \equiv (((\mu_0 \wedge \Theta(\mathbf{N})))^+)^\sim$  follows. The second point follows directly from the definition  $\text{TrapInv}(\mathcal{N})$ .  $\square$

The computation of a trap invariant consists of the following steps: (1) convert the propositional formula  $\mu_0 \wedge \Theta(\mathbf{N})$  in DNF, (2) for each conjunctive clause, remove the negative literals and (3) dualize the result. Importantly, the first two steps can be replaced by any transformation on formulae whose result is a positive formula that is minimally equivalent to the input, because only the minimal traps are important for the trap invariant. Moreover, the negative literals do not occur in the propositional definition of a set of places, which is why we require the input of dualization to be a positive

formula<sup>6</sup>. These two properties of positivation constitute the basis of the definition of positivation for quantified IL1S formulae, next in §3.2.

In the rest of this section we focus on computing trap invariants for 1-safe marked PNs obtained from parametric systems consisting of  $O(n)$  components, where  $n \geq 1$  is an unknown parameter. We write parametric trap constraints using the same logic IL1S, used to describe interaction formulae. Namely, if  $\Gamma$  is an interaction formula consisting of a disjunction of formulae of the form (1), then  $\Theta(\Gamma)$  is the conjunction of formulae of the form below (2), one for each (1) formula in the disjunction:

$$\begin{aligned} \forall x_1 \dots \forall x_\ell . \varphi \wedge \left[ \bigvee_{j=1}^\ell \bullet p_j(i_j) \vee \bigvee_{j=\ell+1}^{\ell+m} \exists x_j . \psi_j \wedge \bullet p_j(i_j) \right] \\ \rightarrow \bigvee_{j=1}^\ell p_j^\bullet(i_j) \vee \bigvee_{j=\ell+1}^{\ell+m} \exists x_j . \psi_j \wedge p_j^\bullet(i_j) \end{aligned} \quad (2)$$

where, for a port  $p \in \mathbb{P}^k$  of some component type  $C^k$ ,  $\bullet p(x)$  and  $p(x)^\bullet$  denote the unique predicate atoms  $s(x)$  and  $s'(x)$ , such that  $s \xrightarrow{p} s' \in \mathcal{A}^k$  is the unique transition involving  $p$ , or  $\perp$  if there is no such rule. Note that  $\Theta(\Gamma)$  is the generalization of the trap constraint  $\Theta(N)$  for a given fixed size PN, to the case of a parametric system described by an interaction formula  $\Gamma$ . For instance, the trap constraint of the Dining Philosophers example from Fig. 1, with interaction formula  $\Gamma_{\text{philo}} = \exists i . [g(i) \wedge t(i) \wedge t(\text{succ}(i))] \vee [p(i) \wedge \ell(i) \wedge \ell(\text{succ}(i))]$  is  $\Theta(\Gamma_{\text{philo}}) = \forall i . w(i) \vee f(i) \vee f(\text{succ}(i)) \leftrightarrow e(i) \vee b(i) \vee b(\text{succ}(i))$ .

In order to define a trap invariant computation method for parametric systems described using IL1S interaction formulae, we need counterparts of the propositional positivation and dualization operations, obtained as follows: (1) we translate IL1S trap constraints into equivalent formulae of weak monadic second order logic of one successor (WS1S), and (2) we leverage from the standard automata theoretic two-way translation between WS1S and finite Rabin-Scott automata to define positivation and dualization directly on automata. For presentation purposes, we define first dualization on WS1S formulae, however for efficiency, our implementation applies it on automata directly. We have not been able to define a semantic equivalent of positivation as an operation on WS1S formulae, thus we need to work with automata for this purpose.

### 3.1 From IL1S to WS1S

We introduce the standard second order logic WS1S interpreted over finite words, by considering an infinite countable set  $\text{SVar}$  of set variables, denoted as  $X, Y, \dots$  in the following. The syntax of WS1S is the following:

$$\begin{aligned} t &:= \bar{e} \mid x \mid \text{succ}(t) && \text{terms} \\ \phi &:= t_1 = t_2 \mid \text{pr}(t) \mid X(t) \mid \phi_1 \wedge \phi_2 \mid \neg \phi_1 \mid \exists x . \phi_1 \mid \exists X . \phi_1 && \text{formulae} \end{aligned}$$

Note that the syntax of WS1S is the syntax of IL1S, extended with the constant symbol  $\bar{e}$ , atoms  $X(t)$  and monadic second order quantifiers  $\exists X . \phi$ . As discussed below, we consider w.l.o.g. equality atoms  $t_1 = t_2$  instead of inequalities  $t_1 \leq t_2$  in IL1S.

WS1S formulae are interpreted over structures  $\mathcal{S} = ([n], \iota, \nu, \mu)$ , where  $\iota$  and  $\nu$  are as for IL1S and  $\mu : \text{SVar} \rightarrow 2^{[n]}$  is an interpretation of the set variables. Moreover, the constant symbol  $\bar{0}$  is interpreted as the integer zero and the successor function is

<sup>6</sup> If the DNF is  $(p \wedge q) \vee (p \wedge \neg r)$ , the dualization would give  $(p \vee q) \wedge (p \vee \neg r)$ . The first clause corresponds to the trap  $\{p, q\}$  (either  $p$  or  $q$  is marked), but the second does not directly define a trap. However, by first removing the negative literals, we obtain the traps  $\{p, q\}$  and  $\{r\}$ .

interpreted differently, by the function  $\mathbf{s}_{WS1S}(x) \stackrel{\text{def}}{=} x + 1$  if  $x < n - 1$  and  $\mathbf{s}_{WS1S}(n - 1) \stackrel{\text{def}}{=} n - 1$ <sup>7</sup>. Inequalities  $t_1 \leq t_2$  can be defined in the usual way, using second-order transitive closure of the successor relation and  $t_1 < t_2$  stands for  $t_1 \leq t_2 \wedge \neg t_1 = t_2$ . Moreover,  $\bar{0}$  can be defined using inequality and is considered as part of the syntax mainly for the conciseness of the presentation.

Next, we define an embedding of IL1S formulae into WS1S. W.l.o.g. we consider IL1S formulae that have been previously flattened, i.e the successor function occurs only within atomic propositions of the form  $x = \text{succ}(y)$ . Roughly, this is done by replacing each atomic proposition  $\text{succ}^i(x) = y$  by the formula  $\forall x_1 \dots \forall x_{i-1} . x_1 = \text{succ}(x) \wedge \bigwedge_{j=1}^{i-2} x_{j+1} = \text{succ}(x_j) \rightarrow \text{succ}(x_{i-1}) = y$ , the result being a formula  $\phi_{\text{flat}}$  in which only atoms of the form  $s(x) = y$  occur. Moreover, any constant symbol  $c \in \text{Const}$  from the input IL1S formula is replaced by a fresh free variable  $x_c$ . Let  $Tr(\phi) \stackrel{\text{def}}{=} \exists \xi . \forall y . y \leq \xi \wedge tr(\phi)$ , where  $tr(\phi)$  is defined recursively on the structure of  $\phi$ :

$$\begin{aligned} tr(\text{succ}(x) = y) &\stackrel{\text{def}}{=} (x < \xi \wedge \text{succ}(x) = y) \vee (x = \xi \wedge y = \bar{0}) & tr(x \leq y) &\stackrel{\text{def}}{=} x \leq y \\ tr(\text{pr}(x)) &\stackrel{\text{def}}{=} \text{pr}(x) & tr(\phi_1 \wedge \phi_2) &\stackrel{\text{def}}{=} tr(\phi_1) \wedge tr(\phi_2) \\ tr(\neg \phi_1) &\stackrel{\text{def}}{=} \neg tr(\phi_1) & tr(\exists x . \phi_1) &\stackrel{\text{def}}{=} \exists x . tr(\phi_1) \end{aligned}$$

and  $\xi$  is not among the free variables of  $\phi$ .

**Lemma 3.** *Given an IL1S formula  $\phi$ , the following are equivalent:*

1.  $([n], \nu, \iota) \models \phi$ ,
2.  $([n], \nu, \iota, \mu) \models Tr(\phi)$ , for any  $\mu : \text{SVar} \rightarrow 2^{[n]}$ .

*Proof:* “(1)  $\Rightarrow$  (2)” First, it is routine to check that, for any WS1S-structure, we have  $([n], \iota, \nu, \mu) \models \forall y . y \leq x \iff \nu(x) = n - 1$ . Suppose that  $\phi$  has a model  $\mathcal{I} = ([n], \iota, \nu)$  and the interpretation of  $s$  is  $\mathbf{s}_{IL1S}^n$ . Then we show that  $\mathcal{S} = ([n], \iota, \nu, \mu)$  is a model of  $Tr(\phi)$ , for any  $\mu : \text{SVar} \rightarrow 2^{[n]}$ . For this, it is enough to show that  $([n], \iota, \nu[\mu \leftarrow n - 1], \mu) \models tr(\phi)$ , by induction on the structure of  $\phi$ . The base cases are:

- $s(x) = y$ : in this case  $\mathbf{s}_{IL1S}^n(\nu(x)) = \nu(y)$  and thus  $\nu(y) = (\nu(x) + 1) \bmod n$ , by the definition of  $\mathbf{s}_{IL1S}^n$ . But then either  $\nu(x) < n - 1$  and  $\nu(y) = \nu(x) + 1$  or  $\nu(x) = n - 1$  and  $\nu(y) = 0$ , thus  $\mathcal{S} \models tr(s(x) = y)$ , as required.
- $\text{pr}(x)$ : in this case  $\nu(x) \in \iota(p)$  and  $\mathcal{S} \models tr(\text{pr}(x))$  by the definition.

The induction cases are immediate.

“(1)  $\Leftarrow$  (2)” If  $([n], \nu, \iota, \mu) \models Tr(\phi)$ , for some arbitrary mapping  $\mu : \text{SVar} \rightarrow 2^{[n]}$ , then we have  $([n], \nu[\nu \leftarrow n - 1], \iota, \mu) \models tr(\phi)$  and we show  $([n], \nu, \iota) \models \phi$  by induction on the structure of  $\phi$ . The most interesting case is when  $\phi$  is  $s(x) = y$ , in which case either:

- $\nu(x) < n - 1$  and  $\nu(y) = \nu(x) + 1$ , or
- $\nu(x) = n - 1$  and  $\nu(y) = 0$ .

In each case, we have  $\mathbf{s}_{IL1S}^n(\nu(x)) = \nu(y)$ , hence  $([n], \nu, \iota) \models s(x) = y$ , as required.  $\square$

**Remark** The above translation can be easily generalized to the case where IL1S contains any WS1S-definable relation, such as the *even*( $x$ ) predicate, defined below:

$$\begin{aligned} \text{even}(x) &\stackrel{\text{def}}{=} \exists X \exists Y . X(x) \wedge X(\bar{0}) \wedge \forall y . X(y) \leftrightarrow \neg Y(y) \wedge \\ &\quad \forall y . X(y) \wedge y \neq \text{succ}(y) \rightarrow Y(\text{succ}(y)) \wedge \forall y . Y(y) \wedge y \neq \text{succ}(y) \rightarrow X(\text{succ}(y)) \end{aligned}$$

<sup>7</sup> By classical convention, the successor on a finite domain is a total function that loops on the greatest element [26, Example 2.10.3].

Analogously, we can include any modulo constraint of the form  $x \equiv_k \ell$ , where  $k > 0$  and  $0 \leq \ell < k$  are integer constants. ■

Next, we define the *dualization*  $\phi^\sim$  of a WS1S formula  $\phi$ , in negative normal form:

$$\begin{aligned} (t_1 = t_2)^\sim &\stackrel{\text{def}}{=} \neg t_1 = t_2 & (\neg t_1 = t_2)^\sim &\stackrel{\text{def}}{=} t_1 = t_2 & (\phi_1 \wedge \phi_2)^\sim &\stackrel{\text{def}}{=} \phi_1^\sim \vee \phi_2^\sim & (\phi_1 \vee \phi_2)^\sim &\stackrel{\text{def}}{=} \phi_1^\sim \wedge \phi_2^\sim \\ \text{pr}(t)^\sim &\stackrel{\text{def}}{=} \neg \text{pr}(t) & (\neg \text{pr}(t))^\sim &\stackrel{\text{def}}{=} \text{pr}(t) & (\exists x . \phi_1)^\sim &\stackrel{\text{def}}{=} \forall x . \phi_1^\sim & (\forall x . \phi_1)^\sim &\stackrel{\text{def}}{=} \exists x . \phi_1^\sim \\ X(t)^\sim &\stackrel{\text{def}}{=} \neg X(t) & (\neg X(t))^\sim &\stackrel{\text{def}}{=} X(t) & (\exists X . \phi_1)^\sim &\stackrel{\text{def}}{=} \forall X . \phi_1^\sim & (\forall X . \phi_1)^\sim &\stackrel{\text{def}}{=} \exists X . \phi_1^\sim \end{aligned}$$

Note that dualization acts differently on predicate literals of the form  $\text{pr}(t)$  and  $\neg \text{pr}(t)$  than on literals involving a set variable  $X(t)$  and  $\neg X(t)$ . Namely, the former are left unchanged, whereas the latter are negated. Its formal property is stated below:

**Lemma 4.** *Given a WS1S formula  $\phi$ , for every structure  $\mathcal{S} = ([n], \nu, \iota, \mu)$  we have  $\mathcal{S} \models \phi \iff \bar{\mathcal{S}} \models \neg(\phi^\sim)$ , where  $\bar{\mathcal{S}} \stackrel{\text{def}}{=} ([n], \nu, \bar{\iota}, \mu)$  and for each  $\text{pr} \in \text{Pred}$ ,  $\bar{\iota}(\text{pr}) \stackrel{\text{def}}{=} [n] \setminus \iota(\text{pr})$ .*

*Proof:* By induction on the structure of  $\phi$ :

- $t_1 = t_2$  and  $\neg t_1 = t_2$ : the truth value of this atom is the same in  $\mathcal{S}$  and  $\bar{\mathcal{S}}$  and moreover  $t_1 = t_2$  and  $\neg(t_1 = t_2)^\sim$  are equivalent.
- $X(t)$  and  $\neg X(t)$ : same as above.
- $\text{pr}(t)$ : the interpretation of  $t$  is the same in  $\mathcal{S}$  and  $\bar{\mathcal{S}}$ , because it depends only on  $\nu$ . Let  $k \in [n]$  be this value. Then we obtain:

$$\mathcal{S} \models \text{pr}(t) \iff k \in \iota(\text{pr}) \iff k \notin \bar{\iota}(\text{pr}) \iff \bar{\mathcal{S}} \models \neg \text{pr}(t) .$$

- $\neg \text{pr}(t)$ : a consequence of the equivalence  $\mathcal{S} \models \text{pr}(t) \iff \bar{\mathcal{S}} \models \neg \text{pr}(t)$ , established at the previous point.

