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PRESERVATION OF UNIFORM CONTINUITY UNDER POINTWISE PRODUCT

A. BOUZIAD AND E. SUKHACHEVA

ABSTRACT. Let X be a uniform space and $U(X)$ the linear space of real-valued uniformly continuous functions on X . Our main objective is to give a number of properties characterizing the fact that $U(X)$ is stable under pointwise product in case X is a metric space. Some of these characterizations hold in much more general circumstances.

1. INTRODUCTION AND PRELIMINARIES

Let (X, \mathcal{U}) be a Hausdorff uniform space and let $U(X)$ (respectively, $U^*(X)$) stand for the set of all (bounded) real-valued uniformly continuous on X , the reals \mathbb{R} being equipped with the usual uniformity. In this paper, we are interested in a description, possibly easy-to-handle, of the category of uniform spaces for which the linear space $U(X)$ is a ring, that is, $U(X)$ is closed under pointwise product. As the product of two bounded uniformly continuous functions is uniformly continuous, if $U(X) = U^*(X)$ then $U(X)$ is a ring. This is also true if $U(X)$ is locally fine (see [30] and [1]) or if $U(X)$ coincides with $C(X)$, the ring of real continuous functions on X ; that is, if (X, \mathcal{U}) belongs to the class of u -normal uniform spaces in Nagata's sense [29], nowadays called UC-spaces or Atsugi spaces. Let us mention that the study of metric UC-spaces can be traced back at least to 1947 (Doss [11], Monteiro and Peixoto [27]). For more information about UC-spaces, we refer to Beer's book [3]; see also [6], [17], [26] and the references therein.

The problem of finding intrinsic conditions that characterize those metric spaces for which $U(X)$ is ring is explicitly stated by Nadler in [28]. The present note was originally motivated by the recent work [10] of J. Cabello Sánchez, where Nadler's question is solved as follows: for every metric space (X, d) , $U(X)$ is a ring if and only if each set $B \subset X$ which is not Bourbaki bounded in X contains an infinite uniformly isolated subset. Here, a set $A \subset X$ is said to be *Bourbaki bounded* in the uniform space (X, \mathcal{U}) if each $f \in U(X)$ is bounded on A and A is *uniformly isolated* in (X, \mathcal{U}) if there is $U \in \mathcal{U}$ such that $U[x] = \{x\}$ for each $x \in A$ (A is then said to be *uniformly U -isolated*). In Section 2, we shall extend Cabello Sánchez's criterion to a large class of uniform spaces which includes arbitrary products of metric spaces.

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In an earlier paper, Artico, Le Donne and Moresco [1] proved that for any uniform space (X, \mathcal{U}) , $U(X)$ is a ring if and only if every $f \in U(X)$ remains uniformly continuous when \mathbb{R} is endowed with the uniformity generated by polynomial dominated continuous functions. In the same vein, a very recent result established by Beer, Garrido and Meroño [7, Theorem 2.2] asserts that a metric space (X, d) is UC-space iff $1/f \in U(X)$ for each never zero function $f \in U(X)$ (i.e., $U(X)$ is inversion closed [13]). Along the same lines, we have that for any metric space (X, d) , $U(X)$ is a ring if and only if for every $f \in U(X)$ and any $g \in C(\mathbb{R})$, $g \circ f \in U(X)$ (that is, (X, d) is \mathbb{R} -fine in the sense of [14]). This is obtained as consequence of the following result (combining Proposition 2.1 and 2.4): $U(X)$ is a ring iff for each $f \in U(X)$ there is $k \geq 0$ such that the set $\{x \in X : |f(x)| \geq k\}$ is uniformly isolated.

By Nagata's theorem [29, Theorem 3], a metric space (X, d) is a UC-space iff the set X' of limit points is compact and for each $\varepsilon > 0$, $X \setminus B(X', \varepsilon)$ is uniformly isolated (here, as usual, $B(X', \varepsilon)$ is the ε -enlargement of X' with respect to the metric d). See also [23] (accordingly to [2, Theorem 1]), [20] (accordingly to [4]) and [26] (for proofs and more details). Motivated by this characterization of UC-spaces, Cabello Sánchez conjectured in his paper [10] that for every metric space (X, d) , $U(X)$ is a ring if and only if there is a Bourbaki bounded set $F \subset X$ such that for any $\varepsilon > 0$, the set $X \setminus B(F, \varepsilon)$ is uniformly isolated. The "if" part of this conjecture is established in [10]. We prove in Propositions 2.4 and 2.7 below that the following slight correction of this conjecture turns out to be true: $U(X)$ is a ring iff there is a Bourbaki bounded set $F \subset X$ such that for every $\varepsilon > 0$, there is $n \in \mathbb{N}$ such that $X \setminus B^n(F, \varepsilon)$ is uniformly isolated.

In section 3, we show in Theorem 3.5 that Cabello Sánchez's conjecture becomes, however, true for any metric space (X, d) in which Bourbaki bounded sets are totally bounded (e.g, if (X, d) is non-Archimedean). Along the way, we prove that for any metric space (X, d) , the following conditions are equivalent (Corollary 3.6): (a) $U(X)$ is a ring and every Bourbaki bounded set in X is precompact, (b) there is a precompact set $K \subset X$ such that for $\varepsilon > 0$, $X \setminus B(K, \varepsilon)$ is uniformly isolated and (c) the completion of (X, d) is a UC-space. The class of metric spaces having a completion which is a UC-space was studied by Beer [5] and has been deeply investigated by T. Jain and S. Kundu in their paper [24]. No less than twenty-nine equivalent characterizations for a metric space to have a UC completion are presented in [24]. The equivalence between (a) and (c) was recently established in [7, Theorem 3.11] and (b) appears to be new.

We also give in Section 3 a counter-example to the "only if" part of Cabello Sánchez's conjecture. We learned from Professor J. Cabello Sánchez that a counter-example has been given by Beer, Garrido and Meroño in [7], where his result was subsequently described in terms of the coincidence of the bornology of Bourbaki bounded subsets with a larger bornology. At first sight, the example in [7] seems more complicated than the one proposed here.

In the final part of Section 3, we investigate the following natural question: *What are the metrizable spaces X that admit a compatible metric d such $U(X, d)$ is a ring?* We show in Theorem 3.13 that such a metric exists provided that the set of limit points of X is contained in a closed finitely chainable subspace of X .

2. SOME PROPERTIES OF UNIFORM SPACES

Our goal in this section is to propose various properties (items (ii) to (xii) below) and show that most of them characterize the fact that $U(X)$ is a ring for every metric spaces (X, d) . Some of these criteria also apply to a class of uniform spaces introduced below by means of a certain ω -length game; this class is broad enough to include arbitrary Cartesian products of metric spaces.

