



HAL
open science

Integral p -adic étale cohomology of Drinfeld symmetric spaces

Pierre Colmez, Gabriel Dospinescu, Wiesława Nizioł

► **To cite this version:**

Pierre Colmez, Gabriel Dospinescu, Wiesława Nizioł. Integral p -adic étale cohomology of Drinfeld symmetric spaces. *Duke Mathematical Journal*, 2021, 170 (3), 10.1215/00127094-2020-0084 . hal-02387669

HAL Id: hal-02387669

<https://hal.science/hal-02387669v1>

Submitted on 27 May 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

INTEGRAL p -ADIC ÉTALE COHOMOLOGY OF DRINFELD SYMMETRIC SPACES

PIERRE COLMEZ, GABRIEL DOSPINESCU, AND WIESŁAWA NIZIOŁ

ABSTRACT. We compute the integral p -adic étale cohomology of Drinfeld symmetric spaces of any dimension. This refines the computation of the rational p -adic étale cohomology from [9]. The main tools are: the computation of the integral de Rham cohomology from [9] and the integral p -adic comparison theorems of Bhatt-Morrow-Scholze and Česnavičius-Koshikawa which replace the quasi-integral comparison theorem of Tsuji used in [9].

CONTENTS

1. Introduction	1
2. Preliminaries	3
2.1. Derived completions and the décalage functor	3
2.2. The complexes $A\Omega_{\mathfrak{X}}$ and $\tilde{\Omega}_{\mathfrak{X}}$	5
2.3. The Hodge-Tate and de Rham specializations	6
2.4. p -adic nearby cycles and A_{inf} -cohomology	9
3. A_{inf} -symbol maps	11
3.1. The construction of symbol maps	11
3.2. Compatibility with the Hodge-Tate symbol map	12
4. The A_{inf} -cohomology of Drinfeld symmetric spaces	14
4.1. Generalized Steinberg representations and their duals	14
4.2. Integral de Rham cohomology of Drinfeld symmetric spaces	15
4.3. Integrating symbols	16
4.4. The A_{inf} -cohomology of Drinfeld symmetric spaces	19
5. Integral p -adic étale cohomology of Drinfeld symmetric spaces	20
References	21

1. INTRODUCTION

Let p be a prime number, K a finite extension of \mathbf{Q}_p , and C the p -adic completion of an algebraic closure \overline{K} of K . Drinfeld's symmetric space of dimension d over K is the rigid analytic variety

$$\mathbb{H}_K^d := \mathbb{P}_K^d \setminus \cup_{H \in \mathcal{H}} H,$$

where \mathcal{H} is the space of K -rational hyperplanes in K^{d+1} . It is equipped with an action of $G = \text{GL}_{d+1}(K)$. One of the main results of [9] is the description of the $G \times \mathcal{G}_K$ -modules $H_{\text{ét}}^i(\mathbb{H}_C^d, \mathbf{Q}_p(i))$, where $\mathbb{H}_C^d := \mathbb{H}_{K,C}^d$ and $\mathcal{G}_K = \text{Gal}(\overline{K}/K)$. The analogous result for ℓ -adic étale cohomology, $\ell \neq p$, is a classical result of Schneider and Stuhler [19]. It relies on the fact that ℓ -adic étale cohomology satisfies a homotopy property with respect to the open ball (a fact that is false for p -adic étale cohomology).

The goal of this paper is to refine our result, by describing the integral p -adic étale cohomology groups $H_{\text{ét}}^i(\mathbb{H}_C^d, \mathbf{Z}_p(i))$. Recall that, for $i \geq 0$, there is a natural generalized Steinberg representation $\text{Sp}_i(\mathbf{Z}_p)$ of G (see Section 4.1 for the precise definition). We endow it with the trivial action of \mathcal{G}_K and we write $\text{Sp}_i(\mathbf{Z}_p)^*$ for its \mathbf{Z}_p -dual.

This research was partially supported by the project ANR-14-CE25 and the NSF grant No. DMS-1440140.

The main result of this paper is the following:

Theorem 1.1. *Let $i \geq 0$. There are compatible topological isomorphisms of $G \times \mathcal{G}_K$ -modules*

$$H_{\text{ét}}^i(\mathbb{H}_C^d, \mathbf{Z}_p(i)) \simeq \mathrm{Sp}_i(\mathbf{Z}_p)^*, \quad H_{\text{ét}}^i(\mathbb{H}_C^d, \mathbf{F}_p(i)) \simeq \mathrm{Sp}_i(\mathbf{F}_p)^*,$$

compatible with the isomorphism $H_{\text{ét}}^i(\mathbb{H}_C^d, \mathbf{Q}_p(i)) \simeq \mathrm{Sp}_i(\mathbf{Z}_p)^ \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ from [9]. In particular, for $i > d$ these cohomology groups are trivial.*

If $d = 1$ and $i = 1$, this result is due to Drinfeld [10] (with a shaky proof corrected in [12]; see also [8, Th. 1.7]).

Étale cohomology and A_{inf} -cohomology. We will describe now the key ideas and difficulties occurring in the proof of Theorem 1.1. As in [9], a key input is the pro-ordinarity of the standard semistable formal model $\mathfrak{X}_{\mathcal{O}_K}$ of \mathbb{H}_K^d , a result due to Grosse-Klönne [13]. More precisely, he proved that

$$(1.2) \quad H^i(\mathfrak{X}_{\mathcal{O}_K}, \Omega_{\mathfrak{X}_{\mathcal{O}_K}}^j) = 0, \quad i \geq 1, j \geq 0,$$

where $\Omega_{\mathfrak{X}_{\mathcal{O}_K}}^\bullet$ is the logarithmic de Rham complex of $\mathfrak{X}_{\mathcal{O}_K}$ over \mathcal{O}_K (for the canonical log-structures of $\mathfrak{X}_{\mathcal{O}_K}$ and \mathcal{O}_K). One easily infers from this that $\mathfrak{X}_{\mathcal{O}_K}$ is ordinary in the usual sense [9]. The strongest (and easiest) integral p -adic comparison theorems are available for ordinary varieties, making it natural to try to adapt them to $\mathfrak{X}_{\mathcal{O}_K}$. Nevertheless, the fact that $\mathfrak{X}_{\mathcal{O}_K}$ is not quasi-compact seems to be a serious obstacle in implementing the usual strategy [3, Ch. 7] to our setup. The syntomic method, suitably adapted [9], works well only up to some absolute constants, and reduces the computation of $H_{\text{ét}}^i(\mathbb{H}_C^d, \mathbf{Q}_p(i))$ to that of the (integral) Hyodo-Kato cohomology of the special fiber of $\mathfrak{X}_{\mathcal{O}_K}$, which was done in [9]. The latter computation can be done integrally and also shows that the de Rham cohomology of $\mathfrak{X}_{\mathcal{O}_K}$ is p -torsion-free.

The results of Bhatt-Morrow-Scholze [5] (adapted to the semistable reduction setting by Česnavičius-Koshikawa [7]) show that, for proper rigid analytic varieties with semistable reduction, if the de Rham cohomology of the semistable integral model is p -torsion free (equivalently, if the integral Hyodo-Kato cohomology of the special fiber is p -torsion free) so is the p -adic étale cohomology of the generic fiber. Combined with [9] and with the rigidity of G -invariant lattices in $\mathrm{Sp}_i(\mathbf{Q}_p)$ (a result due to Grosse-Klönne [16]), this would yield our main result. The problem is that the proofs in [5] and [7] rely on the properness of the varieties and it is not clear how to adapt them to our context. However, the key actor in *loc. cit.* makes perfect sense: the A_{inf} -cohomology. One then needs a way to read the p -adic étale cohomology in terms of the A_{inf} -cohomology, which can be done even for non quasi-compact varieties thanks to a remarkable (especially due to its simplicity!) formula in [6] (the way p -adic étale cohomology and A_{inf} -cohomology are related in [5] is rather different and does not seem to be very useful in our case). This reduces the proof of our main theorem to the computation of the A_{inf} -cohomology.

More precisely, let $A_{\text{inf}} = W(\mathcal{O}_C^b)$ be Fontaine's ring associated to C . The choice of a compatible system of primitive p -power roots of unity $(\zeta_{p^n})_n$ gives rise to an element $\mu = [\varepsilon] - 1 \in A_{\text{inf}}$ (where ε corresponds to $(\zeta_{p^n})_n$ under the identification $\mathcal{O}_C^b = \varprojlim_{x \mapsto x^p} \mathcal{O}_C$). This, in turn, induces a modified Tate twist $M \rightarrow M\{i\} := M \otimes_{A_{\text{inf}}} A_{\text{inf}}\{i\}$, $i \geq 0$, on the category of A_{inf} -modules, where $A_{\text{inf}}\{1\} := \frac{1}{\mu} A_{\text{inf}}(1)$, $A_{\text{inf}}\{i\} := A_{\text{inf}}\{1\}^{\otimes i}$. Let $X = \mathbb{H}_C^d$ and $\mathfrak{X} = \mathfrak{X}_{\mathcal{O}_K} \otimes_{\mathcal{O}_K} \mathcal{O}_C$. Using the projection from the pro-étale site of X to the étale site of \mathfrak{X} and a relative version of Fontaine's construction of the ring A_{inf} , one constructs in [5], [7] a complex of sheaves of A_{inf} -modules $A\Omega_{\mathfrak{X}}$ on the étale site of \mathfrak{X} , which allows one to interpolate between étale, crystalline, and de Rham cohomology of X and \mathfrak{X} .

The technical result we prove is then:

Theorem 1.3. *Let $i \geq 0$. There is a topological φ^{-1} -equivariant isomorphism of $G \times \mathcal{G}_K$ -modules*

$$H_{\text{ét}}^i(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\}) \simeq A_{\text{inf}} \widehat{\otimes}_{\mathbf{Z}_p} \mathrm{Sp}_i(\mathbf{Z}_p)^*.$$

Theorem 1.1 is now obtained from this and the description of p -adic nearby cycles in [6] in terms of $A\Omega_{\mathfrak{X}}$ (a twisted version of the Artin-Schreier exact sequence): an exact sequence

$$(1.4) \quad 0 \rightarrow H_{\text{ét}}^{i-1}(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\})/(1 - \varphi^{-1}) \rightarrow H_{\text{ét}}^i(X, \mathbf{Z}_p(i)) \rightarrow H_{\text{ét}}^i(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\})^{\varphi^{-1}=1} \rightarrow 0.$$

Proof of Theorem 1.3. We end the introduction by briefly explaining the key steps in the proof of Theorem 1.3. Fix $i \geq 0$ and write for simplicity $M = H^i(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\})$. This is an A_{inf} -module, which is derived $\tilde{\xi}$ -complete, for $\tilde{\xi} = \varphi(\mu)/\mu$.

In the first step, we interpret (following Schneider-Stuhler [19] and Iovita-Spiess [17]) $\text{Sp}_i(\mathbf{Z}_p)^*$ as a suitable quotient of the space of \mathbf{Z}_p -valued measures on \mathcal{H}^{i+1} (recall that \mathcal{H} is the space of K -rational hyperplanes in K^{d+1}). This allows us to construct an étale regulator (an "integration of étale symbols") map

$$r_{\text{ét}} : \text{Sp}_i(\mathbf{Z}_p)^* \rightarrow H_{\text{ét}}^i(X, \mathbf{Z}_p(i))$$

which induces a regulator map

$$(1.5) \quad r_{\text{inf}} : A_{\text{inf}} \widehat{\otimes}_{\mathbf{Z}_p} \text{Sp}_i(\mathbf{Z}_p)^* \rightarrow H_{\text{ét}}^i(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\}).$$

To prove that r_{inf} is an isomorphism we use the derived Nakayama Lemma: since both sides of (1.5) are derived $\tilde{\xi}$ -complete it suffices to show that r_{inf} is a quasi-isomorphism when reduced modulo $\tilde{\xi}$ (in the derived sense). That is, that the morphism

$$r_{\text{inf}} \otimes^{\mathbf{L}} \text{Id}_{A_{\text{inf}}/\tilde{\xi}} : (A_{\text{inf}} \widehat{\otimes}_{\mathbf{Z}_p} \text{Sp}_i(\mathbf{Z}_p)^*) \otimes_{A_{\text{inf}}}^{\mathbf{L}} (A_{\text{inf}}/\tilde{\xi}) \rightarrow H_{\text{ét}}^i(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\}) \otimes_{A_{\text{inf}}}^{\mathbf{L}} (A_{\text{inf}}/\tilde{\xi})$$

is a quasi-isomorphism. To compute the naive reduction \bar{r}_{inf} modulo $\tilde{\xi}$ of (1.5) we use the Hodge-Tate specialization of $A\Omega_{\mathfrak{X}}$, which identifies $H^i(A\Omega_{\mathfrak{X}}/\tilde{\xi})$ with the (twisted) sheaf of i 'th logarithmic differential forms on \mathfrak{X} . And, globally, those are well controlled by the acyclicity result (1.2). Combined with a compatibility between the étale and the Hodge-Tate Chern class maps and the Hodge-Tate specialization this implies that \bar{r}_{inf} is isomorphic to the Hodge-Tate regulator

$$r_{\text{HT}} : \mathcal{O}_C \widehat{\otimes}_{\mathbf{Z}_p} \text{Sp}_i(\mathbf{Z}_p)^* \rightarrow H_{\text{ét}}^0(\mathfrak{X}, \Omega_{\mathfrak{X}}^i).$$

And this we have shown to be an isomorphism in [9].

Along the way we also compute that the target $H_{\text{ét}}^i(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\})$ of r_{inf} is $\tilde{\xi}$ -torsion free. Since the domain $A_{\text{inf}} \widehat{\otimes}_{\mathbf{Z}_p} \text{Sp}_i(\mathbf{Z}_p)^*$ of r_{inf} is also $\tilde{\xi}$ -torsion free this shows that $r_{\text{inf}} \otimes^{\mathbf{L}} \text{Id}_{A_{\text{inf}}/\tilde{\xi}} \simeq \bar{r}_{\text{inf}}$ and hence, by the above, it is a quasi-isomorphism, as wanted.

Acknowledgments. W.N. would like to thank MSRI, Berkeley, for hospitality during Spring 2019 semester when parts of this paper were written. We would like to thank Bhargav Bhatt for suggesting that derived completions could simplify our original proof (which they did!). We thank Kęstutis Česnavičius and Matthew Morrow for helpful discussions related to the subject of this paper.

Notation and conventions. Throughout the paper p is a fixed prime. K is a finite extension of \mathbf{Q}_p with the ring of integers \mathcal{O}_K ; C is the p -adic completion of an algebraic closure \bar{K} of K .

All formal schemes are p -adic. A formal scheme over \mathcal{O}_K is called *semistable* if, locally for the Zariski topology, it admits étale maps to the formal spectrum $\text{Spf}(\mathcal{O}_K\{X_1, \dots, X_n\}/(X_1 \cdots X_r - \varpi))$, $1 \leq r \leq n$, where ϖ is a uniformizer of K . We equip it with the log-structure coming from the special fiber.