The rest of the cases are easy applications of the induction hypothesis. ■

For technical reasons, we also introduce a *booleanization* operation that, given a WS1S formula  $\phi$  and a positive constant  $n > 0$ , produces a propositional formula  $B_n(\phi)$  with the property that each model  $([n], \nu, \iota, \mu)$  of  $\phi$  can be turned into a satisfying boolean valuation for  $B_n(\phi)$  and viceversa, from every boolean model of  $B_n(\phi)$  one can extract a model of  $\phi$ .

First, given an integer  $i \geq 0$  and a WS1S formula  $\phi(x)$ , we denote by  $\phi[i/x]$  (resp.  $t[i/x]$ ) the formula (term) obtained from  $\phi$  (resp.  $t$ ) by replacing every occurrence of  $x$  with the term  $s^i(\bar{0})$ , where  $s^i$  denotes  $i$  successive applications of the successor function. Second, for a set  $S$  of positive integers, the formula  $\phi[S/X]$  is defined homomorphically, starting with the base case  $X(t)[S/X] \stackrel{\text{def}}{=} \bigvee_{i \in S} t = s^i(\bar{0})$ .

$$\begin{aligned} B_n(s^i(\bar{0}) = s^j(\bar{0})) &\stackrel{\text{def}}{=} i = j \vee (i \geq n-1 \wedge j \geq n-1) & B_n(\text{pr}(s^i(\bar{0}))) &\stackrel{\text{def}}{=} \text{pr}_{\min(i, n-1)} \\ B_n(\phi_1 \wedge \phi_2) &\stackrel{\text{def}}{=} B_n(\phi_1) \wedge B_n(\phi_2) & B_n(\neg \phi_1) &\stackrel{\text{def}}{=} \neg B_n(\phi_1) \\ B_n(\exists x . \phi) &\stackrel{\text{def}}{=} \bigvee_{i \in [n]} B_n(\phi[i/x]) & B_n(\exists X . \phi) &\stackrel{\text{def}}{=} \bigvee_{S \subseteq [n]} B_n(\phi[S/X]) \end{aligned}$$

where, for any  $\text{pr} \in \text{Pred}$  and  $j \in [n]$ ,  $\text{pr}_j$  is a propositional variable ranging over the boolean values  $\top$  (true) and  $\perp$  (false). Moreover, we relate WS1S structures with boolean valuations as follows. Given a structure  $\mathcal{S} = ([n], \nu, \iota, \mu)$  we define the boolean valuation  $\beta_{\mathcal{S}}(\text{pr}_j) \stackrel{\text{def}}{=} \top \iff s_{\text{WS1S}}^j(0) \in \iota(\text{pr})$ , for all  $\text{pr} \in \text{Pred}$  and  $j \in [n]$ . The following lemma states the formal property of booleanization:

**Lemma 5.** *Given a WS1S sentence  $\phi$  and  $n > 0$ , for every structure  $S = ([n], \nu, \iota, \mu)$ , we have  $S \models \phi \iff \beta_S \models B_n(\phi)$ .*

*Proof:* We prove the following more general statement. Let  $\phi(x_1, \dots, x_k, X_1, \dots, X_m)$  be a WS1S formula with free variables  $x_1, \dots, x_k \in \text{Var}$  and  $X_1, \dots, X_m \in \text{SVar}$ ,  $i_1, \dots, i_k \in [n]$  and  $S_1, \dots, S_m \subseteq [n]$ . Then we show that:

$$\begin{aligned} S \models \phi[i_1/x_1, \dots, i_k/x_k, S_1/X_1, \dots, S_m/X_m] \\ \iff \\ \beta_S \models B_n(\phi[i_1/x_1, \dots, i_k/x_k, S_1/X_1, \dots, S_m/X_m]) \end{aligned}$$

by induction on the structure of  $\phi$ :

- $t_1 = t_2$ : since  $\phi[i_1/x_1, \dots, i_k/x_k, S_1/X_1, \dots, S_m/X_m]$  is a sentence, it must be the case that  $t_1 = s^{i_1}(\bar{0})$  and  $t_2 = s^{i_2}(\bar{0})$ , for some  $i_1, i_2 \geq 0$ . Then we have:

$$\begin{aligned} S \models s^{i_1}(\bar{0}) = s^{i_2}(\bar{0}) &\iff s_{WS1S}^{i_1}(0) = s_{WS1S}^{i_2}(0) \\ &\iff i_1 = i_2 \vee (i_1 \geq n-1 \wedge i_2 \geq n-1) \\ &\iff \beta_S \models B_n(s^{i_1}(\bar{0}) = s^{i_2}(\bar{0})) . \end{aligned}$$

- $\text{pr}(t)$ : since  $\phi[i_1/x_1, \dots, i_k/x_k, S_1/X_1, \dots, S_m/X_m]$  is a sentence, it must be the case that  $t = s^i(\bar{0})$ , for some  $i \geq 0$ . We obtain:

$$\begin{aligned} S \models \text{pr}(s^i(\bar{0})) &\iff s_{WS1S}^i(\bar{0}) \in \iota(\text{pr}) \\ &\iff s_{WS1S}^{\min(i, n-1)} \in \iota(\text{pr}) \\ &\iff \beta_S \models \text{pr}_{\min(i, n-1)} . \end{aligned}$$

The rest of the cases are easy applications of the induction hypothesis.  $\square$

Finally, we relate WS1S dualization, booleanization and propositional dualization:

**Lemma 6.** *Given a WS1S formula  $\phi$  and an integer  $n > 0$ , we have  $B_n(\phi^\sim) \equiv B_n(\phi)^\sim$ .*

*Proof:* Let  $\beta : \{\text{pr}_k \mid \text{pr} \in \text{Pred}, k \in [n]\} \rightarrow \{\top, \perp\}$  be an arbitrary boolean valuation and let  $S = ([n], \nu, \iota, \mu)$  be a structure such that, for each  $\text{pr} \in \text{Pred}$ , we have  $\iota(\text{pr}) = \{k \in [n] \mid \beta(\text{pr}_k) = \top\}$  and  $\nu, \mu$  are picked at random. Obviously, we have that  $\beta = \beta_S$ , hence by Lemma 10,  $\beta \models B_n(\phi^\sim) \iff S \models \phi^\sim$  and by Lemma 4 we get  $S \models \phi^\sim \iff \bar{S} \models \neg\phi \iff \beta_{\bar{S}} \models \neg B_n(\phi)$  again, by Lemma 10 and the definition of  $B_n(\neg\phi) = \neg B_n(\phi)$ . Let  $\bar{\beta}$  be the boolean valuation defined as  $\bar{\beta}(\text{pr}_k) = \neg\beta(\text{pr}_k)$  for all  $\text{pr} \in \text{Pred}$  and  $k \in [n]$ . Then clearly  $\bar{\beta} = \beta_{\bar{S}}$  and  $\bar{\beta} \models \neg B_n(\phi) \iff \beta \models B_n(\phi)^\sim$  follows.  $\square$

### 3.2 Trap Invariants as Automata

The purpose of introducing automata is the definition of a positivation operator for IL1S or, equivalently, for WS1S formulae. Recall that, given a formula  $\phi$ , the result of positivation is a formula  $(\phi)^\oplus$  in which all predicate symbols occur under an even number of negations and, moreover  $\phi \equiv^{\min} (\phi)^\oplus$ .

Unlike dualization, positivation is not defined on formulae but on equivalent automata on finite words, obtained via the classical two-way translation between WS1S

and Rabin-Scott automata, described next. Let us fix a structure  $\mathcal{S} = ([n], \nu, \iota, \mu)$  such that  $\text{dom}(\nu) = \{x_1, \dots, x_k\}$ ,  $\text{dom}(\iota) = \{\text{pr}_1, \dots, \text{pr}_\ell\}$  and  $\text{dom}(\mu) = \{X_1, \dots, X_m\}$  are all finite. Each such structure is viewed as a word  $w_{\mathcal{S}} = \sigma_0 \dots \sigma_{n-1}$  of length  $n$  over the alphabet  $\{0, 1\}^{k+\ell+m}$ , where, for all  $i \in [n]$ , we have:

- $\sigma_i(j) = 1$  if  $\nu(x_j) = i$  and  $\sigma_i(j) = 0$  otherwise, for all  $1 \leq j \leq k$ ,
- $\sigma_i(j) = 1$  if  $i \in \iota(\text{pr}_{j-k})$  and  $\sigma_i(j) = 0$  otherwise, for all  $k < j \leq k + \ell$ ,
- $\sigma_i(j) = 1$  if  $i \in \mu(X_{j-k-\ell})$  and  $\sigma_i(j) = 0$  otherwise, for all  $k + \ell < j \leq k + \ell + m$ .

In other words, the  $j$ -th track of  $w$  encodes (i) the unique value  $w(x_j) \stackrel{\text{def}}{=} \nu(x_j)$ , if  $1 \leq j \leq k$ , (ii) the set  $w(\text{pr}_j) \stackrel{\text{def}}{=} \iota(\text{pr}_{j-k})$ , if  $k < j \leq k + \ell$ , or (iii) the set  $w(X_j) \stackrel{\text{def}}{=} \mu(X_{j-k-\ell})$ , if  $k + \ell < j \leq k + \ell + m$ . For an alphabet symbol  $\sigma \in \{0, 1\}^{k+\ell+m}$ , we write  $\sigma(\text{pr}_j)$  for  $\sigma(j + k)$ . *Example 2.* Consider the structure  $\mathcal{S} = ([6], \nu, \iota, \mu)$ , where  $\nu(x_1) = 3$ ,  $\iota(\text{pr}_1) = \{0, 2, 5\}$  and  $\mu(X_1) = \{1, 3\}$ . Moreover, assume that  $\nu$ ,  $\iota$  and  $\mu$  are undefined elsewhere. The word  $w_{\mathcal{S}}$  is given below:

	0	1	2	3	4	5
$x_1$	0	0	0	1	0	0
$\text{pr}_1$	1	0	1	0	0	1
$X_1$	0	1	0	1	0	0

■

Given  $x_1, \dots, x_k \in \text{Var}$ ,  $\text{pr}_1, \dots, \text{pr}_\ell \in \text{Pred}$  and  $X_1, \dots, X_m \in \text{SVar}$ , a nondeterministic finite automaton over the alphabet  $\{0, 1\}^{k+\ell+m}$  is a tuple  $A = (Q, I, F, \delta)$ , where  $Q$  is the finite set of states,  $I \subseteq Q$  is the set of initial states,  $F \subseteq Q$  is the set of final states and  $\delta \subseteq Q \times \{0, 1\}^{k+\ell+m} \times Q$  is the transition relation. A given a word  $w = \sigma_0 \dots \sigma_{n-1}$  as before, a run of  $A$  over  $w$  is a sequence of states  $\rho = s_0 \dots s_n$ , such that  $s_0 \in I$  and  $(s_i, \sigma_i, s_{i+1}) \in \delta$ , for all  $i \in [n]$ . The run is *accepting* if  $s_n \in F$ , in which case we say that  $A$  accepts the word  $w$ . The language of  $A$ , denoted by  $\mathcal{L}(A)$ , is the set of words accepted by  $A$ . The following theorem is automata-theoretic folklore<sup>8</sup>:

**Theorem 1.** *For each WS1S formula  $\phi(x_1, \dots, x_k, \text{pr}_1, \dots, \text{pr}_\ell, X_1, \dots, X_m)$  there exists an automaton  $A_\phi$  over the alphabet  $\{0, 1\}^{k+\ell+m}$  such that  $\mathcal{S} \models \phi \iff w_{\mathcal{S}} \in \mathcal{L}(A_\phi)$ , for each structure  $\mathcal{S}$ . Conversely, for each automaton  $A$  over the alphabet  $\{0, 1\}^\ell$ , there exists a WS1S formula  $\Phi_A(\text{pr}_1, \dots, \text{pr}_\ell)$  such that  $w_{\mathcal{S}} \in \mathcal{L}(A) \iff \mathcal{S} \models \Phi_A$ , for each structure  $\mathcal{S} = ([n], \nu, \iota, \mu)$  such that  $\text{dom}(\iota) = \{\text{pr}_1, \dots, \text{pr}_\ell\}$  and  $\text{dom}(\nu) = \text{dom}(\mu) = \emptyset$ .*

The construction of  $A_\phi$  for the first point (logic to automata) is by induction on the structure of  $\phi$ . The main consequence of this construction is the decidability of the satisfiability problem for the WS1S logic, implied by the decidability of emptiness for finite automata. Incidentally, this also proves the decidability of IL1S, as a consequence of Lemma 3. The second point (automata to logic) is a bit less known and deserves presentation. Given  $A = (Q, I, F, \delta)$  with alphabet  $\{0, 1\}^\ell$  and states  $Q = \{s_1, \dots, s_q\}$ , we define a formula  $\Psi_A(\text{pr}_1, \dots, \text{pr}_\ell, X_1, \dots, X_q) \stackrel{\text{def}}{=} \psi_{\text{cover}} \wedge \psi_I \wedge \psi_\delta \wedge \psi_F$ , where:

$$\begin{aligned}
\psi_{\text{cover}} &\stackrel{\text{def}}{=} \forall x. \bigvee_{i=1}^q X_i(x) \wedge \bigvee_{1 \leq i < j \leq q} \neg X_i(x) \vee \neg X_j(x) \\
\psi_I &\stackrel{\text{def}}{=} \bigvee_{s_i \in I} X_i(\bar{0}) & \psi_F &\stackrel{\text{def}}{=} \exists x \forall y. y \leq x \wedge \bigvee_{s_i \in F} X_i(x) \\
\psi_\delta &\stackrel{\text{def}}{=} \forall x \forall y. y \leq x \vee \bigvee_{(s_i, \sigma, s_j) \in \delta} X_i(x) \wedge X_j(\text{succ}(x)) \wedge \bigwedge_{\substack{1 \leq k \leq \ell \\ \sigma(\text{pr}_k)=1}} \text{pr}_k(x) \wedge \bigwedge_{\substack{1 \leq k \leq \ell \\ \sigma(\text{pr}_k)=0}} \neg \text{pr}_k(x)
\end{aligned}$$

<sup>8</sup> See e.g. [26, Theorem 2.10.1 and 2.10.3].