For our purpose we shall consider uniformities as were introduced by Weil, instead of Tukey's coverings approach; so in this paper uniformities will be systematically manipulated by means of the set \mathcal{U} of their filters of entourages. Generally, for undefined concepts about uniform spaces, we refer to Isbell's book [22]. Throughout the paper, all considered $U \in \mathcal{U}$ are assumed to be open and symmetric (that is, $(x, y) \in U$ iff $(y, x) \in U$). If $U \in \mathcal{U}$, $F \subset X$ and $n \in \mathbb{N}$, then U^n stands for the composition $U \circ \cdots \circ U$, n times, and $U[F]$ denotes the set of $x \in X$ such that $(x, y) \in U$ for some $y \in F$.

Recall that a set $B \subset X$ is Bourbaki bounded in (X, \mathcal{U}) if for every $f \in U(X)$, $f(B)$ is bounded in \mathbb{R} . If no confusion can arise, then "Bourbaki bounded" will be simplified to "bounded". It is well-known that B is bounded in X iff for each $U \in \mathcal{U}$, B is U -bounded, that is, there are $n \in \mathbb{N}$ and a finite set $F \subset X$ such that $B \subset U^n[F]$, see for instance [19]. The set B is said to be *totally bounded* (or *precompact*) if it is always possible to take $n = 1$ in this criterion. Let us mention that in the metric context, the Bourbaki bounded subsets reduce to precompact sets iff each Bourbaki-Cauchy sequence as defined by Garrido and Meroño in their paper [15] has a Cauchy subsequence.

We denote the positive integers by \mathbb{N} and \mathbb{R} stand for the real line with its usual uniformity. Now we give the definitions of the properties we are going to examine for a given uniform space (X, \mathcal{U}) :

- (i) $U(X)$ is a ring,
- (ii) for every $f \in U(X)$ and $g \in U^*(X)$, fg is proximally continuous (the definition is given below),
- (iii) for every $f \in U(X)$, there is $k \geq 0$ such that $\{x \in X : |f(x)| \geq k\}$ is uniformly isolated in X ,
- (iv) for every $f \in U(X)$, there is $k \geq 0$ such that f is *uniformly locally constant* on $I = \{x \in X : |f(x)| \geq k\}$ (i.e., there is $U \in \mathcal{U}$ such that for every $x \in I$, $f(U[x]) = \{f(x)\}$),
- (v) (X, \mathcal{U}) is \mathbb{R} -*fine* (that is, for every $f \in U(X)$ and $g \in C(\mathbb{R})$, $g \circ f \in U(X)$),
- (vi) every unbounded set in X contains an infinite uniformly isolated set,

- (vii) for every unbounded set $B \subset X$, there are $U \in \mathcal{U}$ and an infinite set $A \subset B$ such that $U[A] \subset B$.
- (viii) for every $U \in \mathcal{U}$, there are a bounded set $B \subset X$ and $n \in \mathbb{N}$ such that $X \setminus U^n[B]$ is uniformly discrete,
- (ix) for every $U \in \mathcal{U}$, there are a bounded set $B \subset X$ and $n \in \mathbb{N}$ such that $X \setminus U^n[B]$ is uniformly isolated,
- (x) there is a bounded set $B \subset X$ such that for every $U \in \mathcal{U}$, there is $n \in \mathbb{N}$ such that $X \setminus U^n[B]$ is uniformly isolated,
- (xi) for every $U \in \mathcal{U}$, there is a bounded set $B \subset X$ such that $X \setminus U[B]$ is uniformly isolated,
- (xii) there is a bounded subset B of X such that for every $U \in \mathcal{U}$, $X \setminus U[B]$ is uniformly isolated.

Properties (iii) should be compared with the characterizations of UC metric spaces given by condition (6) of [2, Theorem 1]. Cabello Sánchez [10] has an example of a metric space (X, d) such that $U(X)$ is a ring but for which there is no bounded set $B \subset X$ such that $X \setminus B$ is uniformly isolated. So, we can not go further by eliminating the enlargement of the bounded set B in (xii). As said above, Cabello Sánchez also proved for metric spaces the equivalence (vi) \Leftrightarrow (i), the implication (xii) \Rightarrow (i) and conjectured that, conversely, (i) implies (xii). Note that this conjecture is at least as strong as the equivalence between (xi) and (i). The equivalence between (iv) and (v) is established (at least in one direction) in [16] for metric spaces.

The following implications are obvious: (i) \Rightarrow (ii), (v) \Rightarrow (i), (iii) \Rightarrow (iv), (x) \Rightarrow (ix), (xii) \Rightarrow (xi) and (xii) \Rightarrow (x). The next tables summarizes the link between the statements proved in the remaining of this section and the relationships between the properties (i)-(x). The implications that are valid in the class of β -defavorable uniform spaces (defined below) are indicated by (*), whereas the one indicated by (\dagger) holds mainly in the context of metric spaces. The remaining ones hold for arbitrary Hausdorff uniform spaces.

(ii) \Rightarrow^* (iii)	(iii) \Rightarrow (vi)	(vi) \Leftrightarrow (vii)	(iv) \Rightarrow (v)
2.1	2.2	2.3	2.4

(viii) \Leftrightarrow (ix)	(ix) \Rightarrow (iii) \Rightarrow (vi) \Rightarrow^* (ix)	(ix) \Rightarrow^\dagger (x)
2.5	2.6	2.7

As a consequence, the first ten properties are equivalent for metric spaces (this is summarized in Theorem 3.1). On the other hand, as said before, conditions (xi) and (xii) cannot be added to this list (see Example 3.7); however, the twelve properties turn out to be equivalent in every metric space for which Bourbaki bounded sets are totally bounded (Theorem 3.5).

Let (X, \mathcal{U}) be a uniform space and D a dense subset of the product space $X \times X$ when X is given the the topology induced by \mathcal{U} . We shall consider the following game $\mathcal{J}(D)$ between tow Players α and β . Player α is the first to move

and gives $W_0 \in \mathcal{U}$, and the answer of Player β is a point (x_0, y_0) in $W_0 \cap D$. At stage $n \in \mathbb{N}$, Player α chooses $W_n \in \mathcal{U}$ and then Player β gives $(x_n, y_n) \in W_n \cap D$. The game is of length ω , and Player α is declared to be the winner of the game $(W_n, (x_n, y_n))_{n \in \mathbb{N}}$ if for any infinite set $I \subset \mathbb{N}$ and $U \in \mathcal{U}$, there are $n, m \in \mathbb{N}$ such that $\{n, m\} \cap I \neq \emptyset$ and $(x_n, y_m) \in U$. Otherwise Player β wins.

We say that the uniform space (X, \mathcal{U}) is β -defavorable if there is a dense set $D \subset X \times X$ such that Player β has no winning strategy in the game $\mathcal{J}(D)$.