If A is a ring and $f \in A$ is a regular element (i.e., nonzero divisor) and $T \in D(A)$, we will often write T/f for $T \otimes_A^{\mathbf{L}} A/f$ if there is no confusion.

2. PRELIMINARIES

2.1. Derived completions and the décalage functor.

2.1.1. *Derived completions.* We will need the following derived version of completion¹:

Definition 2.1. ([22, 091S]) Let I be an ideal of a ring A . We say that $M \in D(A)$ is *derived I -complete* if for all $f \in I$

$$\mathrm{holim}(\cdots \rightarrow M \xrightarrow{f} M \xrightarrow{f} M \xrightarrow{f} M) = 0.$$

Let A be a ring and let $I \subset A$ be an ideal. We list the following basic properties of derived completions [22, 091N]:

- (1) Let M be an A -module. If M is classically I -complete, i.e., the map $M \rightarrow \varprojlim_n M/I^n$ is an isomorphism, then M is also derived I -complete; the converse is true if M is I -adically separated.
- (2) The collection of all derived I -complete A -complexes forms a full triangulated subcategory of $D(A)$.
- (3) $M \in D(A)$ is derived I -complete if and only if so are its cohomology groups $H^i(M)$, $i \in \mathbf{Z}$.
- (4) (*Derived Nakayama Lemma*) A derived I -complete complex $M \in D(A)$ is 0 if and only if $M \otimes_A^L A/I \simeq 0$.
- (5) If I is generated by $x_1, \dots, x_n \in A$, then $M \in D(A)$ is derived I -complete if and only if M is derived (x_i) -complete for $1 \leq i \leq n$.
- (6) If f is a morphism of ringed topoi, the functor $\mathrm{R}f_*$ commutes with derived completions [22, 0944].

2.1.2. *The Berthelot-Deligne-Ogus décalage functor.* For any ring A and any regular element $f \in A$ there is a functor $\mathrm{L}\eta_f : D(A) \rightarrow D(A)$ (which is **not** exact) with the key property [5, Lemma 6.4] that there is a functorial isomorphism²

$$H^i(\mathrm{L}\eta_f(T)) \simeq H^i(T)/(H^i(T)[f]),$$

where $M[f] := \{x \in M \mid fx = 0\}$. Concretely, choose a representative T^\bullet of $T \in D(A)$ such that $T^i[f] = 0$ for all i , and consider the sub-complex $\eta_f(T^\bullet) \subset T^\bullet[1/f]$ defined by

$$\eta_f(T^\bullet)^i = \{x \in f^i T^i \mid dx \in f^{i+1} T^{i+1}\}.$$

Its image $\mathrm{L}\eta_f(T)$ in $D(A)$ depends only on T .

We list the following properties of the above construction (sometimes extended naturally to ringed topoi):

- (1) $\mathrm{L}\eta_f$ commutes with truncations and with restriction of scalars³. Moreover, $\mathrm{L}\eta_f(\mathrm{L}\eta_g(T)) \simeq \mathrm{L}\eta_{fg}(T)$ for $f, g \in A$ regular elements and $T \in D(A)$.
- (2) For all $T \in D(A)$, we have $\mathrm{L}\eta_f(T)[1/f] \simeq T[1/f]$ and there is a canonical isomorphism

$$\mathrm{L}\eta_f(T)/f = \mathrm{L}\eta_f(T) \otimes_A^L A/f \simeq (H^*(T/f), \beta_f),$$

where $(H^*(T/f), \beta_f)$ is the Bockstein complex equal to $H^i(T \otimes_A^L (f^i A/f^{i+1} A))$ in degree i , the differential being the boundary map associated to the triangle

$$T \otimes_A^L (f^{i+1} A/f^{i+2} A) \rightarrow T \otimes_A^L (f^i A/f^{i+2} A) \rightarrow T \otimes_A^L (f^i A/f^{i+1} A).$$

This is discussed in [5, Chapter 6] and [4, Lemma 5.9].

¹The terminology here is misleading. The derived I -completion is not given by $M \mapsto \mathrm{holim}_n (M \otimes_A^L A/I^n)$, as one would naturally guess.

²Depending on f , not only on the ideal fA . If we want to avoid this, the "correct" isomorphism is

$$H^i(\mathrm{L}\eta_f(T)) \simeq (H^i(T)/H^i(T)[f]) \otimes_A (f^i),$$

where $(f^i) \subset A[1/f]$ is the fractional A -ideal generated by f^i .

³The latter means that $\alpha_*(\mathrm{L}\eta_{\alpha(f)}(M)) \simeq \mathrm{L}\eta_f(\alpha_* M)$ for $M \in D(B)$ and $\alpha : A \rightarrow B$ a map of rings such that $\alpha(f) \in B$ is regular.

- (3) If $T \rightarrow L \rightarrow M$ is an exact triangle in $D(A)$, then $L\eta_f(T) \rightarrow L\eta_f(L) \rightarrow L\eta_f(M)$ is also an exact triangle **if** the boundary map $H^i(M/f) \rightarrow H^{i+1}(T/f)$ is the zero map for all i . For a regular element $g \in A$ and $T \in D(A)$, the natural map $L\eta_f(T)/g \rightarrow L\eta_f(T/g)$ is an isomorphism **if** $H^*(T/f)$ has no g -torsion.
- (4) If $I \subset A$ is a finitely generated ideal and $T \in D(A)$ is derived I -complete, then so is $L\eta_f(T)$ [5], [4, Lemma 5.19].
- (5) If $T \in D^{[0,d]}(A)$ and $H^0(T)$ is f -torsion-free then there are natural maps $L\eta_f(T) \rightarrow T$ and $T \rightarrow L\eta_f(T)$ whose compositions are f^d . More precisely, if T^\bullet is a representative concentrated in degrees $0, \dots, d$ and with f -torsion-free terms, then the first map is induced by $\eta_f(T^\bullet) \subset T^\bullet$. Multiplication by f^d on each of the two complexes factors over this inclusion map. When $T \in D^{\geq 0}(A)$, we will refer to the map $L\eta_f(T) \rightarrow T$ as the *canonical map*.

2.2. The complexes $A\Omega_{\mathfrak{X}}$ and $\tilde{\Omega}_{\mathfrak{X}}$.

2.2.1. *Fontaine rings.* Let

$$\mathcal{O}_C^b := \varprojlim_{x \rightarrow x^p} \mathcal{O}_C \simeq \varprojlim_{x \rightarrow x^p} \mathcal{O}_C/p$$

be the tilt of \mathcal{O}_C (so that $C^b = \text{Frac}(\mathcal{O}_C^b)$ is an algebraically closed field of characteristic p). Let $A_{\text{inf}} = W(\mathcal{O}_C^b)$ and choose once and for all a compatible sequence $(1, \zeta_p, \zeta_{p^2}, \dots)$ of primitive p -power roots of 1, giving rise to $\varepsilon = (1, \zeta_p, \zeta_{p^2}, \dots) \in \mathcal{O}_C^b$. Letting φ be the natural Frobenius automorphism of A_{inf} , define

$$\mu := [\varepsilon] - 1, \quad \xi := \frac{\mu}{\varphi^{-1}(\mu)} = \frac{[\varepsilon] - 1}{[\varepsilon^{1/p}] - 1} \in A_{\text{inf}}.$$

The natural surjective map $\mathcal{O}_C^b \rightarrow \mathcal{O}_C/p$ lifts to a map $\theta : A_{\text{inf}} \rightarrow \mathcal{O}_C$ with kernel generated by ξ ; the map θ , in turn, lifts to a map $\theta_\infty : A_{\text{inf}} \rightarrow W(\mathcal{O}_C)$ with kernel generated by μ (however, contrary to θ , θ_∞ is not always surjective, see [5, Lemma 3.23]). The kernel of the twisted map $\tilde{\theta} := \theta\varphi^{-1} : A_{\text{inf}} \rightarrow \mathcal{O}_C$ is generated by

$$\tilde{\xi} := \varphi(\xi) = \frac{\varphi(\mu)}{\mu} = \frac{[\varepsilon^p] - 1}{[\varepsilon] - 1}.$$

We have $\tilde{\theta}(\mu) = \zeta_p - 1$.

We list the following properties [4, 2.25]

- (1) $\tilde{\xi}$ modulo μ is equal to p .
- (2) Since A_{inf} and its reduction mod p are integral domains and since $\xi, \tilde{\xi}, \mu$ are not 0 modulo p , $(p, \xi), (p, \tilde{\xi}), (p, \mu)$ are regular sequences, and so is the sequence $(\tilde{\xi}, \mu)$.
- (3) The ideals $(p, \xi), (p, \tilde{\xi})$, and $(\tilde{\xi}, \mu)$ define the same topology on A_{inf} .

The above constructions naturally generalize to the case when \mathcal{O}_C is replaced by a perfectoid ring.

2.2.2. *Modified Tate twists.* The compatible sequence of roots of unity $(\zeta_{p^n})_n$ gives a trivialization $\mathbf{Z}_p(1) \simeq \mathbf{Z}_p$, and we will write $\zeta = (\zeta_{p^n})_n$ for the corresponding basis of $\mathbf{Z}_p(1)$. By Fontaine's theorem [11], the \mathcal{O}_C -module

$$\mathcal{O}_C\{1\} := T_p(\Omega_{\mathcal{O}_C/\mathbf{Z}_p}^1)$$

is free of rank 1 and the natural map $\text{dlog} : \mu_{p^\infty} \rightarrow \Omega_{\mathcal{O}_C/\mathbf{Z}_p}^1$ induces an \mathcal{O}_C -linear injection

$$\text{dlog} : \mathcal{O}_C(1) \rightarrow \mathcal{O}_C\{1\}, \quad \text{dlog}(\zeta) = (\text{dlog}(\zeta_{p^n}))_{n \geq 1}.$$

The \mathcal{O}_C -module $\mathcal{O}_C\{1\}$ is generated by

$$\omega := \frac{1}{\zeta_p - 1} \text{dlog}(\zeta),$$

thus the annihilator of $\text{coker}(\text{dlog})$ is $(\zeta_p - 1)$. For any \mathcal{O}_C -module M , let $M\{1\} := M \otimes_{\mathcal{O}_C} \mathcal{O}_C\{1\}$, and we will often write $m\{1\}$ for the element of $M\{1\}$ corresponding to $m \in M$ (in particular, $a\{1\} = a \cdot \omega$ in $\mathcal{O}_C\{1\}$).

Finally, define

$$A_{\text{inf}}\{1\} := \frac{1}{\mu} A_{\text{inf}}(1),$$

and let $a\{1\} = \frac{1}{\mu} a(1) \in A_{\text{inf}}\{1\}$, if $a \in A_{\text{inf}}$. The Frobenius φ on $W(C^b)(1)$ induces an isomorphism

$$\varphi : A_{\text{inf}}\{1\} \xrightarrow{\sim} (1/\xi)A_{\text{inf}}\{1\}.$$

Its inverse defines a map

$$\varphi^{-1} : A_{\text{inf}}\{1\} \rightarrow A_{\text{inf}}\{1\}.$$

There is a natural map

$$\tilde{\theta} := \theta \circ \varphi^{-1} : A_{\text{inf}}\{1\} \rightarrow \mathcal{O}_C\{1\}$$

sending $a\{1\}$, for $a \in A_{\text{inf}}$, to $\theta(\varphi^{-1}(a))\omega$.

If M is an A_{inf} -module, let $M\{i\} := M \otimes_{A_{\text{inf}}} A_{\text{inf}}\{1\}^{\otimes i}$, $i \in \mathbf{Z}$. The map $\tilde{\theta} : A_{\text{inf}}\{1\} \rightarrow \mathcal{O}_C\{1\}$ induces a map $\tilde{\theta} : M\{1\} \rightarrow (M/\xi)\{1\}$ of A_{inf} -modules (via the map $A_{\text{inf}} \rightarrow A_{\text{inf}}/\xi$).

2.2.3. The complexes $A\Omega_{\mathfrak{X}}$ and $\tilde{\Omega}_{\mathfrak{X}}$. Let \mathfrak{X} be a flat formal scheme over \mathcal{O}_C , with smooth generic fibre X , seen as an adic space over C . There is a natural morphism of sites

$$\nu : X_{\text{proét}} \rightarrow \mathfrak{X}_{\text{ét}},$$

as well as a sheaf $\mathbb{A}_{\text{inf}} := \mathbb{A}_{\text{inf}, X} := W(\varprojlim_{\varphi} \mathcal{O}_X^+/p)$ of A_{inf} -modules on $X_{\text{proét}}$, obtained by sheafifying Fontaine's construction $R \rightarrow A_{\text{inf}}(R)$ on the basis of affinoid perfectoids of $X_{\text{proét}}$. This sheaf is endowed with a bijective Frobenius φ as well as with a surjective map

$$\theta : \mathbb{A}_{\text{inf}} \rightarrow \widehat{\mathcal{O}}_X^+ := \varprojlim \mathcal{O}_X^+/p^n,$$

with kernel generated by the non-zero divisor ξ .

Define

$$A\Omega_{\mathfrak{X}} := L\eta_{\mu}(R\nu_* \mathbb{A}_{\text{inf}, X}) \in D^{\geq 0}(\mathfrak{X}, A_{\text{inf}}),$$

a commutative algebra object, as well as

$$\tilde{\Omega}_{\mathfrak{X}} := L\eta_{\zeta_p - 1}(R\nu_* \widehat{\mathcal{O}}_X^+) \in D(\mathfrak{X}_{\text{ét}}),$$

a commutative $\mathcal{O}_{\mathfrak{X}}$ -algebra object in $D(\mathfrak{X}_{\text{ét}})$.

2.3. The Hodge-Tate and de Rham specializations.

2.3.1. The smooth case. Suppose first that \mathfrak{X} is smooth over \mathcal{O}_C . The following result is proved in [5] (for the Zariski site, but the proof is identical in our case).

Theorem 2.2. (Bhatt-Morrow-Scholze, [5, Th. 8.3]) *There is a natural isomorphism of $\mathcal{O}_{\mathfrak{X}}$ -modules on $\mathfrak{X}_{\text{ét}}$*

$$H^i(\tilde{\Omega}_{\mathfrak{X}}) \simeq \Omega_{\mathfrak{X}/\mathcal{O}_C}^i\{-i\}.$$

We will recall the key relevant points since we will need some information about the construction of this isomorphism.