Intuitively, each  $X_i$  keeps the positions that are labeled by the state  $s_i$  during the run of  $A$  over some input word  $w$  of length  $n$ . First, each position between 0 and  $n-1$  must be labeled with exactly one state from  $Q$  ( $\psi_{cover}$ ). The initial ( $\psi_I$ ) and final ( $\psi_F$ ) positions are labeled with states from  $I$  and  $F$ , respectively. Next, each pair of adjacent positions is labeled with a pair of states that is compatible with the transition relation of  $A$ , on the corresponding input symbol, encoded as the tuple  $(w(\text{pr}_1), \dots, w(\text{pr}_\ell)) \in \{0, 1\}^\ell$  ( $\psi_\delta$ ). Finally, we define  $\Phi_A \stackrel{\text{def}}{=} \exists X_1 \dots X_q . \Psi_A$ , to capture the fact that a word  $w$  is accepted by  $A$  if and only if there exists an accepting run of  $A$  over  $w$ .

Given a WS1S formula  $\phi$ , we define a positivation operation  $(\phi)^\oplus$  by translating first  $\phi$  into an automaton  $A_\phi$ . Then we *saturate*  $A_\phi$  by adding new transitions to it, such that the language of the new automaton  $A_\phi^*$  contains  $\mathcal{L}(A_\phi)$  and the words corresponding to minimal structures are the same in both  $\mathcal{L}(A_\phi)$  and  $\mathcal{L}(A_\phi^*)$ . Then we obtain  $(\phi)^\oplus$  by a slightly modified translation of  $A_\phi^*$  into WS1S, which is guaranteed to produce positive formulae only. Note that the result  $\Phi_{A_\phi}$  of the above translation is not positive, due to the formula  $\psi_\delta$  which introduces negative predicates.

The saturation of an automaton  $A = (Q, I, F, \delta)$  over the alphabet  $\{0, 1\}^\ell$  is defined next. For each transition  $(s, \sigma, s') \in \delta$  the set  $\delta^*$  contains all transitions  $(s, \tau, s')$  such that  $\tau \in \{0, 1\}^\ell$  and  $\sigma(j) \leq \tau(j)$ , for all  $1 \leq j \leq \ell$ . Moreover, nothing else is in  $\delta^*$  and  $A^* \stackrel{\text{def}}{=} (Q, I, F, \delta^*)$ . In other words,  $A^*$  is obtained by adding to  $A$ , for each transition whose  $j$ -th track is 0, another transition in which this track is 1.

To state the formal relation between  $A$  and  $A^*$ , we define a partial order on words over the alphabet  $\{0, 1\}^\ell$ , encoding the interpretations of the predicates  $\text{pr}_1, \dots, \text{pr}_\ell$ :  $w_1 \leq w_2 \iff w_1(\text{pr}_j) \subseteq w_2(\text{pr}_j)$  for all  $1 \leq j \leq \ell$ . The minimal language of  $A$  is  $\mathcal{L}^{\min}(A) \stackrel{\text{def}}{=} \{w \in \mathcal{L}(A) \mid \forall w' . w' \leq w \wedge w' \neq w \Rightarrow w' \notin \mathcal{L}(A)\}$ .

**Lemma 7.** *Given an automaton  $A$  over the alphabet  $\{0, 1\}^\ell$ , we have  $\mathcal{L}^{\min}(A) = \mathcal{L}^{\min}(A^*)$ .*

*Proof:* We start from the observation that  $\mathcal{L}(A) \subseteq \mathcal{L}(A^*)$  because  $A^* = (Q, I, F, \delta^*)$  is obtained by adding transitions to  $A = (Q, I, F, \delta)$ . Moreover, given a run  $\rho = s_0, \dots, s_m$  of  $A^*$  over some word  $\sigma_0 \dots \sigma_{m-1}$ , there exists a word  $\sigma'_0 \dots \sigma'_{m-1}$  such that for each  $i \in [m]$  and  $1 \leq j \leq \ell$ , we have  $\sigma_i(j) \leq \sigma'_i(j)$  and  $(s_i, \sigma'_i, s_{i+1}) \in \delta^*$ . This is because we only add to  $A^*$  transitions  $(q_i, \sigma', q_j)$  such that  $\sigma(j) \leq \sigma'(i)$ , for all  $1 \leq j \leq \ell$ , where  $(q_i, \sigma, q_j) \in \delta$ . " $\subseteq$ " Let  $w \in \mathcal{L}^{\min}(A)$ , then  $w \in \mathcal{L}(A^*)$  because  $\mathcal{L}(A) \subseteq \mathcal{L}(A^*)$ . Let  $w'$  be a word such that  $w' \leq w$  and  $w' \neq w$  and suppose, for a contradiction that  $w' \in \mathcal{L}^{\min}(A^*)$ . Then  $A^*$  has an accepting run  $\rho = s_0, \dots, s_m$  over  $w'$ , thus  $\rho$  is also an accepting run of  $A$  over another word  $w'' \leq w'$ . Since  $w \in \mathcal{L}^{\min}(A)$  and  $w'' \leq w' \leq w$  and  $w' \neq w$ , we obtain a contradiction. Thus,  $w \in \mathcal{L}^{\min}(A^*)$ , as required.

" $\supseteq$ " Let  $w \in \mathcal{L}^{\min}(A^*)$  and let  $\rho = s_0, \dots, s_m$  be an accepting run of  $A^*$  over  $w$ . Then there exists a word  $w' \leq w$  such that  $\rho$  is an accepting run of  $A$ . Since  $w' \in \mathcal{L}(A) \subseteq \mathcal{L}(A^*)$ , we obtain that  $w' = w$ , thus  $w \in \mathcal{L}(A)$ . Now suppose, for a contradiction, that there exists  $w'' \leq w$  such that  $w'' \neq w$  and  $w'' \in \mathcal{L}(A)$ . Then  $w'' \in \mathcal{L}(A^*)$  and since  $w'' \leq w$  and  $w'' \neq w$ , this contradicts the fact that  $w \in \mathcal{L}^{\min}(A^*)$ . Thus,  $w \in \mathcal{L}^{\min}(A)$ , as required.  $\square$

Finally, we define  $(\phi)^\oplus \stackrel{\text{def}}{=} \exists X_1 \dots X_q . \psi_{cover} \wedge \psi_I \wedge \psi_\delta^* \wedge \psi_F$  as the formula obtained from  $A_\phi = (Q, I, F, \delta)$  by applying the translation scheme above in which, instead of  $\psi_\delta$ ,

we use the following formula:

$$\psi_\delta^* \stackrel{\text{def}}{=} \forall x \forall y . y \leq x \vee \bigvee_{(s_i, \sigma, s_j) \in \delta} X_i(x) \wedge X_j(\text{succ}(x)) \wedge \bigwedge_{\substack{1 \leq k \leq \ell \\ \sigma(\text{pr}_k)=1}} \text{pr}_k(x)$$

Note that  $(\phi)^\oplus$  is a positive formula, independently of whether  $\phi$  is positive or not. The following lemma proves the required property of this positivation operation.

**Lemma 8.** *Given a WS1S sentence  $\phi(\text{pr}_1, \dots, \text{pr}_\ell)$ , the following hold:*

1.  $\mathcal{S} \models (\phi)^\oplus \iff w_{\mathcal{S}} \in \mathcal{L}(A_\phi^*)$ , for each structure  $\mathcal{S} = ([n], \nu, \iota, \mu)$  such that  $\text{dom}(\iota) = \{\text{pr}_1, \dots, \text{pr}_\ell\}$  and  $\text{dom}(\nu) = \text{dom}(\mu) = \emptyset$ .
2.  $\phi \equiv^{\min} (\phi)^\oplus$ .

*Proof:* (1) It is sufficient to show that  $(\phi)^\oplus \equiv \Phi_{A_\phi^*}$  and apply Theorem 1. Denoting  $A = (Q, I, F, \delta)$ , with  $Q = \{s_1, \dots, s_q\}$  and  $A^* = (Q, I, F, \delta^*)$  as before, we only show that  $\psi_\delta^* \equiv \psi_{\delta^*}$ . Because  $(\phi)^\oplus = \exists X_1 \dots \exists X_q . \psi_{\text{cover}} \wedge \psi_I \wedge \psi_\delta^* \wedge \psi_F$  and  $\Phi_{A_\phi^*} = \exists X_1 \dots \exists X_q . \psi_{\text{cover}} \wedge \psi_I \wedge \psi_{\delta^*} \wedge \psi_F$ , we immediately obtain the result. We have the following equivalence, for each  $\sigma \in \{0, 1\}^\ell$ :

$$\bigwedge_{\substack{1 \leq k \leq \ell \\ \sigma(\text{pr}_k)=1}} \text{pr}_k(x) \equiv \bigvee_{\sigma \leq \tau} \left( \bigwedge_{\substack{1 \leq k \leq \ell \\ \tau(\text{pr}_k)=1}} \text{pr}_k(x) \wedge \bigwedge_{\substack{1 \leq k \leq \ell \\ \tau(\text{pr}_k)=0}} \neg \text{pr}_k(x) \right)$$

where  $\sigma \leq \tau$  stands for  $\forall j . 1 \leq j \leq \ell \Rightarrow \sigma(j) \leq \tau(j)$ . This immediately implies that  $\psi_\delta^* \equiv \psi_{\delta^*}$ , by the definitions of these formulae and the construction of  $\delta^*$ .

(2) For an arbitrary structure  $\mathcal{S} = ([n], \nu, \iota, \mu)$  we have  $\iota(\text{pr}_k) = w_{\mathcal{S}}(\text{pr}_k)$ , for any  $1 \leq k \leq \ell$ , by the definition of  $w_{\mathcal{S}}$ . Then  $\mathcal{S}_1 \sqsubseteq \mathcal{S}_2 \iff w_{\mathcal{S}_1} \leq w_{\mathcal{S}_2}$ , for any two structures  $\mathcal{S}_i = ([n], \nu_i, \iota_i, \mu_i)$ , where  $i = 1, 2$ . Hence a structure  $\mathcal{S}$  is a minimal model of  $\phi$  if and only if  $w_{\mathcal{S}} \in \mathcal{L}^{\min}(A_\phi)$ . By Lemma 7, we have  $\mathcal{L}^{\min}(A) = \mathcal{L}^{\min}(A^*)$ . Then the result follows from Theorem 1 and point (1) of this Lemma.  $\square$

Positivation and booleanization are related via the following property:

**Lemma 9.** *Given a WS1S formula  $\phi$  and a constant  $n > 0$ , we have  $(B_n(\phi))^+ \equiv B_n((\phi)^\oplus)$ .*

*Proof:* First, note that, for any propositional formulae  $f$  and  $g$ , whose variables occur under even number of negations, we have  $f \equiv g \iff f \equiv^{\min} g$ . Since both  $(B_n(\phi))^+$  and  $B_n((\phi)^\oplus)$  are positive propositional formulae, it is sufficient to prove  $(B_n(\phi))^+ \equiv^{\min} B_n((\phi)^\oplus)$ , by showing  $\llbracket (B_n(\phi))^+ \rrbracket^{\min} \subseteq \llbracket B_n((\phi)^\oplus) \rrbracket$  and  $\llbracket B_n((\phi)^\oplus) \rrbracket^{\min} \subseteq \llbracket (B_n(\phi))^+ \rrbracket$ , respectively, which establishes  $\llbracket (B_n(\phi))^+ \rrbracket^{\min} = \llbracket B_n((\phi)^\oplus) \rrbracket^{\min}$  (the latter step is left to the reader).

“ $\llbracket (B_n(\phi))^+ \rrbracket^{\min} \subseteq \llbracket B_n((\phi)^\oplus) \rrbracket$ ” Let  $\beta \in \llbracket (B_n(\phi))^+ \rrbracket^{\min}$  be a valuation. Then, we also have  $\beta \in \llbracket B_n(\phi) \rrbracket^{\min}$ , since  $(\varphi)^+ \equiv^{\min} \varphi$ , in general for any propositional formula  $\varphi$ . Then, by Lemma 10, there exists a structure  $\mathcal{S} \in \llbracket \phi \rrbracket^{\min}$  such that  $\beta = \beta_{\mathcal{S}}$ . Hence we obtain  $\mathcal{S} \in \llbracket (\phi)^\oplus \rrbracket^{\min} \subseteq \llbracket (\phi)^\oplus \rrbracket$ . But then  $\beta \in \llbracket B_n((\phi)^\oplus) \rrbracket$ , by Lemma 10.



“ $\llbracket B_n((\phi)^\oplus) \rrbracket^{\min} \subseteq \llbracket (B_n(\phi))^+ \rrbracket$ ” Let  $\beta \in \llbracket B_n((\phi)^\oplus) \rrbracket^{\min}$  be a boolean valuation. By Lemma 10, we obtain a structure  $S \in \llbracket (\phi)^\oplus \rrbracket^{\min}$  such that  $\beta = \beta_S$ . But then  $S \in \llbracket \phi \rrbracket^{\min}$  and  $\beta \in \llbracket B_n(\phi) \rrbracket^{\min}$ , by Lemma 10. Hence  $\beta \in \llbracket (B_n(\phi))^+ \rrbracket$ .  $\square$

We are now ready to state the main result of the paper, concerning the computation of trap invariants for parametric component-based systems.