As an illustration, let (X_i, \mathcal{U}_i) , $i \in I$, be a family of nonempty uniform spaces and denote by (X, \mathcal{U}) their uniform product. Choose $a_i \in X_i$ for each $i \in I$, and let D be the set of $x \in X$ for which the set $\{i \in I : x_i \neq a_i\}$ is finite. Then D is dense in X . It is not difficult to show that if each (X_i, \mathcal{U}_i) has a countable basis for its uniformity, then Player α has a strategy σ in the game $\mathcal{J}(D \times D)$ such that for any σ -compatible game $(U_n, (x_n, y_n))_{n \in \mathbb{N}}$, the sequence $(x_n, y_n)_{n \in \mathbb{N}}$ converges to the diagonal of $X \times X$ (i.e., every $U \in \mathcal{U}$ contains all but finitely many (x_n, y_n)). In particular, any uniform space which is the product of metric spaces is β -defavorable.

Recall that a function $f : X \rightarrow \mathbb{R}$ is said to be *proximally continuous* if for any $A \subset X$ and any $\varepsilon > 0$, there is $U \in \mathcal{U}$ such that $f(U[A]) \subset B(f(A), \varepsilon)$. Every uniformly continuous function is proximally continuous. The converse holds if X is a metric space [12].

Our first statement makes the connection between (ii) and (iii):

Proposition 2.1. *Let (X, \mathcal{U}) be a β -defavorable uniform space and $f \in U(X)$. Suppose that for every $h \in U^*(X)$, the product function fh is proximally continuous. Then, there is $n \in \mathbb{N}$ such that the set $\{x \in X : |f(x)| \geq n\}$ is uniformly isolated in (X, \mathcal{U}) .*

Proof. Let $D \subset X \times X$ be a dense set such that there is no winning strategy for Player β in the game $\mathcal{J}(D)$. Suppose, on the contrary, that for every $n \in \mathbb{N}$ and $U \in \mathcal{U}$, there are $y, z \in X$ such that $(y, z) \in U$, $|f(y)| > n$ and $y \neq z$. It is possible to choose such (y, z) in D , since f is continuous and D is dense in $X \times X$. Consider the following strategy σ for Player β in the game $\mathcal{J}(D)$: At stage $n \geq 0$, let U_n be the n th move of Player α and assume that $\sigma(U_0, \dots, U_k)$, $k < n$, has been defined. Player β chooses $(y_n, z_n) \in U_n \cap D$ such that $|f(y_{n+1})| > n$ and $y_n \neq z_n$. Since f is uniformly continuous, one can assume that $|f(y_{n+1})| > 1 + |f(y_n)|$ and $|f(y_n) - f(z_n)| < 1/n$. Note that $\{y_n : n \in \mathbb{N}\} \cap \{z_n : n \in \mathbb{N}\} = \emptyset$. Finally, define $\sigma(U_0, \dots, U_n) = (y_n, z_n)$.

Let $(U_n, (y_n, z_n))_{n \in \mathbb{N}}$ be a winning game for Player α against the strategy σ . Following an idea from [10] (see also [2]), for each $n \in \mathbb{N}$, put $g(y_n) = 1/(n+1)$ and $g(z_n) = 0$. In view of how the sequence $(y_n, z_n)_{n \in \mathbb{N}}$ has been selected, g is well defined and uniformly continuous on the subspace $\{y_n : n \in \mathbb{N}\} \cup \{z_n : n \in \mathbb{N}\}$ of (X, \mathcal{U}) . According to Katetov's theorem [25], g has a uniformly continuous extension $h : X \rightarrow [0, 1]$. Let $A = \{y_n : n \in \mathbb{N}\}$. Since fh is

proximally continuous, there is $U \in \mathcal{U}$ such that $fh(U[A]) \subset B(f(A), 1)$. Since $(U_n, (y_n, z_n))_{n \in \mathbb{N}}$ is a winning game for Player α , there are $n, m \in \mathbb{N}$ such that $(y_n, z_m) \in U$. It follows that $|fh(y_n) - fh(z_m)| = |fh(y_n)| \geq 1$, a contradiction. \square

For each $U \in \mathcal{U}$, let $I_U = \{x \in X : U[x] = \{x\}\}$ and let X' stand for the set of limit points of X .

The implication (iii) \Rightarrow (vi) is a consequence of the following:

Proposition 2.2. *Let (X, \mathcal{U}) be a uniform space such that for every $f \in U(X)$ there is $k \geq 0$ such that $\{x \in X : |f(x)| \geq k\}$ is uniformly isolated. Let $B \subset X$. Then, B is bounded if and only if for each $U \in \mathcal{U}$, $B \cap I_U$ is finite. In particular, the set X' of limit points of X is bounded in X .*

Proof. If B is unbounded, then there is $f \in U(X)$ such that $f(B)$ is unbounded. Let $k \geq 0$ and $U \in \mathcal{U}$ be such that the set $I = \{x \in X : |f(x)| \geq k\}$ is uniformly U -isolated. Then $I \subset I_U$ and $B \cap I$ is infinite, hence $B \cap I_U$ is infinite.

The converse is obvious, because if $B \cap I_U$ is infinite for some $U \in \mathcal{U}$, then $B \cap I_U$ (hence B) is unbounded in X . \square

The following corresponds to the equivalence (vi) \Leftrightarrow (vii):

Proposition 2.3. *The following are equivalent for every uniform space (X, \mathcal{U}) :*

- (a) *every unbounded set contains an infinite uniformly isolated set,*
- (b) *for every unbounded set $A \subset X$, there are $U \in \mathcal{U}$ and an infinite set $B \subset A$ such that $U[B] \subset A$.*

Proof. (a) implies (b) obviously. Suppose that (b) holds and let $A \subset X$ be an unbounded set. There are $f \in U(X)$ and a sequence $(a_n)_{n \in \mathbb{N}} \subset A$ such that for each $n \in \mathbb{N}$, $|f(a_{n+1})| > 1 + |f(a_n)|$. Let $U \in \mathcal{U}$ be such that $|f(x) - f(y)| < 1$ whenever $(x, y) \in U$. By (b), there are an infinite $I \subset \mathbb{N}$ and $V \subset U$ such that $V[\{a_n : n \in I\}] \subset \{a_n : n \in \mathbb{N}\}$. Then $\{a_n : n \in I\}$ is uniformly V -isolated. \square

As said above, the fact that (iv) implies (v) is established in [16] for metric spaces. This also follows from the next general fact, the proof of which uses the following result from [8]: for every compact set $K \subset \mathbb{R}$, every continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ is *strongly uniformly continuous at K* , that is, for every $\varepsilon > 0$, there is $\eta > 0$ such that $|f(x) - f(y)| \leq \varepsilon$ whenever $|x - y| < \eta$ and $\{x, y\} \cap K \neq \emptyset$.