Let R be a formally smooth \mathcal{O}_C -algebra, such that $\text{Spf}(R)$ is connected, together with an étale map $A := \mathcal{O}_C\{T_1^{\pm 1}, \dots, T_d^{\pm 1}\} \rightarrow R$. We will simply say that R is a *small algebra* and call the map $A \rightarrow R$ a *framing*. Let \widehat{R} be the (perfectoid) completion of the normalization \overline{R} of R in the maximal pro-finite étale extension of $R[1/p]$, and let $\Delta := \text{Gal}(\overline{R}[1/p]/R[1/p])$. Define

$$A_{\infty} := \mathcal{O}_C\{T_1^{\pm 1/p^{\infty}}, \dots, T_d^{\pm 1/p^{\infty}}\}, \quad R_{\infty} = R \widehat{\otimes}_A A_{\infty}.$$

We have $\Gamma := \text{Gal}(R_\infty/R) \simeq \mathbf{Z}_p(1)^d = \bigoplus_{i=1}^d \mathbf{Z}_p \gamma_i$, where γ_i sends T_i^{1/p^n} to $\zeta_{p^n} T_i^{1/p^n}$ and fixes T_j^{1/p^n} for $j \neq i$. By the almost purity theorem of Faltings, the natural map (group cohomology is always continuous below) $\text{R}\Gamma(\Gamma, R_\infty) \rightarrow \text{R}\Gamma(\Delta, \widehat{\widetilde{R}})$ is an almost quasi-isomorphism. We have the following more precise results:

Theorem 2.3. (Bhatt-Morrow-Scholze, [5, Cor. 8.13, proof of prop. 8.14]) *Let R be a small algebra together with a framing, as above, and let $X = \text{Sp}(R[1/p])$ and $\mathfrak{X} = \text{Spf}(R)$.*

a) *The natural maps*

$$\text{L}\eta_{\zeta_p-1} \text{R}\Gamma(\Gamma, R_\infty) \rightarrow \text{L}\eta_{\zeta_p-1} \text{R}\Gamma(X_{\text{proét}}, \widehat{\mathcal{O}}_X^+) \rightarrow \text{R}\Gamma(\mathfrak{X}, \widetilde{\Omega}_{\mathfrak{X}})$$

are quasi-isomorphisms.

b) *Writing $\widetilde{\Omega}_R$ for any of these objects, the map $\widetilde{\Omega}_R \otimes_R \mathcal{O}_{\mathfrak{X}} \rightarrow \widetilde{\Omega}_{\mathfrak{X}}$ is a quasi-isomorphism in $D(\mathfrak{X}_{\text{ét}})$.*

c) *If $R \rightarrow S$ is a formally étale map of small algebras, the natural map $\widetilde{\Omega}_R \otimes_R^L S \rightarrow \widetilde{\Omega}_S$ is a quasi-isomorphism.*

Note that

$$H^i(\widetilde{\Omega}_R) \simeq H^i(\text{L}\eta_{\zeta_p-1} \text{R}\Gamma(\Gamma, R_\infty)) \simeq \frac{H^i(\Gamma, R_\infty)}{H^i(\Gamma, R_\infty)[\zeta_p - 1]} \simeq H^i(\Gamma, R),$$

the last isomorphism⁴ being a standard decompletion result ([5, Prop. 8.9]).

The key result (not obvious since one needs to define the isomorphisms canonically, independent of coordinates!) is then:

Theorem 2.4. (Bhatt-Morrow-Scholze, [5, Chapter 8]) *Let R be a small algebra.*

a) *There is a natural R -linear isomorphism $H^1(\widetilde{\Omega}_R) \simeq \Omega_{R/\mathcal{O}_C}^1 \{-1\}$.*

b) *The cup-product maps induce natural R -linear isomorphisms $\wedge^i H^1(\widetilde{\Omega}_R) \simeq H^i(\widetilde{\Omega}_R)$ and hence isomorphisms $H^i(\widetilde{\Omega}_R) \simeq \Omega_{R/\mathcal{O}_C}^i \{-i\}$.*

The isomorphism in a) is constructed in [5, Prop. 8.15] using completed cotangent complexes. We will make it explicit, as follows: consider a framing $A \rightarrow R$ (recall that $A = \mathcal{O}_C\{T_1^{\pm 1}, \dots, T_d^{\pm 1}\}$). By compatibility with base change from A to R of all objects involved, it suffices to construct the isomorphism for $R = A$. Moreover we may reduce to describing the isomorphism for $A = \mathcal{O}_C\{T^{\pm 1}\}$, i.e., for $d = 1$. Then the twisted map

$$\alpha : \Omega_{R/\mathcal{O}_C}^1 \simeq H^1(\widetilde{\Omega}_R)\{1\} \simeq \frac{H^1(\Gamma, R_\infty)}{H^1(\Gamma, R_\infty)[\zeta_p - 1]}\{1\} \xrightarrow{x \mapsto (\zeta_p - 1)x} (\zeta_p - 1)H^1(\Gamma, R_\infty)\{1\}$$

is an isomorphism, described explicitly by

$$\alpha \left(\frac{dT}{T} \right) = (\gamma \mapsto 1 \otimes \text{dlog}(\zeta_\gamma)) = (\gamma \mapsto (\zeta_p - 1) \otimes \frac{1}{\zeta_p - 1} \text{dlog}(\zeta_\gamma)),$$

where $\zeta_\gamma = (\zeta_{\gamma,n})_n$, for $\gamma \in \Gamma$, is defined by the formula $\zeta_{\gamma,n} := \gamma(T^{1/p^n})/T^{1/p^n}$.

2.3.2. The semistable case. Suppose now that \mathfrak{X} is semistable. This means that, locally for the étale topology, $\mathfrak{X} = \text{Spf}(R)$, where R admits an étale morphism of \mathcal{O}_C -algebras

$$A := \mathcal{O}_C\{T_0, \dots, T_r, T_{r+1}^{\pm 1}, \dots, T_d^{\pm 1}\} / (T_0 \cdots T_r - p^q) \rightarrow R$$

for some $d \geq 0$, $r \in \{0, 1, \dots, d\}$ and some rational number $q > 0$ (we fix once and for all an embedding $p^{\mathbf{Q}} \subset C$). Equip \mathcal{O}_C with the log-structure $\mathcal{O}_C \setminus \{0\} \rightarrow \mathcal{O}_C$ and \mathfrak{X} with the canonical log-structure, i.e. given by the sheafification of the subsheaf $\mathcal{O}_{\mathfrak{X}, \text{ét}} \cap (\mathcal{O}_{\mathfrak{X}, \text{ét}}[1/p])^*$ of $\mathcal{O}_{\mathfrak{X}, \text{ét}}$. Let $\Omega_{\mathfrak{X}/\mathcal{O}_C}$ be the finite locally free $\mathcal{O}_{\mathfrak{X}}$ -module of logarithmic differentials on \mathfrak{X} . We have the following result:

⁴Induced by the natural map $H^i(\Gamma, R) \rightarrow H^i(\Gamma, R_\infty)$.

Theorem 2.5. (Česnavičius-Koshikawa, [7, Th. 4.2, Cor. 4.6, Prop. 4.8, Th. 4.11])

a) *There is a natural $\mathcal{O}_{\mathfrak{X}, \text{ét}}$ -module isomorphism $H^1(\tilde{\Omega}_{\mathfrak{X}}) \simeq \Omega_{\mathfrak{X}/\mathcal{O}_C}^1\{-1\}$ whose restriction to the smooth locus \mathfrak{X}^{sm} is the one given by Theorem 2.2.*

b) *The natural map $\wedge^i(H^1(\tilde{\Omega}_{\mathfrak{X}})) \rightarrow H^i(\tilde{\Omega}_{\mathfrak{X}})$ is an isomorphism and so there is a natural $\mathcal{O}_{\mathfrak{X}, \text{ét}}$ -module isomorphism*

$$H^i(\tilde{\Omega}_{\mathfrak{X}}) \simeq \Omega_{\mathfrak{X}/\mathcal{O}_C}^i\{-i\}.$$

Remark 2.6. 1) A few words about a). The same arguments as in [5] (using completed cotangent complexes) give a map $\Omega_{\mathfrak{X}/\mathcal{O}_C}^{1, \text{cl}}\{-1\} \rightarrow \mathbb{R}^1\nu_*(\widehat{\mathcal{O}}_X^+)$, where we denoted by the superscript $(-)^{\text{cl}}$ the classical, non logarithmic, differential forms. The results in [5] ensure that the resulting map (the second map being the natural projection)

$$(2.7) \quad \Omega_{\mathfrak{X}/\mathcal{O}_C}^{1, \text{cl}}\{-1\} \rightarrow \mathbb{R}^1\nu_*(\widehat{\mathcal{O}}_X^+) \rightarrow \frac{\mathbb{R}^1\nu_*(\widehat{\mathcal{O}}_X^+)}{\mathbb{R}^1\nu_*(\widehat{\mathcal{O}}_X^+)[\zeta_p - 1]} \simeq H^1(\tilde{\Omega}_{\mathfrak{X}})$$

restricts to an isomorphism $\Omega_{\mathfrak{X}^{\text{sm}}/\mathcal{O}_C}^1\{-1\} \simeq (\zeta_p - 1)H^1(\tilde{\Omega}_{\mathfrak{X}})|_{\mathfrak{X}^{\text{sm}}}$. Moreover, one shows that $H^1(\tilde{\Omega}_{\mathfrak{X}})$ is a vector bundle. Hence one can divide the map (2.7) by $\zeta_p - 1$ to obtain a map

$$\Omega_{\mathfrak{X}/\mathcal{O}_C}^1\{-1\} \rightarrow H^1(\tilde{\Omega}_{\mathfrak{X}})$$

which is an isomorphism over \mathfrak{X}^{sm} . One shows that this extends to the isomorphism in a).

2) A few words about the maps in b). Letting $K = \mathbb{R}\nu_*(\widehat{\mathcal{O}}_X^+)$ and using the identifications

$$H^1(\tilde{\Omega}_{\mathfrak{X}}) \simeq \frac{H^1(K)}{H^1(K)[\zeta_p - 1]}, \quad H^i(\tilde{\Omega}_{\mathfrak{X}}) \simeq \frac{H^i(K)}{H^i(K)[\zeta_p - 1]},$$

they are induced by the product maps $H^1(K)^{\otimes i} \rightarrow H^i(K)$, which, in turn, are induced by the product maps

$$H^j(K) \otimes_{\mathcal{O}_{\mathfrak{X}, \text{ét}}} H^k(K) \rightarrow H^{j+k}(K \otimes_{\mathcal{O}_{\mathfrak{X}, \text{ét}}}^L K) \rightarrow H^{j+k}(K).$$

We continue assuming that \mathfrak{X} is semistable. Recall that the map $\tilde{\theta} = \theta \circ \varphi^{-1} : \mathbb{A}_{\text{inf}} \rightarrow \widehat{\mathcal{O}}_X^+$ is surjective, with kernel generated by $\tilde{\xi} = \varphi(\xi)$, and it sends μ to $\zeta_p - 1$, inducing therefore a morphism

$$A\Omega_{\mathfrak{X}}/\tilde{\xi} := A\Omega_{\mathfrak{X}} \otimes_{A_{\text{inf}}, \tilde{\theta}}^L \mathcal{O}_C \rightarrow \tilde{\Omega}_{\mathfrak{X}}.$$

Theorem 2.8. (Česnavičius-Koshikawa, [7, Th. 4.2, Th. 4.17, Cor. 4.6])

- (1) *The above morphism $A\Omega_{\mathfrak{X}}/\tilde{\xi} \rightarrow \tilde{\Omega}_{\mathfrak{X}}$ is a quasi-isomorphism.*
- (2) *There is a natural quasi-isomorphism $A\Omega_{\mathfrak{X}}/\xi \xrightarrow{\sim} \Omega_{\mathfrak{X}/\mathcal{O}_C}^\bullet$.*
- (3) *The complex $A\Omega_{\mathfrak{X}}$ is derived $\tilde{\xi}$ -complete. Hence so is $\text{R}\Gamma_{\text{ét}}(\mathfrak{X}, A\Omega_{\mathfrak{X}})$ (and its cohomology groups).*

For $i \geq 0$, (using the above theorems) we define

- the *Hodge-Tate specialization map*

- (1) (on sheaves) as the composition

$$\tilde{\iota}_{\text{HT}} : A\Omega_{\mathfrak{X}} \rightarrow A\Omega_{\mathfrak{X}}/\tilde{\xi} \rightarrow \tilde{\Omega}_{\mathfrak{X}}.$$

- (2) (on cohomology) $\iota_{\text{HT}} : H_{\text{ét}}^i(\mathfrak{X}, A\Omega_{\mathfrak{X}}) \rightarrow H_{\text{ét}}^0(\mathfrak{X}, \Omega_{\mathfrak{X}/\mathcal{O}_C}^i\{-i\})$ as the composition

$$\iota_{\text{HT}} : H_{\text{ét}}^i(\mathfrak{X}, A\Omega_{\mathfrak{X}}) \xrightarrow{\tilde{\iota}_{\text{HT}}} H_{\text{ét}}^i(\mathfrak{X}, \tilde{\Omega}_{\mathfrak{X}}) \rightarrow H_{\text{ét}}^0(\mathfrak{X}, H^i(\tilde{\Omega}_{\mathfrak{X}})) \simeq H_{\text{ét}}^0(\mathfrak{X}, \Omega_{\mathfrak{X}/\mathcal{O}_C}^i\{-i\}),$$

where the second map is the edge morphism in the spectral sequence

$$E_2^{i,j} = H_{\text{ét}}^i(\mathfrak{X}, H^j(\tilde{\Omega}_{\mathfrak{X}})) \Rightarrow H_{\text{ét}}^i(\mathfrak{X}, \tilde{\Omega}_{\mathfrak{X}}).$$

- the *de Rham specialization map* as the composition $\tilde{\iota}_{\text{dR}} : A\Omega_{\mathfrak{X}} \rightarrow A\Omega_{\mathfrak{X}}/\xi \xrightarrow{\sim} \Omega_{\mathfrak{X}/\mathcal{O}_C}^\bullet$ yielding on cohomology a map

$$\iota_{\text{dR}} : H_{\text{ét}}^i(\mathfrak{X}, A\Omega_{\mathfrak{X}}) \xrightarrow{\tilde{\iota}_{\text{dR}}} H_{\text{dR}}^i(\mathfrak{X}).$$

2.4. p -adic nearby cycles and A_{inf} -cohomology. We review here a result from [6], which describes integral p -adic étale cohomology in terms of the complex $A\Omega_{\mathfrak{X}}$. Let X be a smooth adic space over C and let \mathfrak{X} be a flat formal model of X (not necessarily semistable !). Fix an integer $i \geq 0$. Recall that there is a natural endomorphism (as abelian sheaf, **not** as A_{inf} -module !)

$$\xi^i \varphi^{-1} : \tau_{\leq i} A\Omega_{\mathfrak{X}} \rightarrow \tau_{\leq i} A\Omega_{\mathfrak{X}}$$

defined as the composition

$$\tau_{\leq i} A\Omega_{\mathfrak{X}} = L\eta_{\mu} \tau_{\leq i} R\nu_* \mathbb{A}_{\text{inf}} \xrightarrow{\varphi^{-1}} L\eta_{\varphi^{-1}(\mu)} \tau_{\leq i} R\nu_* \mathbb{A}_{\text{inf}} \xrightarrow{\xi^i} L\eta_{\xi} L\eta_{\varphi^{-1}(\mu)} \tau_{\leq i} R\nu_* \mathbb{A}_{\text{inf}} = \tau_{\leq i} A\Omega_{\mathfrak{X}}.$$

The following result is proved in [6] in the good reduction case. As we show below the proof goes through in a more general setting.

