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Calculating the Euler characteristic of the moduli space of curves

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Abstract

The orbifold Euler characteristic of the moduli space $\mathcal{M}_{g,1}$ of genus g smooth curves with one marked point ($g \geq 1$) was calculated by Harer and Zagier: $\chi(\mathcal{M}_{g,1}) = \zeta(1-2g) = -B_{2g}/(2g)$, where ζ is the Riemann zeta function and B_{2g} is the $(2g)$ th Bernoulli number. We want to make their proof shorter and more combinatorial.

1 Introduction

Harer and Zagier [15] have shown that calculating the orbifold Euler characteristic of the moduli space $\mathcal{M}_{g,1}$ of genus g smooth curves with one marked point is a purely combinatorial problem:

Let us consider a (convex regular) $(2m)$ -gon ($m \geq 1$), orient each of its $2m$ edges counter-clockwise and label them from 1 to $2m$ following the orientation. There are

$$(2m-1)!! = (2m-1) \cdot (2m-3) \cdots 5 \cdot 3 \cdot 1 = \frac{(2m)!}{2^m m!} \quad (1.1)$$

perfect matchings on a set of $2m$ elements providing $(2m-1)!!$ possible gluings in pairs of the oriented sides of our $(2m)$ -gon, such that two oppositely oriented arcs become an unoriented edge. Of course the gluing of two oppositely oriented arcs implies the identification of their endpoints reducing the number of vertices from $2m$ to some integer v . The embedded graph (or Riemann surface) obtained in this way has one face (the $2m$ -gon), m edges, v vertices and therefore the Euler characteristic g determined by

$$v - m + 1 = 2 - 2g \quad \Leftrightarrow \quad v + 2g = m + 1. \quad (1.2)$$

Of course,

$$g \geq 0 \Leftrightarrow v \leq m + 1 \quad \text{and} \quad v \geq 1 \Leftrightarrow 2g \leq m. \quad (1.3)$$

We suppose that among our $(2m-1)!!$ possible identifications $\varepsilon_g(m)$ give an embedded graph of genus g , whereas $\lambda_g(m)$ of those embedded graphs of genus g do not contain any vertex of degree 1 or 2. If every vertex degree has to be at least 3, our identity $3v + 6g = 3m + 3$ implies the additional inequality

$$3v \leq 2m \Leftrightarrow m \leq 6g - 3. \quad (1.4)$$

Harer and Zagier [15] have shown that the orbifold Euler characteristic of the moduli space $\mathcal{M}_{g,1}$ of genus g smooth curves with one marked point ($g \geq 1$) is equal to

$$\chi(\mathcal{M}_{g,1}) = \sum_{m=2g}^{6g-3} (-1)^{m-1} \frac{\lambda_g(m)}{2m}. \quad (1.5)$$

This theorem can be proved using the isomorphism, as real orbifolds, of the decorated moduli space $\mathcal{M}_{g;1} \times \mathbb{R}_+$ and the combinatorial moduli space $\mathcal{M}_{g,1}^{comb}$ whose m -dimensional part consists of the embedded graphs counted by $\lambda_g(m)$ (in order to calculate the orbifold Euler characteristic we have to divide by $2m$, the number of automorphisms of our $(2m)$ -gons, and to multiply by $(-1)^{codim} = (-1)^{(6g-3)-m} = (-1)^{m-1}$). Indeed, if we fix a positive real number (which will be the perimeter of our $(2m)$ -gon) for our marked point, then there is a unique quadratic differential (Strebel differential), and we can look at its critical graph. It is embedded, its edges have fixed lengths, and its vertices are the zeroes of the differential, where a zero of order i gives a vertex of degree $i+2 \geq 3$ (see [20, 22, 36] for more details). In particular, the real dimension of $\mathcal{M}_{g,1}^{comb}$ and $\mathcal{M}_{g;1} \times \mathbb{R}_+$ is $6g - 3$ giving a complex dimension of $3g - 2$ for $\mathcal{M}_{g;1}$. If $g = 1$, the identity (1.5) implies

$$\chi(\mathcal{M}_{1;1}) = -\frac{\lambda_1(2)}{4} + \frac{\lambda_1(3)}{6} = -\frac{1}{4} + \frac{1}{6} = -\frac{1}{12}, \quad (1.6)$$