**Theorem 2.** *Given a parametric component-based system  $S = \langle C^1, \dots, C^K, \Gamma \rangle$ , where  $C^k = \langle P^k, S^k, s_0^k, \Delta^k \rangle$ , for all  $k = 1, \dots, K$ , for any integer  $n > 0$  we have:*

$$\text{TrapInv}(\mathcal{N}_S^n) \equiv B_n\left(\left((\text{Init}(S) \wedge \text{Tr}(\Theta(\Gamma)))^\oplus\right)^\sim\right)$$

where  $\text{Init}(S) \stackrel{\text{def}}{=} \exists x. \bigvee_{k=1}^K s_0^k(x)$ .

*Proof:* Let  $\mathcal{N}_S^n = (N, m_0)$  and  $\mu_0 = \bigvee_{m_0(s)=1} s$ . By Lemma 2, we have  $\text{TrapInv}(\mathcal{N}_S) \equiv ((\mu_0 \wedge \Theta(N))^+)$ . From the definition of  $\mathcal{N}_S$ , it is not difficult to show that  $\mu_0 \equiv B_n(\text{Init}(S))$  and  $\Theta(N) \equiv B_n(\text{Tr}(\Theta(\Gamma)))$ , hence  $\mu_0 \wedge \Theta(N) \equiv B_n(\text{Init}(S) \wedge \text{Tr}(\Theta(\Gamma)))$ . By Lemma 11, we obtain  $(\mu_0 \wedge \text{Tr}(\Theta(N)))^+ \equiv B_n\left(\left((\text{Init}(S) \wedge \text{Tr}(\Theta(\Gamma)))^\oplus\right)^\sim\right)$  and, by Lemma 12, we obtain  $((\mu_0 \wedge \Theta(N))^+)^\sim \equiv B_n\left(\left(\left((\text{Init}(S) \wedge \text{Tr}(\Theta(\Gamma)))^\oplus\right)^\sim\right)^\sim\right)$ , as required.  $\square$

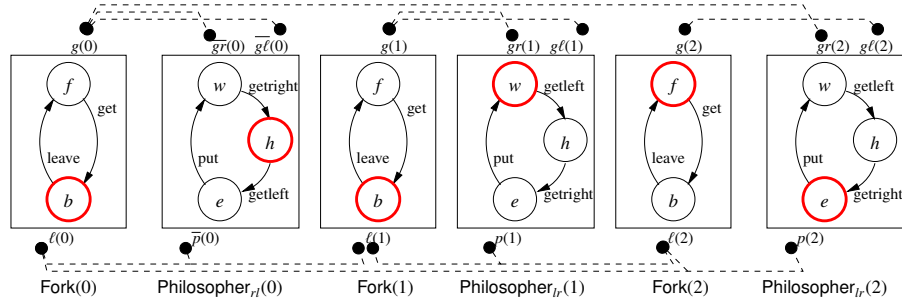
In practice, it is more efficient to perform dualization directly on the saturated automaton  $A_\phi^*$  for a given WS1S formula  $\phi$  with predicate symbols  $\text{pr}_1, \dots, \text{pr}_\ell$ . To this end, we swap the 0's and 1's on the tracks corresponding to  $\text{pr}_1, \dots, \text{pr}_\ell$  in the transition rules of  $A_\phi^*$  and complement the resulting automaton, call it  $A_\phi$ . Using Lemma 4, it is not difficult to show that the complement of  $\tilde{A}_\phi$  corresponds to the formula  $(\phi)^\oplus$ , needed to compute the trap invariant of a system. A further optimization, that avoids complementation of  $\tilde{A}_\phi$ , is to check the inclusion of the automaton  $A_\psi$ , obtained from the safety property to be checked (i.e.  $\psi$  may encode the states where a deadlock or mutual exclusion violation occurs) into  $\tilde{A}_\phi$ , using state-of-the-art antichain or simulation-based inclusion checkers. For this reasons, our experiments were carried out using the VATA [28] tree automata library as a decision procedure for inclusion.

**Remark** We argue that the trap invariant synthesis method given by Theorem 2 can be easily extended to handle unbounded tree-like (hierarchical) systems. To this end, we consider a variant of IL1S equipped with a countably infinite set of successor functions  $\text{succ}_0, \text{succ}_1, \dots$  ( $\text{succ}_0$  being the leftmost successor) interpreted over the set  $\mathbb{N}^*$  of strings of natural numbers, that identify positions in a tree as  $\text{succ}_k(k_0 \dots k_m) \stackrel{\text{def}}{=} k_0 \dots k_m k$ . Also, the inequality is interpreted by the prefix relation between strings. This logic is embedded into  $\text{WS}\omega\text{S}$ , the weak monadic second order logic of countably many successors. Akin to the finite word case,  $\text{WS}\omega\text{S}$  formulae can be translated into (bottom-up nondeterministic) tree automata over finite trees with symbolic (binary) alphabet, on which positivation and dualization can be implemented similar to the word case. Moreover, efficient antichain/simulation-based inclusion checks are also available for tree automata [23], thus expensive complementation can be avoided in this case too. In §5 we present an example involving a parametric hierarchical tree architecture. A detailed workout of this generalization is left for the future.  $\blacksquare$

## 4 Refining Invariants

Since the safety verification problem is undecidable for parametric systems [5], the trap invariants method cannot be complete. As an example, consider the alternating dining philosophers system, of which an instance (for  $n = 3$ ) is shown in Fig. 3. The system consists of two philosopher component types, namely  $\text{Philosopher}_{rl}$ , which takes its right fork before its left fork, and  $\text{Philosopher}_{lr}$ , taking the left fork before the right one. Each philosopher has two interaction ports for taking the forks, namely  $g\ell$  (get left) and  $gr$  (get right) and one port for releasing the forks  $p$  (put). The ports of the  $\text{Philosopher}_{rl}$  component type are overlined, in order to be distinguished. The Fork component type is the same as in Fig. 1. The interaction formula for this system  $\Gamma_{philo}^{alt}$ , shown in Fig. 3, implicitly states that only the 0-index philosopher component is of type  $\text{Philosopher}_{rl}$ , whereas all other philosophers are of type  $\text{Philosopher}_{lr}$ . Note that the interactions on ports  $\overline{g\ell}$ ,  $\overline{gr}$  and  $\overline{p}$  are only allowed if  $\text{inf}(x)$  holds, i.e.  $x = 0$ .

Fig. 3: Alternating Dining Philosophers



$$\Gamma_{philo}^{alt} = \exists x . \text{inf}(x) \wedge [(\overline{g\ell}(x) \wedge g(x)) \vee (\overline{gr}(x) \wedge g(s(x))) \vee (\overline{p}(x) \wedge \ell(x) \wedge \ell(s(x)))] \vee \\ \neg \text{inf}(x) \wedge [(g\ell(x) \wedge g(x)) \vee (gr(x) \wedge g(s(x))) \vee (p(x) \wedge \ell(x) \wedge \ell(s(x)))]$$

It is well-known that any instance of the parametric alternating dining philosophers system consisting of at least one  $\text{Philosopher}_{rl}$  and one  $\text{Philosopher}_{lr}$  is deadlock-free. However, trap invariants are not enough to prove deadlock freedom, as shown by the global state  $\{b(0), h(0), b(1), w(1), f(2), e(2)\}$ , marked with thick red lines in Fig. 3. Note that no interaction is enabled in this state. Moreover, this state intersects with any trap of the marked PN that defines the executions of this particular instance, as proved below. Consequently, the trap invariant contains a deadlock configuration, and the system cannot be proved deadlock-free by this method.

**Proposition 1.** *Consider an instance of the alternating dining philosophers system in Fig. 3, consisting of components Fork(0),  $\text{Philosopher}_{rl}(0)$ , Fork(1),  $\text{Philosopher}_{lr}(1)$ , Fork(2) and  $\text{Philosopher}_{lr}(2)$  placed in a ring, in this order. Then each nonempty trap of this system contains one of the places  $(b, 0)$ ,  $(h, 0)$ ,  $(b, 1)$ ,  $(w, 1)$ ,  $(f, 2)$  or  $(e, 2)$ .*

*Proof:* Let  $C = \{b(0), h(0), b(1), w(1), f(2), e(2)\}$  in the following. We shall try to build a nonempty trap  $T$  that avoids every state in  $C$ . If such a trap can be found, the counterexample is shown to be spurious (unreachable). Below is the list of states allowed in

$T$ , indexed by component (using other states that the ones listed below would result in a trap that is satisfied by the counterexample  $C$ , which is exactly the opposite of what we want):

Fork(0)	Philosopher <sub><math>r_l</math></sub> (0)	Fork(1)	Philosopher <sub><math>l_r</math></sub> (1)	Fork(2)	Philosopher <sub><math>l_r</math></sub> (2)
$f(0)$	$w(0), e(0)$	$f(1)$	$h(1), e(1)$	$b(2)$	$w(2), h(2)$

Assume that  $f(0) \in T$ . Then  $T$  must contain  $b(0)$  or  $e(2)$  (constraint  $gr(2) \wedge g(0)$ ). However neither is allowed, thus  $f(0) \notin T$ . Assume that  $f(1) \in T$ . Then  $T$  must contain  $b(1)$  or  $h(0)$  (constraint  $gr(0) \wedge g(1)$ ), contradiction, thus  $f(1) \notin T$ . Assume that  $b(2) \in T$ . Then  $T$  must contain  $f(1), w(1)$  or  $f(2)$  (constraint  $p(1) \wedge \ell(1) \wedge \ell(2)$ ), contradiction, thus  $b(2) \notin T$ . Then  $T$  contains only philosopher states, except for  $h(0), w(1)$  and  $e(2)$ . One can prove that there is no such trap, for instance, for  $\text{Philosopher}_{l_r}(1)$  we have:

$$\begin{aligned} h(1) \in T &\Rightarrow e(1) \in T \\ e(1) \in T &\Rightarrow w(1) \in T \end{aligned}$$

since  $f(1), b(1), f(2), b(2) \notin T$ . Since  $w(1) \notin T$ , we obtain that  $h(1), e(1) \notin T$ . Then the only possibility is  $T = \emptyset$ .  $\square$

However, the configuration is unreachable by a real execution of the PN, started in the initial configuration  $\bigwedge_{i=0}^2 f(i) \wedge w(i)$ . An intuitive reason is that, in any reachable configuration, each fork is in state  $f(\text{ree})$  only if none of its neighbouring philosophers is in state  $e(\text{ating})$ . In order to prove deadlock freedom, one must learn this and other similar constraints. Next, we present a heuristic method for strenghtening the trap invariant, that learns such universal constraints, involving a fixed set of components.

#### 4.1 Ashcroft Invariants

*Ashcroft invariants* (AI) [6] are a classical method for proving safety properties of parallel programs, in which a global program state is viewed as an array consisting of the local states of each thread. Typically, an AI is an universally quantified assertion  $\forall x_1 \dots x_m . \phi$  that relates the local states of at most  $m$  distinct threads, which is, moreover, an invariant of the execution of the parallel program.

Next, we define a variant of AI tuned for our purposes. We first consider a finite *window* of the parametric system, by identifying a fixed set of components, together with their interactions, and abstracting away all interactions among components outside of this window. The crux is that the indices of the components from the window are not numbers but Skolem constants  $c_1, \dots, c_w$  and the window is defined by a logical formula  $\psi(c_1, \dots, c_w)$  over the vocabulary of these constants, involving inequalities between terms of the form  $s^i(c_j)$ , for some  $i \geq 0$ . This allows to slide the window inside a certain range, without changing the view (i.e. the sub-systems observed by sliding the window are all isomorphic). Each view is a finite component-based system, whose set of reachable states is computable by enumerating the (finite set of) reachable markings of a 1-safe PN of fixed size. Let  $\Phi(c_1, \dots, c_w)$  be the formula defining this set. Then we show that  $\forall x_1 \dots \forall x_w . \psi(x_1, \dots, x_w) \rightarrow \Phi(x_1, \dots, x_w)$  defines an AI of the parametric system  $\mathcal{S}$ , that can be used to strenghten the trap invariant and converge towards a proof of the given safety property. Before entering the formal details, we provide an example.

*Example 3.* Consider the alternating dining philosophers system, with interaction formula  $\Gamma_{\text{philo}}^{\text{alt}}$ , given in Fig. 3. We fix three adjacent components, namely  $\text{Philosopher}_{lr}(c_1)$ ,  $\text{Fork}(c_2)$  and  $\text{Philosopher}_{lr}(c_3)$ , such that the window constraint  $\psi(c_1, c_2, c_3) \stackrel{\text{def}}{=} \exists \zeta . \inf(\zeta) \wedge \zeta < c_1 \wedge c_1 < c_2 \wedge c_2 = \text{succ}(c_1) \wedge c_2 = c_3$  holds. The interactions specified by  $\Gamma_{\text{philo}}^{\text{alt}}$  involving nothing but these components are  $gr(c_1) \wedge g(c_2)$ ,  $p(c_1) \wedge \ell(c_2) \wedge p(c_3)$  and  $g\ell(c_3) \wedge g(c_2)$ . In addition,  $\text{Philosopher}_{lr}(c_1)$  interacts with its left fork, not present in the window defined by  $\psi$ . We abstract this interaction to  $g\ell(c_1)$ . The other partial interactions are  $gr(c_3)$  and  $p(c_3) \wedge \ell(c_2)$ , where the fork to the right of  $\text{Philosopher}_{lr}(c_3)$  is missing from the window. The marked PN corresponding to the window is given in Fig. 4, with the initial marking highlighted in blue. Let  $\Phi(c_1, c_2, c_3)$  be the ground formula describing the set of reachable markings of this PN. The AI corresponding to this window is  $\forall x_1 \forall x_2 \forall x_3 . \exists \zeta . \inf(\zeta) \wedge \zeta < x_1 \wedge x_1 < x_2 \wedge x_2 = x_3 \wedge x_2 = \text{succ}(x_1) \rightarrow \Phi(x_1, x_2, x_3)$ . In particular, this invariant excludes the spurious deadlock counterexample of Fig. 3 by ensuring that a fork is in state  $f(\text{ree})$  only if none of its neighbouring  $\text{Philosopher}_{lr}$ 's is in state  $e(\text{ating})$ . ■

Let  $\mathcal{S} = \langle C^1, \dots, C^K, \Gamma \rangle$  be a parametric component-based system with component types  $C^k = \langle P^k, S^k, s_0^k, \Delta^k \rangle$ , for all  $k = 1, \dots, K$  and an existential interaction formula:

$$\Gamma = \exists x_1 \dots \exists x_m \bigvee_{i=1}^{\ell} . \varphi_i(x_1, \dots, x_m) \wedge \bigwedge_{j=1}^{k_i} p_{ij}(x_{ij}) \quad (3)$$

where  $\varphi_i$  is a quantifier-free IL1S formula not involving predicate atoms and  $x_{ij} \in \{x_1, \dots, x_m\}$ , for all  $i \in \{1, \dots, \ell\}$  and all  $j \in \{1, \dots, k_i\}$ . For example, the interaction formulae  $\Gamma_{\text{philo}}$  (Fig. 1) and  $\Gamma_{\text{philo}}^{\text{alt}}$  (Fig. 3) are both inside this class. In order to define the notion of a window, we fix a set of constant symbols  $\mathbf{c} = \{c_1, \dots, c_w\}$ , each having an associated component type, denoted by  $\text{type}(c_i) \in \{C^1, \dots, C^K\}$ , for all  $i = 1, \dots, w$ . Note that we overload the  $\text{type}(\cdot)$  notation to handle both predicate and constant symbol arguments. For instance, in Example 3, we have  $\text{type}(c_1) = \text{type}(c_3) = \text{Philosopher}_{lr}$  and  $\text{type}(c_2) = \text{Fork}$ .