Theorem 2.9. (Bhatt-Morrow-Scholze, [6, Chapter 10]) *Let X be a smooth adic space over C with a flat formal model \mathfrak{X} . Let $i \geq 0$. There is a natural quasi-isomorphism*

$$\mu^i : \tau_{\leq i} R\nu_* \widehat{\mathbf{Z}}_p \xrightarrow{\sim} \tau_{\leq i} [\tau_{\leq i} A\Omega_{\mathfrak{X}} \xrightarrow{1 - \xi^i \varphi^{-1}} \tau_{\leq i} A\Omega_{\mathfrak{X}}],$$

where $\widehat{\mathbf{Z}}_p := \varprojlim_n \mathbf{Z}/p^n$ and $[\cdot]$ denotes the homotopy fiber.

Remark 2.10. We warn the reader that this quasi-isomorphism is not Galois equivariant (in the case X is defined over K). For an equivariant version see Corollary 2.13 below.

Proof. We follow [6] faithfully, but work directly on the p -adic level. Let $\psi_i = \xi^i \varphi^{-1}$, seen as an endomorphism of $\tau_{\leq i} A\Omega_{\mathfrak{X}}$ (as explained above) or of $T := R\nu_* \mathbb{A}_{\text{inf}}$ (defined in the obvious way). These two endomorphisms are compatible with the canonical map $\text{can} : A\Omega_{\mathfrak{X}} \rightarrow T$.

We start with the following simple fact:

Lemma 2.11. a) *For $i \geq j$, $1 - \psi_i$ induces an automorphism of the sheaf $\mathbb{A}_{\text{inf}}/\mu^j$.*

b) *There is an exact sequence of sheaves on $X_{\text{proét}}$*

$$0 \rightarrow \widehat{\mathbf{Z}}_p \xrightarrow{\mu^i} \mathbb{A}_{\text{inf}} \xrightarrow{1 - \psi_i} \mathbb{A}_{\text{inf}} \rightarrow 0.$$

Proof. a) This is proved by Morrow in [18, Lemma 3.5 (iii)].

b) Clearly, this is a sequence. For surjectivity use a) to deduce surjectivity modulo μ and then use the fact that \mathbb{A}_{inf} is μ -adically complete. To describe $\ker(1 - \psi_i)$, note that we have $\mu^i \widehat{\mathbf{Z}}_p \subset \ker(1 - \psi_i)$. It suffices thus to show that $\ker(1 - \psi_i) \subset \mu^i \widehat{\mathbf{Z}}_p$. But by part a) we have

$$\ker(\mathbb{A}_{\text{inf}} \xrightarrow{1 - \psi_i} \mathbb{A}_{\text{inf}}) \xleftarrow{\sim} \mu^i \ker(\mathbb{A}_{\text{inf}} \xrightarrow{1 - \varphi^{-1}} \mathbb{A}_{\text{inf}}) \xleftarrow{\sim} \mu^i \widehat{\mathbf{Z}}_p.$$

The last quasi-isomorphism follows from the Artin-Schreier exact sequence [18, Lemma 3.5 (ii)]

$$0 \rightarrow \widehat{\mathbf{Z}}_p \rightarrow \mathbb{A}_{\text{inf}} \xrightarrow{1 - \varphi^{-1}} \mathbb{A}_{\text{inf}} \rightarrow 0.$$

This finishes the proof of the lemma. \square

Write $U^{\psi_i=1}$ for the homotopy fiber of $\psi_i - 1 : U \rightarrow U$ for $U \in \{A\Omega_{\mathfrak{X}}, T\}$. The above lemma gives an exact triangle

$$R\nu_* \widehat{\mathbf{Z}}_p \xrightarrow{\mu^i} T \xrightarrow{\psi_i - 1} T,$$

inducing a quasi-isomorphism

$$\mu^i : \tau_{\leq i} R\nu_* \widehat{\mathbf{Z}}_p \xrightarrow{\sim} \tau_{\leq i} T^{\psi_i=1}.$$

To finish the proof of our theorem, it remains to show (and this is the hard part) that the natural map (induced by the natural maps $\text{can} : A\Omega_{\mathfrak{X}} \rightarrow T$ and $\tau_{\leq i} A\Omega_{\mathfrak{X}} \rightarrow A\Omega_{\mathfrak{X}}$)

$$\tau_{\leq i} (\tau_{\leq i} A\Omega_{\mathfrak{X}})^{\psi_i=1} \rightarrow \tau_{\leq i} T^{\psi_i=1}$$

is a quasi-isomorphism.

By easy homological algebra, this happens if ψ_i acts bijectively on the kernel and cokernel of $\text{can}_j : H^j(A\Omega_{\mathfrak{X}}) \rightarrow H^j(T)$ for $j < i$, bijectively on the kernel for $j = i$, and injectively on the cokernel for $j = i$. This is clear for $j = 0$: the map can_0 is then bijective as $H^0(A\Omega_{\mathfrak{X}}) = H^0(T)$ since $H^0(A\Omega_{\mathfrak{X}}) \simeq H^0(T)/H^0(T)[\mu]$ and $H^0(T)[\mu] = 0$ as \mathbb{A}_{inf} is μ -torsion-free.

Assume now that $j > 0$. For $i \geq 0$, write $M_i = H^i(T)$. Recall [5, Lemma 6.4] that the map $\mu^j : M_j/M_j[\mu] \rightarrow H^j(A\Omega_{\mathfrak{X}})$ is an isomorphism. It follows that, for $0 < j \leq i$, the canonical map can_j fits into a natural exact sequence

$$0 \rightarrow M_j[\mu] \rightarrow M_j[\mu^j] \rightarrow H^j(A\Omega_{\mathfrak{X}}) \xrightarrow{\text{can}_j} M_j \rightarrow M_j/\mu^j \rightarrow 0.$$

This sequence is compatible with the operators $\psi_{i-j}, \psi_{i-j}, \psi_i, \psi_i, \psi_i$, respectively. Thus it suffices to show that ψ_{i-j} is bijective on $M_j[\mu^j]/M_j[\mu]$, that ψ_i is bijective on M_j/μ^j for $j < i$, and is injective for $j = i$. This follows from Lemma 2.12 below. \square

Lemma 2.12. ([6, Lemma 10.5]) *Let $j \geq 1, i \geq 0$.*

- ψ_{l+j} is bijective on M_i/μ^j for $l > 0$ and is injective for $l = 0$.
- ψ_l is bijective on $M_i[\mu^j]$ for $l > 0$, surjective for $l = 0$.
- ψ_l is bijective on $M_i[\mu^j]/M_i[\mu]$, for $l \geq 0$.

Proof. We prove first that ψ_l is injective on $M_i[\mu^j]$ for $l > 0$. If $\psi_l(x) = 0$ and $\mu^j x = 0$, then $\psi_{l+1}(\mu x) = 0$ and $\mu x \in M_i[\mu^{j-1}]$, thus, arguing by induction on j , we may assume that $j = 1$. Suppose that $\mu x = 0$ and $\psi_l(x) = 0$, i.e., $x - \xi^l \varphi^{-1}(x) = 0$. Since $\xi \equiv p \pmod{\varphi^{-1}(\mu)}$ in \mathbb{A}_{inf} and $\varphi^{-1}(\mu)$ kills $\varphi^{-1}(x)$, we deduce that $(1 - p\xi^{l-1}\varphi^{-1})(x) = 0$, which forces $x = 0$, since $1 - p\xi^{l-1}\varphi^{-1}$ is an automorphism of \mathbb{A}_{inf} (as \mathbb{A}_{inf} is p -adically complete), thus also of $T = \text{R}\nu_*\mathbb{A}_{\text{inf}}$ and M_i . This proves the first step.

Next, the commutative diagram of distinguished triangles

$$\begin{array}{ccccc} T & \xrightarrow{\mu^j} & T & \longrightarrow & T/\mu^j \\ \downarrow \psi_l & & \downarrow \psi_{l+j} & & \downarrow \psi_{l+j} \\ T & \xrightarrow{\mu^j} & T & \longrightarrow & T/\mu^j \end{array}$$

gives a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & M_i/\mu^j & \longrightarrow & H^i(T/\mu^j) & \longrightarrow & M_{i+1}[\mu^j] \longrightarrow 0 \\ & & \downarrow \psi_{l+j} & & \downarrow \psi_{l+j} & & \downarrow \psi_l \\ 0 & \longrightarrow & M_i/\mu^j & \longrightarrow & H^i(T/\mu^j) & \longrightarrow & M_{i+1}[\mu^j] \longrightarrow 0 \end{array}$$

Since ψ_{l+j} is bijective on $H^i(T/\mu^j)$ (Lemma 2.11 shows that ψ_{l+j} is an automorphism of T/μ^j), we deduce that ψ_{l+j} is injective on M_i/μ^j , ψ_l is surjective on $M_{i+1}[\mu^j]$ and the cokernel of ψ_{l+j} on M_i/μ^j identifies with the kernel of ψ_l on $M_{i+1}[\mu^j]$. This last kernel is 0 for $l > 0$ (by the first step), thus ψ_l is bijective on $M_{i+1}[\mu^j]$ (this holds trivially on $M_0[\mu^j] = 0$) and ψ_{l+j} is bijective on M_i/μ^j for $l > 0$.

Finally, we need to show that ψ_l is an automorphism of $M_i[\mu^j]/M_i[\mu]$. Surjectivity follows from that of ψ_l on $M_i[\mu^j]$. For injectivity, note that if $\mu\psi_l(x) = 0$, then $\psi_{l+1}(\mu x) = 0$ and, since ψ_{l+1} is injective on $M_i[\mu^{j-1}]$, we obtain $x \in M_i[\mu]$, as needed. \square

Using the modified Tate twists we can write the statement of the above theorem in the following way:

Corollary 2.13. *Let X be a smooth adic space over C with a flat formal model \mathfrak{X} . Let $i \geq 0$. There is a natural quasi-isomorphism*

$$\gamma : \tau_{\leq i} \text{R}\nu_* \widehat{\mathbf{Z}}_p(i) \xrightarrow{\sim} \tau_{\leq i} [\tau_{\leq i} A\Omega_{\mathfrak{X}}\{i\}] \xrightarrow{1-\varphi^{-1}} \tau_{\leq i} A\Omega_{\mathfrak{X}}\{i\}.$$

It is Galois equivariant in the case X is defined over K .

Proof. Immediate from Theorem 2.9 and the following commutative diagram (which also defines the operator φ^{-1} on $\tau_{\leq i} A\Omega_{\mathfrak{X}}\{i\}$ in the corollary):

$$\begin{array}{ccccc} \tau_{\leq i} R\nu_* \widehat{\mathbf{Z}}_p & \xrightarrow{\mu^i} & \tau_{\leq i} A\Omega_{\mathfrak{X}} & \xrightarrow{1-\xi^i \varphi^{-1}} & \tau_{\leq i} A\Omega_{\mathfrak{X}} \\ \parallel & & \downarrow \mu^{-i} & & \downarrow \mu^{-i} \\ \tau_{\leq i} R\nu_* \widehat{\mathbf{Z}}_p(i) & \xrightarrow{\gamma=\text{can}} & \tau_{\leq i} A\Omega_{\mathfrak{X}}\{i\} & \xrightarrow{1-\varphi^{-1}} & \tau_{\leq i} A\Omega_{\mathfrak{X}}\{i\} \end{array}$$

□

3. A_{inf} -SYMBOL MAPS

Let X be a smooth adic space over C and let \mathfrak{X} be a flat p -adic formal model of X over \mathcal{O}_C . Let $\nu : X_{\text{proét}} \rightarrow \mathfrak{X}_{\text{ét}}$ be the natural map.

3.1. The construction of symbol maps. We will define compatible continuous pro-étale and A_{inf} -symbol maps⁵

$$(3.1) \quad r_{\text{proét}} : \mathcal{O}(X)^{*, \otimes i} \rightarrow H_{\text{proét}}^i(X, \widehat{\mathbf{Z}}_p(i)), \quad r_{\text{inf}} : \mathcal{O}(X)^{*, \otimes i} \rightarrow H_{\text{ét}}^i(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\}).$$

For $i = 1$, we will construct below compatible maps of sheaves

$$(3.2) \quad c_1^{\text{proét}} : \tau_{\leq 1}(R\nu_* \mathbb{G}_m[-1]) \rightarrow \tau_{\leq 1} \widehat{\mathbf{Z}}_p(1), \quad c_1^{\text{inf}} : \tau_{\leq 1}(R\nu_* \mathbb{G}_m[-1]) \rightarrow \tau_{\leq 1} A\Omega_{\mathfrak{X}}\{1\}.$$

Applying $H_{\text{ét}}^1(\mathfrak{X}, -)$ and observing that

$$H_{\text{ét}}^1(\mathfrak{X}, \tau_{\leq 1}(R\nu_* \mathbb{G}_m[-1])) \xrightarrow{\sim} H_{\text{ét}}^1(\mathfrak{X}, R\nu_* \mathbb{G}_m[-1]) = H_{\text{ét}}^0(\mathfrak{X}, R\nu_* \mathbb{G}_m) \simeq \mathcal{O}(X)^*,$$

we get that the maps $c_1^{\text{proét}}, c_1^{\text{inf}}$ induce global symbol maps

$$r_{\text{proét}} : \mathcal{O}(X)^* \rightarrow H_{\text{proét}}^1(X, \widehat{\mathbf{Z}}_p(1)), \quad r_{\text{inf}} : \mathcal{O}(X)^* \rightarrow H_{\text{ét}}^1(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{1\}).$$

For $i \geq 1$, we define the symbol maps (3.1) using cup product: $x_1 \otimes \cdots \otimes x_i \mapsto r_*(x_1) \cup \cdots \cup r_*(x_i)$.

The construction of the first map in (3.2) uses the Kummer exact sequence on $X_{\text{proét}}$

$$0 \rightarrow \widehat{\mathbf{Z}}_p(1) \rightarrow \varprojlim_{x \mapsto x^p} \mathbb{G}_m \rightarrow \mathbb{G}_m \rightarrow 0,$$

inducing, by projection to $\mathfrak{X}_{\text{ét}}$, the Chern class map

$$c_1^{\text{proét}} : R\nu_* \mathbb{G}_m[-1] \rightarrow R\nu_* \widehat{\mathbf{Z}}_p(1).$$

The construction of the second map in (3.2) uses the above Kummer exact sequence and the *twisted* Artin-Schreier exact sequence⁶ on $X_{\text{proét}}$

$$0 \rightarrow \widehat{\mathbf{Z}}_p(1) \xrightarrow{\gamma} \mathbb{A}_{\text{inf}}\{1\} \xrightarrow{1-\varphi^{-1}} \mathbb{A}_{\text{inf}}\{1\} \rightarrow 0,$$

where the map γ is defined by $x(1) \mapsto \mu x\{1\}$, $x \in \widehat{\mathbf{Z}}_p$. Pushing down to $\mathfrak{X}_{\text{ét}}$ we obtain a map

$$\gamma : \tau_{\leq 1} R\nu_* \widehat{\mathbf{Z}}_p(1) \rightarrow \tau_{\leq 1} R\nu_* \mathbb{A}_{\text{inf}}\{1\}.$$

⁵We refer the reader to [9, Sec. 2.2] for a discussion of topology on cohomologies of rigid analytic varieties and formal schemes. Integrally, we work in the category of pro-discrete modules, rationally – in the category of locally convex topological vector spaces over \mathbf{Q}_p . But, in this paper, we work with the naive topology on cohomology groups, i.e., the quotient topology, as opposed to the refined cohomology groups (denoted \tilde{H} in [9]) taken in the derived category of pro-discrete modules.