because for a 4-gon and a 6-gon, it is necessary to glue each side with its opposite one in order to avoid the creation of vertices of degree 1 or 2, and in this case we always get $g = 1$. For $g \geq 2$, however, calculating $\lambda_g(m)$ becomes more complicated, a problem which Harer and Zagier [15] solved in three steps: First, they proved recurrence relations allowing to calculate the numbers $\lambda_g(m)$ with the help of the numbers $\varepsilon_g(m)$ (not involving restrictions on vertex degrees). Then they established the following beautiful formula.

Theorem 1. (Harer-Zagier formula) *For fixed $m \in \mathbb{N}^*$ and for every $N \in \mathbb{N}^*$ we have*

$$\sum_{g=0}^{\lfloor m/2 \rfloor} \varepsilon_g(m) \cdot N^{m+1-2g} = \sum_{n=1}^N \binom{N}{n} \cdot (2m-1)!! \cdot 2^{n-1} \cdot \binom{m}{n-1}. \quad (1.7)$$

And finally, they put everything together to obtain their main result.

Theorem 2. *The orbifold Euler characteristic of the moduli space $\mathcal{M}_{g;1}$ of genus g smooth curves with 1 marked point ($g \geq 1$) equals*

$$\chi(\mathcal{M}_{g;1}) = \zeta(1-2g) = -\frac{B_{2g}}{2g}, \quad (1.8)$$

where ζ is the Riemann zeta function and B_{2g} is the $(2g)$ th Bernoulli number, i.e. the coefficient of z^{2g} in the expansion

$$\frac{z}{e^z - 1} = 1 - \frac{1}{2}z + B_2 \frac{z^2}{2!} + B_4 \frac{z^4}{4!} + B_6 \frac{z^6}{6!} + B_8 \frac{z^8}{8!} + B_{10} \frac{z^{10}}{10!} + B_{12} \frac{z^{12}}{12!} + B_{14} \frac{z^{14}}{14!} \dots, \quad (1.9)$$

$$B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, B_8 = -\frac{1}{30}, B_{10} = \frac{5}{66}, B_{12} = -\frac{691}{2730}, B_{14} = \frac{7}{6}, \dots$$

The most difficult part of Harer and Zagier's approach was probably their proof of the Harer-Zagier formula (1.7). In particular, they used integration over a Gaussian set of random matrices. Itzykson and Zuber [16] have simplified this with the help of harmonic oscillators and the Baker-Campbell-Hausdorff formula. A further simplification can be obtained by using matching polynomials (see [40], VI-34, remark 21, or [11], chapter 1) allowing to solve the integral (3.5) of [16] directly. Matrix integration is maybe the most popular method for this problem, see [30, 18, 19, 42, 22, 14, 1], because those matrices can be diagonalized. As we have shown in [24],

however, it is possible to replace the integrals by differential operators. Usually, differentiation is easier than integration, and in this particular case, it is even possible to diagonalize the differential operators in a combinatorial way using the so called BEST theorem (for counting directed Eulerian tours), as shown in [23], where we have given the first combinatorial proof of the Harer-Zagier formula (1.7). Goulden and Nica's proof [12] of this formula also uses implicitly the BEST theorem, but finishes by counting planar arborescences, not arborescences with marked vertices as [23].

Another technique useful for proving the Harer-Zagier formula (1.7) is representation theory, see [17, 31, 16, 41] and Zagier's appendix in [22]. This method, however, is based on the regular representation of the symmetric group S_n , in $n!$ dimensions instead of n dimensions. Therefore it is not necessarily the easiest approach.