**Definition 3.** Given two IL1S formulae  $\phi_1$  and  $\phi_2$ , we write  $\phi_1 \models \phi_2$  for  $\llbracket \phi_1 \rrbracket \subseteq \llbracket \phi_2 \rrbracket$ . Then  $\phi_1$  is non-overlapping with  $\phi_2$  if and only if either  $\phi_1 \models \phi_2$  or  $\phi_1 \models \neg \phi_2$  holds.

A window constraint for the interaction formula  $\Gamma$  as before (3), is a formula  $\psi$  that is non-overlapping with each of the formulae:

$$\phi_i(c_{i_1}, \dots, c_{i_k}) \stackrel{\text{def}}{=} \exists x_1 \dots \exists x_p . \bigwedge_{i=1}^p \bigwedge_{u=1}^w x_i \neq c_u \wedge \varphi_i(y_1, \dots, y_m)$$

where  $0 \leq p \leq w$  is an integer,  $\{c_{i_1}, \dots, c_{i_k}\} \subseteq \{c_1, \dots, c_w\}$  and  $\{y_1, \dots, y_m\}$  is a reindexing of the set  $\{x_1, \dots, x_p\} \cup \{c_{i_1}, \dots, c_{i_k}\}$ . Since there are finitely many such formulae<sup>9</sup>, it is possible to build window constraints, by taking conjunctions in which each  $\phi_i(c_{i_1}, \dots, c_{i_k})$  formula occurs either positively or under negation.

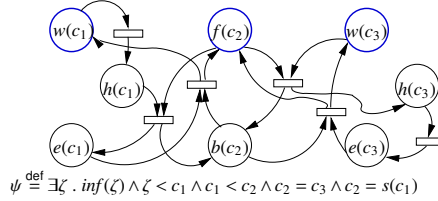
**Definition 4.** Given a set of constant symbols  $\mathbf{c}$ , two IL1S-structures  $([n], \iota_1, \nu_1)$  and  $([n], \iota_2, \nu_2)$  are  $\mathbf{c}$ -isomorphic, denoted  $([n], \iota_1, \nu_1) \approx_{\mathbf{c}} ([n], \iota_2, \nu_2)$  if and only if  $\iota_1(c) \in \iota_1(\text{pr}) \iff \iota_2(c) \in \iota_2(\text{pr})$  for all  $c \in \mathbf{c}$  and all  $\text{pr} \in \text{Pred}$ . For a structure  $I$ , we denote by  $[I]_{\mathbf{c}}$  its  $\approx_{\mathbf{c}}$ -equivalence class.

<sup>9</sup> The set  $\{\phi_i(c_{i_1}, \dots, c_{i_k}) \mid 1 \leq i \leq \ell, c_{i_1}, \dots, c_{i_k} \in \mathbf{c}\}$  is determined by  $\Gamma$  and  $\mathbf{c}$ .

**Definition 5.** Given a window constraint  $\psi$ , the view of  $\Gamma$  via  $\psi$  is the ground formula  $V_\Gamma^\psi \stackrel{\text{def}}{=} \bigvee_{i=1}^\ell \bigwedge_{j=1}^{k_i} \pi_{ij}$ , where, for each  $1 \leq i \leq \ell$  and each  $1 \leq j \leq k_i$ , we have  $\pi_{ij} \stackrel{\text{def}}{=} p_{ij}(c_{ij})$  if  $\psi \models \phi_i(c_{i_1}, \dots, c_{i_{k_i}})$ , for some  $c_{i_1}, \dots, c_{i_{k_i}} \in \mathbf{c}$  and  $\pi_{ij} \stackrel{\text{def}}{=} \top$ , otherwise.

Intuitively, the view specifies the complete interactions between the components identified by  $c_1, \dots, c_w$  and their respective types as well as all the partial interactions from which some component is missing from the window, i.e. when  $\{c_{i_1}, \dots, c_{i_{k_i}}\} \neq \mathbf{c}$ . Note that each interaction is unambiguously specified by the window constraint, because either (i)  $p_{ij}(c_{ij})$  is part of the interaction then  $\psi \models \phi_i$ , thus the component identified by  $c_{ij}$  and the component type  $\text{type}(c_{ij})$  is always in the window (no matter what value does  $c_{ij}$  take), or (ii)  $\psi \not\models \phi_i$ , and since  $\psi$  is non-overlapping with  $\phi_i$ , we have  $\psi \models \neg\phi_i$ , in which case the  $c_{ij}$  component is never within the  $\psi$  window.

Fig. 4: Window Petri Net for the Alternating Dining Philosophers Example



*Example 4.* For the system in Fig. 3 and the window constraint  $\psi$  from Example 3, we obtain the view  $V_\Gamma^\psi = (gr(c_1) \wedge g(c_2)) \vee (p(c_1) \wedge \ell(c_2) \wedge p(c_3)) \vee (g\ell(c_3) \wedge g(c_2)) \vee g\ell(c_1) \vee gr(c_3) \vee (p(c_3) \wedge \ell(c_2))$ . The interaction  $gr(c_1) \wedge g(c_2)$  occurs between components inside the window only, since  $\psi \models \neg \text{inf}(c_1) \wedge c_2 = \text{succ}(c_1)$ . On the other hand,  $p(c_3) \wedge \ell(c_2)$  is a partial interaction, because  $\psi \models \exists z_1. c_2 = c_3 \wedge z_1 = \text{succ}(c_3) \wedge p(c_3) \wedge \ell(c_2) \wedge \ell(z_1)$ , thus ports  $p(c_3)$  and  $\ell(c_2)$  are kept inside and  $\ell(\text{succ}(c_3))$  is abstracted away. ■

A view  $V_\Gamma^\psi$  becomes the interaction formula of a system with a constant number of components, whose marked PN is denoted by  $\mathcal{N}_S^\psi$ . Formally, we define  $\mathcal{N}_S^\psi = (\mathbf{N}^\psi, \mathbf{m}_0^\psi)$ , where  $\mathbf{N}^\psi = (S^\psi, T^\psi, E^\psi)$  and:

- $S^\psi \stackrel{\text{def}}{=} \{\text{succ}(c) \mid s \in \mathbf{S}(\text{type}(c)), c \in \mathbf{c}\}$ ,
- for each equivalence class  $[I]_c$  of some  $I = ([n], \iota, \nu) \in \llbracket V_\Gamma^\psi \rrbracket$ , there exists  $t \in T^\psi$  and  $(\text{succ}(c), t), (t, s'(c)) \in E^\psi$  iff  $s \xrightarrow{p} s' \in \mathcal{A}(\text{type}(c))$  and  $\iota(c) \in \iota(p)$ , for all  $\text{succ}(c) \in S^\psi$ ,
- for all  $\text{succ}(c) \in S^\psi$ , we have  $\mathbf{m}_0(\text{succ}(c)) = 1$  iff  $s = s_0(\text{type}(c))$ .

Since this is a 1-safe marked PN of known size, it is possible to compute its reachable markings by exhaustive enumeration and compute a ground formula  $\Phi_\Gamma^\psi(c_1, \dots, c_w)$  that defines this set. For instance, the marked PN for the view of the system in Fig. 3 via the window constraint  $\psi$  from Example 3, is given in Fig. 4. Its set of reachable markings is defined by the formula<sup>10</sup>:

$$\begin{aligned} & (w(c_1) \wedge f(c_2) \wedge w(c_3)) \vee (h(c_1) \wedge f(c_2) \wedge w(c_3)) \vee (w(c_1) \wedge b(c_2) \wedge h(c_3)) \\ & \vee (e(c_1) \wedge b(c_2) \wedge w(c_3)) \vee (h(c_1) \wedge b(c_2) \wedge h(c_3)) \\ & \vee (w(c_1) \wedge b(c_2) \wedge e(c_3)) \vee (h(c_1) \wedge b(c_2) \wedge e(c_3)) \end{aligned}$$

<sup>10</sup> We intentionally left out the negative literals, as they play no role in proving deadlock freedom.

Finally, this formula is used to build the AI  $\forall x_1 \dots \forall x_w . \psi(x_1, \dots, x_w) \rightarrow \Phi_F^\psi(x_1, \dots, x_w)$ , where  $\psi(x_1, \dots, x_w)$  and  $\Phi_F^\psi(x_1, \dots, x_w)$  are obtained from  $\psi$  and  $\Phi_F^\psi$ , respectively, by replacing each constant symbol  $c_i$  with a variable  $x_i$ . The following result states that this formula defines an invariant of the system:

**Proposition 2.** *Let  $S = \langle C^1, \dots, C^K, \Gamma \rangle$  be a system and let  $\psi(c_1, \dots, c_w)$  be a window constraint. Then  $\mathcal{A}_F^{\psi, n} \stackrel{\text{def}}{=} \llbracket \forall x_1 \dots \forall x_w . \psi(x_1, \dots, x_w) \rightarrow \Phi_F^\psi(x_1, \dots, x_w) \rrbracket^n$  is an invariant of  $\mathcal{N}_S^n$ , for any  $n > 0$ , where  $\llbracket \phi \rrbracket^n \stackrel{\text{def}}{=} \{([n], \iota, \nu) \mid ([n], \iota, \nu) \models \phi\}$  for any  $\text{LL1S}$  formula  $\phi$ .*

*Proof:* Let  $n > 0$  be an arbitrary positive integer. The component types of  $S$  are  $C^k = \langle \mathbf{P}^k, \mathbf{S}^k, s_0^k, \mathcal{A}^k \rangle$ , for all  $k = 1, \dots, K$  and its marked PN  $\mathcal{N}_S^n = (\mathbf{N}, m_0)$ , where:

- $\mathbf{N} = (\bigcup_{k=1}^K \mathbf{S}^k \times [n], T, E)$  and,
- for all  $1 \leq k \leq K$  and all  $s \in \mathbf{S}^k$ , we have  $m_0((s, i)) = 1$ , if  $s = s_0^k$  and  $m_0((s, i)) = 0$ , otherwise.

Moreover, for a marking  $m : \bigcup_{k=1}^K \mathbf{S}^k \times [n] \rightarrow \{0, 1\}$  of  $\mathcal{N}_S^n$  and a formula  $\phi$ , we write  $m \in \llbracket \phi \rrbracket^n$  iff  $([n], \iota, \nu) \models \phi$ , where  $\iota$  is such that  $\iota(s) = \{i \in [n] \mid m((s, i)) = 1\}$  and  $\nu$  is an arbitrary valuation.

We prove first that  $m_0 \in \mathcal{A}_F^{\psi, n}$ . Let  $i_1, \dots, i_w \in [n]$  be arbitrary integers. Clearly,  $m_0((s_0^{k_1}, i_1)) = \dots = m_0((s_0^{k_w}, i_w)) = 1$ , thus  $([n], \iota[s_0^{k_1} \leftarrow \{i_1\}] \dots [s_0^{k_w} \leftarrow \{i_w\}], \nu[x_1 \leftarrow i_1] \dots [x_w \leftarrow i_w]) \models \Phi_F^\psi$ , since  $\Phi_F^\psi$  is the set of reachable markings of  $\mathcal{N}_S^\psi$  and  $m_0$  subsumes the initial marking thereof.