⁶Note that, for $x \in \mathbb{A}_{\text{inf}}$,

$$(1 - \varphi^{-1})(x\{1\}) = [(1 - \xi\varphi^{-1})(x)]\{1\}$$

and this is 0 precisely when $x = \mu y$ with $y \in \widehat{\mathbf{Z}}_p$.

On the other hand, Corollary 2.13 gives us a natural map

$$\gamma : \tau_{\leq 1} R\nu_* \widehat{\mathbf{Z}}_p(1) \rightarrow \tau_{\leq 1} A\Omega_{\mathfrak{X}}\{1\}.$$

It is easy to check that the above two maps are compatible via the canonical map $\tau_{\leq 1} A\Omega_{\mathfrak{X}}\{1\} \rightarrow \tau_{\leq 1} R\nu_* \mathbb{A}_{\text{inf}}\{1\}$ and that the composition

$$\tau_{\leq 1} R\nu_* \widehat{\mathbf{Z}}_p(1) \xrightarrow{\gamma} \tau_{\leq 1} A\Omega_{\mathfrak{X}}\{1\} \xrightarrow{1-\varphi^{-1}} \tau_{\leq 1} A\Omega_{\mathfrak{X}}\{1\}$$

is the zero map.

Now, our symbol map is simply the composition

$$c_1^{\text{inf}} : \tau_{\leq 1} R\nu_* \mathbb{G}_m[-1] \xrightarrow{c_1^{\text{proét}}} \tau_{\leq 1} R\nu_* \widehat{\mathbf{Z}}_p(1) \xrightarrow{\gamma} \tau_{\leq 1} A\Omega_{\mathfrak{X}}\{1\}.$$

3.2. Compatibility with the Hodge-Tate symbol map. Let $\mathfrak{X}_{\mathcal{O}_K}$ be a semistable formal schemes over \mathcal{O}_K . Let M be the sheaf of monoids on $\mathfrak{X}_{\mathcal{O}_K}$ defining the log-structure, M^{gp} its group completion. This log-structure is canonical, in the terminology of Berkovich [2, 2.3], i.e., $M(U) = \{x \in \mathcal{O}_{\mathfrak{X}_{\mathcal{O}_K}}(U) \mid x_K \in \mathcal{O}_{\mathfrak{X}_K}^*(U_K)\}$. This is shown in [2, Th. 2.3.1], [1, Th. 5.3] and applies also to semistable formal schemes with self-intersections. It follows that $M^{\text{gp}}(U) = \mathcal{O}_{\mathfrak{X}_K}^*(U_K)$. Set $X_K := \mathfrak{X}_{\mathcal{O}_K, K}$, $\mathfrak{X} := \mathfrak{X}_{\mathcal{O}_C}$, $X := X_{K, C}$.

For $i \geq 1$, the Hodge-Tate symbol maps

$$r_{\text{HT}} : \mathcal{O}(X_K)^{*, \otimes i} \rightarrow H_{\text{ét}}^0(\mathfrak{X}, \Omega_{\mathfrak{X}}^i)$$

are defined by taking cup products of the Chern class maps

$$c_1^{\text{HT}} : \mathcal{O}(X_K)^* \rightarrow \tau_{\leq 1}(R\nu_* \mathbb{G}_m[-1]) \rightarrow \Omega_{\mathfrak{X}}[-1], \quad x \mapsto \text{dlog}(x).$$

The purpose of this section is to prove the following fact:

Proposition 3.3. *Let $i \geq 1$. The symbol maps r_{inf} and r_{HT} are compatible under the Hodge-Tate specialization map ι_{HT} , i.e., $\iota_{\text{HT}} r_{\text{inf}} | \mathcal{O}(X_K)^{*, \otimes i} = r_{\text{HT}}$.*

Proof. The case $i = 1$. Consider the composition $\iota_{\text{HT}} c_1^{\text{inf}}$:

$$\mathcal{O}(X_K)^* \rightarrow \mathcal{O}(X)^* \simeq H_{\text{ét}}^1(\mathfrak{X}, \tau_{\leq 1}(R\nu_* \mathbb{G}_m[-1])) \xrightarrow{c_1^{\text{inf}}} H_{\text{ét}}^1(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{1\}) \xrightarrow{\iota_{\text{HT}}} H_{\text{ét}}^0(\mathfrak{X}, \Omega_{\mathfrak{X}}^1).$$

We need to show that

Lemma 3.4. *The above composition $\iota_{\text{HT}} c_1^{\text{inf}}$ is equal to the map $c_1^{\text{HT}} = \text{dlog} : \mathcal{O}(X_K)^* \rightarrow H_{\text{ét}}^0(\mathfrak{X}, \Omega_{\mathfrak{X}}^1)$.*

Proof. From the definitions, it suffices to check that the composition

$$\mathcal{O}_X^* \longrightarrow R^1 \nu_* \widehat{\mathbf{Z}}_p(1) \xrightarrow{\gamma} H^1(A\Omega_{\mathfrak{X}}\{1\}) \xrightarrow{\tilde{\theta}} H^1(\tilde{\Omega}_{\mathfrak{X}}\{1\}) \simeq \Omega_{\mathfrak{X}}^1$$

is the map dlog . To do this we will study the diagram

$$\begin{array}{ccccccc} & & \frac{R^1 \nu_* \mathbb{A}_{\text{inf}}\{1\}}{R^1 \nu_* \mathbb{A}_{\text{inf}}[\mu]\{1\}} & \xrightarrow{\tilde{\theta}} & \frac{R^1 \nu_* \widehat{\mathcal{O}}^+\{1\}}{R^1 \nu_* \widehat{\mathcal{O}}^+[\zeta_p-1]\{1\}} & & \\ & & \downarrow \gamma_1 \wr & & \downarrow \gamma_2 \wr & & \\ \mathcal{O}_X^* & \xrightarrow{c_1^{\text{proét}}} & R^1 \nu_* \widehat{\mathbf{Z}}_p(1) & \xrightarrow{\gamma} & H^1(A\Omega_{\mathfrak{X}}\{1\}) & \xrightarrow{\tilde{\theta}} & H^1(\tilde{\Omega}_{\mathfrak{X}}\{1\}) \xrightarrow{\sim} \Omega_{\mathfrak{X}}^1 \\ & \searrow \beta & \downarrow \text{can} & & \downarrow \text{can} & & \downarrow \iota \\ & & R^1 \nu_* \mathbb{A}_{\text{inf}}\{1\} & \xrightarrow{\tilde{\theta}} & R^1 \nu_* \widehat{\mathcal{O}}^+\{1\} & \longrightarrow & R^1 \nu_* \widehat{\mathcal{O}}(1) \xrightarrow{\sim} \varepsilon_* \Omega_X^1 \end{array}$$

Here $\varepsilon : X_{\text{ét}} \rightarrow \mathfrak{X}_{\text{ét}}$ is the canonical map. A few words about this diagram:

- the map $R^1 \nu_* \widehat{\mathcal{O}}^+\{1\} \rightarrow R^1 \nu_* \widehat{\mathcal{O}}(1)$ is induced by the natural map $\widehat{\mathcal{O}}^+ \rightarrow \widehat{\mathcal{O}}$ and the map $\mathcal{O}_C\{1\} \rightarrow C(1)$ is given by $x\{1\} \rightarrow \frac{x}{\zeta_p-1}(1)$.

- let $\nu_X : X_{\text{proét}} \rightarrow X_{\text{ét}}$ be the canonical projection. There is an isomorphism

$$(3.5) \quad \alpha_2 : \mathbb{R}^1 \nu_{X,*} \widehat{\mathcal{O}}(1) \xrightarrow{\sim} \Omega_X^1$$

due to Scholze ([21, Lemma 3.24]). It is uniquely characterized by the property that its inverse is the unique \mathcal{O}_X -linear map $\alpha_2^{-1} : \Omega_X^1 \rightarrow \mathbb{R}^1 \nu_{X,*} \widehat{\mathcal{O}}(1)$ making the following diagram commute

$$(3.6) \quad \begin{array}{ccc} \mathcal{O}_X^* & \xrightarrow{c_1^{\text{proét}}} & \mathbb{R}^1 \nu_{X,*} \widehat{\mathbf{Z}}_p(1) , \\ \text{dlog} \downarrow & & \downarrow \\ \Omega_X^1 & \xrightarrow{\alpha_2^{-1}} & \mathbb{R}^1 \nu_{X,*} \widehat{\mathcal{O}}(1) \end{array}$$

the right vertical map being the obvious one.

- The map $\alpha_2 : \mathbb{R}^1 \nu_* \widehat{\mathcal{O}}(1) \rightarrow \varepsilon_* \Omega_X^1$ is defined as the composition

$$(3.7) \quad \mathbb{R}^1 \nu_* \widehat{\mathcal{O}}(1) \simeq \mathbb{R}^1 (\varepsilon \nu_X)_* \widehat{\mathcal{O}}(1) \xrightarrow{\sim} \varepsilon_* \mathbb{R}^1 \nu_{X,*} \widehat{\mathcal{O}}(1) \simeq \varepsilon_* \Omega_X^1,$$

where the last map is the isomorphism (3.5). Recall that this last isomorphism generalizes to isomorphisms [21, Prop. 3.23]:

$$\mathbb{R}^i \nu_{X,*} \widehat{\mathcal{O}}(i) \simeq \Omega_X^i, \quad i \geq 0.$$

This combined with Tate's acyclicity theorem allows us to infer that the second map in (3.7) is an isomorphism.

We claim that the composite

$$\mathcal{O}_X^* \rightarrow \mathbb{R}^1 \nu_{X,*} \widehat{\mathbf{Z}}_p(1) \rightarrow \mathbb{R}^1 \nu_{X,*} \mathbb{A}_{\text{inf}}\{1\} \rightarrow \mathbb{R}^1 \nu_{X,*} \widehat{\mathcal{O}}^+\{1\} \rightarrow \mathbb{R}^1 \nu_{X,*} \widehat{\mathcal{O}}(1) \rightarrow \Omega_X^1$$

is the dlog map. Using the characterization of Scholze's isomorphism (3.5), this comes down to checking that the map $\mathbb{R}^1 \nu_{X,*} \widehat{\mathbf{Z}}_p(1) \rightarrow \mathbb{R}^1 \nu_{X,*} \mathbb{A}_{\text{inf}}\{1\} \rightarrow \mathbb{R}^1 \nu_{X,*} \widehat{\mathcal{O}}^+\{1\} \rightarrow \mathbb{R}^1 \nu_{X,*} \widehat{\mathcal{O}}(1)$ is the obvious one. But, by construction, this map is induced by the map

$$\widehat{\mathbf{Z}}_p(1) \rightarrow \mathbb{A}_{\text{inf}}\{1\} \rightarrow \widehat{\mathcal{O}}^+\{1\} \rightarrow \widehat{\mathcal{O}}(1)$$

sending $x(1)$ to $x\mu\{1\}$, then to $\tilde{\theta}(x\mu\{1\}) = x(\zeta_p - 1)\{1\}$, then to $x(1)$, as wanted.

To finish the proof of the lemma, it remains to check that the above big diagram commutes. The only nonobvious commutativity is that of the right-bottom trapezoid, i.e., we need to check the compatibility of the maps α_1 and α_2 . Call

$$(3.8) \quad \rho : \Omega_{\mathfrak{X}}^1 \xrightarrow{\alpha_1^{-1}} H^1(\widetilde{\Omega}_{\mathfrak{X}}\{1\}) \xrightarrow{\text{can}} \mathbb{R}^1 \nu_* \widehat{\mathcal{O}}^+\{1\} \rightarrow \mathbb{R}^1 \nu_* \widehat{\mathcal{O}}(1) \xrightarrow{\alpha_2} \varepsilon_* \Omega_X^1.$$

We want to show that $\rho = \iota$. It suffices to check this on the smooth locus of \mathfrak{X} , which reduces us to the case when \mathfrak{X} is smooth. Note that the maps ρ, ι are $\mathcal{O}_{\mathfrak{X}}$ -linear. This is clear for ι ; for ρ we look at the individual maps in the composition (3.8) that defines it: the second and the third map are clearly $\mathcal{O}_{\mathfrak{X}}$ -linear, for the first map we use Theorem 2.5, and for the last map we use the discussion above. Now, the claim that $\rho = \iota$ is local, so we may assume that \mathfrak{X} is associated to a small algebra R with a framing $A = \mathcal{O}_C\{T_i^{\pm 1}\} \rightarrow R$. By functoriality, we may reduce to the case when $R = A$ and $A = \mathcal{O}_C\{T^{\pm 1}\}$.

Now, the desired compatibility follows from the very construction of the isomorphism α_1 . More precisely, since $\text{can} \circ \gamma_2$ is the multiplication by $\zeta_p - 1$, we have

$$(\zeta_p - 1)\gamma_2^{-1}(\alpha_1^{-1}(dT/T)) = \text{can}(\alpha_1^{-1}(dT/T)).$$

As we have already seen (cf. discussion after Theorem 2.4) this corresponds to the element $(\gamma \mapsto (\zeta_p - 1) \otimes \frac{1}{\zeta_p - 1} \text{dlog}(\zeta_\gamma))$ in $(\zeta_p - 1)H^1(\Gamma, A_\infty)\{1\}$. Now the compatibility of the map α_2 with the Kummer map (see the diagram (3.6)) shows that $\rho(dT/T) = dT/T$, as wanted. \square

The case $i \geq 1$. Take now the symbol maps

$$r_{\text{inf}} : \mathcal{O}(X)^{*, \otimes i} \rightarrow H_{\text{ét}}^i(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\})$$

and consider the composition $\iota_{\text{HT}} r_{\text{inf}}$:

$$\mathcal{O}(X)^{*, \otimes i} \rightarrow H_{\text{ét}}^i(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\}) \rightarrow H_{\text{ét}}^i(\mathfrak{X}, \tilde{\Omega}_{\mathfrak{X}}\{i\}) \rightarrow H_{\text{ét}}^0(\mathfrak{X}, H^i(\tilde{\Omega}_{\mathfrak{X}}\{i\})) \simeq H_{\text{ét}}^0(\mathfrak{X}, \Omega_{\mathfrak{X}}^i).$$

To finish the proof of our proposition, in view of Lemma 3.4, it suffices to check that this composition is compatible with products. But, the third map in the composition is clearly compatible with products and the first map is compatible with products by definition. The second map is induced by the map $\tilde{\theta} : A\Omega_{\mathfrak{X}} \rightarrow \tilde{\Omega}_{\mathfrak{X}}$ hence it is also compatible with products. Finally, the last map is the isomorphism given by Theorem 2.5 hence is compatible with products by its very definition. \square

4. THE A_{inf} -COHOMOLOGY OF DRINFELD SYMMETRIC SPACES

Let $\mathcal{H} = \mathbb{P}((K^{d+1})^*) \simeq \mathbb{P}^d(K)$ be the space of K -rational hyperplanes in K^{d+1} . Let

$$\mathbb{H}_K^d := \mathbb{P}_K^d \setminus \cup_{H \in \mathcal{H}} H$$

be the Drinfeld symmetric space of dimension d . It is a rigid analytic space. Let $\mathfrak{X}_{\mathcal{O}_K}$ be the standard semistable formal model over \mathcal{O}_K of \mathbb{H}_K^d (see [14, Section 6.1]). Let $\mathfrak{X} := \mathfrak{X}_{\mathcal{O}_K} \widehat{\otimes}_{\mathcal{O}_K} \mathcal{O}_C$, let $X := \mathbb{H}_K^d \widehat{\otimes}_K C$ be the rigid analytic generic fiber of \mathfrak{X} , and let $X_K = \mathbb{H}_K^d$. The group $G = \text{GL}_{d+1}(K)$ acts naturally on all these objects.