Proposition 2.4. *Let (X, \mathcal{U}) be uniform space, $f \in U(X)$ and $g \in C(\mathbb{R})$. Suppose that for each $U \in \mathcal{U}$, there are $B \subset X$ and $n \in \mathbb{N}$ such that $f(B)$ is bounded and the composition $g \circ f$ is uniformly continuous on $X \setminus U^n[B]$. Then $g \circ f \in U(X)$.*

Proof. Let $\varepsilon > 0$ and choose $U \in \mathcal{U}$ such that $(x, y) \in U$ implies $|f(x) - f(y)| < \varepsilon$. Let $B \subset X$ and $n \in \mathbb{N}$ be such that $f(B)$ is bounded and $g \circ f$ is uniformly continuous on $X \setminus U^n[B]$. Note that f is bounded on $U^n[B]$, so by the strong

uniform continuity of g at $f(U^n[B])$ [8], there is $\eta > 0$ such that $|g(a) - g(b)| < \varepsilon$ for every $a, b \in \mathbb{R}$ satisfying $\{a, b\} \cap f(U^n[B]) \neq \emptyset$ and $|a - b| < \eta$. Let $V_1 \in \mathcal{U}$ be such that $|g(f(x)) - g(f(y))| < \varepsilon$ whenever $(x, y) \in V_1$ and $\{x, y\} \subset X \setminus U^n[B]$. Since f is uniformly continuous, there is $V_2 \in \mathcal{U}$ such that $|f(x) - f(y)| < \eta$ for every $(x, y) \in V_2$. Then, for every $(x, y) \in V_1 \cap V_2$, we have $|g(f(x)) - g(f(y))| < \varepsilon$. \square

Recall that a subset A of a uniform space (X, \mathcal{U}) is said to be *uniformly discrete* if there is $U \in \mathcal{U}$ such that for each $x \in A$, $U[x] \cap A = \{x\}$. Clearly, every uniformly isolated set is uniformly discrete but not conversely. However, as stated in the next remark, it is possible to replace “uniform isolatedness” in (iii) by the weaker condition “uniform discreteness”. Condition (viii) is derived from (ix) in the same way. This will bring some simplification in the proof of Proposition 2.6 below.

Remarks 2.5. Let (X, \mathcal{U}) be a uniform space, $B \subset X$ and $n \in \mathbb{N}$.

1) If the set $I = X \setminus U^n[B]$ is uniformly discrete, then the set $J = X \setminus U^{n+1}[B]$ is uniformly isolated. Indeed, let $V \in \mathcal{U}$ be such that for every $x \in I$, $I \cap V[x] = \{x\}$. We assume that $V \subset U$. Let $x \in J$ and $y \in V[x]$. If $y \neq x$, then $y \in U^n[B]$, hence $x \in U^{n+1}[B]$, a contradiction.

2) Similarly, if $f \in U(X)$ and $k \geq 0$ are such that the set $\{x \in X : |f(x)| \geq k\}$ is uniformly discrete, then for every $\delta > 0$, the set $\{x \in X : |f(x)| \geq k + \delta\}$ is uniformly isolated.

Proposition 2.6. *Let (X, \mathcal{U}) be uniform space. Consider the following:*

- (a) *For each $U \in \mathcal{U}$, there are a bounded set $B \subset X$ and $n \in \mathbb{N}$ such that $X \setminus U^n[B]$ is uniformly isolated.*
- (b) *For every $f \in U(X)$, there is $k \geq 0$ such that $\{x \in X : |f(x)| \geq k\}$ is uniformly isolated.*
- (c) *Every unbounded set in X contains an infinite uniformly isolated subset.*

Then (a) \Rightarrow (b) \Rightarrow (c). If (X, \mathcal{U}) is β -defavorable, then the three conditions are equivalent.

Proof. To show the implication (a) \Rightarrow (b), let $f \in U(X)$ and choose $U \in \mathcal{U}$ such that $(x, y) \in U$ implies $|f(x) - f(y)| < 1$. Let $B \subset X$ be a bounded set and $n \in \mathbb{N}$ such that $X \setminus U^n[B]$ is uniformly isolated. Then f is bounded on $U^n[B]$, so there exists $k \geq 0$ such that $\{x \in X : |f(x)| \geq k\} \subset X \setminus U^n(B)$. Thus $\{x \in X : |f(x)| \geq k\}$ is uniformly isolated.

Condition (b) implies (c) by Proposition 2.2.

Assume now that (X, \mathcal{U}) is β -defavorable and let us show that (c) \Rightarrow (a). Let D be a dense subset of $X \times X$ such that (X, \mathcal{U}) is β -defavorable for the game $\mathcal{J}(D)$. In view of Proposition 2.2 and Remark 2.5, it suffices to show that there are a finite set $F \subset X$ and $n \in \mathbb{N}$ such that $X \setminus U^n[X' \cup F]$ is uniformly discrete, where X' is the set of limit points of the space X . To do that, we proceed by

contradiction, so suppose that this is not possible. We will define a strategy σ for Player β in the game $\mathcal{J}(D)$ as follows. Let U_0 be the first move of Player α . Let $x_0, y_0 \notin U^2[X']$ with $y_0 \in U_0[x_0]$ such that $x_0 \neq y_0$ (we will soon see that x_0 and y_0 can be chosen in D). Put $F_1 = \{x_0, y_0\}$ and $\sigma(U_0) = (x_0, y_0)$. Let U_n be the n th move of Player α and put $F_n = \{x_i : i < n\} \cup \{y_i : i < n\}$. Since $X \setminus U^{n+1}[X' \cup F_n]$ is not uniformly discrete, there is $(x_n, y_n) \in U_n$ such that $x_n \neq y_n$ and $x_n, y_n \notin U^{n+1}[X' \cup F_n]$. Then x_n and y_n are not in the closure of $U^n[X' \cup F_n]$, so we can assume that $x_n, y_n \in D$ and $x_n, y_n \notin U^n[X' \cup F_n]$. Moreover, since (X, \mathcal{U}) is Hausdorff and $x_i, y_i \notin X'$ for each $i < n$, by modifying U_n if necessary, we assume that $U_n[x_i] = \{x_i\}$ and $U_n[y_i] = \{y_i\}$ for each $i < n$. Define $\sigma(U_0, \dots, U_n) = (x_n, y_n)$.