Second, we show that  $\mathcal{A}_F^{\psi, n}$  is inductive, i.e. for each move  $m \xrightarrow{t} m'$  of  $\mathcal{N}_S^n$ , such that  $m \in \mathcal{A}_F^{\psi, n}$ , we must show that  $m' \in \mathcal{A}_F^{\psi, n}$ . First, notice that, for any interpretation  $\iota$ , any valuation  $\nu$  and any  $i_1, \dots, i_w \in [n]$ , we have:

$$\begin{aligned} ([n], \iota[c_1 \leftarrow i_1] \dots [c_w \leftarrow i_w], \nu) \models \phi(c_1, \dots, c_w) \\ \iff ([n], \iota, \nu[x_1 \leftarrow i_1] \dots [x_w \leftarrow i_w]) \models \phi(x_1, \dots, x_w) \end{aligned} \tag{4}$$

for an arbitrary formula  $\phi$ . In the following, we define, for all  $s \in \bigcup_{k=1}^K \mathbf{S}^k$ :

$$\begin{aligned} \iota_m(s) &\stackrel{\text{def}}{=} \{i \in [n] \mid m((s, i)) = 1\} \\ \iota_{m'}(s) &\stackrel{\text{def}}{=} \{i \in [n] \mid m'((s, i)) = 1\} \end{aligned}$$

Let  $i_1, \dots, i_w \in [n]$  be integers such that  $([n], \iota_{m'}, \nu[x_1 \leftarrow i_1] \dots [x_w \leftarrow i_w]) \models \psi(x_1, \dots, x_w)$ . We compute as follows:

$$\begin{aligned} ([n], \iota_{m'}, \nu[x_1 \leftarrow i_1] \dots [x_w \leftarrow i_w]) \models \psi(x_1, \dots, x_w) &\iff \text{since } \psi \text{ has only atoms } s^k(x_i) \leq s^\ell(x_j) \\ ([n], \iota_m, \nu[x_1 \leftarrow i_1] \dots [x_w \leftarrow i_w]) \models \psi(x_1, \dots, x_w) &\iff \text{by (5)} \\ ([n], \iota_m[c_1 \leftarrow i_1] \dots [c_w \leftarrow i_w], \nu) \models \psi(c_1, \dots, c_w) &\implies \text{since } m \in \mathcal{A}_F^{\psi, n} \\ ([n], \iota_m[c_1 \leftarrow i_1] \dots [c_w \leftarrow i_w], \nu) \models \Phi_F^\psi(c_1, \dots, c_w) &\implies (\dagger) \\ ([n], \iota_{m'}[c_1 \leftarrow i_1] \dots [c_w \leftarrow i_w], \nu) \models \Phi_F^\psi(c_1, \dots, c_w) &\iff \text{by (5)} \\ ([n], \iota_{m'}, \nu[x_1 \leftarrow i_1] \dots [x_w \leftarrow i_w]) \models \Phi_F^\psi(x_1, \dots, x_w) &\text{thus } m' \in \mathcal{A}_F^{\psi, n}, \text{ as required.} \end{aligned}$$

We are left with proving the step  $(\dagger)$  above. Because  $\mathcal{N}_S^n$  is  $S$ -decomposable, by Lemma 13, and since  $m \xrightarrow{t} m'$  by the hypothesis, there are states  $s_1, s'_1 \in \mathbf{S}^{k_1} \dots s_m, s'_m \in \mathbf{S}^{k_m}$ , with  $k_1, \dots, k_m$  pairwise disjoint, integers  $j_1, \dots, j_m \in [n]$  and edges  $((s_i, j_i), t), (t, (s'_i, j_i)) \in E$ , for all  $i = 1, \dots, m$ . For each  $i = 1, \dots, m$ , we distinguish the cases:

- if  $s_i = s'_i$  then  $m((s_i, j_i)) = m'((s_i, j_i)) = 1$ ,
- else, if  $s_i \neq s'_i$  then  $m((s_i, j_i)) = m'((s'_i, j_i)) = 1$  and  $m((s'_i, j_i)) = m'((s_i, j_i)) = 0$ .

Observe now that  $\mathcal{N}_S^\psi$  has the same structure as the subnet obtained by restricting  $\mathcal{N}_S^n$  to the states in  $\{s_1, \dots, s_m, s'_1, \dots, s'_m\} \times \{j_1, \dots, j_m\}$ . Moreover, there exists a transition  $t \downarrow_\psi$  in  $\mathcal{N}_S^\psi$  and edges  $(s_i(c_i), t \downarrow_\psi), (t \downarrow_\psi, s'_i(c_i)) \in E^\psi$  only if  $((s_i, j_i), t), (t, (s'_i, j_i)) \in E$ . Let  $m \downarrow_\psi$  and  $m' \downarrow_\psi$  be the projections of  $m$  and  $m'$  on  $\{s_1, \dots, s_m, s'_1, \dots, s'_m\} \times \{j_1, \dots, j_m\}$ , respectively.

Since  $m \xrightarrow{t} m'$ , we obtain that  $m \downarrow_\psi \xrightarrow{t \downarrow_\psi} m' \downarrow_\psi$ , thus  $([n], \iota_{m'}[c_1 \leftarrow i_1] \dots [c_w \leftarrow i_w], \nu) \models \Phi_T^\psi(c_1, \dots, c_w)$ , as required.  $\square$

## 5 Experiments

We carried out a preliminary evaluation of our parametric verification method, using a number of textbook examples, shown in Table 1. The table reports the size of the example (number of states/transition per component type) and the running times (in seconds) needed to check deadlock freedom (D-freedom) and mutual exclusion (Mutex). We used the MONA tool [22] to generate the automata from WS1S formulae and the VATA tree automata library [28] to check the verification condition on automata. The running times from the table are relative to a x86\_64bit Ubuntu virtual machine with 4GB of RAM. The files needed to reproduce the results are available online<sup>11</sup>. All examples were successfully verified for deadlock freedom by our method using trap invariants. However, not all experiments with mutual exclusion were conclusive, as the intersection of the invariant with the bad states was not empty in some cases. The dash from the Mutex column indicates that mutual exclusion checking is not applicable for the considered example.

Example	States/Transitions	D-freedom (sec)	Mutex (sec)
Dining Philosophers I	3/3 + 2/2	0.252	–
Dining Philosophers II	3/3 + 3/4	0.496	–
Dining Philosophers III	3/3 + 2/2	21.640	–
Exclusive Tasks	2/3	0.004	0.004
Preemptive Tasks I	4/5	0.020	1.612
Preemptive Tasks II	4/5	0.020	1.564
Burns	6/8	0.012	0.012
Szymanski	12/13	2.892	not empty
Dijkstra-Scholten I	4/4	0.012	–
Dijkstra-Scholten II	4/4	0.064	–

Table 1: Running times deadlock-freedom and mutual exclusion checking

Dining Philosophers I is the alternating dinning philosophers protocol where all but one philosopher are taking the forks in the same order. This example requires an additional Ashcroft invariant for deadlock freedom. Dining Philosophers II is a refinement of the previous model, where the behavior of the forks remembers which philosopher is

<sup>11</sup> <http://nts.imag.fr/images/0/06/Cav19.tar.gz>

handling them (using two busy states  $b_{left}$  and  $b_{right}$ ). Dining Philosophers III is a variant of the protocol, where the philosophers are sharing two global forks, taken in the same order. In these two cases, the trap invariant is sufficient to prove deadlock freedom.

Exclusive Tasks is a mutual exclusion protocol in which every task can be waiting or executing. A task moves from waiting to executing only if all other tasks are waiting, whereas an executing task can move back from execution to waiting at any time. Preemptive Tasks I is a concurrent system in which every task can be ready, waiting, executing or preempting. Initially, one task is executing, while the others are ready. A task moves from ready to waiting at any time. A task begins execution by preempting the currently executing tasks. When a task finishes it becomes ready and one of the preempted tasks resumes back to execution. Preemptive Tasks II is same as before, except that the task which resumes back to execution is always the one with the highest identifier.

Burns [24] and Szymanski [31] are classical mutual exclusion protocols taken from literature. Dijkstra-Scholten I is an algorithm used to detect termination in a distributed computation. It organizes the computational nodes into a tree and propagates a message from the root to all the leaves and back, once the computation is finished. In the first variant, we consider the degenerate case where the tree is a list. Dijkstra-Scholten II is the full version of the algorithm on an arbitrary binary tree. This example required using MONA and VATA in tree mode, on  $WS\omega S$  and finite nondeterministic bottom-up tree automata, respectively.

## 6 Conclusions and Future Work

We presented a method for checking safety properties of parametric systems, in which the number of components is not known a priori. The method is based on a synthesis of trap invariants from the interaction formula of the system and relies on two logical operations (positivation and dualization) that are implemented using the automata-theoretic connection between  $WS1S$  and finite Rabin-Scott automata. We show that trap invariants, strengthened with Ashcroft invariants, produced by an orthogonal method are, in general, strong enough to prove deadlock freedom.

As future work, we plan on developing a toolbox integrating the existing tools used to generate trap and Ashcroft invariants, supporting the interactive application of the method to real-life architectures (controllers, autonomous cyber-physical systems, etc.)

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## A Proofs

### A.1 Proof of Lemma 2

Let  $N = (S, T, E)$  and  $W \subseteq S$  be a trap of  $N$ . We have the following equivalences:

$$\begin{aligned}
W^\bullet &\subseteq \bullet W && \iff \\
\bigwedge_{p \in S} [p \in W \rightarrow \{t \in T \mid (p, t) \in E\} \subseteq \{t \in T \mid \bigvee_{q \in S} (t, q) \in E\}] &&& \iff \\
\bigwedge_{p \in S} [p \in W \rightarrow (\bigwedge_{t \in T} p \in \bullet t \rightarrow \bigvee_{q \in S} q \in W \wedge q \in t^\bullet)] &&& \iff \\
\bigwedge_{p \in S} \bigwedge_{t \in T} (p \in W \wedge p \in \bullet t \rightarrow \bigvee_{q \in S} q \in W \wedge q \in t^\bullet) &&& \iff \\
\bigwedge_{t \in T} (\bigvee_{p \in \bullet t} p \in W \rightarrow \bigvee_{q \in t^\bullet} q \in W) &&& 
\end{aligned}$$

If we use propositional variables  $p$  and  $q$  to denote  $p \in W$  and  $q \in W$ , respectively, we obtain the trap constraint  $\Theta(N)$  from the last formula. Hence, any boolean valuation  $\beta \in \llbracket \mu_0 \wedge \Theta(N) \rrbracket$  corresponds to an initially marked trap  $W_\beta \stackrel{\text{def}}{=} \{p \in S \mid \beta(p) = \top\}$ . Further, since  $\mu_0 \wedge \Theta(N)$  is a propositional formula, each satisfying valuation corresponds to a conjunctive clause of its DNF. Hence the set of propositional variables in each conjunctive clause in the DNF of  $(\mu_0 \wedge \Theta(N))^+$  corresponds to an IMT and, moreover, every IMT has a corresponding conjunctive clause. Thus  $\text{TrapInv}(N) \equiv ((\mu_0 \wedge \Theta(N))^+)^{\sim}$  follows. The second point follows directly from the definition  $\text{TrapInv}(N)$ .  $\square$

### A.2 Proof of Lemma 3

“(1)  $\Rightarrow$  (2)” First, it is routine to check that, for any WS1S-structure, we have  $([n], \iota, \nu, \mu) \models \forall y . y \leq x \iff \nu(x) = n - 1$ . Suppose that  $\phi$  has a model  $\mathcal{I} = ([n], \iota, \nu)$  and the interpretation of  $s$  is  $\mathbf{s}_{LLIS}^n$ . Then we show that  $\mathcal{S} = ([n], \iota, \nu, \mu)$  is a model of  $\text{Tr}(\phi)$ , for any  $\mu : \text{SVar} \rightarrow 2^{[n]}$ . For this, it is enough to show that  $([n], \iota, \nu[\mu \leftarrow n - 1], \mu) \models \text{tr}(\phi)$ , by induction on the structure of  $\phi$ . The base cases are:

- $s(x) = y$ : in this case  $\mathbf{s}_{LLIS}^n(\nu(x)) = \nu(y)$  and thus  $\nu(y) = (\nu(x) + 1) \bmod n$ , by the definition of  $\mathbf{s}_{LLIS}^n$ . But then either  $\nu(x) < n - 1$  and  $\nu(y) = \nu(x) + 1$  or  $\nu(x) = n - 1$  and  $\nu(y) = 0$ , thus  $\mathcal{S} \models \text{tr}(s(x) = y)$ , as required.
- $\text{pr}(x)$ : in this case  $\nu(x) \in \iota(p)$  and  $\mathcal{S} \models \text{tr}(\text{pr}(x))$  by the definition.

The induction cases are immediate.

“(1)  $\Leftarrow$  (2)” If  $([n], \nu, \iota, \mu) \models \text{Tr}(\phi)$ , for some arbitrary mapping  $\mu : \text{SVar} \rightarrow 2^{[n]}$ , then we have  $([n], \nu[\nu \leftarrow n - 1], \iota, \mu) \models \text{tr}(\phi)$  and we show  $([n], \nu, \iota) \models \phi$  by induction on the structure of  $\phi$ . The most interesting case is when  $\phi$  is  $s(x) = y$ , in which case either:

- $\nu(x) < n - 1$  and  $\nu(y) = \nu(x) + 1$ , or
- $\nu(x) = n - 1$  and  $\nu(y) = 0$ .

In each case, we have  $\mathbf{s}_{LLIS}^n(\nu(x)) = \nu(y)$ , hence  $([n], \nu, \iota) \models s(x) = y$ , as required.  $\square$

### A.3 Proof of Lemma 4

By induction on the structure of  $\phi$ :

- $t_1 = t_2$  and  $\neg t_1 = t_2$ : the truth value of this atom is the same in  $\mathcal{S}$  and  $\overline{\mathcal{S}}$  and moreover  $t_1 = t_2$  and  $\neg(t_1 = t_2)^{\sim}$  are equivalent.
- $X(t)$  and  $\neg X(t)$ : same as above.

- $\text{pr}(t)$ : the interpretation of  $t$  is the same in  $S$  and  $\bar{S}$ , because it depends only on  $v$ . Let  $k \in [n]$  be this value. Then we obtain:

$$S \models \text{pr}(t) \iff k \in \iota(\text{pr}) \iff k \notin \bar{\iota}(\text{pr}) \iff \bar{S} \models \neg \text{pr}(t) .$$

- $\neg \text{pr}(t)$ : a consequence of the equivalence  $S \models \text{pr}(t) \iff \bar{S} \models \neg \text{pr}(t)$ , established at the previous point.

The rest of the cases are easy applications of the induction hypothesis.  $\square$

#### A.4 Proof of Lemma 7

We start from the observation that  $\mathcal{L}(A) \subseteq \mathcal{L}(A^*)$  because  $A^* = (Q, I, F, \delta^*)$  is obtained by adding transitions to  $A = (Q, I, F, \delta)$ . Moreover, given a run  $\rho = s_0, \dots, s_m$  of  $A^*$  over some word  $\sigma_0 \dots \sigma_{m-1}$ , there exists a word  $\sigma'_0 \dots \sigma'_{m-1}$  such that for each  $i \in [m]$  and  $1 \leq j \leq \ell$ , we have  $\sigma'_i(j) \leq \sigma_i(j)$  and  $(s_i, \sigma'_i, s_{i+1}) \in \delta$ . This is because we only add to  $A^*$  transitions  $(q_i, \sigma, q_j)$  provided that there exists  $(q_i, \sigma', q_j) \in \delta$  such that  $\sigma'(j) \leq \sigma(i)$ , for all  $1 \leq j \leq \ell$ .