Having calculated $\chi(\mathcal{M}_{g;1})$, the orbifold Euler characteristic of the moduli space $\mathcal{M}_{g;n}$ of genus g smooth curves with n marked points can be obtained using the recurrence relation

$$\chi(\mathcal{M}_{g;n+1}) = \chi(\mathcal{M}_{g;n})(2 - 2g - n) \quad (1.10)$$

for stable (g, n) . It follows from the facts that the forgetful mapping $\mathcal{M}_{g;n+1} \rightarrow \mathcal{M}_{g;n}$ is a fiber bundle whose fiber is a genus g curve punctured at n points, that the Euler characteristic of the fiber is $2 - 2g - n$ and that the Euler characteristic of the total space (of an orbifold fiber bundle) is the product of the Euler characteristics of the base and of the fiber. This recurrence relation was exploited by Kontsevich [20] (see also [30, 22]) in order to reduce the matrix integration to the one-dimensional case. Finally (1.8) follows from Stirling's formula for $\log \Gamma(t)$.

We want to give a short proof of (1.8) using only formal power series and classical combinatorics but no integration or representation theory. We hope that this will make combinatorial approaches even more popular than they are already [4, 5, 6, 7, 8, 9, 12, 13, 21, 26, 27, 28, 29, 32, 33, 37, 38, 39]. We start in the same way as Harer and Zagier: We color the v vertices of our embedded graph (obtained after gluing in pairs the oriented sides of our $(2m)$ -gon) in order to allow additional identifications among those vertices (vertices getting the same color). Harer and Zagier used N colors and had to evaluate (1.7):

$$\sum_{g=0}^{\lfloor m/2 \rfloor} \varepsilon_g(m) \cdot N^{m+1-2g} = \sum_{g=0}^{\lfloor m/2 \rfloor} \varepsilon_g(m) \cdot N^v. \quad (1.11)$$

Inspired by [35, 25], we use an infinite number of colors corresponding to an infinite number of variables x_1, x_2, x_3, \dots , and each variable appears to the power of the degree of the vertex colored with it. In other words, if our $(2m - 1)!!$ gluings lead to $\varepsilon_{d_1, d_2, d_3, \dots}(m)$ embedded graphs having d_1 vertices of degree 1, d_2 vertices of degree 2, d_3 vertices of degree 3, etc., then we have to evaluate

$$\sum_{d_1, d_2, d_3, \dots \geq 0} \varepsilon_{d_1, d_2, d_3, \dots}(m) \cdot p_1^{d_1} p_2^{d_2} p_3^{d_3} \dots, \quad (1.12)$$

where

$$p_1 = \sum_{i=1}^{\infty} x_i, \quad p_2 = \sum_{i=1}^{\infty} x_i^2, \quad p_3 = \sum_{i=1}^{\infty} x_i^3, \quad \dots \quad (1.13)$$

are the classical power symmetric functions. The elementary symmetric functions e_1, e_2, e_3, \dots can be defined by the identity

$$1 + \sum_{n=1}^{\infty} e_n t^n = \prod_{i=1}^{\infty} (1 + x_i t) \quad (1.14)$$

whereas the complete homogeneous symmetric functions h_1, h_2, h_3, \dots are given by

$$1 + \sum_{n=1}^{\infty} h_n t^n = \prod_{i=1}^{\infty} (1 + x_i t + x_i^2 t^2 + x_i^3 t^3 + \dots) = \prod_{i=1}^{\infty} \frac{1}{1 - x_i t}. \quad (1.15)$$

For every $m \geq 1$, h_m admits a natural decomposition

$$h_m = \sum_{n=1}^m h_{m,n}, \quad h_{m,1} = p_m, \quad h_{m,m} = e_m, \quad (1.16)$$

into parts $h_{m,n}$ using exactly n different variables (and thus interpolating between p_m and e_m). Their generating function is

$$\begin{aligned} 1 + \sum_{1 \leq n \leq m} h_{m,n} u^n t^m &= \prod_{i=1}^{\infty} (1 + x_i u t + x_i^2 u t^2 + x_i^3 u t^3 + \dots) = \prod_{i=1}^{\infty} \frac{1 - x_i(t - ut)}{1 - x_i t} \\ &= \exp \left[\sum_{i=1}^{\infty} \log(1 - x_i(t - ut)) - \log(1 - x_i t) \right] \\ &= \exp \left[\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_i^j \left(\frac{t^j}{j} - \frac{(t - ut)^j}{j} \right) \right] \\ &= \exp \left[\sum_{j=1}^{\infty} p_j \left(\frac{t^j}{j} - \frac{(t - ut)^j}{j} \right) \right] \end{aligned} \quad (1.17)$$

and allows to develop the symmetric functions $h_{m,n}$ in the basis of power symmetric functions p_j . We explain in the next section that our combinatorial proof [23] of the Harer-Zagier formula (1.7) also implies the following result.