The main goal of this section is to prove the following (here and elsewhere in the paper, the completed tensor product is taken in the category of pro-discrete modules):

Theorem 4.1. *Let $i \geq 0$. There is a $G \times \mathcal{G}_K$ -equivariant isomorphism of topological A_{inf} -modules*

$$A_{\text{inf}} \widehat{\otimes}_{\mathbf{Z}_p} \text{Sp}_i(\mathbf{Z}_p)^* \simeq H_{\text{ét}}^i(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\}),$$

where $\text{Sp}_i(\mathbf{Z}_p)^*$ is the \mathbf{Z}_p -dual of a generalized Steinberg representation (see Section 4.1 for a definition). This isomorphism is compatible with the operator φ^{-1} .

4.1. Generalized Steinberg representations and their duals.

4.1.1. *Generalized Steinberg representations.* Let B be the upper triangular Borel subgroup of G and $\Delta = \{1, 2, \dots, d\}$, identified with the set of simple roots associated to B . For each subset J of Δ we let P_J be the corresponding parabolic subgroup of G and set $X_J = G/P_J$, a compact topological space.

If A is an abelian group and $J \subset \Delta$, let

$$\text{Sp}_J(A) = \frac{\text{LC}(X_J, A)}{\sum_{i \in \Delta \setminus J} \text{LC}(X_{J \cup \{i\}}, A)},$$

where LC means locally constant (automatically with compact support). This is a smooth G -module over A and we have a canonical isomorphism $\text{Sp}_J(A) \simeq \text{Sp}_J(\mathbf{Z}) \otimes A$. For $J = \emptyset$ we obtain the usual Steinberg representation with coefficients in A , while for $J = \Delta$ we have $\text{Sp}_J(A) = A$. For $r \in \{0, 1, \dots, d\}$ we use the simpler notation

$$\text{Sp}_r := \text{Sp}_{\{1, 2, \dots, d-r\}}.$$

We will need the following result:

Theorem 4.2. (Grosse-Klönne, [16, Cor. 4.3]) *If A is a field of characteristic p then $\text{Sp}_J(A)$ (for varying J) are the irreducible constituents of $\text{LC}(G/B, A)$, each occurring with multiplicity 1.*

4.1.2. *Duals of generalized Steinberg representations.* If Λ is a topological ring, then $\mathrm{Sp}_J(\Lambda)$ has a natural topology: the space X_J being profinite, we can write $X_J = \varprojlim_n X_{n,J}$ for finite sets $X_{n,J}$ and then $\mathrm{LC}(X_J, \Lambda) = \varinjlim_n \mathrm{LC}(X_{n,J}, \Lambda)$, each $\mathrm{LC}(X_{n,J}, \Lambda)$ being a finite free Λ -module endowed with the natural topology.

Let $M^* := \mathrm{Hom}_{\mathrm{cont}}(M, \Lambda)$ for any topological Λ -module M , and equip M^* with the weak topology. Then $\mathrm{Sp}_J(\Lambda)^*$ is naturally isomorphic to $\varprojlim_n \mathrm{LC}(X_{n,J}, \Lambda)^*$, i.e. a countable inverse limit of finite free Λ -modules. In particular, suppose that L is a finite extension of \mathbf{Q}_p . Then $\mathrm{Sp}_J(\mathcal{O}_L)^*$ is a compact \mathcal{O}_L -module, which is torsion-free.

If S is a profinite set and A an abelian group, let

$$D(S, A) = \mathrm{Hom}(\mathrm{LC}(S, \mathbf{Z}), A) = \mathrm{LC}(S, A)^*$$

be the space of A -valued locally constant distributions on S . We recall the interpretation of $\mathrm{Sp}_i(\mathbf{Z}_p)^*$ in terms of distributions. Recall that \mathcal{H} denotes the compact space of K -rational hyperplanes in K^{d+1} . If $H \in \mathcal{H}$, let ℓ_H be a unimodular equation for H (thus ℓ_H is a linear form with integer coefficients, at least one of them being a unit). Let $\mathrm{LC}^c(\mathcal{H}^{i+1}, \mathbf{Z})$ be the space of locally constant functions $f : \mathcal{H}^{i+1} \rightarrow \mathbf{Z}$ such that, for all $H_0, \dots, H_{i+1} \in \mathcal{H}$,

$$f(H_1, \dots, H_{i+1}) - f(H_0, H_2, \dots, H_{i+1}) + \dots + (-1)^{i+1} f(H_0, \dots, H_i) = 0$$

and, moreover, if ℓ_{H_j} , $0 \leq j \leq i$, are linearly dependent, then $f(H_0, \dots, H_i) = 0$. The work of Schneider-Stuhler [19] gives a G -equivariant isomorphism

$$\mathrm{Sp}_i(\mathbf{Z}) \simeq \mathrm{LC}^c(\mathcal{H}^{i+1}, \mathbf{Z}).$$

It follows that the inclusion $\mathrm{LC}^c(\mathcal{H}^{i+1}, \mathbf{Z}) \subset \mathrm{LC}(\mathcal{H}^{i+1}, \mathbf{Z})$ gives rise to a strict exact sequence

$$(4.3) \quad 0 \rightarrow D(\mathcal{H}^{i+1}, A)_{\mathrm{deg}} \rightarrow D(\mathcal{H}^{i+1}, A) \rightarrow \mathrm{Hom}(\mathrm{Sp}_i(\mathbf{Z}), A) \rightarrow 0,$$

where $D(\mathcal{H}^{i+1}, A)_{\mathrm{deg}}$ is the space of degenerate distributions (which is defined via the exact sequence above).

4.2. **Integral de Rham cohomology of Drinfeld symmetric spaces.** Recall the following acyclicity result of Grosse-Klönne, which played a crucial role in [9].

Theorem 4.4. (Grosse-Klönne, [13, Th. 4.5], [15, Prop. 4.5]) *For $i > 0$, $j \geq 0$, we have $H_{\mathrm{ét}}^i(\mathfrak{X}, \Omega_{\mathfrak{X}}^j) = 0$ and $d = 0$ on $H_{\mathrm{ét}}^0(\mathfrak{X}, \Omega_{\mathfrak{X}}^j)$. In particular, we have a natural quasi-isomorphism*

$$\mathrm{R}\Gamma_{\mathrm{dR}}(\mathfrak{X}) \simeq \mathrm{R}\Gamma_{\mathrm{ét}}(\mathfrak{X}, \Omega_{\mathfrak{X}}^\bullet) \simeq \bigoplus_{i \geq 0} \Gamma_{\mathrm{ét}}(\mathfrak{X}, \Omega_{\mathfrak{X}}^i)[-i].$$

Using it and some extra work, we have obtained the following description of $H_{\mathrm{dR}}^i(\mathfrak{X}_{\mathcal{O}_K})$:

Theorem 4.5. (Colmez-Dospinescu-Nizioł, [9, Th. 6.26]) *There are natural de Rham and Hodge-Tate regulator maps*

$$\begin{aligned} r_{\mathrm{dR}} &: D(\mathcal{H}^{i+1}, \mathcal{O}_K) \rightarrow H_{\mathrm{dR}}^i(\mathfrak{X}_{\mathcal{O}_K}), \\ r_{\mathrm{HT}} &: D(\mathcal{H}^{i+1}, \mathcal{O}_K) \rightarrow H_{\mathrm{HT}}^0(\mathfrak{X}_{\mathcal{O}_K}, \Omega_{\mathfrak{X}_{\mathcal{O}_K}}^i) \end{aligned}$$

that induce topological $G \times \mathcal{G}_K$ -equivariant isomorphisms in the commutative diagram:

$$\begin{array}{ccc} \mathrm{Sp}_i(\mathcal{O}_K)^* & \xrightarrow{\sim} & H_{\mathrm{dR}}^i(\mathfrak{X}_{\mathcal{O}_K}) \\ & \searrow \sim & \uparrow \wr \\ & & H^0(\mathfrak{X}_{\mathcal{O}_K}, \Omega_{\mathfrak{X}_{\mathcal{O}_K}}^i) \end{array}$$

r_{HT}

Proof. (Sketch) Our starting point was the computation of Schneider-Stuhler [19]: a G -equivariant topological isomorphism

$$\mathrm{Sp}_r(K)^* \xrightarrow[\sim]{\alpha_S} H_{\mathrm{dR}}^i(X_K).$$

Iovita-Spiess [17] made this isomorphism explicit: they have proved that there is a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & D(\mathcal{H}^{i+1}, K)_{\mathrm{deg}} & \longrightarrow & D(\mathcal{H}^{i+1}, K) & \longrightarrow & \mathrm{Sp}_i(K)^* \longrightarrow 0 \\ & & & & \searrow^{r_{\mathrm{dR}}} & & \downarrow \alpha_S \\ & & & & & & H_{\mathrm{dR}}^i(X_K) \end{array}$$

With a help from a detailed analysis of the integral Hyodo-Kato cohomology of the special fiber of $\mathfrak{X}_{\mathcal{O}_K}$ and some representation theory⁷ this computation can be lifted to \mathcal{O}_K . \square

The following computation follows immediately:

Corollary 4.6. (1) *The de Rham regulator r_{dR} induces a topological G -equivariant isomorphism*

$$r_{\mathrm{dR}} : \mathrm{Sp}_i(\mathcal{O}_K)^* \widehat{\otimes}_{\mathcal{O}_K} \mathcal{O}_C \xrightarrow{\sim} H_{\mathrm{dR}}^i(\mathfrak{X}_{\mathcal{O}_K}) \widehat{\otimes}_{\mathcal{O}_K} \mathcal{O}_C \xrightarrow{\sim} H_{\mathrm{dR}}^i(\mathfrak{X}).$$

(2) *The Hodge-Tate regulator r_{HT} induces a topological G -equivariant isomorphism*

$$r_{\mathrm{HT}} : \mathrm{Sp}_i(\mathcal{O}_K)^* \widehat{\otimes}_{\mathcal{O}_K} \mathcal{O}_C \xrightarrow{\sim} H_{\mathrm{ét}}^0(\mathfrak{X}_{\mathcal{O}_K}, \Omega_{\mathfrak{X}_{\mathcal{O}_K}}^i) \widehat{\otimes}_{\mathcal{O}_K} \mathcal{O}_C \xrightarrow{\sim} H_{\mathrm{ét}}^0(\mathfrak{X}, \Omega_{\mathfrak{X}}^i).$$

4.3. Integrating symbols. Let $i \geq 1$. In this section, our goal is to construct natural compatible regulator maps

$$r_{\mathrm{ét}} : \mathrm{Sp}_i(\mathbf{Z}_p)^* \rightarrow H_{\mathrm{ét}}^i(X, \mathbf{Z}_p(i)), \quad r_{\mathrm{inf}} : \mathrm{Sp}_i(\mathbf{Z}_p)^* \rightarrow H_{\mathrm{ét}}^i(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\})$$

that are compatible with the classical étale and A_{inf} -regulators. We will show later that (the linearizations of) both regulators are $G \times \mathcal{G}_K$ -equivariant isomorphisms. The maps $r_{\mathrm{ét}}$, r_{inf} are constructed by interpreting elements of $\mathrm{Sp}_i(\mathbf{Z}_p)^*$ as suitable distributions (see the discussion in Section 4.1.2), and integrating étale and A_{inf} -symbols of invertible functions on \mathbb{H}_K^d against them. This idea appears in Iovita-Spiess [17] and was also heavily used in [9].

4.3.1. Integrating étale symbols. We start with the construction of the étale regulator map

$$r_{\mathrm{ét}} : \mathrm{Sp}_i(\mathbf{Z}_p)^* \rightarrow H_{\mathrm{ét}}^i(X, \mathbf{Z}_p(i)).$$

Fix a cohomological degree i and set $M := H_{\mathrm{ét}}^i(X, \mathbf{Z}_p(i))$. For $H_0, \dots, H_i \in \mathcal{H}$, let

$$\psi_{\mathrm{ét}}(H_0, \dots, H_i) := r_{\mathrm{ét}} \left(\frac{\ell_{H_1}}{\ell_{H_0}} \otimes \dots \otimes \frac{\ell_{H_i}}{\ell_{H_0}} \right) \in M,$$

where $r_{\mathrm{ét}} : \mathcal{O}(\mathbb{H}_C^d)^{*, \otimes i} \rightarrow M$ is the étale regulator map. It is clear that this definition is independent of the choice of the unimodular equations for H_0, \dots, H_i .

Proposition 4.7. *Let $i \geq 1$.*

(1) *Let δ_x denote the Dirac distribution at x . There is a unique continuous \mathbf{Z}_p -linear map*

$$r_{\mathrm{ét}} : D(\mathcal{H}^{i+1}, \mathbf{Z}_p) \rightarrow H_{\mathrm{ét}}^i(X, \mathbf{Z}_p(i))$$

such that $r_{\mathrm{ét}}(\delta_{(H_0, \dots, H_i)}) = \psi_{\mathrm{ét}}(H_0, \dots, H_i)$ for all $H_0, \dots, H_i \in \mathcal{H}$.

(2) *The map $r_{\mathrm{ét}}$ factors through the quotient $\mathrm{Sp}_i(\mathbf{Z}_p)^*$ of $D(\mathcal{H}^{i+1}, \mathbf{Z}_p)$ and induces a natural map of \mathbf{Z}_p -modules*

$$r_{\mathrm{ét}} : \mathrm{Sp}_i(\mathbf{Z}_p)^* \rightarrow H_{\mathrm{ét}}^i(X, \mathbf{Z}_p(i)).$$

⁷We used here two facts: (a) $\mathrm{Sp}_i(\mathcal{O}_K)$ is, up to a K^* -homothety, the unique G -stable lattice in $\mathrm{Sp}_i(K)$; (b) $\mathrm{Sp}_i(k)$ is irreducible.

Proof. Uniqueness in (1) is clear since the \mathbf{Z}_p -submodule of $D(\mathcal{H}^{i+1}, \mathbf{Z}_p)$ spanned by the Dirac distributions is dense.