Let $(U_n, (x_n, y_n))_{n \in \mathbb{N}}$ be a winning game for Player α against the strategy σ . Then, for every infinite set $I \subset \mathbb{N}$, the sets $\{x_n : n \in I\}$ and $\{y_n : n \in I\}$ are U -unbounded, hence by (c) there are an infinite $J \subset \mathbb{N}$ and $V \in \mathcal{U}$ such that $\{x_n : n \in J\}$ and $\{y_n : n \in J\}$ are uniformly V -isolated. On the other hand, there are $n \in \mathbb{N}$ and $m \in \mathbb{N}$, with $n \in J$ or $m \in J$, such that $(x_n, y_m) \in V$. Then $y_m = x_n$, thus $m > n$ and $x_n \in U_m[x_m]$. Since $U_m[x_n] = \{x_n\}$, it follows that $x_m = x_n$, which is impossible. \square

We now come to the proof of the implication (ix) \Rightarrow (x) for metric spaces. This is proved in the next assertion in a somewhat more general framework.

Proposition 2.7. *Let (X, \mathcal{U}) be a uniform space such that for every $U \in \mathcal{U}$, there are a bounded set $B \subset X$ and $n \in \mathbb{N}$ such that $X \setminus U^n[B]$ is uniformly isolated. Suppose further that there is a sequence $(U_n)_{n \in \mathbb{N}} \subset \mathcal{U}$ such that:*

- (1) *every infinite uniformly isolated set in X contains an infinite set which is uniformly U_n -isolated for some $n \in \mathbb{N}$,*
- (2) *for every bounded set $B \subset X$ and $U \in \mathcal{U}$, there are $n, m \in \mathbb{N}$ such that the set $I = U_n[B] \setminus U^m[B]$ is uniformly isolated and $U_n[I] \subset I$.*

Then, there is a bounded set $F \subset X$ such that for each $U \in \mathcal{U}$, there is $n \in \mathbb{N}$ such that $X \setminus U^n[F]$ is uniformly isolated.

Proof. We suppose without loss of generality that the sequence $(U_n)_{n \in \mathbb{N}}$ is decreasing. For each $n \in \mathbb{N}$, let $L_n \subset X$ be a finite set and $k_n \in \mathbb{N}$ such that $X \setminus U_n^{k_n}[L_n]$ is uniformly isolated. We may suppose that each L_n is minimal in the sense that for every $x \in L_n$, $U_n[x] \neq \{x\}$. For if it happens that $U_n[x] = \{x\}$ for some $x \in L_n$, then $U_n^{k_n}[x] = \{x\}$, hence $X \setminus U_n^{k_n}[L_n \setminus \{x\}]$ is uniformly isolated, so x could be removed from the finite set L_n .

We claim that the set $F = \bigcup_{n \in \mathbb{N}} L_n$ is bounded in X . To prove this, suppose that F is not bounded in X and choose by Proposition 2.6 and (1) an infinite set $C \subset F$ and $p \in \mathbb{N}$ such that C is uniformly U_p -isolated. Since C is infinite, there is $q \geq p$ such that $C \cap L_q \neq \emptyset$; choose x in $C \cap L_q$. Since $U_q \subset U_p$, we have $U_q[x] \subset U_p[x] = \{x\}$, hence $U_q^{k_q}[x] = \{x\}$, which is in contradiction to the minimality of L_q . Hence F must be bounded in X as claimed.

To conclude, let $U \in \mathcal{U}$ and let us show that $X \setminus U^n[F]$ is uniformly isolated for some $n \in \mathbb{N}$. By (2), there are $p, q \in \mathbb{N}$ such that $U_p[F] \subset U^q[F] \cup I$, where I is uniformly isolated and $U_p[I] \subset I$. Since $U_p \circ U = U \circ U_p$ (U and U_p being symmetric), we have $U_p^{k_p}[F] \subset U^{qk_p}[F] \cup I$. Since I and $X \setminus U_p^{k_p}[F]$ are uniformly isolated, we obtain that $X \setminus U^{qk_p}[F]$ is uniformly isolated too. \square

Remark 2.8. Let (X, \mathcal{U}) be a uniform space and let \mathcal{I} be the set of all uniformly isolated subsets of X . The property (iii) in Section 2 suggests to consider the extended pseudonorm $\|\cdot\| : U(X) \rightarrow [0, +\infty]$ defined by

$$\|f\| = \inf\{\varepsilon \geq 0 : \{x \in X : |f(x)| \geq \varepsilon\} \in \mathcal{I}\}.$$

Let us note that, in view of Remark 2.5, an equivalent definition of $\|\cdot\|$ is obtained if \mathcal{I} is replaced by the set of uniformly discrete subsets of X .

It is easy to check that if $f, g \in U(X)$ and if for some $r \geq 0$, the sets $\{x \in X : |f(x)| \geq r\}$ and $\{x \in X : |g(x)| \geq r\}$ are uniformly isolated, then $fg \in U(X)$. Hence the subspace $U^\#(X)$ of $U(X)$ given by all $f \in U(X)$ such that $\|f\| < +\infty$ is a ring on which $\|\cdot\|$ is finite (hence a true pseudonorm). Furthermore, one can prove that $U^\#(X)$ is closed in $U(X)$ and complete with respect to $\|\cdot\|$.

Finally, according to Proposition 2.1, if (X, \mathcal{U}) is β -defavorable, then $U^\#(X)$ is the maximal ring containing $U^*(X)$.

3. THE METRIC CASE

In view of the results established in Section 2 and since every metric space is β -defavorable, we have:

Theorem 3.1. *The ten conditions (i)-(x) are equivalent for every metric space.*

Nadler proved in [28, Theorem 5.2] that every metric space (X, d) for which $U(X)$ is a ring and in which every metrically bounded set is Bourbaki bounded, is the union of a Bourbaki bounded in itself subspace and a uniformly isolated set. So it is possible to add condition (xii) and, a fortiori, condition (xi) in Theorem 3.1 for any metric space in which every metrically bounded subspace is Bourbaki bounded in itself. We shall show in Theorem 3.5 below that the same result holds for metric spaces in which Bourbaki bounded sets are precompact.

The proof of Theorem 3.5 uses the following three lemmas.