Theorem 3. (Weighted Harer-Zagier formula) *For fixed $m \in \mathbb{N}^*$ we have*

$$\frac{1}{2m} \sum_{d_1, d_2, d_3, \dots \geq 0} \varepsilon_{d_1, d_2, d_3, \dots}(m) \cdot p_1^{d_1} p_2^{d_2} p_3^{d_3} \dots = \sum_{n=1}^{m+1} h_{2m,n} \cdot \frac{(2m-n)!}{2^{m+1-n} (m+1-n)!}. \quad (1.18)$$

Since 2001 we have presented this weighted version instead of the classical one in every seminar enamored of symmetric functions. Another proof for it can be found in [9]. In the last section we finally show how the weighted Harer-Zagier formula can be used to calculate the Euler characteristic of the moduli space of curves (1.8).

2 Proof of the weighted Harer-Zagier formula

Each coloration considered in the left hand side of (1.18) uses only a finite number n of different colors, and without loss of generality, we can consider that the set of those colors really used is $V = \{1, 2, 3, \dots, n\}$, the set of vertices of a multigraph (loops and multiple edges are allowed) with m unoriented edges each of which was obtained by gluing two oppositely oriented arcs of our $2m$ -gon (as suggested by Berge [3], preface). This $2m$ -gon defines an oriented Eulerian circuit of the graph. The division of the left hand side by $2m$ (the number of rotations of our

$2m$ -gon) shows that we should consider that the arc of our $2m$ -gon labeled 1 starts at our vertex labeled 1 and uses a fixed outgoing arc from it. For every $v \in V \setminus \{1\}$ let us consider the unique last arc of our Eulerian circuit (the arc with the highest label) leaving it. Those $b = n - 1$ arcs define an (oriented) arborescence

$$a : V \setminus \{1\} \rightarrow V \quad (2.1)$$

with root 1 (a has no cycles and iterating a leads to 1, see [10], page 61). Moreover, our oriented Eulerian circuit defines for every $v \in V$ an order among the other arcs (not belonging to the arborescence a) leaving v : the order in which it uses those arcs. On the other hand, given the arborescence a and, for every $v \in V$, the order of other departs, we uniquely reconstruct our Eulerian circuit (this is the content of the theorem called BEST, see [34], theorem 5.6.2 or [3], chapter 11.3, theorem 8).

Let us associate to each $v \in V$ the variable x_v (used in our symmetric function) and the additional variable y_v (used for counting purposes only). It is classical to associate to the arborescence $a : V \setminus \{1\} \rightarrow V$ the monomial

$$a(\mathbf{x}, \mathbf{y}) = \prod_{v \in V \setminus \{1\}} x_{a(v)} y_{a(v)}. \quad (2.2)$$

According to Prüfer and Foata, it is natural to code the function $a : V \setminus \{1\} \rightarrow V$ by the sequence of its $b = n - 1$ values $a(v_1), a(v_2), \dots, a(v_b)$. For v_1 we can take $v_1 = \min[V \setminus a(V \setminus \{1\})]$. Since $a' = a|_{V \setminus v_1}$ is still an arborescence, we can apply the same procedure to a' obtaining finally a bijection between arborescences $a : V \setminus \{1\} \rightarrow V$ and number sequences $a(v_1) \in V, a(v_2) \in V, \dots, a(v_{b-1}) \in V, a(v_b) = 1$ such that $a(\mathbf{x}, \mathbf{y}) = x_{a(v_1)} y_{a(v_1)} \cdots x_{a(v_b)} y_{a(v_b)}$. This proves Cayley's theorem on the weighted counting of arborescences (see [34], chapter 5.3):

$$\sum_{a: V \setminus \{1\} \rightarrow V} a(\mathbf{x}, \mathbf{y}) = (x_1 y_1 + \cdots + x_n y_n)^{b-1} \cdot x_1 y_1. \quad (2.3)$$