“ $\subseteq$ ” Let  $w \in \mathcal{L}^{\min}(A)$ , then  $w \in \mathcal{L}(A^*)$  because  $\mathcal{L}(A) \subseteq \mathcal{L}(A^*)$ . Let  $w'$  be a word such that  $w' \leq w$  and  $w' \neq w$  and suppose, for a contradiction that  $w' \in \mathcal{L}^{\min}(A^*)$ . Then  $A^*$  has an accepting run  $\rho = s_0, \dots, s_m$  over  $w'$ , thus  $\rho$  is also an accepting run of  $A$  over another word  $w'' \leq w'$ . Since  $w \in \mathcal{L}^{\min}(A)$  and  $w'' \leq w' \leq w$  and  $w' \neq w$ , we obtain a contradiction. Thus,  $w \in \mathcal{L}^{\min}(A^*)$ , as required.

“ $\supseteq$ ” Let  $w \in \mathcal{L}^{\min}(A^*)$  and let  $\rho = s_0, \dots, s_m$  be an accepting run of  $A^*$  over  $w$ . Then there exists a word  $w' \leq w$  such that  $\rho$  is an accepting run of  $A$ . Since  $w' \in \mathcal{L}(A) \subseteq \mathcal{L}(A^*)$ , we obtain that  $w' = w$ , thus  $w \in \mathcal{L}(A)$ . Now suppose, for a contradiction, that there exists  $w'' \leq w$  such that  $w'' \neq w$  and  $w'' \in \mathcal{L}(A)$ . Then  $w'' \in \mathcal{L}(A^*)$  and since  $w'' \leq w$  and  $w'' \neq w$ , this contradicts the fact that  $w \in \mathcal{L}^{\min}(A^*)$ . Thus,  $w \in \mathcal{L}^{\min}(A)$ , as required.  $\square$

#### A.5 Proof of Lemma 8

(1) It is sufficient to show that  $(\phi)^\oplus \equiv \Phi_{A_\phi}^*$  and apply Theorem 1. Denoting  $A_\phi = (Q, I, F, \delta)$ , with  $Q = \{s_1, \dots, s_q\}$  and  $A_\phi^* = (Q, I, F, \delta^*)$  as before, we only show that  $\psi_\delta^* \equiv \psi_{\delta^*}$ . Because  $(\phi)^\oplus = \exists X_1 \dots \exists X_q . \psi_{\text{cover}} \wedge \psi_I \wedge \psi_\delta^* \wedge \psi_F$  and  $\Phi_{A_\phi}^* = \exists X_1 \dots \exists X_q . \psi_{\text{cover}} \wedge \psi_I \wedge \psi_{\delta^*} \wedge \psi_F$ , we immediately obtain the result. We have the following equivalence, for each  $\sigma \in \{0, 1\}^\ell$ :

$$\bigwedge_{\substack{1 \leq k \leq \ell \\ \sigma(\text{pr}_k)=1}} \text{pr}_k(x) \equiv \bigvee_{\sigma \leq \tau} \left( \bigwedge_{\substack{1 \leq k \leq \ell \\ \tau(\text{pr}_k)=1}} \text{pr}_k(x) \wedge \bigwedge_{\substack{1 \leq k \leq \ell \\ \tau(\text{pr}_k)=0}} \neg \text{pr}_k(x) \right)$$

where  $\sigma \leq \tau$  stands for  $\forall j . 1 \leq j \leq \ell \Rightarrow \sigma(j) \leq \tau(j)$ . This immediately implies that  $\psi_\delta^* \equiv \psi_{\delta^*}$ , by the definitions of these formulae respectively, and the construction of  $\delta^*$ .

(2) For an arbitrary structure  $\mathcal{S} = ([n], \nu, \iota, \mu)$  we have  $\iota(\text{pr}_k) = w_{\mathcal{S}}(\text{pr}_k)$ , for any  $1 \leq k \leq \ell$ , by the definition of  $w_{\mathcal{S}}$ . Then  $\mathcal{S}_1 \sqsubseteq \mathcal{S}_2 \iff w_{\mathcal{S}_1} \leq w_{\mathcal{S}_2}$ , for any two structures  $\mathcal{S}_i = ([n], \nu_i, \iota_i, \mu_i)$ , where  $i = 1, 2$ . Hence a structure  $\mathcal{S}$  is a minimal model of  $\phi$  if and only if  $w_{\mathcal{S}} \in \mathcal{L}^{\min}(A_\phi)$ . By Lemma 7, we have  $\mathcal{L}^{\min}(A) = \mathcal{L}^{\min}(A^*)$ . Then the result follows from Theorem 1 and point (1) of this Lemma.  $\square$

## A.6 Proof of Theorem 2

We define first booleanization formally. First, given an integer  $i \geq 0$  and a WS1S formula  $\phi(x)$ , we denote by  $\phi[i/x]$  (resp.  $t[i/x]$ ) the formula (term) obtained from  $\phi$  (resp.  $t$ ) by replacing every occurrence of  $x$  with the term  $s^i(\bar{0})$ , where  $s^i$  denotes  $i$  successive applications of the successor function. Second, for a set  $S$  of positive integers, the formula  $\phi[S/X]$  is defined homomorphically, starting with the base case  $X(t)[S/X] \stackrel{\text{def}}{=} \bigvee_{i \in S} t = s^i(\bar{0})$ .

$$\begin{aligned} B_n(s^i(\bar{0}) = s^j(\bar{0})) &\stackrel{\text{def}}{=} i = j \vee (i \geq n-1 \wedge j \geq n-1) & B_n(\text{pr}(s^i(\bar{0}))) &\stackrel{\text{def}}{=} \text{pr}_{\min(i, n-1)} \\ B_n(\phi_1 \wedge \phi_2) &\stackrel{\text{def}}{=} B_n(\phi_1) \wedge B_n(\phi_2) & B_n(\neg \phi_1) &\stackrel{\text{def}}{=} \neg B_n(\phi_1) \\ B_n(\exists x . \phi) &\stackrel{\text{def}}{=} \bigvee_{i \in [n]} B_n(\phi[i/x]) & B_n(\exists X . \phi) &\stackrel{\text{def}}{=} \bigvee_{S \subseteq [n]} B_n(\phi[S/X]) \end{aligned}$$

where, for any  $\text{pr} \in \text{Pred}$  and  $j \in [n]$ ,  $\text{pr}_j$  is a propositional variable ranging over the boolean values  $\top$  (true) and  $\perp$  (false). Moreover, we relate WS1S structures with boolean valuations as follows. Given a structure  $\mathcal{S} = ([n], \nu, \iota, \mu)$  we define the boolean valuation  $\beta_{\mathcal{S}}(\text{pr}_j) \stackrel{\text{def}}{=} \top \iff s_{\text{WS1S}}^j(0) \in \iota(\text{pr})$ , for all  $\text{pr} \in \text{Pred}$  and  $j \in [n]$ . The following lemma states the formal property of booleanization:

**Lemma 10.** *Given a WS1S sentence  $\phi$  and  $n > 0$ , for every structure  $\mathcal{S} = ([n], \nu, \iota, \mu)$ , we have  $\mathcal{S} \models \phi \iff \beta_{\mathcal{S}} \models B_n(\phi)$ .*

*Proof.* We prove the following more general statement. Let  $\phi(x_1, \dots, x_k, X_1, \dots, X_m)$  be a WS1S formula with free variables  $x_1, \dots, x_k \in \text{Var}$  and  $X_1, \dots, X_m \in \text{SVar}$ ,  $i_1, \dots, i_k \in [n]$  and  $S_1, \dots, S_m \subseteq [n]$ . Then we show that:

$$\begin{aligned} \mathcal{S} \models \phi[i_1/x_1, \dots, i_k/x_k, S_1/X_1, \dots, S_m/X_m] \\ \iff \\ \beta_{\mathcal{S}} \models B_n(\phi[i_1/x_1, \dots, i_k/x_k, S_1/X_1, \dots, S_m/X_m]) \end{aligned}$$

by induction on the structure of  $\phi$ :

- $t_1 = t_2$ : since  $\phi[i_1/x_1, \dots, i_k/x_k, S_1/X_1, \dots, S_m/X_m]$  is a sentence, it must be the case that  $t_1 = s^{i_1}(\bar{0})$  and  $t_2 = s^{i_2}(\bar{0})$ , for some  $i_1, i_2 \geq 0$ . Then we have:

$$\begin{aligned} \mathcal{S} \models s^{i_1}(\bar{0}) = s^{i_2}(\bar{0}) &\iff s_{\text{WS1S}}^{i_1}(0) = s_{\text{WS1S}}^{i_2}(0) \\ &\iff i_1 = i_2 \vee (i_1 \geq n-1 \wedge i_2 \geq n-1) \\ &\iff \beta_{\mathcal{S}} \models B_n(s^{i_1}(\bar{0}) = s^{i_2}(\bar{0})) . \end{aligned}$$

- $\text{pr}(t)$ : since  $\phi[i_1/x_1, \dots, i_k/x_k, S_1/X_1, \dots, S_m/X_m]$  is a sentence, it must be the case that  $t = s^i(\bar{0})$ , for some  $i \geq 0$ . We obtain:

$$\begin{aligned} \mathcal{S} \models \text{pr}(s^i(\bar{0})) &\iff s_{\text{WS1S}}^i(\bar{0}) \in \iota(\text{pr}) \\ &\iff s_{\text{WS1S}}^{\min(i, n-1)} \in \iota(\text{pr}) \\ &\iff \beta_{\mathcal{S}} \models \text{pr}_{\min(i, n-1)} . \end{aligned}$$

The rest of the cases are easy applications of the induction hypothesis.  $\square$

**Lemma 11.** *Given a WS1S formula  $\phi$  and a constant  $n > 0$ , we have  $(B_n(\phi))^+ \equiv B_n((\phi)^{\oplus})$ .*

**Proof.** First, note that, for any propositional formulae  $f$  and  $g$ , whose variables occur under even number of negations, we have  $f \equiv g \iff f \equiv^{\min} g$ . Since both  $(B_n(\phi))^+$  and  $B_n((\phi)^\oplus)$  are positive propositional formulae, it is sufficient to prove  $(B_n(\phi))^+ \equiv^{\min} B_n((\phi)^\oplus)$ , by showing  $\llbracket (B_n(\phi))^+ \rrbracket^{\min} \subseteq \llbracket B_n((\phi)^\oplus) \rrbracket$  and  $\llbracket B_n((\phi)^\oplus) \rrbracket^{\min} \subseteq \llbracket (B_n(\phi))^+ \rrbracket$ , respectively, which establishes  $\llbracket (B_n(\phi))^+ \rrbracket^{\min} = \llbracket B_n((\phi)^\oplus) \rrbracket^{\min}$  (the latter step is left to the reader).

“ $\llbracket (B_n(\phi))^+ \rrbracket^{\min} \subseteq \llbracket B_n((\phi)^\oplus) \rrbracket$ ” Let  $\beta \in \llbracket (B_n(\phi))^+ \rrbracket^{\min}$  be a valuation. Then, we also have  $\beta \in \llbracket B_n(\phi) \rrbracket^{\min}$ , since  $(\phi)^+ \equiv^{\min} \phi$ , in general for any propositional formula  $\phi$ . Then, by Lemma 10, there exists a structure  $\mathcal{S} \in \llbracket \phi \rrbracket^{\min}$  such that  $\beta = \beta_{\mathcal{S}}$ . Hence we obtain  $\mathcal{S} \in \llbracket (\phi)^\oplus \rrbracket^{\min} \subseteq \llbracket (\phi)^\oplus \rrbracket$ . But then  $\beta \in \llbracket B_n((\phi)^\oplus) \rrbracket$ , by Lemma 10.

“ $\llbracket B_n((\phi)^\oplus) \rrbracket^{\min} \subseteq \llbracket (B_n(\phi))^+ \rrbracket$ ” Let  $\beta \in \llbracket B_n((\phi)^\oplus) \rrbracket^{\min}$  be a boolean valuation. By Lemma 10, we obtain a structure  $\mathcal{S} \in \llbracket (\phi)^\oplus \rrbracket^{\min}$  such that  $\beta = \beta_{\mathcal{S}}$ . But then  $\mathcal{S} \in \llbracket \phi \rrbracket^{\min}$  and  $\beta \in \llbracket B_n(\phi) \rrbracket^{\min}$ , by Lemma 10. Hence  $\beta \in \llbracket (B_n(\phi))^+ \rrbracket$ .  $\square$

Finally, we relate WS1S dualization, booleanization and propositional dualization:

**Lemma 12.** *Given a WS1S formula  $\phi$  and an integer  $n > 0$ , we have  $B_n(\phi^\sim) \equiv B_n(\phi)^\sim$ .*

**Proof.** Let  $\beta : \{\text{pr}_k \mid \text{pr} \in \text{Pred}, k \in [n]\} \rightarrow \{\top, \perp\}$  be an arbitrary boolean valuation and let  $\mathcal{S} = ([n], \nu, \iota, \mu)$  be a structure such that, for each  $\text{pr} \in \text{Pred}$ , we have  $\iota(\text{pr}) = \{k \in [n] \mid \beta(\text{pr}_k) = \top\}$  and  $\nu, \mu$  are picked at random. Obviously, we have that  $\beta = \beta_{\mathcal{S}}$ , hence by Lemma 10,  $\beta \models B_n(\phi^\sim) \iff \mathcal{S} \models \phi^\sim$  and by Lemma 4 we get  $\mathcal{S} \models \phi^\sim \iff \bar{\mathcal{S}} \models \neg\phi \iff \beta_{\bar{\mathcal{S}}} \models \neg B_n(\phi)$  again, by Lemma 10 and the definition of  $B_n(\neg\phi) = \neg B_n(\phi)$ . Let  $\bar{\beta}$  be the boolean valuation defined as  $\bar{\beta}(\text{pr}_k) = \neg\beta(\text{pr}_k)$  for all  $\text{pr} \in \text{Pred}$  and  $k \in [n]$ . Then clearly  $\bar{\beta} = \beta_{\bar{\mathcal{S}}}$  and  $\bar{\beta} \models \neg B_n(\phi) \iff \beta \models B_n(\phi)^\sim$  follows.  $\square$