Lemma 3.2. *Let (Y, d) be a metric space and X a dense subset of Y . If every Bourbaki bounded set in (X, d) is precompact, then the same property holds in (Y, d) .*

Proof. Let $B \subset Y$ and suppose that B is not precompact or, equivalently, that there is a sequence $(y_n)_{n \in \mathbb{N}} \subset B$ without any Cauchy subsequence. For each $n \in \mathbb{N}$, let $x_n \in X$ be such that $d(x_n, y_n) < 1/n$. Clearly, the sequence $(x_n)_{n \in \mathbb{N}}$ has no Cauchy subsequence. It follows that the set $A = \{x_n : n \in \mathbb{N}\}$ is not Bourbaki bounded in X , hence there exists a uniformly continuous function $f : X \rightarrow \mathbb{R}$

which is not bounded on A . Let $g : Y \rightarrow \mathbb{R}$ be the uniformly continuous extension of f to Y . Then g is not bounded on $\{y_n : n \in \mathbb{N}\}$, hence B is not Bourbaki bounded in Y . \square

Lemma 3.3. *Let X be a dense subset of the metric space (Y, d) . Then Y has the property that there exists a precompact subset K of Y such that for each $\varepsilon > 0$, $Y \setminus B(K, \varepsilon)$ is uniformly isolated iff X has the same property.*

Proof. Suppose that Y has such a precompact set K . Since X is dense in Y , for each $n \in \mathbb{N}$ there is a finite set $F_n \subset X$ such that $K \subset B(F_n, 1/n)$. We suppose that each F_n is minimal, so that $F_n \subset B(K, 1/n)$. Since K is precompact and $\cup_{k \geq n} F_k \subset B(K, 1/n)$ for each $n \in \mathbb{N}$, the set $F = \cup_{n \in \mathbb{N}} F_n$ is precompact. It follows that the set $L = (K \cap X) \cup F$ is precompact. To conclude, let $n \in \mathbb{N}$ and let us show by contradiction that $X \setminus B(L, 1/n)$ is uniformly isolated. In the opposite case, we could find two sequences $(a_k)_{k \in \mathbb{N}}$ and $(b_k)_{k \in \mathbb{N}}$ in $X \setminus B(L, 1/n)$ such that $\lim d(a_k, b_k) = 0$ and $a_k \neq b_k$ for each $k \in \mathbb{N}$. Since $Y \setminus B(K, 1/2n)$ is uniformly isolated, there is $k \in \mathbb{N}$ such that, say, $a_k \in B(K, 1/2n)$. Since $K \subset B(F_{2n}, 1/2n)$, we get $a_k \in B(F, 1/n)$, which is a contradiction.

Conversely, if L is a precompact set satisfying the required property for X , then the closure K of L in Y works for Y . The straightforward proof is omitted. \square

Following Waterhouse [32], a *C-sequence* is a sequence of pairs $(x_n, y_n)_{n \in \mathbb{N}}$ of distinct points in a metric space (X, d) such that $\lim d(x_n, y_n) = 0$. C-sequences are considered by Nadler in [28], where the involved sequences $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ are called *twin sequences*. It is proved in [28, Lemma 5.3] that if $U(X)$ is a ring then twin sequences are metrically bounded. When we turn to Doss's article [11], we see that a sequence $(x_n)_{n \in \mathbb{N}} \subset X$ is said to be *accessible* if there is a sequence $(y_n)_{n \in \mathbb{N}} \subset X$ disjoint from $(x_n)_{n \in \mathbb{N}}$ such that $\lim d(x_n, y_n) = 0$. Doss proved in [11, Theorem I], among other things, that (X, d) is a UC-space iff every accessible sequence in X has a convergent subsequence. The following improvement of Nadler's result can be considered as the counterpart for $U(X)$ to be a ring of Doss's criterion. This lemma is not new because once expressed in terms of the so-called isolation functional (see [7]), we see that it corresponds to the very recent result [7, Theorem 3.9].

Lemma 3.4. *Let (X, d) be a metric space. Then $U(X)$ is a ring if and only if twin sequences in (X, d) are Bourbaki bounded in X .*

Proof. Suppose that $U(X)$ is a ring. By Theorem 3.1, condition (iii) is thus satisfied by X . Let $(x_n, y_n)_{n \in \mathbb{N}}$ be a C-sequence of X . We may suppose that both the sets $\{x_n : n \in \mathbb{N}\}$ and $\{y_n : n \in \mathbb{N}\}$ are infinite. Since $\lim d(x_n, y_n) = 0$, for every $\varepsilon > 0$, the set $\{n \in \mathbb{N} : B(x_n, \varepsilon) = \{x_n\}\}$ is finite. It follows from Proposition 2.2 that the sequence $(x_n)_{n \in \mathbb{N}}$ is Bourbaki bounded in X . Clearly, $(y_n)_{n \in \mathbb{N}}$ is also Bourbaki bounded in X .

To show the converse, we proceed by contradiction. So suppose that there are $f, g \in U(X)$ such that $fg \notin U(X)$. Then there are a C-sequence $(x_n, y_n)_{n \in \mathbb{N}}$

living in X and $\varepsilon > 0$ such that for each $n \in \mathbb{N}$, $|fg(x_n) - fg(y_n)| \geq \varepsilon$. In particular, fg is not uniformly continuous on the metric subspace of X given by $A = \{x_n : n \in \mathbb{N}\} \cup \{y_n : n \in \mathbb{N}\}$. It follows that at least one of the two functions f or g is not bounded on A . Consequently, the sequences $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ are not Bourbaki bounded in X . \square

In view of Theorem 3.1, the following shows that conditions (i)-(xii) are equivalent for every metric space in which Bourbaki bounded sets are precompact.

Theorem 3.5. *Let (X, d) be a metric space in which Bourbaki bounded sets are precompact. If $U(X)$ is a ring, then there is a precompact set $L \subset X$ such that for each $\varepsilon > 0$, $X \setminus B(L, \varepsilon)$ is uniformly isolated.*

Proof. Suppose that $U(X)$ is a ring and let (Y, d) be the completion of (X, d) . Then $U(Y)$ is a ring, since each $f \in U(X)$ is the restriction to X of a unique $g \in U(Y)$. Consequently, by Proposition 2.2 and Lemma 3.2, the closed subspace Y' of Y is precompact, hence compact.

Let $\varepsilon > 0$ and suppose that $Y \setminus B(Y', \varepsilon)$ is not uniformly isolated. Then $Y \setminus B(Y', \varepsilon)$ contains twin sequences $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$. Since condition (iii) is satisfied by the metric space (Y, d) (Theorem 3.1), it follows from Lemma 3.4 that the sequence $(x_n)_{n \in \mathbb{N}}$ is Bourbaki bounded in Y . By Lemma 3.2, twin sequences in (Y, d) are precompact, hence $(x_n)_{n \in \mathbb{N}}$ has a cluster point in Y' , which is impossible. Consequently, $Y \setminus B(Y', \varepsilon)$ is uniformly isolated. We can now apply Lemma 3.3 to conclude the proof. \square

To prove the next corollary we shall make use of the “if” part of the following characterization of metric UC-spaces: $(*)$ (Y, d) is a UC-space if and only if there is a compact set $K \subset Y$ such that for every $\varepsilon > 0$, the set $Y \setminus B(K, \varepsilon)$ is uniformly isolated. This natural statement does not seem to appear in the literature, although it is not difficult to obtain: the sufficiency follows from the fact that twin sequences must have cluster points in the compact K and the necessity follows from [29, Theorem 3].