Among our m unoriented edges, the arborescence a uses (and distinguishes) $b = n - 1$. If they create at the vertex $v \in V$ the degree $d_B(v)$, then

$$\frac{a(\mathbf{x}, \mathbf{y})}{x_1 y_1} = \prod_{v \in V} (x_v y_v)^{d_B(v)-1}. \quad (2.4)$$

Let us code the $s = m - b$ supplementary undirected edges by a multiset $S : V \cup \binom{V}{2} \rightarrow \mathbb{N}$, where $S(v)$ denotes the number of loops at $v \in V$ and $S(\{u, v\})$ denotes the number of edges between the two distinct vertices $u, v \in V$. Of course,

$$|S| = \sum_{v \in V} S(v) + \sum_{\{u, v\} \in \binom{V}{2}} S(\{u, v\}) = s = m - b = m + 1 - n, \quad (2.5)$$

and the supplementary degrees for each $v \in V$ are

$$d_S(v) = 2 \cdot S(v) + \sum_{u \in V \setminus \{v\}} S(\{u, v\}). \quad (2.6)$$

The contribution for our symmetric function (1.18) is now

$$\prod_{v \in V} x_v^{d_B(v) + d_S(v)}, \quad (2.7)$$

but, since an Eulerian circuit is coded by its arborescence and, for every $v \in V$, the order of other departs, we have to multiply by

$$\prod_{v \in V} (d_B(v) - 1 + d_S(v))! = \sum_{\substack{i_1 + \dots + i_n = b-1+2s \\ i_1, \dots, i_n \geq 0}} \partial_{y_1}^{i_1} \dots \partial_{y_n}^{i_n} \prod_{v \in V} y_v^{d_B(v)-1+d_S(v)}. \quad (2.8)$$

Indeed, at every vertex different from 1 there is one edge of the arborescence which must not be taken into account in the order of other departs, and at the vertex 1, there is the edge 1 which must not be taken into account. Finally, the contribution of $V = \{1, \dots, n\}$ to (1.18) is

$$\prod_{v \in V} x_v \sum_{\substack{i_1 + \dots + i_n = b-1+2s \\ i_1, \dots, i_n \geq 0}} \partial_{y_1}^{i_1} \dots \partial_{y_n}^{i_n} \frac{a(\mathbf{x}, \mathbf{y})}{x_1 y_1} \prod_{v \in V} \frac{\left[\frac{(x_v y_v)^2}{2} \right]^{S(v)}}{S(v)!} \prod_{\{u, v\} \in \binom{V}{2}} \frac{[(x_u y_u)(x_v y_v)]^{S(\{u, v\})}}{S(\{u, v\})!} \quad (2.9)$$

summed over all arborescences $a : V \setminus \{1\} \rightarrow V$ and over all multisets $S : V \cup \binom{V}{2} \rightarrow \mathbb{N}$ with $|S| = s$ (the divisions by $S(\{u, v\})!$, by $S(v)!$ and by 2 reflect the facts that we must not distinguish Eulerian circuits obtained from each other by permutations of supplementary multiple edges, supplementary multiple loops or arcs within a loop). Using (2.3) and

$$\sum_{\substack{S: V \cup \binom{V}{2} \rightarrow \mathbb{N} \\ |S|=s}} \prod_{v \in V} \frac{\left[\frac{(x_v y_v)^2}{2} \right]^{S(v)}}{S(v)!} \prod_{\{u, v\} \in \binom{V}{2}} \frac{[(x_u y_u)(x_v y_v)]^{S(\{u, v\})}}{S(\{u, v\})!} = \frac{\left[\frac{(x_1 y_1 + \dots + x_n y_n)^2}{2} \right]^s}{s!} \quad (2.10)$$

the contribution of $V = \{1, \dots, n\}$ to (1.18) becomes

$$\begin{aligned} & \prod_{v \in V} x_v \sum_{\substack{i_1 + \dots + i_n = b-1+2s \\ i_1, \dots, i_n \geq 0}} \partial_{y_1}^{i_1} \dots \partial_{y_n}^{i_n} \frac{(x_1 y_1 + \dots + x_n y_n)^{b-1+2s}}{2^s s!} \\ &= \frac{(b-1+2s)!}{2^s s!} \sum_{\substack{i_1 + \dots + i_n = b-1+2s \\ i_1, \dots, i_n \geq 0}} x_1^{i_1+1} \dots x_n^{i_n+1}. \end{aligned} \quad (2.11)$$

This finishes the proof of (1.18) because $(i_1 + 1) + \dots + (i_n + 1) = n + b - 1 + 2s = 2(b + s) = 2m$, $b - 1 + 2s = 2m - n$ and $s = m - b = m + 1 - n$.