Existence claim in (1) requires more work. Let $\{U_n\}_{n \geq 1}$ be the standard admissible affinoid covering of X . Let $\Pi(n)$ be the fundamental group of U_n . Denote by $\mathrm{R}\Gamma(\Pi(n), \mathbf{Z}_p(i))$ the complex of nonhomogenous continuous cochains representing the continuous group cohomology of $\Pi(n)$. By the $K(\pi, 1)$ -Theorem of Scholze [20, Th. 1.2] this complex also represents $\mathrm{R}\Gamma_{\acute{\mathrm{e}}\mathrm{t}}(U_n, \mathbf{Z}_p(i))$. Since the action of $\Pi(n)$ on $\mathbf{Z}_p(1)$ is trivial the local étale Chern class map factors as

$$c_{1,n}^{\acute{\mathrm{e}}\mathrm{t}} : \mathcal{O}(U_n)^* \rightarrow \mathrm{Hom}(\Pi(n), \mathbf{Z}_p(1)) \rightarrow \mathrm{R}\Gamma(\Pi(n), \mathbf{Z}_p(1))[1].$$

The global étale Chern class is represented by the composition

$$c_{1,n}^{\acute{\mathrm{e}}\mathrm{t}} : \mathcal{O}(X)^* \rightarrow \varprojlim_n \mathcal{O}(U_n)^* \rightarrow \mathrm{holim}_n \mathrm{Hom}(\Pi(n), \mathbf{Z}_p(1)) \rightarrow \mathrm{holim}_n \mathrm{R}\Gamma(\Pi(n), \mathbf{Z}_p(1))[1] \xleftarrow{\sim} \mathrm{R}\Gamma_{\acute{\mathrm{e}}\mathrm{t}}(X, \mathbf{Z}_p(1))[1].$$

The étale regulator $r_{\acute{\mathrm{e}}\mathrm{t}} : \mathcal{O}(X)^{*, \otimes i} \rightarrow \mathrm{R}\Gamma_{\acute{\mathrm{e}}\mathrm{t}}(X, \mathbf{Z}_p(i))[i]$ is then represented by the cup product: $r_{\acute{\mathrm{e}}\mathrm{t}} := c_1^{\acute{\mathrm{e}}\mathrm{t}} \cup \dots \cup c_1^{\acute{\mathrm{e}}\mathrm{t}}$.

The composition

$$(4.8) \quad \Psi_i : \mathcal{H}^{i+1} \rightarrow \mathcal{O}(X)^{*, \otimes i} \xrightarrow{r_{\acute{\mathrm{e}}\mathrm{t}}} \mathrm{R}\Gamma_{\acute{\mathrm{e}}\mathrm{t}}(X, \mathbf{Z}_p(i))[i]$$

represents the map $\psi_{\acute{\mathrm{e}}\mathrm{t}}$. We claim that it is continuous. Indeed, it suffices to show that so are the induced maps $\Psi_{i,n} : \mathcal{H}^{i+1} \rightarrow \mathrm{R}\Gamma(\Pi(n), \mathbf{Z}_p(i))[i]$, for $n \geq 1$. Or, by continuity of the cup product that so are the maps $\Psi_{1,n}$. Or, simplifying further, that so are the maps

$$(4.9) \quad \Psi_{1,n} : \mathcal{H}^2 \rightarrow \mathcal{O}(X)^* \xrightarrow{r_{\acute{\mathrm{e}}\mathrm{t}}} \mathrm{Hom}(\Pi(n), \mathbf{Z}_p(1)).$$

To show this, write $\mathcal{H} = \varprojlim_m \mathcal{H}_m$, where \mathcal{H}_m is the set of m -equivalence classes of K -rational hyperplanes⁸ and set $M_n := \mathrm{Hom}(\Pi(n), \mathbf{Z}_p(1))$. It suffices to show that, for each $k \geq 1$, there is an m such that the map

$$\Psi_{1,n,k} : \mathcal{H}^2 \xrightarrow{\Psi_1} M_n \rightarrow M_n/p^k M_n$$

factors through the projection $\mathcal{H}^2 \rightarrow \mathcal{H}_m^2$. Taking into account the construction of $\Psi_{1,n}$, it suffices to show that, for m large enough, if two hyperplanes H_0, H_1 are m -equivalent, then $r_{\acute{\mathrm{e}}\mathrm{t}}(\ell_{H_1}/\ell_{H_0}) \in p^k M_n$. But this is clear, since in this case ℓ_{H_1}/ℓ_{H_0} has a $p^{r_n m}$ 'th root in $\mathcal{O}(U_n)^*$, for some constant $r_n > 0$ depending only on U_n , and since $r_{\acute{\mathrm{e}}\mathrm{t}}$ is a homomorphism, we have $r_{\acute{\mathrm{e}}\mathrm{t}}(\ell_{H_1}/\ell_{H_0}) \in p^{r_n m} M_n$.

Since $\mathrm{Hom}(\Pi(n), \mathbf{Z}_p(1))$ is a Banach space and the map $\Psi_{1,n}$ from (4.8) is continuous on \mathcal{H}^{i+1} , it defines, by integration against distributions, a continuous map

$$r_{\acute{\mathrm{e}}\mathrm{t},n} : D(\mathcal{H}^{i+1}, \mathbf{Z}_p) \rightarrow \mathrm{R}\Gamma_{\acute{\mathrm{e}}\mathrm{t}}(U_n, \mathbf{Z}_p(i))[i]$$

such that $r_{\acute{\mathrm{e}}\mathrm{t},n}(\delta_{(H_0, \dots, H_i)}) = \psi_{\acute{\mathrm{e}}\mathrm{t},n}(H_0, \dots, H_i)$, for all $H_0, \dots, H_i \in \mathcal{H}$, where $\psi_{\acute{\mathrm{e}}\mathrm{t},n}$ is the analog of $\psi_{\acute{\mathrm{e}}\mathrm{t}}$ for U_n . The construction being compatible with the change of n we get the existence of the map in (1) by setting $r_{\acute{\mathrm{e}}\mathrm{t}} := \varprojlim_n r_{\acute{\mathrm{e}}\mathrm{t},n}$ and passing to cohomology.

For (2) we need to check the factorization of the regulator $r_{\acute{\mathrm{e}}\mathrm{t}}$ from (1) through the quotient by the degenerate distributions. That is, we need to show that, for any $\mu \in D(\mathcal{H}^{i+1}, \mathbf{Z}_p)_{\mathrm{deg}}$, we have $r_{\acute{\mathrm{e}}\mathrm{t}}(\mu) = 0$. For that, by the construction of $r_{\acute{\mathrm{e}}\mathrm{t}}(\mu)$, it suffices to check that:

(1) for all $H_0, \dots, H_{i+1} \in \mathcal{H}$, we have

$$(4.10) \quad \psi_{\acute{\mathrm{e}}\mathrm{t}}(H_1, \dots, H_{i+1}) - \psi_{\acute{\mathrm{e}}\mathrm{t}}(H_0, H_2, \dots, H_{i+1}) + \dots + (-1)^{i+1} \psi_{\acute{\mathrm{e}}\mathrm{t}}(H_0, \dots, H_i) = 0$$

(2) and, if the ℓ_{H_j} , $0 \leq j \leq i$, are linearly dependent, then $\psi_{\acute{\mathrm{e}}\mathrm{t}}(H_0, \dots, H_i) = 0$.

⁸Recall that two hyperplanes H_1, H_2 are called m -equivalent (i.e., $[H_1] = [H_2] \in \mathcal{H}_m$) if they have unimodular equations ℓ_1, ℓ_2 such that $\ell_1 = \ell_2$ modulo ϖ^m , where ϖ is a uniformizer of K .

To see (1) note that (a) if one permutes H_j with H_{j+1} in the expression in (4.10) the latter will change its sign, (b) the coefficient of $r_{\text{ét}}(\delta_{H_1, \dots, H_{i+1}})$ in the development of the expression in (4.10) is zero.

(2) follows from the fact that the étale regulator satisfies the Steinberg relations. More precisely, if $x_j = \ell_j/\ell_0$, $0 \leq j \leq i$, where ℓ_0, \dots, ℓ_i are linear equations of K -rational hyperplanes, it suffices to show that the symbol

$$\{x_1, \dots, x_i, 1 + a_1x_1 + \dots + a_ix_i\} = 0, \quad a_j \in K,$$

in the Milnor K -theory group $K_{i+1}^M(\mathcal{O}(X)^*)$. Note that the symbol $\{x_1, \dots, x_i, 1\} = 0$. We will reduce to this case by the following algorithm. Step 1: up to reordering we may assume that $y_1 := (1 + a_1x_1) \neq 0$ (otherwise we are done). Then, using the Steinberg relations $\{z, 1 - z\} = 0$ and the fact that $\{x, a\} = 0$, for $a \in K^*$, we compute

$$\{x_1, x_2, \dots, x_i, 1 + a_1x_1 + \dots + a_ix_i\} = \{x_1, x_2/y_1, \dots, x_i/y_1, 1 + a_2x_2/y_1 + \dots + a_ix_i/y_1\}.$$

Note that this makes sense since $x_j/y_1 \in \mathcal{O}(X)^*$. In fact, $x_j/y_1 = \ell_j/(\ell_0 + a_1\ell_1)$ is again a quotient of two linear equations of K -hyperplanes. Step 2: reorder the terms in the last symbol to make x_2/y_1 appear first and repeat. □

4.3.2. *Integrating A_{inf} -symbols.* Let $i \geq 1$. We pass now to the \mathbf{A}_{inf} -regulator map

$$r_{\text{inf}} : A_{\text{inf}} \widehat{\otimes}_{\mathbf{Z}_p} \text{Sp}_i(\mathbf{Z}_p)^* \rightarrow H_{\text{ét}}^i(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\})$$

that is compatible with the classical A_{inf} -regulator as well as with the étale regulator

$$r_{\text{ét}} : \text{Sp}_i(\mathbf{Z}_p)^* \rightarrow H_{\text{ét}}^i(X, \mathbf{Z}_p(i))$$

defined above. To start, we define the regulators

$$r_{\text{inf}} : D(\mathcal{H}^{i+1}, \mathbf{Z}_p) \rightarrow H_{\text{ét}}^i(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\}), \quad r_{\text{inf}} : \text{Sp}_i(\mathbf{Z}_p)^* \rightarrow H_{\text{ét}}^i(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\})$$

by setting $r_{\text{inf}} := \gamma r_{\text{ét}}$, where $\gamma : H_{\text{ét}}^i(X, \mathbf{Z}_p(i)) \rightarrow H_{\text{ét}}^i(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\})$ is the canonical map and the étale regulator

$$(4.11) \quad r_{\text{ét}} : D(\mathcal{H}^{i+1}, \mathbf{Z}_p) \rightarrow H_{\text{ét}}^i(X, \mathbf{Z}_p(i))$$

is the map defined above.

Corollary 4.12. *Let $i \geq 1$. The above regulators extend uniquely to compatible continuous A_{inf} -linear maps*

$$r_{\text{inf}} : A_{\text{inf}} \widehat{\otimes}_{\mathbf{Z}_p} D(\mathcal{H}^{i+1}, \mathbf{Z}_p) \rightarrow H_{\text{ét}}^i(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\}), \quad r_{\text{inf}} : A_{\text{inf}} \widehat{\otimes}_{\mathbf{Z}_p} \text{Sp}_i(\mathbf{Z}_p)^* \rightarrow H_{\text{ét}}^i(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\})$$

that are compatible with the étale regulators.

Proof. Uniqueness is clear. To show the existence, let $\{U_n\}_{n \in \mathbf{N}}$ be the standard admissible affinoid covering of X . For $n \in \mathbf{N}$, set

$$r_{\text{inf},n} : D(\mathcal{H}^{i+1}, \mathbf{Z}_p) \rightarrow \text{R}\Gamma_{\text{ét}}(\mathcal{U}_n, A\Omega_{\mathcal{U}_n}\{i\})[i], \quad r_{\text{inf},n} := \gamma r_{\text{ét},n},$$

where \mathcal{U}_n is the standard semistable formal model of U_n and the map

$$r_{\text{ét},n} : D(\mathcal{H}^{i+1}, \mathbf{Z}_p) \rightarrow \text{R}\Gamma_{\text{ét}}(U_n, \mathbf{Z}_p(i))[i]$$

was constructed above. The map $r_{\text{inf},n}$ factors as

$$r_{\text{inf},n} : D(\mathcal{H}^{i+1}, \mathbf{Z}_p) \rightarrow M_n \rightarrow \text{R}\Gamma_{\text{ét}}(\mathcal{U}_n, A\Omega_{\mathcal{U}_n}\{i\})[i],$$

where $M_n := \text{Hom}(\Pi(n), \mathbf{Z}_p(i))$ for the fundamental group $\Pi(n)$ of U_n .

Since $r_{\text{ét}} = \text{holim}_n r_{\text{ét},n}$, we have the factorization

$$r_{\text{ét}} : D(\mathcal{H}^{i+1}, \mathbf{Z}_p) \rightarrow \varprojlim_n M_n \xrightarrow{\text{holim}_n r_{\text{ét},n}} \text{holim}_n \text{R}\Gamma_{\text{ét}}(U_n, \mathbf{Z}_p(i))[i] \simeq \text{R}\Gamma_{\text{ét}}(X, \mathbf{Z}_p(i))[i].$$

This induces the following factorization

$$\begin{aligned} r_{\text{inf}} = \text{holim}_n r_{\text{inf},n} : A_{\text{inf}} \widehat{\otimes}_{\mathbf{Z}_p} D(\mathcal{H}^{i+1}, \mathbf{Z}_p) &\rightarrow A_{\text{inf}} \widehat{\otimes}_{\mathbf{Z}_p} \varprojlim_n M_n \xrightarrow{\sim} \varprojlim_n A_{\text{inf}} \widehat{\otimes}_{\mathbf{Z}_p} M_n \\ &\rightarrow \text{holim}_n \text{R}\Gamma_{\text{ét}}(\mathcal{U}_n, A\Omega_{\mathfrak{X}}\{i\})[i] \xleftarrow{\sim} \text{R}\Gamma_{\text{ét}}(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\})[i]. \end{aligned}$$

The existence of the first map is clear and so is the following isomorphism. The third map exists because both $A_{\text{inf}} \widehat{\otimes}_{\mathbf{Z}_p} M_n$ and $\text{R}\Gamma_{\text{ét}}(\mathcal{U}_n, A\Omega_{\mathfrak{X}}\{i\})$ are derived (p, μ) -adically complete. This proves the existence of the first regulator in the corollary. The existence of the second follows immediately from the fact that the map (4.11) factors through $\text{Sp}_i(\mathbf{Z}_p)^*$ once we know that the sequence

$$0 \rightarrow A_{\text{inf}} \widehat{\otimes}_{\mathbf{Z}_p} D(\mathcal{H}^{i+1}, \mathbf{Z}_p)_{\text{deg}} \rightarrow A_{\text{inf}} \widehat{\otimes}_{\mathbf{Z}_p} D(\mathcal{H}^{i+1}, \mathbf{Z}_p) \rightarrow A_{\text{inf}} \widehat{\otimes}_{\mathbf{Z}_p} \text{Sp}_i(\mathbf{Z}_p)^* \rightarrow 0$$

is strict exact. This sequence is obtained from the strict exact sequence (4.3) by tensoring with A_{inf} . Hence the only question is the strict surjection on the right, which follows from the fact that the sequence (4.3) is actually split (since all modules are duals of free modules). \square