Corollary 3.6. *For any metric space (X, d) , the following are equivalent:*

- (1) $U(X)$ is a ring and every Bourbaki bounded set in X is precompact,
- (2) there is a precompact set $L \subset X$ such that for each $\varepsilon > 0$, $X \setminus B(L, \varepsilon)$ is uniformly isolated,
- (3) the completion of X is a UC-space.

Proof. The implication (1) \Rightarrow (2) being established in Theorem 3.5, it remains to show that (2) implies (3) and (3) implies (1). Assume (2) and let Y stand for the completion of (X, d) . Then, by Lemma 3.3 and the above criterion $(*)$, Y is a UC-space. To show that (3) implies (1), assume that the completion Y of X is a UC-space. Then $U(X)$ is a ring (since each $f \in U(X)$ has a uniform extension to Y). Let L be a Bourbaki bounded set in X . Then for every $\varepsilon > 0$,

the set $L \setminus B(Y', \varepsilon)$, being uniformly isolated and Bourbaki bounded in X , must be finite. Consequently, since Y' is precompact, L is precompact too. \square

We refer the reader to [5] and [24] where several equivalent characterizations for a metric space to have a UC completion are established. As said above for Lemma 3.4, the equivalence between (1) and (3) in Corollary 3.6 has been established recently in [7, Theorem 3.11]. Condition (2) seems to be new.

The following example shows that it is not possible to add condition (xi) in Theorem 3.1 for arbitrary metric spaces. In particular, (i) and (xii) are not equivalent for metric spaces, which disproves Cabello Sánchez's conjecture mentioned Section 1.

Example 3.7. Let C be the subspace of $\mathbb{R} \times \mathbb{R}$ given by the union of $\{0\} \times [0, 1]$ and all line segments $C_m = [0, 1] \times \{1/m\}$, $m \in \mathbb{N}$. We endow C with the so-called intrinsic metric δ : the distance between two points $x, y \in C$ is the length of the "shortest path" in C between these points. For instance, the metric δ coincides with the Euclidean metric on $\{0\} \times [0, 1]$ and on all lines $[0, 1] \times \{1/m\}$, $m \in \mathbb{N}$. However, for example, if $x = (1, 1/3)$ and $y = (0, 1/4)$, then $\delta(x, y) = 1 + 1/3 - 1/4$. Now, let Y be the subspace of (C, δ) given by the union of

$$\{(0, 0)\} \cup \{0\} \times \{1/m : m \in \mathbb{N}\}$$

and the lines

$$\{k/m : 0 \leq k \leq m\} \times \{1/m\}, \quad m \in \mathbb{N}.$$

The metric space (X, d) that we are looking for is the hedgehog of Y with ω spins, the basis point being $(0, 0)$. More precisely, X is the set of all (n, y) , with $y \in Y$ and $n \in \mathbb{N}$, where all the points $(n, (0, 0))$, $n \in \mathbb{N}$, are identified to a single point 0. More precisely, the metric d of X is given by

- $d(0, (n, y)) = \delta((0, 0), y)$,
- $d((n, y), (n, z)) = \delta(y, z)$,
- $d((n, y), (m, z)) = \delta((0, 0), y) + \delta((0, 0), z)$ if $n \neq m$.

To simplify, the point $(n, (k/m, 1/m))$ is written $(n, k/m, 1/m)$ and the "line" L_m is defined by

$$L_m = \{(n, k/m, 1/m) : n \in \mathbb{N}, 0 \leq k \leq m\}.$$

Then:

1) *A subset of X is uniformly isolated if and only if it is contained in the union of finitely many lines L_m .*

Proof. Clear. \square

2) *$U(X)$ is a ring.*

Proof. By Theorem 3.1, it suffices to check that condition (x) from Section 2 is satisfied if we take $B = \{0\}$. Let $\varepsilon > 0$ and choose a positive integer m_0 such

that $1/m_0 \leq \varepsilon$. Let $m \geq m_0$ and choose $l \in \mathbb{N}$ so that $lm_0 \leq m < (l+1)m_0$. We shall prove that $X \setminus B^{m_0+1}(0, \varepsilon)$ is uniformly isolated. For every $n \in \mathbb{N}$, we have

$$[0, 1] \subset \bigcup_{0 \leq k < m_0} [kl/m, (k+1)l/m]$$

and

$$d((n, kl/m, 1/m), (n, (k+1)l/m, 1/m)) = l/m \leq 1/m_0,$$

from which it follows that

$$\{(n, k/m, 1/m) : k \leq m\} \subset B^{m_0}((n, 0, 1/m), \varepsilon) \subset B^{m_0+1}(0, \varepsilon).$$

Consequently, $L_m \subset B^{m_0+1}(0, \varepsilon)$, hence $\cup_{m \geq m_0} L_m \subset B^{m_0+1}(0, \varepsilon)$. It follows then from 1) that $X \setminus B^{m_0+1}(0, \varepsilon)$ is uniformly isolated as claimed. \square

3) *There is no Bourbaki bounded set $F \subset X$ such that $X \setminus B(F, 1)$ is uniformly isolated.*

Proof. Let $F \subset X$ be a Bourbaki bounded set. By 2) and Theorem 3.1, for each $m \in \mathbb{N}$, there is $k_m \in \mathbb{N}$ such that $F \cap L_m \subset \{(n, k/m, 1/m) : n \leq k_m, 0 \leq k \leq m\}$. Suppose that for some $m \in \mathbb{N}$, $X \setminus B(F, 1) \subset L_1 \cup \dots \cup L_m$. Let $p > m$ and $q > k_p$. Then $(q, 1, 1/p) \notin \cup_{i \leq m} L_i$, so there is $x \in F$ such that $d((q, 1, 1/p), x) < 1$. Since $F \cap L_p \subset \{(n, k/p, 1/p) : n \leq k_p, 0 \leq k \leq p\}$, the point x belongs to the set $Y = X \setminus \{(q, k/p, 1/p) : k \leq p\}$, which is impossible because $d((q, 1, 1/p), Y) \geq 1$. It follows now from 1) that $X \setminus B(F, 1)$ is not uniformly isolated. \square

We would like to conclude with some observations about the following question: *What are the metrizable spaces X that admit a compatible metric d such that $U(X, d)$ is a ring?* We do not know the full answer to this question, but we will show that if X contains a finitely chainable closed subspace $L \subset X$ such that $X' \subset L$, then there is a compatible metric d on X such that $U(X, d)$ is a ring (Theorem 3.13).

Recall that a metrizable space Y is said to be *finitely chainable* if there is a compatible metric d on Y such that (Y, d) is Bourbaki bounded. In what follows, if Y is subset of a metric space (X, d) , then (Y, d) stands for the metric subspace of X .