**Proof of the theorem.** Let  $\mathcal{N}_S^n = (\mathbb{N}, m_0)$  and  $\mu_0 = \bigvee_{m_0(s)=1} s$ . By Lemma 2, we have  $\text{TrapInv}(\mathcal{N}_S) \equiv ((\mu_0 \wedge \Theta(\mathbb{N}))^+)^{\sim}$ . From the definition of  $\mathcal{N}_S$ , it is not difficult to show that  $\mu_0 \equiv B_n(\text{Init}(S))$  and  $\Theta(\mathbb{N}) \equiv B_n(\text{Tr}(\Theta(\Gamma)))$ , hence  $\mu_0 \wedge \Theta(\mathbb{N}) \equiv B_n(\text{Init}(S) \wedge \text{Tr}(\Theta(\Gamma)))$ . By Lemma 11, we obtain  $(\mu_0 \wedge \text{Tr}(\Theta(\mathbb{N})))^+ \equiv B_n((\text{Init}(S) \wedge \text{Tr}(\Theta(\Gamma)))^\oplus)$  and, by Lemma 12, we obtain  $((\mu_0 \wedge \Theta(\mathbb{N}))^+)^{\sim} \equiv B_n(((\text{Init}(S) \wedge \text{Tr}(\Theta(\Gamma)))^\oplus)^\sim)$ , as required.  $\square$

## A.7 Proof of Proposition 1

**Proof.** Let  $C = \{b(0), h(0), b(1), w(1), f(2), e(2)\}$  in the following. We shall try to build a nonempty trap  $T$  that avoids every state in  $C$ . If such a trap can be found, the counterexample is shown to be spurious (unreachable). Below is the list of states allowed in  $T$ , indexed by component (using other states that the ones listed below would result in a trap that is satisfied by the counterexample  $C$ , which is exactly the opposite of what we want):

Fork(0)	Philosopher <sub><math>r_l</math></sub> (0)	Fork(1)	Philosopher <sub><math>l_r</math></sub> (1)	Fork(2)	Philosopher <sub><math>l_r</math></sub> (2)
$f(0)$	$w(0), e(0)$	$f(1)$	$h(1), e(1)$	$b(2)$	$w(2), h(2)$

Assume that  $f(0) \in T$ . Then  $T$  must contain  $b(0)$  or  $e(2)$  (constraint  $gr(2) \wedge g(0)$ ). However neither is allowed, thus  $f(0) \notin T$ . Assume that  $f(1) \in T$ . Then  $T$  must contain

$b(1)$  or  $h(0)$  (constraint  $gr(0) \wedge g(1)$ ), contradiction, thus  $f(1) \notin T$ . Assume that  $b(2) \in T$ . Then  $T$  must contain  $f(1), w(1)$  or  $f(2)$  (constraint  $p(1) \wedge \ell(1) \wedge \ell(2)$ ), contradiction, thus  $b(2) \notin T$ . Then  $T$  contains only philosopher states, except for  $h(0)$ ,  $w(1)$  and  $e(2)$ . One can prove that there is no such trap, for instance, for  $\text{Philosopher}_{\ell}(1)$  we have:

$$\begin{aligned} h(1) \in T &\Rightarrow e(1) \in T \\ e(1) \in T &\Rightarrow w(1) \in T \end{aligned}$$

since  $f(1), b(1), f(2), b(2) \notin T$ . Since  $w(1) \notin T$ , we obtain that  $h(1), e(1) \notin T$ . Then the only possibility is  $T = \emptyset$ .  $\square$

## A.8 Proof of Proposition 2

Below we give a property of the marked PNs that define the semantics of parametric component-based systems.

**Definition 6.** Given a component-based system  $\mathcal{S}$ , a marked PN  $\mathcal{N} = (\mathbf{N}, \mathbf{m}_0)$ , with  $\mathbf{N} = (S, T, E)$ , is  $\mathcal{S}$ -decomposable if and only if there exists an integer  $n > 0$  such that  $S = \bigcup_{k=1}^K \mathbf{S}^k \times [n]$  and in every reachable marking  $\mathbf{m} \in \mathcal{R}(\mathcal{N})$ , for each  $1 \leq i \leq n$  and each  $1 \leq k \leq K$  there exists exactly one state  $s \in \mathbf{S}^k$  such that  $\mathbf{m}((s, i)) = 1$ .

**Lemma 13.** The marked PN  $\mathcal{N}_{\mathcal{S}}^n$  is  $\mathcal{S}$ -decomposable, for each component-based system  $\mathcal{S}$  and each integer  $n > 0$ .

**Proof.** Let  $\mathcal{S} = \langle C^1, \dots, C^K, \Gamma \rangle$  be a system with component types  $C^k = \langle \mathbf{P}^k, \mathbf{S}^k, s_0^k, \Delta^k \rangle$ , for all  $k = 1, \dots, K$ , and let  $n > 0$  be a parameter. Let  $\mathcal{N}_{\mathcal{S}}^n = (\mathbf{N}, \mathbf{m}_0)$  and  $\mathbf{m} \in \mathcal{R}(\mathcal{N}_{\mathcal{S}}^n)$  be a reachable marking. Then  $\mathbf{N} = (\bigcup_{k=1}^K \mathbf{S}^k \times [n], T, E)$ . We prove the property by induction on the length  $\ell$  of the shortest path from  $\mathbf{m}_0$  to  $\mathbf{m}$ . If  $\ell = 0$  the property holds because each component type  $1 \leq k \leq K$  has exactly one initial state  $s_0^k$  and only the states  $(s_0^k, i)$  are initially marked, for all  $1 \leq i \leq n$ . For the induction step  $\ell > 0$ , assume that  $\mathbf{m}' \xrightarrow{t} \mathbf{m}$  and the property of Definition 6 holds for  $\mathbf{m}'$ . Then there exists  $I = ([n], \nu, \iota, \mu) \in \llbracket \Gamma \rrbracket^{\min}$ , such that  $t = t_I$  and, for each  $i \in [n]$  and each  $p \in \mathbf{P}^k$  such that  $s' \xrightarrow{p} s \in \Delta^k$  and  $i \in \iota(p)$ , there are edges  $((s', i), t_I), (t_I, (s, i)) \in E$ . Suppose, for a contradiction, that there exists  $1 \leq i_0 \leq n$  and  $1 \leq k_0 \leq K$  such that  $\mathbf{m}((s, i_0)) = \mathbf{m}((s'', i_0)) = 1$ , for two distinct states  $s, s'' \in \mathbf{S}^{k_0}$ . Then  $(t_I, (s, i_0)), (t_I, (s'', i_0)) \in E$  and  $i_0 \in \iota(p) \cap \iota(q)$ , for two transition rules  $s' \xrightarrow{p} s, s' \xrightarrow{q} s'' \in \Delta^{k_0}$ . However, this comes in contradiction with the assumption that a transition does not involve two different ports from the same component type (1).  $\square$

**Proof of the proposition.** Let  $n > 0$  be an arbitrary positive integer. The component types of  $\mathcal{S}$  are  $C^k = \langle \mathbf{P}^k, \mathbf{S}^k, s_0^k, \Delta^k \rangle$ , for all  $k = 1, \dots, K$  and its marked PN  $\mathcal{N}_{\mathcal{S}}^n = (\mathbf{N}, \mathbf{m}_0)$ , where:

- $\mathbf{N} = (\bigcup_{k=1}^K \mathbf{S}^k \times [n], T, E)$  and,
- for all  $1 \leq k \leq K$  and all  $s \in \mathbf{S}^k$ , we have  $\mathbf{m}_0((s, i)) = 1$ , if  $s = s_0^k$  and  $\mathbf{m}_0((s, i)) = 0$ , otherwise.

Moreover, for a marking  $\mathbf{m} : \bigcup_{k=1}^K \mathbf{S}^k \times [n] \rightarrow \{0, 1\}$  of  $\mathcal{N}_{\mathcal{S}}^n$  and a formula  $\phi$ , we write  $\mathbf{m} \in \llbracket \phi \rrbracket^n$  iff  $([n], \iota, \nu) \models \phi$ , where  $\iota$  is such that  $\iota(s) = \{i \in [n] \mid \mathbf{m}((s, i)) = 1\}$  and  $\nu$  is an arbitrary valuation.

We prove first that  $m_0 \in \mathcal{A}_F^{\psi,n}$ . Let  $i_1, \dots, i_w \in [n]$  be arbitrary integers. Clearly,  $m_0((s_0^{k_1}, i_1)) = \dots = m_0((s_0^{k_w}, i_w)) = 1$ , thus  $([n], \iota[s_0^{k_1} \leftarrow \{i_1\}] \dots [s_0^{k_w} \leftarrow \{i_w\}], \nu[x_1 \leftarrow i_1] \dots [x_w \leftarrow i_w]) \models \Phi_F^\psi$ , since  $\Phi_F^\psi$  is the set of reachable markings of  $\mathcal{N}_S^\psi$  and  $m_0$  subsumes the initial marking thereof.

Second, we show that  $\mathcal{A}_F^{\psi,n}$  is inductive, i.e. for each move  $m \xrightarrow{t} m'$  of  $\mathcal{N}_S^n$ , such that  $m \in \mathcal{A}_F^{\psi,n}$ , we must show that  $m' \in \mathcal{A}_F^{\psi,n}$ . First, notice that, for any interpretation  $\iota$ , any valuation  $\nu$  and any  $i_1, \dots, i_w \in [n]$ , we have:

$$\begin{aligned} ([n], \iota[c_1 \leftarrow i_1] \dots [c_w \leftarrow i_w], \nu) &\models \phi(c_1, \dots, c_w) \\ &\iff \\ ([n], \iota, \nu[x_1 \leftarrow i_1] \dots [x_w \leftarrow i_w]) &\models \phi(x_1, \dots, x_w) \end{aligned} \tag{5}$$

for an arbitrary formula  $\phi$ . In the following, we define, for all  $s \in \bigcup_{k=1}^K \mathbf{S}^k$ :

$$\begin{aligned} \iota_m(s) &\stackrel{\text{def}}{=} \{i \in [n] \mid m((s, i)) = 1\} \\ \iota_{m'}(s) &\stackrel{\text{def}}{=} \{i \in [n] \mid m'((s, i)) = 1\} \end{aligned}$$

Let  $i_1, \dots, i_w \in [n]$  be integers such that  $([n], \iota_{m'}, \nu[x_1 \leftarrow i_1] \dots [x_w \leftarrow i_w]) \models \psi(x_1, \dots, x_w)$ .

We compute as follows:

$$\begin{aligned} ([n], \iota_{m'}, \nu[x_1 \leftarrow i_1] \dots [x_w \leftarrow i_w]) &\models \psi(x_1, \dots, x_w) &\iff \text{since } \psi \text{ has only atoms } s^k(x_i) \leq s^\ell(x_j) \\ ([n], \iota_m, \nu[x_1 \leftarrow i_1] \dots [x_w \leftarrow i_w]) &\models \psi(x_1, \dots, x_w) &\iff \text{by (5)} \\ ([n], \iota_m[c_1 \leftarrow i_1] \dots [c_w \leftarrow i_w], \nu) &\models \psi(c_1, \dots, c_w) &\implies \text{since } m \in \mathcal{A}_F^{\psi,n} \\ ([n], \iota_m[c_1 \leftarrow i_1] \dots [c_w \leftarrow i_w], \nu) &\models \Phi_F^\psi(c_1, \dots, c_w) &\implies (\dagger) \\ ([n], \iota_{m'}[c_1 \leftarrow i_1] \dots [c_w \leftarrow i_w], \nu) &\models \Phi_F^\psi(c_1, \dots, c_w) &\iff \text{by (5)} \\ ([n], \iota_{m'}, \nu[x_1 \leftarrow i_1] \dots [x_w \leftarrow i_w]) &\models \Phi_F^\psi(x_1, \dots, x_w) &\text{thus } m' \in \mathcal{A}_F^{\psi,n}, \text{ as required.} \end{aligned}$$

We are left with proving the step  $(\dagger)$  above. Because  $\mathcal{N}_S^n$  is  $\mathcal{S}$ -decomposable, by

Lemma 13, and since  $m \xrightarrow{t} m'$  by the hypothesis, there are states  $s_1, s'_1 \in \mathbf{S}^{k_1} \dots s_m, s'_m \in \mathbf{S}^{k_m}$ , with  $k_1, \dots, k_m$  pairwise disjoint, integers  $j_1, \dots, j_m \in [n]$  and edges  $((s_i, j_i), t), (t, (s'_i, j_i)) \in E$ , for all  $i = 1, \dots, m$ . For each  $i = 1, \dots, m$ , we distinguish the cases:

- if  $s_i = s'_i$  then  $m((s_i, j_i)) = m'((s_i, j_i)) = 1$ ,
- else, if  $s_i \neq s'_i$  then  $m((s_i, j_i)) = m'((s'_i, j_i)) = 1$  and  $m((s'_i, j_i)) = m'((s_i, j_i)) = 0$ .

Observe now that  $\mathcal{N}_S^\psi$  has the same structure as the subnet obtained by restricting  $\mathcal{N}_S^n$  to the states in  $\{s_1, \dots, s_m, s'_1, \dots, s'_m\} \times \{j_1, \dots, j_m\}$ . Moreover, there exists a transition  $t \downarrow_\psi$  in  $\mathcal{N}_S^\psi$  and edges  $(s_i(c_i), t \downarrow_\psi), (t \downarrow_\psi, s'_i(c_i)) \in E^\psi$  only if  $((s_i, j_i), t), (t, (s'_i, j_i)) \in E$ . Let  $m \downarrow_\psi$  and  $m' \downarrow_\psi$  be the projections of  $m$  and  $m'$  on  $\{s_1, \dots, s_m, s'_1, \dots, s'_m\} \times \{j_1, \dots, j_m\}$ , respectively.

Since  $m \xrightarrow{t} m'$ , we obtain that  $m \downarrow_\psi \xrightarrow{t \downarrow_\psi} m' \downarrow_\psi$ , thus  $([n], \iota_{m'}[c_1 \leftarrow i_1] \dots [c_w \leftarrow i_w], \nu) \models \Phi_F^\psi(c_1, \dots, c_w)$ , as required.  $\square$