3 The Euler characteristic of the moduli space of curves

In order to establish (1.8) we have to prove that

$$\begin{aligned} \chi(z) &= \sum_{g=1}^{\infty} \chi(\mathcal{M}_{g;1}) \frac{z^{2g-1}}{(2g-1)!} = \sum_{g=1}^{\infty} -\frac{B_{2g}}{2g} \frac{z^{2g-1}}{(2g-1)!} \\ &= -\frac{1}{z} \sum_{k=2}^{\infty} B_k \frac{z^k}{k!} = -\frac{1}{z} \left[\frac{z}{e^z - 1} + \frac{1}{2}z - 1 \right] \\ &= -\frac{1}{2} + \frac{1}{z} + \frac{1}{1 - e^z}. \end{aligned} \quad (3.1)$$

Using (1.5) and (1.18) we know that $\chi(z)$ can be obtained from (1.17) if we replace p_1 and p_2 by 0 (in order to forbid vertices of degree 0 or 1) and p_j by x for every $j \geq 3$. This gives the expression

$$\exp \left[x \left(-\log(1-t) + \log(1-t+ut) - ut - ut^2 + \frac{u^2 t^2}{2} \right) \right], \quad (3.2)$$

where we have to replace each coefficient of $x^v u^n t^{2m}$ by

$$(-1)^{m-1} \frac{(2m-n)!}{2^{m+1-n} (m+1-n)!} \cdot \frac{z^{m-v}}{(m-v)!}, \quad (3.3)$$

because $m-v = 2g-1 \geq 1$. Therefore we have

$$\begin{aligned} \chi(z) &= \sum_{m=1}^{\infty} \sum_{n=0}^{m+1} \sum_{v=0}^m (-1)^{m-1} \frac{(2m-n)!}{2^{m+1-n} (m+1-n)!} \cdot \frac{z^{m-v}}{(m-v)!} \langle x^v u^n t^{2m} \rangle \\ &\quad \exp \left[x \left(-\log(1-t) + \log(1-t+ut) - ut - ut^2 + \frac{u^2 t^2}{2} \right) \right] \\ &= \sum_{m=1}^{\infty} \sum_{n=0}^{m+1} (-1)^{m-1} \frac{(2m-n)!}{2^{m+1-n} (m+1-n)!} \langle x^m u^n t^{2m} \rangle \\ &\quad \exp[xz] \cdot \exp \left[x \left(-\log(1-t) + \log(1-t+ut) - ut - ut^2 + \frac{u^2 t^2}{2} \right) \right] \\ &= -\frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=0}^{m+1} \binom{2m-n}{m-1} \frac{1}{m} \langle u^n t^{2m} \rangle \\ &\quad \left[-\frac{z}{2} + \frac{1}{2} \log(1-t) - \frac{1}{2} \log(1-t+2ut) + ut + ut^2 - u^2 t^2 \right]^m \\ &= -\frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=0}^{m+1} \binom{2m-n}{m-1} \frac{1}{m} \langle u^{2m-n} t^{2m} \rangle \\ &\quad \left[-\frac{z}{2} + \frac{1}{2} \log(1-ut) - \frac{1}{2} \log(1-ut+2t) + t + ut^2 - t^2 \right]^m \\ &= -\frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m} \langle u^{m-1} t^{2m} \rangle \\ &\quad \left[-\frac{z}{2} + \frac{1}{2} \log(1-(u+1)t) - \frac{1}{2} \log(1-(u+1)t+2t) + t + (u+1)t^2 - t^2 \right]^m \\ &= -\frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m} \langle u^{m-1} t^{m+1} \rangle \left[-\frac{z}{2} + \frac{1}{2} \log(1-u-t) - \frac{1}{2} \log(1-u+t) + t + ut \right]^m \\ &= -\frac{1}{2} \langle t \rangle \sum_{m=1}^{\infty} \frac{1}{m} \langle u^{-1} \rangle \left[\frac{ut}{-\frac{z}{2} + \frac{1}{2} \log(1-u-t) - \frac{1}{2} \log(1-u+t) + t + ut} \right]^{-m}. \quad (3.4) \end{aligned}$$