Lemma 3.8. *Let L be a bounded subset of a metric space (X, d) and let $M \subset X$ be such that $L \subset M$. If $X \setminus M$ is uniformly isolated, then L is bounded in (M, d) .*

Proof. Let $\eta > 0$ and choose a finite set $F \subset X$ and $n \in \mathbb{N}$ such that $L \subset B^n(F, \eta)$. We may suppose that $X \setminus M$ is η -uniformly isolated. Let $x \in L$. There are $y \in F$ and a finite sequence of distinct points $z_1, \dots, z_n \in X$, with $z_1 = x$ and $z_n = y$, such that $d(z_i, z_{i+1}) < \eta$ for each $1 \leq i < n$. Suppose that $z_i \in M$. Then $z_{i+1} \in M$ because otherwise $z_{i+1} = z_i$, since $z_i \in M$, $d(z_i, z_{i+1}) < \eta$ and $B(z_{i+1}, \eta) = \{z_{i+1}\}$. Since $z_1 \in M$, it follows that $\{z_1, \dots, z_n\} \subset M$. Consequently, L is bounded in (M, d) . \square

Lemma 3.9. *Let (X, d) be a metric space and $M \subset X$. Then (M, d) is bounded if (and only if) for each $\varepsilon > 0$, M is bounded in $(B(M, \varepsilon), d)$.*

Proof. Let $\varepsilon > 0$ and put $Y = B(M, \varepsilon/3)$. There are a finite set $F \subset X$ and $n \in \mathbb{N}$ such that $M \subset B_Y^n(F, \varepsilon/3)$, where $B_Y^n(F, \varepsilon/3)$ is the n -iteration of $B_Y(F, \varepsilon/3)$ in the metric space (Y, d) . We shall show that $M \subset B_M^n(F, \varepsilon)$. For each $z \in B(M, \varepsilon/3)$, select $s(z) \in M$ such that $d(z, s(z)) < \varepsilon/3$. Now let $x \in M$ and choose a finite sequence $z_1, \dots, z_n \in Y$ such that $x = z_1$, $z_n \in F$ and $d(z_i, z_{i+1}) < \varepsilon/3$ for each $i < n$. Let $t_1 = x$, $t_n = z_n$ and $t_i = s(z_i)$ for $1 < i < n$. Then $\{t_1, \dots, t_n\} \subset M$ and $d(t_i, t_{i+1}) < \varepsilon$ for each $i < n$. Hence $M \subset B_M^n(F, \varepsilon)$. \square

Let (X, d) be a metric space and let $L \subset X$. Following Beer [6], let δ_L be the metric on X defined by

$$\delta_L(x, y) = d(x, y) + \max\{d(x, L), d(y, L)\}$$

when $x \neq y$. Observe that for every $x \in \bar{L}$ and $y \in X$, we have $\delta_L(x, y) \leq 2d(x, y)$. Consequently, for every $\varepsilon > 0$, $B_d(L, \varepsilon) \subset B_{\delta_L}(L, 2\varepsilon)$, and if $X' \subset \bar{L}$ then d and δ_L are topologically equivalent. Here and in what follows, $B_d(L, \varepsilon)$ and $B_{\delta_L}(L, \varepsilon)$ stand for the ε -enlargement of the set L with respect to d and δ_L , respectively.

Lemma 3.10. *For every $\eta > 0$, $X \setminus B_{\delta_L}(L, \eta)$ is uniformly $\eta/2$ -isolated in (X, δ_L) .*

Proof. Otherwise, there are $x, y \in X$, with $x \in X \setminus B_{\delta_L}(L, \eta)$, such that $0 < \delta_L(x, y) < \eta/2$. Then $d(x, L) < \eta/2$, hence there is $z \in L$ such that $d(x, z) < \eta/2$. It follows that $\delta_L(x, L) \leq \delta_L(x, z) = d(x, z) + d(x, L) \leq 2d(x, z) < \eta$, a contradiction. \square

In general, the distances d and δ_L are not uniformly equivalent on X , as the following shows.

Proposition 3.11. *The following are equivalent:*

- (a) *For every $\varepsilon > 0$, $X \setminus B_d(L, \varepsilon)$ is uniformly isolated in (X, d) .*
- (b) *d and δ_L are uniformly equivalent on X .*

Proof. Suppose that (a) holds. Let $\varepsilon > 0$. Choose $\eta > 0$ such that $\eta < \varepsilon/3$ and $X \setminus B_d(L, \varepsilon/3)$ is uniformly η -isolated in (X, d) . Let $x, y \in X$ be such that $d(x, y) < \eta$. If x and y are not in $B_d(L, \varepsilon/3)$, then $x = y$, hence $\delta_L(x, y) = 0$. If $\{x, y\} \cap B_d(L, \varepsilon/3) \neq \emptyset$, say $x \in B_d(L, \varepsilon/3)$, then $d(x, L) \leq \varepsilon/3$ and $d(y, L) \leq d(y, x) + \varepsilon/3$, hence $\delta_L(x, y) \leq \varepsilon$.

Conversely, let $\varepsilon > 0$ and suppose that $X \setminus B_d(L, \varepsilon)$ is not uniformly isolated in (X, d) . Then, for each $n \in \mathbb{N}$, there are $x_n, y_n \in X \setminus B_d(L, \varepsilon)$ such that $0 < d(x_n, y_n) < 1/n$. Since $\delta_L(x_n, y_n) \geq \varepsilon$, d and δ_L are not uniformly equivalent. \square

The following answers the natural question of whether $U(X, \delta_L)$ is a ring in case $L = X'$.

Corollary 3.12. *Let $L = X'$. Then $U(X, \delta_L)$ is a ring iff (X', d) is Bourbaki bounded.*

Proof. Suppose that $U(X, \delta_L)$ is a ring. Then, by Proposition 2.2, X' is bounded in (X, δ_L) . By Lemma 3.10, for each $\varepsilon > 0$, $X \setminus B_{\delta_L}(X', \varepsilon)$ is uniformly isolated in (X, δ_L) , hence by Lemma 3.8, X' is bounded in $(B(X', \varepsilon), \delta_L)$. It follows now from Lemma 3.9 that (X', δ_L) is bounded. Since $d = \delta_L$ on X' , we obtain that (X', d) is bounded.

The converse follows from Lemma 3.10 and Theorem 3.1. \square

Theorem 3.13. *Let X be a metrizable space with a finitely chainable closed subspace L such that $X' \subset L$. Then, there is a compatible metric δ on X such that for every $\varepsilon > 0$, $X \setminus B_\delta(L, \varepsilon)$ is uniformly isolated. In particular, $U(X, \delta)$ is a ring.*

Proof. Let d_0 be a compatible metric on the subspace L of X such that (L, d_0) is bounded. By Hausdorff theorem [18] (see [21] for more information), d_0 extends to a compatible metric d on X . To conclude, let $\delta = \delta_L$ be the metric associated to d and L and apply Corollary 3.12. \square

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