4.4. The A_{inf} -cohomology of Drinfeld symmetric spaces. We are now ready to prove Theorem 4.1. Let $i \geq 0$. We will show that the map r_{inf} induces a φ^{-1} -equivariant topological isomorphism of A_{inf} -modules

$$r_{\text{inf}} : A_{\text{inf}} \widehat{\otimes}_{\mathbf{Z}_p} \text{Sp}_i(\mathbf{Z}_p)^* \xrightarrow{\sim} H_{\text{ét}}^i(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\}).$$

Compatibility with the operator φ^{-1} follows from the fact that r_{inf} is A_{inf} -linear and it is induced from $r_{\text{ét}}$ hence maps $D(\mathcal{H}^{i+1}, \mathbf{Z}_p)$ to $H_{\text{ét}}^i(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\})^{\varphi^{-1}=1}$. For the rest of the claim, first, we show that the induced map

$$\bar{r}_{\text{inf}} : (A_{\text{inf}} \widehat{\otimes}_{\mathbf{Z}_p} \text{Sp}_i(\mathbf{Z}_p)^*) / \tilde{\xi} \rightarrow H_{\text{ét}}^i(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\}) / \tilde{\xi}$$

is a topological isomorphism. But this map fits into the following commutative diagram (of A_{inf} -linear continuous maps, where A_{inf} acts on \mathcal{O}_C via $\tilde{\theta}$)

$$\begin{array}{ccc} (A_{\text{inf}} \widehat{\otimes}_{\mathbf{Z}_p} \text{Sp}_i(\mathbf{Z}_p)^*) / \tilde{\xi} & \xrightarrow{\bar{r}_{\text{inf}}} & H_{\text{ét}}^i(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\}) / \tilde{\xi} \\ \downarrow \tilde{\theta} & & \downarrow \alpha \\ \mathcal{O}_C \widehat{\otimes}_{\mathbf{Z}_p} \text{Sp}_i(\mathbf{Z}_p)^* & & H_{\text{ét}}^i(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\}) / \tilde{\xi} \\ & \searrow \tilde{r}_{\text{HT}} & \downarrow \wr \\ & & H_{\text{ét}}^0(\mathfrak{X}, \Omega_{\mathfrak{X}}^i) \end{array} \quad \begin{array}{l} \curvearrowright \\ \tilde{r}_{\text{HK}} \end{array}$$

The map α is the change-of-coefficients map; it is clearly injective. The right vertical map is an isomorphism because we have the local-global spectral sequence

$$E_2^{s,t} = H_{\text{ét}}^s(\mathfrak{X}, H^t(A\Omega_{\mathfrak{X}}\{i\}/\tilde{\xi})) \Rightarrow H_{\text{ét}}^{s+t}(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\}/\tilde{\xi})$$

and, by Theorem 2.8 and Theorem 2.5, the isomorphisms $H^t(A\Omega_{\mathfrak{X}}\{i\}/\tilde{\xi}) \simeq H^t(\tilde{\Omega}_{\mathfrak{X}}\{i\}) \simeq \Omega_{\mathfrak{X}}^t\{i-t\}$. Hence, by Theorem 4.4,

$$E_2^{s,t} = H_{\text{ét}}^s(\mathfrak{X}, \Omega^t\{i-t\}) = 0, \quad s \geq 1.$$

The above diagram commutes by Proposition 3.3. The slanted arrow is a topological isomorphism by Corollary 4.6. It follows that the map α is surjective, hence it is an isomorphism and so is, by the above diagram, the map \bar{r}_{inf} . The latter is also a topological isomorphism because so is the map $\tilde{\theta}$ and the map \tilde{r}_{HK} is a continuous isomorphism.

Next, we will show that \bar{r}_{inf} being a topological isomorphism implies that so is the original map r_{inf} . Let T be the homotopy fiber of r_{inf} . We claim that the complex

$$(4.13) \quad T \otimes_{A_{\text{inf}}}^L A_{\text{inf}}/\tilde{\xi} \simeq 0.$$

Indeed, since \bar{r}_{inf} is an isomorphism, it suffices to show that the domain and the target of r_{inf} are $\tilde{\xi}$ -torsion free. This is clear for the domain. For the target, note that the exact triangle

$$A\Omega_{\mathfrak{X}}\{i\} \xrightarrow{\tilde{\xi}} A\Omega_{\mathfrak{X}}\{i\} \rightarrow A\Omega_{\mathfrak{X}}\{i\}/\tilde{\xi}$$

yields an exact sequence

$$0 \rightarrow H_{\text{ét}}^i(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\})/\tilde{\xi} \xrightarrow{\alpha} H_{\text{ét}}^i(\mathfrak{X}, A\Omega_{\mathfrak{X}}/\tilde{\xi}) \rightarrow H_{\text{ét}}^{i+1}(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\})[\tilde{\xi}] \rightarrow 0.$$

By the above, the first map is an isomorphism, hence $H_{\text{ét}}^{i+1}(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\})[\tilde{\xi}] = 0$. Since $i \geq 0$ was arbitrary, we deduce that, for all $j \geq 1$ and all i , $H_{\text{ét}}^j(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\})$ has no $\tilde{\xi}$ -torsion, and this is clearly true for $j = 0$ as well.

Since T is derived $\tilde{\xi}$ -complete (because so are the domain and the target of r_{inf} , the latter using the derived $\tilde{\xi}$ -completeness of $A\Omega_{\mathfrak{X}}$ and the preservation of this property by derived pushforward and passage to cohomology), by the derived Nakayama Lemma (see Section 2.1.1) we have $T \simeq 0$ as well. This finishes the proof that r_{inf} is an isomorphism.

Since the domain and the target of r_{inf} are $\tilde{\xi}$ -torsion-free and the reduction \bar{r}_{inf} is a topological isomorphism so is r_{inf} . This finishes the proof.

5. INTEGRAL p -ADIC ÉTALE COHOMOLOGY OF DRINFELD SYMMETRIC SPACES

We are now ready to compute the étale cohomology. Let $X_K := \mathbb{H}_K^d$ be the Drinfeld symmetric space of dimension d over K and let $\mathfrak{X}_{\mathcal{O}_K}$ be its standard semistable formal model over \mathcal{O}_K . Let $X := X_K \times_K C$.

Theorem 5.1. *Let $i \geq 0$.*

- (1) *There is a $G \times \mathcal{G}_K$ -equivariant topological isomorphism*

$$r_{\text{ét}} : \text{Sp}_i(\mathbf{Z}_p)^* \xrightarrow{\sim} H_{\text{ét}}^i(X, \mathbf{Z}_p(i)).$$

It is compatible with the rational isomorphism $r_{\text{ét}} : \text{Sp}_i(\mathbf{Z}_p)^ \otimes \mathbf{Q}_p \xrightarrow{\sim} H_{\text{ét}}^i(X, \mathbf{Q}_p(i))$ from [9].*

- (2) *There is a $G \times \mathcal{G}_K$ -equivariant topological isomorphism*

$$\bar{r}_{\text{ét}} : \text{Sp}_i(\mathbf{F}_p)^* \xrightarrow{\sim} H_{\text{ét}}^i(X, \mathbf{F}_p(i)).$$

Proof. Set $\mathfrak{X} := \mathfrak{X}_{\mathcal{O}_K}$. For $i \geq 0$, using the natural isomorphism $H_{\text{ét}}^i(X, \mathbf{Z}_p(i)) \xrightarrow{\sim} H_{\text{proét}}^i(X, \widehat{\mathbf{Z}}_p(i))$ [9, proof of Cor. 3.46], we pass to pro-étale cohomology. Now, by Corollary 2.13, we have a natural short exact sequence

$$(5.2) \quad 0 \rightarrow H_{\text{ét}}^{i-1}(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\})/(1 - \varphi^{-1}) \rightarrow H_{\text{proét}}^i(X, \widehat{\mathbf{Z}}_p(i)) \rightarrow H_{\text{ét}}^i(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\})^{\varphi^{-1}=1} \rightarrow 0.$$

By Theorem 4.1, we have a topological isomorphism $H_{\text{ét}}^i(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\}) \simeq A_{\text{inf}} \widehat{\otimes}_{\mathbf{Z}_p} \text{Sp}_i(\mathbf{Z}_p)^*$ and this isomorphism is compatible with the action of φ^{-1} . We get topological isomorphisms

$$H_{\text{ét}}^i(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\})^{\varphi^{-1}=1} \simeq (A_{\text{inf}} \widehat{\otimes}_{\mathbf{Z}_p} \text{Sp}_i(\mathbf{Z}_p)^*)^{\varphi^{-1}=1} \simeq A_{\text{inf}}^{\varphi^{-1}=1} \widehat{\otimes}_{\mathbf{Z}_p} \text{Sp}_i(\mathbf{Z}_p)^* \simeq \text{Sp}_i(\mathbf{Z}_p)^*,$$

$$H_{\text{ét}}^{i-1}(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\})/(1 - \varphi^{-1}) \simeq (A_{\text{inf}} \widehat{\otimes}_{\mathbf{Z}_p} \text{Sp}_{i-1}(\mathbf{Z}_p)^*)/(1 - \varphi^{-1}) \simeq (A_{\text{inf}}/(1 - \varphi^{-1})) \widehat{\otimes}_{\mathbf{Z}_p} \text{Sp}_i(\mathbf{Z}_p)^* \simeq 0.$$

Hence, by the exact sequence (5.2), we get a natural continuous isomorphism $r_{\text{proét}} : \text{Sp}_i(\mathbf{Z}_p)^* \xrightarrow{\sim} H_{\text{proét}}^i(X, \widehat{\mathbf{Z}}_p(i))$. Since its composition with the natural map $H_{\text{proét}}^i(X, \widehat{\mathbf{Z}}_p(i)) \xrightarrow{\sim} H_{\text{ét}}^i(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\})^{\varphi^{-1}=1}$ is a topological isomorphism so is the map $r_{\text{proét}}$ itself, as wanted in claim (1).

The last sentence of claim (1) of the theorem is clear.

For claim (2), we define the regulator $\bar{r}_{\text{ét}}$ in an analogous way to its integral version $r_{\text{ét}}$ (with which it is compatible by construction). Since $\text{Sp}_i(\mathbf{F}_p)^* \simeq \text{Sp}_i(\mathbf{Z}_p)^* \otimes \mathbf{F}_p$ and $H_{\text{ét}}^i(X, \mathbf{F}_p(i)) \simeq H_{\text{ét}}^i(X, \mathbf{Z}_p(i)) \otimes \mathbf{F}_p$

(the latter isomorphism by claim (1)), we have $\bar{r}_{\text{ét}} \simeq r_{\text{ét}} \otimes \text{Id}_{\mathbf{F}_p}$. Hence, by claim (1), $\bar{r}_{\text{ét}}$ is an isomorphism, as wanted. \square

REFERENCES

- [1] V. Berkovich, *Smooth p -adic analytic spaces are locally contractible*. Invent. Math. 137 (1999), 1–84.
- [2] V. Berkovich, *Complex analytic vanishing cycles for formal schemes*, preprint.
- [3] S. Bloch, K. Kato, *p -adic étale cohomology*, Inst. Hautes Études Sci. Publ. Math. 63 (1986), 107–152.
- [4] B. Bhatt, *Specializing varieties and their cohomology from characteristic 0 to characteristic p* , Algebraic geometry: Salt Lake City 2015, 43–88, Proc. Sympos. Pure Math., 97.2, Amer. Math. Soc., Providence, RI, 2018.
- [5] B. Bhatt, M. Morrow, P. Scholze, *Integral p -adic Hodge Theory*, Inst. Hautes Études Sci. Publ. Math. 128 (2018), 219–397.
- [6] B. Bhatt, M. Morrow, P. Scholze, *Topological Hochschild homology and integral p -adic Hodge theory*, Inst. Hautes Études Sci. Publ. Math. 129 (2019), 199–310.
- [7] K. Česnavičius, T. Koshikawa, *The A_{inf} -cohomology in the semistable case*, arXiv:1710.06145v2 [math.NT].
- [8] P. Colmez, G. Dospinescu, W. Niziol, *Cohomologie p -adique de la tour de Drinfeld: le cas de la dimension 1*, arXiv:arXiv:1704.08928 [math.NT].
- [9] P. Colmez, G. Dospinescu, W. Niziol, *Cohomology of p -adic Stein spaces*, arXiv:1801.06686 [math.NT].
- [10] V. Drinfeld, *Elliptic modules*, Math. Sb. 94 (1974), 594–627.
- [11] J.-M. Fontaine, *Formes différentielles et modules de Tate des variétés abéliennes sur les corps locaux*. Invent. Math. 65 (1981/82), no. 3, 379–409.
- [12] J. Fresnel, M. van der Put, *Rigid analytic geometry and its applications*. Progr. Math. 218, Birkhäuser, 2004.
- [13] E. Grosse-Klönne, *Integral structures in the p -adic holomorphic discrete series*. Represent. Theory 9 (2005), 354–384.
- [14] E. Grosse-Klönne, *Frobenius and monodromy operators in rigid analysis, and Drinfeld’s symmetric space*. J. Algebraic Geom. 14 (2005), 391–437.
- [15] E. Grosse-Klönne, *Sheaves of bounded p -adic logarithmic differential forms*. Ann. Sci. École Norm. Sup. 40 (2007), 351–386.
- [16] E. Grosse-Klönne, *On special representations of p -adic reductive groups*. Duke Math. J. 163 (2014), 2179–2216.
- [17] A. Iovita, M. Spiess, *Logarithmic differential forms on p -adic symmetric spaces*. Duke Math. J. 110 (2001), 253–278.
- [18] M. Morrow, *p -adic vanishing cycles as Frobenius-fixed points*, arXiv:1802.03317 [math.AG].
- [19] P. Schneider, U. Stuhler, *The cohomology of p -adic symmetric spaces*. Invent. Math. 105 (1991), 47–122.
- [20] P. Scholze, *p -adic Hodge theory for rigid-analytic varieties*. Forum Math. Pi 1 (2013), e1, 77 pp.
- [21] P. Scholze, *Perfectoid spaces: a survey. Current developments in mathematics 2012*, 193–227, Int. Press, Somerville, MA, 2013.
- [22] The Stacks Project Authors, Stacks Project, <http://stacks.math.columbia.edu>.

CNRS, IMJ-PRG, SORBONNE UNIVERSITÉ, 4 PLACE JUSSIEU, 75005 PARIS, FRANCE

E-mail address: pierre.colmez@imj-prg.fr

CNRS, UMPA, ÉCOLE NORMALE SUPÉRIEURE DE LYON, 46 ALLÉE D’ITALIE, 69007 LYON, FRANCE

E-mail address: gabriel.dospinescu@ens-lyon.fr

CNRS, UMPA, ÉCOLE NORMALE SUPÉRIEURE DE LYON, 46 ALLÉE D’ITALIE, 69007 LYON, FRANCE

E-mail address: wieslawa.niziol@ens-lyon.fr