The formal power series in the variable u

$$f_{z,t}(u) = \frac{ut}{-\frac{z}{2} + \frac{1}{2} \log(1-u-t) - \frac{1}{2} \log(1-u+t) + t + ut} \quad (3.5)$$

whose coefficients are formal power series in z (with coefficients in the field of formal Laurent series in t) has valuation 1, i.e. $\langle u^0 \rangle f_{z,t}(u) = 0$ and $\langle u^1 \rangle f_{z,t}(u)$ is invertible. Therefore it admits a unique inverse $g_{z,t}(u)$ with respect to composition. Indeed, every formal power series $f(u)$ with $\langle u^0 \rangle f(u) = 0$ and $\langle u^1 \rangle f(u)$ invertible has a unique left inverse and a unique right inverse with respect to composition as can be seen by comparing coefficients or by using [2]. By associativity of composition, these inverses are identical, i.e. we have a unique $g(u)$ such that $f(g(u)) = g(f(u)) = u$. Moreover, it is classical from residue calculus and easy to see directly that for any formal Laurent series $l(u)$ and any formal power series $p(u)$ of valuation $d > 0$ (such that $\langle u^d \rangle p(u)$ is invertible) we have

$$d \cdot \langle u^{-1} \rangle l(u) = \langle u^{-1} \rangle [l(p(u)) \cdot p'(u)]. \quad (3.6)$$

Using these results we can continue our calculation:

$$\chi(z) = -\frac{1}{2} \langle t \rangle \sum_{m=1}^{\infty} \frac{1}{m} \langle u^{-1} \rangle u^{-m} \cdot \frac{\partial}{\partial u} g_{z,t}(u) = -\frac{1}{2} \langle t \rangle \sum_{m=1}^{\infty} \langle u^m \rangle g_{z,t}(u) = -\frac{1}{2} \langle t \rangle g_{z,t}(1). \quad (3.7)$$

But we know

$$\begin{aligned} f_{z,t}(g_{z,t}(1)) &= 1 \Leftrightarrow \\ tg_{z,t}(1) &= -\frac{z}{2} + \frac{1}{2} \log(1 - g_{z,t}(1) - t) - \frac{1}{2} \log(1 - g_{z,t}(1) + t) + t + tg_{z,t}(1) \Leftrightarrow \\ e^{z-2t} &= \frac{1 - g_{z,t}(1) - t}{1 - g_{z,t}(1) + t} \Leftrightarrow 1 - e^{z-2t} = \frac{2t}{1 - g_{z,t}(1) + t} \Leftrightarrow \\ g_{z,t}(1) &= 1 + t - \frac{2t}{1 - e^{z-2t}}. \end{aligned} \quad (3.8)$$

Therefore

$$\begin{aligned} \chi(z) &= -\frac{1}{2} + \langle t^0 \rangle \frac{1}{1 - e^{z-2t}} = -\frac{1}{2} + \langle t^0 \rangle \left[\frac{1}{z - 2t} + \frac{1}{1 - e^{z-2t}} \right] \\ &= -\frac{1}{2} + \frac{1}{z} + \frac{1}{1 - e^z}, \end{aligned} \quad (3.9)$$

because

$$\frac{1}{z - 2t} = -\frac{1}{2t} \cdot \frac{1}{1 - \frac{z}{2t}} = -\frac{1}{2t} - \frac{z}{4t^2} - \frac{z^2}{8t^3} - \dots \quad (3.10)$$

as a formal power series in z has no constant term in t , but

$$\frac{1}{z - 2t} + \frac{1}{1 - e^{z-2t}} \quad (3.11)$$

is a formal power series in both z and t , so its constant term in t is obtained by putting $t = 0$. This concludes our proof.

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